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Thomas-Fermi-Dirac Theory with Correlation Correction*

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Using the formula of the correlation energy due to Pines, we modify the Thomas-Fermi-Dirac model with the Fermi-Amaldi correction for free atom or ion to include correlations between electrons. By the use of the present model, electron distributions for free Rb^+ , Kr and Br^- are calculated, the results are shown graphically, and further the energy components of the total energy are given numerically.

§ 1. Introduction

In the theory of ionic crystals it becomes sometimes necessary for us to know the electronic density distribution in the constituent ions. We have the Hartree or Hartree-Fock solutions not for all the ions at present, therefore we must be contented with some approximate solutions for the electronic density distributions. In many approximate methods, the statistical one founded by Thomas¹⁾ and Fermi²⁾ and later modified by Dirac³⁾ is very useful, especially for the atoms or ions with large atomic number. The full account of this statistical method with its many applications is shown in the text book by Gombás^{4,5)} or the review article by March⁶⁾.

As is well known, the so-called Thomas-Fermi-Dirac model (abbreviated as TFD hereafter) has two main failures. First, the interactions between electrons are not correctly taken into account: A mean screening effect of the electron cloud for the potential field by the nucleus is corrected by the fact that the electrons with parallel spin have a tendency to keep away from each other due to their statistical nature, in other words, by including the exchange energy. However, it is not taken into account that whether the spins of the electrons being parallel or antiparallel, there is a tendency to keep away from each other by the Coulomb repulsion between them. The energy depression due to this tendency is well known as the correlation energy. Second, the electrons unavoidably interact with their own field in the formulation of the TFD theory.

Although the statistical model of the atom or ion is a very crude approximation in its character, it is naturally desirable to correct such fundamental failures in the theory. An approach to attack at the latter has been made by Fermi and Amaldi⁷⁾,

* The results were presented at the Hiroshima Meeting of the Chugoku-Shikoku Branch of the Physical Society of Japan, in February 1959.

and another to correct the former* by Gombás⁸⁾ using the expression for the correlation energy obtained by Wigner⁹⁾ and taking Fermi-Amaldi correction into account too.

Recently, the correlation energy for a free electron gas has been calculated by many investigators. Especially, Nozières and Pines¹⁰⁾ have discussed the limit of validity of the various correlation energy calculation and showed that the high density calculation of Gell-Mann and Brueckner¹¹⁾,

$$\varepsilon_c = -0.096 + 0.0622 \ln r_s \text{ (Ry)}, \quad (1.1)^{**}$$

is valid for $r_s \lesssim 1$, while the low density calculation of Wigner⁹⁾,

$$\varepsilon_c = -0.88/r_s \text{ (Ry)}, \quad (1.2)$$

is valid for $r_s \gtrsim 20$. Further for the region of actual metallic densities ($1.8 \lesssim r_s \lesssim 5.6$), the result obtained by Pines¹²⁾,

$$\varepsilon_c = -0.115 + 0.0313 \ln r_s \text{ (Ry)}, \quad (1.3)$$

on the basis of the collective description of electron gas due to Bohm and Pines¹³⁾ is useful.

On the other hand, the statistical calculation of the correlation energy by Cowan and Kirkwood¹⁴⁾ is not practicable owing to the fact that the results are given only numerically and further the correlation energy cannot be broken up from the exchange energy.

In determining the electronic density distribution in atoms, the correlation energy is effective only near the atomic surface and the density at the surface of the atoms in the TFD model is finite ($r_s \sim 4.2$, cf. eq. (2.15)). The latter fact has been considered to be more favourable for various problems than the TF model for which the electron density decreases as inverse six power of the distance from the nucleus as the distance increases. Therefore, we will use here Pines' expression of the correlation energy.

In this paper, the TFD equation for an atom or ion is derived by taking account of the correlation correction with Pines' expression, as well as the Fermi-Amaldi correction. By approximately solving the above equation, the electron density distribution for Br^- , Kr and Rb^+ in free state all having 36 electrons are given, especially for Kr the electron density distribution due to the ordinary TFD model and that due to the Hartree-Fock method are compared with the present

* When we had finished the numerical work of this paper, we found the work with respect to this problem by H. W. Lewis in *Physical Review* **111** (1958), 1554, in which numerical results were not reported. He has used the expression for the correlation energy obtained by interpolation from Gell-Mann and Brueckner's. Fermi-Amaldi correction has not been taken into account.

** ε_c means the correlation energy per electron of a free electron gas. r_s is the mean inter-electronic distance in a.u. (atomic unit), relating to the electronic density ρ in a.u. by the equation $r_s = (3/4\pi)^{1/3} \rho^{-1/3}$.

result. Lastly, total kinetic, potential, exchange and correlation energies in these ions are calculated and compared with the results obtained by the ordinary TFD method.

§ 2. Basic equations

We shall treat a free atom or ion with atomic number Z and electron number N . If we use *atomic units* throughout this paper, and express the electron density at \mathbf{r} measured from the nucleus by $\rho(\mathbf{r})$, the total energy of the system, in the TFD model, may be given by the following equation.

$$E = E_k + E_e + E_p^n + E_p^e + E_c, \quad (2.1)$$

where

$$E_k = \kappa_k \int \rho^{5/3} dv, \quad (2.2)$$

$$E_e = -\kappa_e \int \rho^{4/3} dv, \quad (2.3)$$

$$E_p^n = - \int \rho V_n dv, \quad (2.4)$$

$$E_p^e = -\frac{1}{2}(1-1/N) \int \rho V_e dv, \quad (2.5)$$

$$E_c = - \int \rho g(\rho) dv, \quad (2.6)$$

and

$$V_n(\mathbf{r}) = Z/|\mathbf{r}|, \quad (2.4a)$$

$$V_e(\mathbf{r}) = - \int \rho(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'| \cdot dv', \quad (2.5a)$$

$$g(\rho) = \alpha_1 + \alpha_2 \ln \rho, \quad (2.6a)$$

further

$$\left. \begin{aligned} \kappa_k &= (3/10) (3\pi^2)^{2/3} = 2.871, \\ \kappa_e &= (3/4) (3/\pi)^{1/3} = 0.7386, \\ \alpha_1 &= 0.065, \\ \alpha_2 &= 0.0052. \end{aligned} \right\} \quad (2.7)$$

E_k , E_e , E_p^n , E_p^e and E_c are the total kinetic, exchange, potential due to nucleus, potential due to electrons and correlation energies respectively. In eq. (2.5), $(1-1/N)$ is the Fermi-Amaldi correction factor⁷⁾ to exclude the self-interaction of the electrons. Here we use the correlation energy per electron given by Pines¹²⁾, $\varepsilon_c = -g(\rho)$ (cf. eq. (1.3)). In the interior of the atom where the electron density

is very high, expression (2.6a) is not correct, as has been pointed out by Nozières and Pines¹⁰⁾. However, it would not bring a serious error into the determination of electron distribution to use this approximate formula instead of a more exact one even in the high density region, because the large kinetic energy almost determines the distribution there. Further, if we estimate E_c by using this formula, as we shall do in § 4, a correct order of magnitude of correlation energy will be given. (cf. eqs. (1.1) and (1.3))

At the absolute zero of temperature, the total energy of the system E must be minimum, subject to the subsidiary condition

$$N = \int \rho dv = \text{constant.} \quad (2.8)$$

Therefore, the variation equation with respect to $\rho(\mathbf{r})$

$$\delta(E + V_0 N) = 0, \quad (2.9)$$

that is,

$$\frac{5}{3} \kappa_k \rho^{2/3} - \frac{4}{3} \kappa_e \rho^{1/3} - \alpha_2 - g(\rho) = V^* - V_0, \quad (2.10)$$

where V_0 is the Lagrange multiplier, and

$$V^* = V_n + (1 - 1/N) V_e, \quad (2.11)$$

may determine ρ as a function of \mathbf{r} , in connection with Poisson's equation

$$\Delta V^* = \frac{N-1}{N} 4\pi\rho. \quad (2.12)$$

Boundary conditions:

Now, we shall make the usual assumption of spherical symmetry of the electron distribution in the atom. We should first determine the atomic radius r_0 and the density ρ at $r=r_0$ which will be expressed as ρ_0 hereafter, since the solution of the simultaneous equations (2.10) and (2.12) can be determined uniquely only for the definite r_0 and ρ_0 . Therefore, we shall determine r_0 and ρ_0 , according to Jensen¹⁵⁾, by the following minimum condition of the total energy,

$$dE/dr_0 = 0, \quad (2.13)$$

which corresponds to the zero pressure at the surface of a free atom. In calculating dE/dr_0 , it should be noticed that all the integrals in E are to be extended only in the sphere of radius r_0 , since outside this sphere $\rho=0$. Eliminating $\partial V_e/\partial r_0$ and $\partial \rho/\partial r_0$ in the expression of dE/dr_0 by using (2.5a) and (2.8), we have the following equation,

$$dE/dr_0 = -\frac{2}{3} \kappa_k \rho_0^{2/3} + \frac{1}{3} \kappa_e \rho_0^{1/3} + \alpha_2 = 0. \quad (2.14)$$

Therefore,

$$\rho_0^{1/3} = \frac{1}{4\kappa_k} (\kappa_e + \sqrt{\kappa_e^2 + 24\kappa_k \alpha_2}) = 0.1471. \quad (2.15)$$

For the determination of r_0 , we will use the following boundary conditions.

First, we insert the value ρ_0 given by (2.15) into (2.10), then we have

$$(V^* - V_0)_{r=r_0} = \frac{5}{3} \kappa_k \rho_0^{2/3} - \frac{4}{3} \kappa_e \rho_0^{1/3} - \alpha_2 - g_0 = \frac{3}{2} \alpha_2 - (\kappa_e/2) \rho_0^{1/3} - g_0, \quad (2.16)$$

where g_0 is the value of g for $\rho = \rho_0$. Second, making use of the fact that at the surface, $r = r_0$, the electric field due to the atom or ion under consideration in the spherical approximation must be just as the field by the point charge $(Z - N)$ at the origin, we may set up the boundary condition at the surface as follows,

$$-(dV^*/dr)_{r=r_0} = (Z - N + 1)/r_0^2. \quad (2.17)$$

Third, near the origin, the field due to the charge of the nucleus being absolutely dominant, we have

$$\lim_{r \rightarrow 0} r V^* = Z. \quad (2.18)$$

The three conditions stated above, (2.16), (2.17) and (2.18), can determine uniquely the solution ρ of the second order simultaneous differential equation (2.10) and (2.12), together with the atomic radius r_0 .

§ 3. Simplification of the basic equation and its solution

It is very difficult to solve eqs. (2.10) and (2.12) exactly because of the logarithmic term of ρ in (2.10). As stated before, however, the term originating from the correlation correction is essential to the determination of ρ only near the atomic surface, where the electron density is small, so we may replace $g(\rho)$ by another function which is easy to treat and varies with ρ similarly to $g(\rho)$ near the surface. After the manner of Gombás⁽⁸⁾, we will expand $g(\rho)$ as a power series with respect to $\rho^{1/3}$ around $\rho = \rho_0$, and take only up to the first order, then we have an approximate function $g^*(\rho)$, such as

$$g^*(\rho) = \alpha_1' + \alpha_2' \rho^{1/3}, \quad (3.1)$$

$$\text{where} \quad \alpha_1' = \alpha_1 + \alpha_2 \ln \rho_0 - 3\alpha_2 = 0.0195, \quad (3.1a)$$

$$\text{and} \quad \alpha_2' = 3\alpha_2/\rho_0^{1/3} = 0.1061. \quad (3.1b)$$

In Fig. 1, $g(\rho)$ and $g^*(\rho)$ are shown against $\rho^{1/3}$. If we use $g^*(\rho)$ instead of $g(\rho)$, the correlation correction can be included into the exchange correction term, so the treatment of the differential equation becomes simpler. This simplification would not bring a serious error into the determination of electron distribution, while the correlation energy of the system should be calculated by using the expression $g(\rho)$ rather, as will be done in § 4.

Replacing $g(\rho)$ by $g^*(\rho)$ in the expression of E_c (2.6), and inserting it into the variation equation (2.9), we have the following equation instead of (2.10) :

$$\frac{5}{3} \kappa_k \rho^{2/3} - \frac{4}{3} \kappa_e' \rho^{1/3} - (V^* - V_0 + \alpha_1') = 0, \quad (3.2)$$

$$\text{where} \quad \kappa_e' = \kappa_e + \alpha_2' = 0.8447, \quad (3.2a)$$

which, in connection with Poisson's equation (2.12), may determine ρ as a function of r .

Replacement of $g(\rho)$ by $g^*(\rho)$ does not completely change the value derived by the condition (2.13), as may be verified by a calculation similar to that in § 2. Therefore, the expression of ρ_0 , (2.15), may be held correctly in this case. Using the new notation κ'_e , we may write

$$\rho_0^{1/3} = \kappa'_e / 2\kappa_k. \quad (3.3)$$

Accordingly, the boundary condition (2.16) is to be

$$(V^* - V_0 + \alpha'_1)_{r=r_0} = -\frac{1}{2} \kappa_e'^2 / \kappa_k. \quad (3.4)$$

The other two conditions (2.17) and (2.18) should be held also for this case.

Equation (3.2) may be written as

$$V^* - V_0 + \alpha'_1 = \frac{5}{3} \kappa_k \left(\rho^{1/3} - \frac{2}{5} \frac{\kappa_e'}{\kappa_k} \right)^2 - \frac{4}{15} \frac{\kappa_e'^2}{\kappa_k}. \quad (3.2a)$$

Now, we shall introduce the new function w and the new variable y instead of V^* or ρ and r respectively by the following relations, which are a slight modification of the ones due to Thomas¹⁶⁾,

$$\frac{5}{3} \kappa_k \left(\rho^{1/3} - \frac{2}{5} \frac{\kappa_e'}{\kappa_k} \right)^2 = \nu \frac{9\pi^2}{128} \frac{w}{r^{1/2} r_0^{7/2}}, \quad (3.5)$$

$$\text{and} \quad r = r_0 \exp(-y), \quad (3.6)$$

$$\text{where} \quad \nu \equiv (N/N-1)^2. \quad (3.7)$$

Then the simultaneous equations determining ρ , (3.2) and (2.12), can be reduced to the following equation for w as a function of y ,

$$\frac{d^2 w}{dy^2} = \frac{1}{4} w + e^{-(5/2)y} (e^{y/4} w^{1/2} + \gamma)^3, \quad (3.8)$$

$$\text{where} \quad \gamma \equiv \frac{32}{3^{7/3} \pi^{5/3}} \kappa_e' \nu^{-1/2} r_0^2. \quad (3.9)$$

And the boundary conditions (3.4), (2.17) and (2.18) change the form respectively as follows:

At the boundary $y=0$ ($r=r_0$)

$$(w^{1/2})_{y=0} = \frac{1}{4} \gamma, \quad (3.10)$$

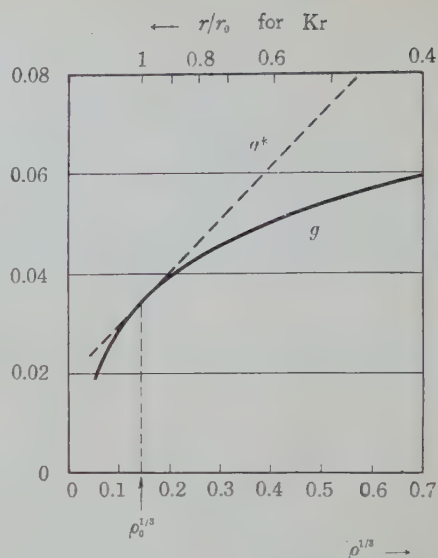


Fig. 1. Functions $g(\rho)$ and $g^*(\rho)$ versus $\rho^{1/3}$. The upper abscissa indicates the ratio of the distance from the nucleus to the atomic radius r_0 for Kr in free state.

$$\text{and} \quad \left(\frac{dw}{dy} + \frac{1}{2} w \right)_{y=0} = \nu^{-1/4} \frac{128}{9\pi^2} \left(\frac{9\pi}{32} (3\pi^2)^{1/3} \frac{1}{\kappa'_e} \right)^{3/2} \gamma^{3/2} (Z-N+1), \quad (3.11)$$

while as $y \rightarrow \infty$ ($r \rightarrow 0$)

$$\lim_{y \rightarrow \infty} e^{-y/2} w = \nu^{-1/4} \frac{128}{9\pi^2} \left(\frac{9\pi}{32} (3\pi^2)^{1/3} \frac{1}{\kappa'_e} \right)^{3/2} \gamma^{3/2} Z. \quad (3.12)$$

Eq. (3.8), subject to the boundary conditions (3.10), (3.11) and (3.12), may be solved directly by the method of Gauss, Jackson and Numerov¹⁷⁾. However, more simply, it would be allowed to determine w by the double interpolation with respect to the initial value of w i.e. γ and the order of ionization $Z-N$ from the solutions of the ordinary TFD equation due to Thomas¹⁶⁾. For a particular atom

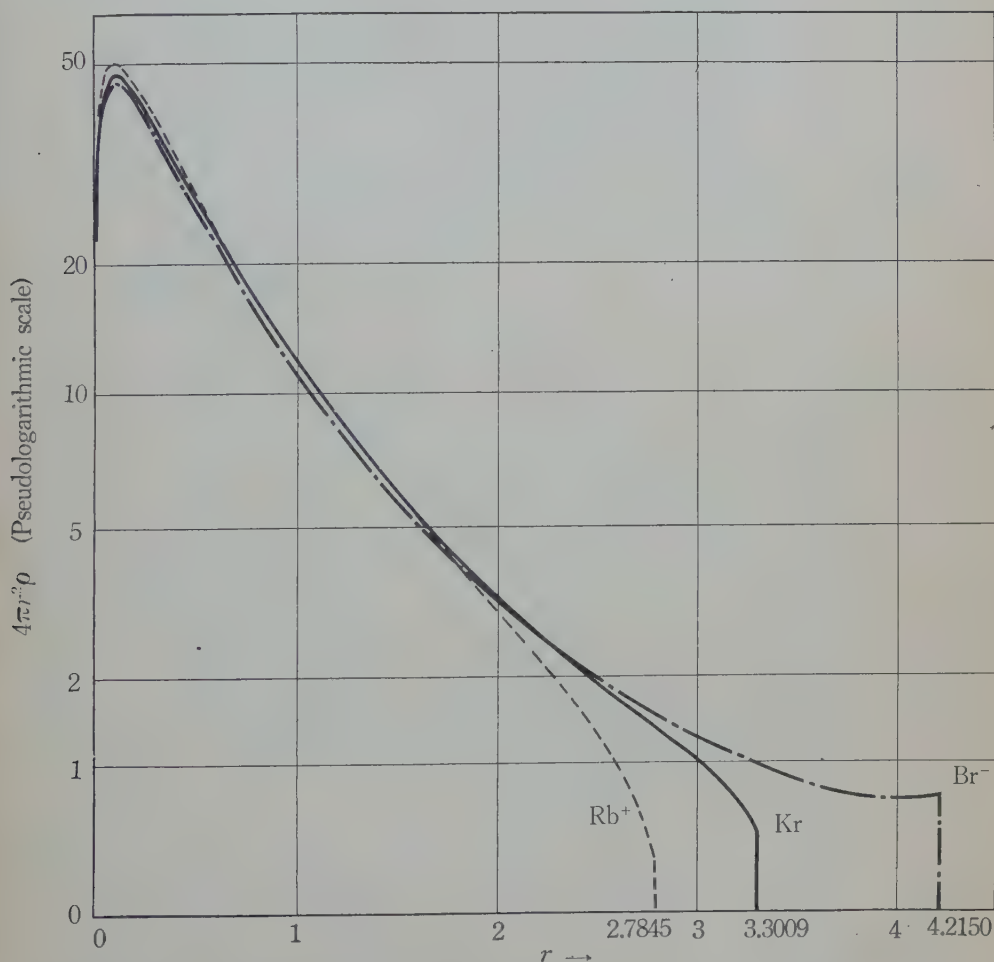


Fig. 2. Radial density distribution $4\pi r^2 \rho$ versus r , for Br^- , Kr and Rb^+ in free state.
(in a. u.)

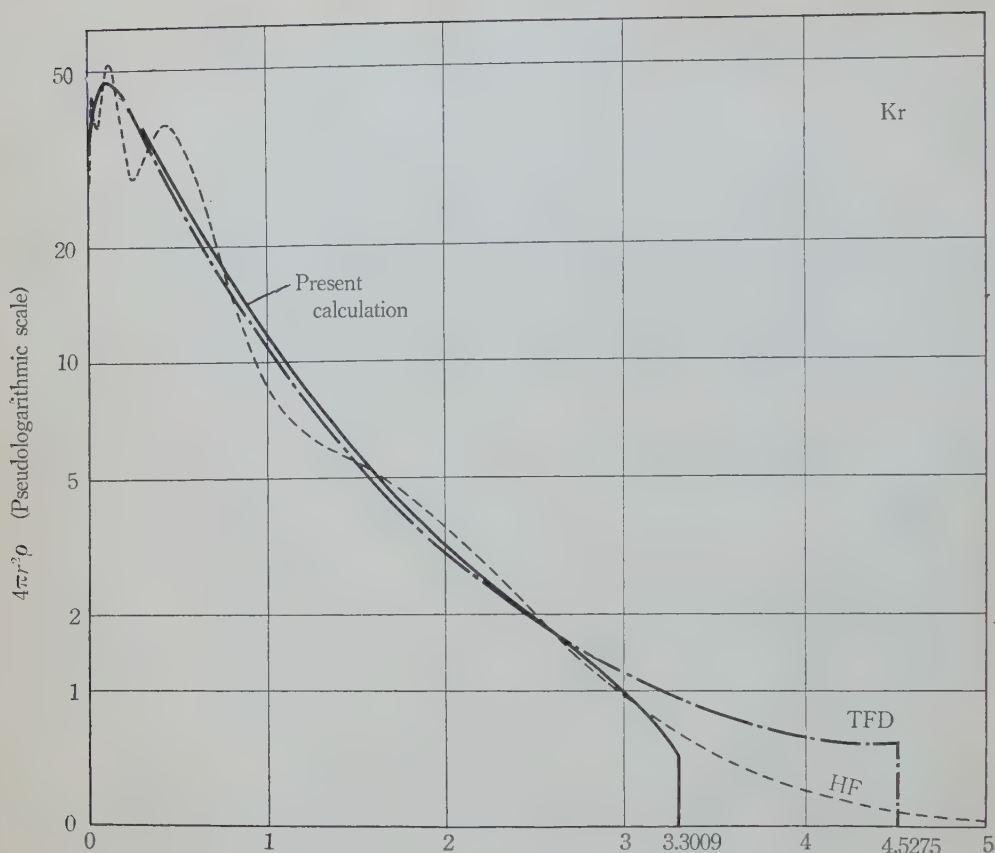


Fig. 3. Radial density distribution $4\pi r^2 \rho$ versus r for Kr in free state. (in a. u.)

- Present calculation.
 - - - TFD method due to Thomas¹⁶.
 . . . Hartree-Fock method due to Worsley²⁰.

or ion, in other words for given Z and N , the value of γ , for which the solution w of eq. (3.8), subject to the initial conditions (3.10) and (3.11), satisfies the boundary condition (3.12), gives the radius r_0 of this atom or ion in free state according to (3.9), and further inserting the solution $w(y)$ thus obtained into (3.5), we may have the electron distribution ρ as a function of y . Such an interpolation procedure would presumably be more correct than the perturbation like a treatment due to Gombás. (cf. Table I)

In Table I, r_0 -values in the present model for Rb^+ , Kr and Br^- are compared with the values due to Thomas¹⁶) (the ordinary TFD method), due to Gombás¹⁸) (the TFD method including Wigner's correlation and the Fermi-Amaldi corrections) and due to Pauling¹⁹) (the semi-empirical method). The fact that the r_0 -values by the present calculation are very close to those by Pauling, would give a theoretical basis to the hard ion model which is a fairly good approximation in the theory of ionic crystals.

Table I. r_0 values in a. u.

	Rb ⁺	Kr	Br ⁻
Present calculation	2.7845	3.3009	4.2150
Thomas*	3.4211	4.5275	—*
Gombás**	3.55	4.22	5.19
Pauling***	2.80	3.19	3.69

* Ordinary TFD method, in which negative ions cannot be treated.

** TFD method including Wigner's correlation and Fermi-Amaldi corrections.

*** Semi-empirical method.

The radial density distributions $4\pi r^2 \rho$'s obtained above, are shown in Fig. 2 for Rb⁺, Kr and Br⁻, all having the identical electron number $N=36$. Especially for Kr the present calculation is compared with that by the ordinary TFD method¹⁶⁾ and that by the Hartree-Fock method²⁰⁾ in Fig. 3. From this figure, we see that the electron distribution contracts toward the nucleus as a result of inclusion of the correlation and Fermi-Amaldi corrections, and with increasing distance from the nucleus the electron density decreases faster than in the case of the ordinary TFD model, more similarly to that due to the Hartree-Fock method. This means that the introduction of these corrections weakens the screening effect of the electron cloud as is expected.

§ 4. Energy relations

We shall derive the relations realized between the energy components in (2.1). After the Fock linear-scale-factor method²¹⁾, we shall assume that the distances of electron-electron as well as electron-nucleus are contracted by a factor $1/\lambda$ compared with the case of the minimum total energy, the total electron number N being unchanged. We express the density at the distance r from the nucleus in the contracted distribution by $\rho_\lambda(r)$, then by the normalization condition (2.8) we have

$$\rho_\lambda(r) = \lambda^3 \rho(\lambda r), \quad (4.1)$$

where ρ is the density to minimize the total energy. We distinguish each energy component for the contracted distribution by the upper suffix λ , as

$$E^\lambda = E_k^\lambda + E_e^\lambda + E_p^{n\lambda} + E_p^{e\lambda} + E_c^\lambda, \quad (4.2)$$

where

$$E_k^\lambda = \lambda^2 E_k, \quad (4.3)$$

$$E_e^\lambda = \lambda E_e, \quad (4.4)$$

$$E_p^{n\lambda} = \lambda E_p^n, \quad (4.5)$$

$$E_p^{e\lambda} = \lambda E_p^e, \quad (4.6)$$

and

$$E_a^\lambda = E_c - 3N\alpha_2 \ln \lambda. \quad (4.7)$$

For the value $\lambda=1$, the total energy E^λ should be minimum, therefore

$$\left(\frac{d}{d\lambda} E^\lambda \right)_{\lambda=1} = 0. \quad (4.8)$$

Inserting eqs. (4.2) to (4.7), into (4.8), we find the following energy relation,

$$E_p^n + E_p^e = -2E_k - E_e + 3N\alpha_2. \quad (4.9)$$

The density distribution function $\rho(r)$ obtained in the preceding section, and eqs. (2.2), (2.3), (2.4) and (2.6) enable us to evaluate the values of E_k , E_e , E_p^n and E_p^e , and then eq. (4.9) the value of E_p^e . These values thus obtained are columned in Table II.

Table II. Values of energy components in a. u.

	Rb ⁺	Kr	Br ⁻
E_k	3648.9 (3598.9)*	3409.0 (3377.9)	3229.3
$-E_e$	96.8 (93.6)	93.6 (89.6)	88.3
$-E_p^n$	8431.3 (8322.8)	7887.5 (7820.0)	7433.0
E_p^e	1230.8 (1218.6)	1163.7 (1153.8)	1063.3
$-E_o$	2.8 ₀	2.7 ₇	2.7 ₁
$-E$	3651.2 (3598.9)	3411.2 (3377.9)	3231.4

* The values in brackets are the ones of the ordinary TFD model due to Thomas¹⁶⁾.

For the sake of comparison the energy values of the ordinary TFD model due to Thomas¹⁶⁾ are also reproduced in the same table. From this table we see that each energy value becomes slightly larger than the corresponding one due to Thomas because of the contraction of the electron cloud. Further the correlation energy E_e is very small compared with the other energies E_k , E_e and E_p^n , and since we may neglect the last term $3N\alpha_2$ in (4.9), the Virial theorem for the ordinary TFD theory²²⁾ remains to hold in this case too.

§ 5. Conclusions

In this paper, the basic equations, the Pines correlation and Fermi-Amaldi corrections being taken into account, are simplified to the form which enables us to determine the electron density distributions by the interpolation method from the solutions of the ordinary TFD theory given by Thomas¹⁶⁾, rather simply and correctly for various atoms or ions.

In Fig. 3, the present calculation shows that the electron density distribution, with the correlation and Fermi-Amaldi corrections taken into account, is more contracted towards the nucleus than in the case of the ordinary TFD theory. This means that the screening effect of the inner electrons for the electrostatic field of the nucleus to the outer ones is overestimated in the statistical theory if these corrections are not considered. This is a reasonable result. Since the correlation energy is very small compared with the other energy components—even if we use a more exact formula of correlation energy density at high electron density region of the atom, for example, Gell-Mann and Brueckner's instead of Pines' which is used here, this conclusion would not be changed—the deviation of the density distribution due to the present calculation from that due to the ordinary TFD method would predominantly be originated from the Fermi-Amaldi correction. Therefore, at the present stage of the statistical theory, a correct formulation of the Fermi-Amaldi correction would be most desirable. However, since the effect of these corrections in electron distribution is conspicuous only near the atomic surface, the TFD model is sufficient to treat the problem concerning bulky properties of the atom. For the atomic nature in which the outer electrons play an important role, a statistical theory with sufficient corrections should be used.

As regards the energy values of the atom, the correlation energy may well be ignored and the values of the energy components as well as the Virial relation are practically maintained as those of the ordinary TFD theory.

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Pair Correlations in the Quasichemical Equilibrium Approach to Superconductivity

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It is found that before a quantitative theory of superconductivity can be derived from the quasichemical equilibrium approach, a self-consistent method of evaluating the "quenched" correlation matrix for electron pairs is needed.

The quasichemical equilibrium approach to superconductivity^{1,2} treats the partition function for the Fermi gas of electrons in metals by an approximation which treats statistical correlations exactly, but neglects dynamical correlations between more than two particles. The thermodynamic properties of the system are now given in the form of an equilibrium between the fermions ("atoms") and diatomic "pseudo-molecules", the latter obeying a kind of Bose statistics. Under certain conditions there is a singularity in the partition function—that is to say the thermodynamic properties of the system undergo a transition. Below this transition point, some of the pseudo-molecules will undergo a form of Bose-Einstein condensation, producing a superconducting state³.

Any attempt at a detailed study of this system requires a knowledge of the eigenvalue spectrum, $u_s(\mathbf{K})$, of the "quenched 2-particle U -matrix" which is defined as follows:—The 2-particle U -matrix between the pair of electron states k_1, k_2 and the states k_1', k_2' is

$$\langle k_1 k_2 | U | k_1' k_2' \rangle = \langle k_1 k_2 | e^{-\alpha H} - e^{-\alpha H_0} | k_1' k_2' \rangle \quad (1)$$

where H is the 2-electron hamiltonian, complete with interactions, and H_0 the hamiltonian for two free electrons. The Fermi statistics of the system introduce "quenching" factors

$$q(k) = \frac{1}{\exp[\alpha(\mu - \epsilon)] + 1} \quad (2)$$

(with μ the chemical potential, $\alpha = 1/kT$), to give the "quenched 2-particle U -matrix", \tilde{U} , as

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$$\langle k_1 k_2 | \tilde{U} | k_1' k_2' \rangle = [q(k_1) q(k_2)]^{1/2} \langle k_1 k_2 | U | k_1' k_2' \rangle [q(k_1') q(k_2')]^{1/2}. \quad (3)$$

An alternative form is obtained by using the occupation numbers $n(k)[n(k) = 1 - q(k)]$:

$$e^{2\alpha\mu} \langle k_1 k_2 | \tilde{U} | k_1' k_2' \rangle = [n(k_1) n(k_2)]^{1/2} \langle k_1 k_2 | e^{\alpha H_0/2} e^{-\alpha H} e^{\alpha H_0/2} - 1 | k_1' k_2' \rangle \times [n(k_1') n(k_2')]^{1/2}. \quad (4)$$

The eigenvalues of the \tilde{U} -matrix may be written $u_s(\mathbf{K})$, where \mathbf{K} is the centre of mass momentum of the electron pair, and S is some "internal quantum number" of the pseudo-molecule.

The partition function contains a singularity, and the system undergoes a phenomenon similar to Bose-Einstein condensation, if and when the largest eigenvalue satisfies the relation

$$\gamma \equiv e^{2\alpha\mu} [u_s(\mathbf{K})]_{max} \approx 1. \quad (5)$$

If the correlation matrix is negative definite ("purely repulsive") there is no transition; for other than purely repulsive correlations, such that $[u_s(\mathbf{K})]_{max}$ is positive, then substitution of γ from Eq. (5) into the partition function of the system gives an estimate of the transition temperature, in terms of the inter-electron interaction parameters.

The actual interaction between electrons in metals consists of a screened Coulomb repulsion, together with the partly attractive Fröhlich interaction due to phonon exchange. Since calculation of \tilde{U} in terms of this interaction is intractable, we employ model interactions, which contain the essential features, to investigate $[u_s(\mathbf{K})]_{max}$.

Calculation of the eigenvalue spectrum of the matrix \tilde{U} as given by Eq. (3) consists of

- (a) first calculating the pair correlation matrix, U , by use of the total and free-particle hamiltonians (H and H_0) for *UNQUENCHED* electrons,
- (b) then finally multiplying by the quenching factors (2) to arrive at the quenched correlation matrix \tilde{U} .

The results, for a variety of model interactions, all indicate that this straightforward evaluation of the quenched correlation matrix, by use of the above prescription, is not adequate, and leads to unphysical results⁴⁾. The trouble stems from the fact that we have first considered the electrons without any quenching effects (step (a)), introducing such effects only as a final stage (step (b)). Thus what is required is some form of self-consistent evaluation, which at no stage refers explicitly to unquenched electrons in the metal⁵⁾.

We also calculated $[u_s(\mathbf{K})]_{max}$ by taking the first step in the semi-classical approximation⁶⁾ which replaces $e^{\alpha H_0/2} e^{-\alpha H} e^{\alpha H_0/2}$ by $e^{-\alpha V}$. Although this approximation is poor at low temperatures (when $T < \hbar^2 \lambda^2 / 2mk$), it is of interest because it satisfies the self-consistency requirement to a large extent: we no longer get unphysical results like Eq. (7). Characterizing our model potentials by a range λ^{-1} ,

and strength parameter β , and with l_T defined by Eq. (9), we find that Eq. (5) is satisfied for $\beta (\lambda l_T)^2$ of order unity. Substituting plausible values of β and λ leads to a reasonable transition temperature. However, eigenvalues obtained by this method lead to far too extreme a transition in the thermodynamics; $C_V(T)$ becomes vanishingly small almost immediately below T_c —this is a fault to be expected in a semi-classical approximation.

We conclude that before a quantitative theory of superconductivity can be derived from the quasichemical equilibrium approach, we need a self-consistent method of evaluating the quenched correlation matrix, which at no stage refers to unquenched electrons.

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$$(2m/\hbar^2) V(x) = -2\beta\lambda^2 e^{-\lambda r} / (1 + \beta e^{-\lambda r})^2 \quad (6)$$

where λ^{-1} is the range and β the interaction strength ($\beta=1$ corresponds to a bound state: we of course take $\beta \ll 1$), we get

$$\gamma \sim \beta^2 \frac{\exp(k_F^2 l_T^2)}{(k_F l_T)^3} \left(\frac{\lambda}{k_F} \right)^6 \left\{ 1 + \mathcal{O} \left(\frac{1}{k_F l_T} \right)^2 + \mathcal{O}(\beta) + \mathcal{O} \left(\frac{\lambda}{k_F} \right) \right\}^2 \quad (7)$$

where k_F and l_T are given by

$$N/V = k_F^3 / 6\pi^2 \quad (8)$$

$$l_T^2 = \hbar^2 / 2mkT. \quad (9)$$

The factor $\exp(k_F^2 l_T^2)$ is vast for all temperatures below the Fermi degeneracy temperature ($\sim 10^4$ °K): i.e. eqs. (5) and (7) imply a transition temperature of order 10^4 °K.

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Nucleon-Antinucleon Annihilation Products

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Three models for nucleon-antinucleon annihilation are considered: the Fermi statistical model, the modification by Pomeranchuk which includes strong π - π interactions, and the Koba-Takeda model of fast "core" annihilation followed by slow emission of "cloud" pions. The parameters of the models are chosen to fit the experimental pion multiplicity $\langle n_\pi \rangle = 5.36$. Then it is shown that the Koba-Takeda model with a small core predicts quite different values than the other models for the average K meson energy and pion multiplicity associated with $K\bar{K}$ events. Data on these quantities would distinguish between models and provide indirect information on the π - π interaction.

§ 1. Introduction

Recent experiments^{1),2),3)} on the annihilation of antiprotons in matter have shown that the average number of pion secondaries is $\langle n_\pi \rangle = 5.36 \pm .3$ and that K mesons are produced in $3.5 \pm 1.5\%$ of the events. Information has also been obtained on the pion energy spectrum, angular correlations, the ratio of charged to neutral pions, and other quantities.

As is shown in references (1) and (2), all the data on pion emission can be fit by the Fermi statistical model⁴⁾, if the large radius $R = 2.5\hbar/m_\pi c$ is used. However, the Fermi model with this choice of radius predicts that K mesons will be emitted in 12.3% of the events. Various proposals have also been made which attempt to explain the large pion multiplicity by consideration of two dynamical effects ignored in the Fermi model:

a) According to a number of authors,^{5),6),7)} the pion multiplicity is enhanced by strong pion-pion interactions.

b) On the other hand Koba and Takeda⁸⁾ suppose that the high frequency core annihilation is non adiabatic with respect to low frequency meson cloud oscillations and that the large pion multiplicity represents the sum of quickly produced core pions and slowly escaping cloud pions.

In the present paper the models of Fermi⁴⁾ (hereafter referred to as F), Pomeranchuk⁷⁾ (P), and Koba and Takeda⁸⁾ (KT) are taken as representative,

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respectively, of the purely statistical viewpoint and the two types of dynamical modification that have been proposed. The parameters of these models are chosen to fit the observed pion multiplicity* $\langle n_\pi \rangle = 5.36$, and then other predictions of the three models are examined to see if presently available experimental information may distinguish between them. We find that if a core radius $\lesssim \frac{2}{3}\hbar/m_\pi c$ is used in KT , then the predictions of KT concerning the pion multiplicity associated with $K\bar{K}$ events and the average K meson kinetic energy differ significantly from the other models, so that a small number of $K\bar{K}$ events may suffice to distinguish between models. The reason for the difference is that $K\bar{K}$ pairs in KT are emitted only from the small, "high temperature" core which produces more energetic K mesons than the larger F and P interaction volumes.

§ 2. Determination of parameters

According to F , the relative probability of various secondary products in nucleon-antinucleon annihilation at rest is given by

$$P(n_\pi, n_k) = G(n_\pi, n_k) (R^3/6\pi^2)^{n_\pi + n_k - 1} \frac{dN}{dW}(n_\pi, n_k, W=2M_N) \quad (2.1)$$

where G takes the isotopic spin and statistics of the final state into account, R is the interaction volume radius in units of $\hbar/m_\pi c$, and dN/dW is the momentum space volume in the center of mass system, in units of $\hbar/m_\pi c$. In this paper we use the product $G \cdot dN/dW$ obtained from Tables 13 and 14 of ref. (2)**, which are based on isotopic spin conservation, strangeness conservation, and the phase space calculations of Fiahlo⁹. Conservation of angular momentum, parity, and charge conjugation are not taken into account. In the process of fitting each model to the observed pion multiplicity, we ignore $K\bar{K}$ emission.

Model P represents the effect of pion-pion interactions by taking the interaction volume to be $n_\pi R^3/6\pi^2$. The factor $n_\pi^{n_\pi - 1}$ thus introduced enables us to fit the pion multiplicity $\langle n_\pi \rangle = 5.36$ with the physically reasonable*** radius $R = 1.08$.

* It should be mentioned that the hydrogen bubble chamber work of Horwitz, Miller, Murray, and Tripp³, which is a more direct determination of $\langle n_\pi \rangle$ than the emulsion work^{1,2} on which we base our conclusions, yields the result $\langle n_\pi \rangle = 4.7 \pm 0.5$. The numbers resulting from $\langle n_\pi \rangle = 4.7$ are quite different from ours. For example the prediction of F for pion multiplicity associated with $K\bar{K}$ events changes from our result, 2.22 (Table II), to 1.87. Thus, although the arguments of our paper remain valid in any case, the specific numbers we obtain are sensitive to the experimental value of $\langle n_\pi \rangle$, which is still in doubt.

** These tables contain the F probabilities $P(n_\pi, n_k)$ for several radii R . To obtain the relative values of $G(n_\pi, n_k) dN/dW(n_\pi, n_k)$ we simply divide the tabulated probabilities by $(R^3/6\pi^2)^{n_\pi + n_k - 1}$.

*** This multiplicity greatly exceeds the result $\langle n_\pi \rangle \sim 3.5$ for $R=1$ obtained in Sudershan, Phys. Rev. **103**, 777, because the Fiahlo phase space approximation enhances the multiplicity relative to the less accurate phase space approximations considered by Sudershan, and because the interaction volume $0.945 R^3$ in Sudershan should be replaced by $1/6\pi^2 (M_N R/m_\pi)^3 = 5.2 R^3$. (ref. (2), footnote 17).

In *KT* the core annihilation within a small radius R is followed by the emission of $\langle n_{\pi \text{ cloud}} \rangle$ low frequency cloud mesons. We will consider two physically reasonable choices for the two parameters R and $\langle n_{\pi \text{ cloud}} \rangle$:

a) Suppose that $R = \frac{2}{3}$, roughly consistent with the Stanford experiments on electron-nucleon scattering¹⁰. Then we must choose $\langle n_{\pi \text{ cloud}} \rangle = 3.28$ to get the total $\langle n_{\pi} \rangle = 5.36$.

b) According to P wave meson theory¹¹, the average number of P state mesons in the combined nucleon clouds is 2.6. If we take $\langle n_{\pi \text{ cloud}} \rangle = 2.6$, then $R = 1.77$ gives $\langle n_{\pi \text{ core}} \rangle = 2.76$ and the correct total number of pions.

Since R represents the radius of a high frequency core in *KT*, the value $R = 1.77$ obtained in *KT*(b) seems unreasonably large unless we suppose an additional complication holds: that R includes the effects of strong pion-pion interactions. To prove this contention, we can estimate the core radius R_c to be expected, in the absence of final state pion-pion interactions, from the scalar nucleon electromagnetic radius $\sim 0.8 \times 10^{-13}$ cm suggested by electron scattering experiments at Stanford¹⁰.

One can show, by using only general conservation laws, that the dispersion relation value for the scalar radius is determined by $3\pi, 5\pi, \dots (2m+1)\pi, \dots, K\bar{K}, \dots, n\bar{n}, \dots$ contributions, where m is an integer. The $3\pi, 5\pi, \dots (2m+1)\pi, \dots$ states can couple to the photon only through baryon loops or K 's (see Fig. 1), because

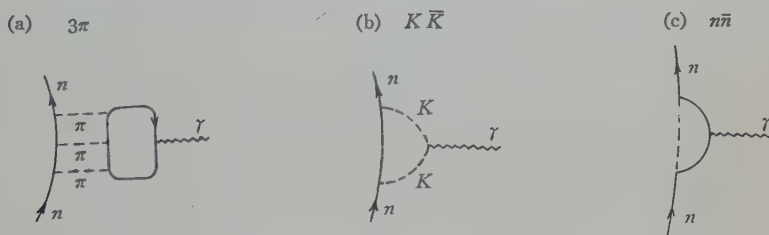


Fig. 1. Representative Feynman diagrams for contributions to the scalar nucleon electromagnetic structure. In each case the photon is absorbed by a baryon or K line; if it is absorbed by a pion then we have a contribution to the *vector* nucleon electromagnetic structure.

couplings such as $H_I \sim \lambda \phi_{\pi}^{2m+1}$ and $H_I \sim e \phi_{\pi}^{2m} \partial_{\mu} \phi_{\pi} A^{\mu}$ are forbidden for pseudoscalar spin zero pions by parity conservation. So the scalar radius is a measure of the spatial extent, R_c , of baryons and K 's in nucleon structure. However, the core may be as small as $R_c \sim \hbar/M_N c$ if the observed radius reflects a breakdown of quantum electrodynamics.¹² We conclude that R_c lies between $\sim 0.2 \times 10^{-13}$ cm. and $\sim 0.8 \times 10^{-13}$ cm. The use of $R = R_c < \frac{2}{3}$ for *KT*(a) would increase the distinction between *KT*(a) and all the other models in Table II.

The numbers for *KT*(a) and (b) were obtained on the assumptions that $n_{\pi \text{ core}}$ is at least two* and that cloud pions have the same momentum and isotopic spin

* Since the core annihilation is non adiabatic with respect to the cloud frequency, the core and cloud do not exchange much energy or momentum during annihilation, except perhaps through

spectrum as core pions. The assumption concerning the momentum and isotopic spin spectrum of the cloud is compatible with experiment and with P wave meson theory, which suggests a cloud pion momentum spectrum $\sim p^2$. In place of this assumption the original KT paper⁸⁾ gave each cloud pion its average energy of 346 Mev for order of magnitude calculations, made the tacit assumption that the cloud has zero isotopic spin and momentum relative to the core, and computed the core pion probabilities according to (2.1) with the substitutions $n_\pi \rightarrow n_{\pi \text{ core}}$, available energy $W \rightarrow \{2M_N - \langle n_{\pi \text{ cloud}} \rangle 346 \text{ Mev}\}^*$.

In the present treatment the percentage probabilities that various n_k , $n_{\pi \text{ core}}$ are associated with a particular $n_{\pi \text{ cloud}}$ are distributed according to the equation:

$$P(n_{\pi \text{ cloud}}, n_{\pi \text{ core}}, n_k) = \frac{P(n_{\pi \text{ cloud}}) P(n_{\pi \text{ cloud}} + n_{\pi \text{ core}}, n_k)}{\sum_{\substack{n_k=0,2 \\ n_{\pi \text{ core}} \geq 2}} P(n_{\pi \text{ cloud}} + n_{\pi \text{ core}}, n_k)} \quad (2.2)$$

Table I. The distribution of cloud pions in KT for the combined nucleon and antinucleon clouds.

	Number of pions	% Probability $P(n_{\pi \text{ cloud}})$	
	$n_{\pi \text{ cloud}}$	(a) $R=\frac{2}{3}$	(b) $R=1.77$
	0	6	9
	1	15	18
	2	17	24
	3	18	21
	4	15	15
	5	14	8
	6	11	4
	7	4	1
Total		100	100
$\langle n_{\pi \text{ cloud}} \rangle$		3.29	2.60

where $n_{\pi \text{ core}} \geq 2$, $n_k = 0$ or 2 , $P(n_{\pi \text{ cloud}})$ is the percentage probability for $n_{\pi \text{ cloud}}$ (see Table I), and $P(n_{\pi \text{ cloud}} + n_{\pi \text{ core}}, n_k)$ is given by (2.1) with the R dependence, in obvious notation, $(R_{\text{cloud}}^3)^{n_{\pi \text{ cloud}}} (R_{KT \text{ core}}^3)^{n_{\pi \text{ core}} + n_k - 1}$. It is not necessary to make

final state interactions while the core pions are on their way out. If we neglect this energy-momentum exchange, then cloud states with the proper relation between energy and momentum to permit $n_{\pi \text{ core}} = 0$ or 1 are a negligible fraction of the cloud states that can yield $n_{\pi \text{ core}} \geq 2$, for which cloud momentum and energy can vary independently. As an exception we must suppose that the small fraction of cloud states, whose energy and momentum are so great that two core pions could not be emitted with energy-momentum conservation, eventually interact with the core or core products.

* We have changed from the original KT assumption here in order to use the accurate values of $GdN/dW(W=2M_N)$ calculated in reference (2) (see explanation following our equation (2.1)). The results obtained with the original KT assumption in case (a) are: 1.44 for the pion multiplicity in $K\bar{K}$ events, and 177 Mev for the K meson kinetic energy. The present assumption leads to the corresponding results 1.53 pions and 171 Mev (Table II (v, vi)), and the qualitative conclusions are unchanged.

any assumption about R_{cloud} because this quantity is a common factor which cancels out of $P(n_{\pi \text{ cloud}}, n_{\pi \text{ core}}, n_k)$. Just as in *KT*, Table I for $P(n_{\pi \text{ cloud}})$ was constructed by distributing cloud pions with the desired average $\langle n_{\pi \text{ cloud}} \rangle$ according to probabilities obtained from an intermediate coupling calculation of Takeda.¹⁸⁾

In case (a) this distribution of cloud pions is too flat and leads to too many 3, 7, 8 and 9 pion annihilations, as can be seen in Table II (i). We can obtain agreement with experiment, with some loss of plausibility for the model, by selecting a more peaked cloud pion distribution. Thus we see that the relative probabilities for the $n_{\pi \text{ cloud}} = 0, 1, 2, 3$ states, which critically influence $K\bar{K}$ events, are not de-

Table II. i) Percentage probabilities for various numbers of pions, if $K\bar{K}$ events are neglected.
 ii) Average pion multiplicity $\langle n_{\pi} \rangle$, if $K\bar{K}$ events are neglected.
 iii) The percentage of events in which a $K\bar{K}$ pair is produced.
 iv) Percentage probabilities for various numbers of pions associated with $K\bar{K}$ events, if events with $n_k = 0$ are not included.
 v) The pion multiplicity associated with events in which a $K\bar{K}$ pair is produced.
 vi) The average kinetic energy $\langle T_k \rangle$ of a K meson in $K\bar{K}$ events, measured in Mev.
 vii) The average total energy $\langle E_{\pi} \rangle$ of a pion detected in a $K\bar{K}$ event, measured in Mev.

		N_{π}	$F(R=2.46)$	$P(R=1.08)$			KT		Experiment
				(a)	(b)	(c)	(a) ($R=\frac{2}{3}$)	(b) ($R=1.77$)	
i)	$p(n_{\pi})$ for $n_k=0$	2	0.0	→	0.1	←	1.4	0.0	consistent with F, P, KT (b)
		3	2.3		4.0		17.5	4.3	
		4	13.4		14.3		18.0	18.9	
		5	40.6		36.8		18.8	36.7	
		6	33.1		31.6		15.2	24.3	
		7	10.6		13.2		14.1	10.8	
		8	0.0		0.0		11.0	4.0	
		9	0.0		0.0		4.0	1.0	
Total			100.0		100.0		100.0	100.0	
ii)	$\langle n_{\pi} \rangle$ for $n_k=0$		5.36	→	5.36	←	5.36	5.36	5.36 ± 3
iii)	% K events		12.3	0.7	1.9	12.9	16.7	13.5	3.5 ± 1.5
iv)	$P(n_{\pi})$ for $n_k=2$	0	0.1	—	2.9	0.8	6.9	0.2	
		1	11.4	—	16.6	19.7	43.0	16.7	
		2	55.1	—	52.9	55.7	39.9	60.5	
		3	33.4	—	27.6	23.8	10.2	22.6	
Total			100.0		100.0	100.0	100.0	100.0	
v)	$\langle n_{\pi} \rangle$ for $n_k=2$		2.22	—	2.05	2.03	1.53	2.06	
vi)	$\langle T_k \rangle$ (Mev)		109	—	125	122	171	117	172 (10 events)
vii)	$\langle E_{\pi} \rangle$ for $n_k=2$ (Mev)		302	—	311	317	357	318	

terminated in $KT(a)$. In the absence of any guide we used the pion cloud distribution of Table I* to calculate what type of $K\bar{K}$ events $KT(a)$ may lead to in Table II.

All three models agree with experimental results¹⁾ on the charged to neutral pion ratio, which is determined mainly by charge independence. Each model is also in rough agreement with the observed distribution¹⁾ of pion multiplicities, and data on this point does not seem accurate enough to distinguish between models. Although any of the models involving specific dynamical effects may give other than statistical angular correlations, present observations on angular correlation¹⁾ are consistent with purely statistical theories. Concerning the pion energy spectrum, F and P agree with experiment¹⁾ while the energy spectrum of cloud pions can be considered to be another adjustable parameter in KT . The statistical KT pion spectrum used in this paper is consistent with experiment. It follows that to distinguish between models we must consider $K\bar{K}$ production using the parameters determined by the pion multiplicity.

§ 3. $K\bar{K}$ abundance

The $K\bar{K}$ abundance according to the various models is listed in Table II (iii).

In KT the result is determined by a competition between the small radius, which tends to increase the $K\bar{K}$ abundance, and the fact that states with more than three cloud pions do not have enough core energy to produce an appreciable number of $K\bar{K}$ pairs, which tends to decrease the $K\bar{K}$ abundance.

The three cases listed for P are:

a) Pions interact strongly and are produced in a volume $n_\pi R^3/6\pi^2$, but K mesons interact weakly and are produced in a volume $R^3/6\pi^2$. Thus the Fermi result is multiplied by $n_\pi^{n_\pi-1}$ (unless $n_\pi=0$; in that case the factor is 1).

b) Pions interact strongly and K mesons do not, but both are produced in the volume $(\delta_0, n_\pi + n_\pi) R^3/6\pi^2$. The multiplicative factor is $n_\pi^{n_\pi+n_k-1} + \delta_0^{n_\pi}$.

c) Pions and K mesons both interact strongly in a volume $(n_\pi + n_k) R^3/6\pi^2$, and the multiplicative factor is $(n_\pi + n_k)^{n_\pi+n_k-1}$.

Cases (a) and (b) of P greatly favor pion emission over $K\bar{K}$ emission, and actually predict a *lower* $K\bar{K}$ abundance than is observed. In these cases the assumption of statistical equilibrium between K and pion production has been dropped, which might perhaps be understood physically on the basis that K interactions are weaker than pion interactions, or that the K is smaller than the pion Compton wavelength. Since cases (b) and (c) of P bracket the observed K abundance, it is clearly possible to fit the K abundance with some version of P . In § 4 we will carry

* If, on the other hand, the effective cloud pion distribution were changed so as to greatly favor $n_{\pi\text{cloud}}=2$ over $n_{\pi\text{cloud}}=0$ or 1, then the predictions of $KT(a)$ for $K\bar{K}$ events would be similar to the predictions of P . Such a cloud pion distribution could arise from pion-pion interactions between escaping cloud pions, and any statistical theory that can be interpreted in terms of strong pion-pion interactions seems to give results similar to those of P .

along (b) and (c) of P to bracket the physical case, and disregard (a) of P which gives too low a K abundance (.7%). When a similar distinction between K and pion production is introduced into the other models they also can fit the K abundance, so the K abundance does not allow a decision between the modified F , P , and KT models.

§ 4. Details of $K\bar{K}$ events

To decide between models, we must consider variables which are associated with $K\bar{K}$ events but are essentially independent of the factor that enhances pion production relative to $K\bar{K}$ production. The pion multiplicity associated with $K\bar{K}$ events, and the average kinetic energy of K 's in $K\bar{K}$ events fulfill these requirements because various $K\bar{K}$ events may depend on the pion enhancement primarily as a common factor* which cancels out of average quantities such as the kinetic energy and pion multiplicity.

The pion multiplicity is listed in Table II (v). Unfortunately, no data has been presented for this quantity.

In order to calculate the average K meson kinetic energy $\langle T_k(n_\pi) \rangle$ for a given number of pions n_π , we used the following momentum spectrum for a K meson:

$$X(n_\pi, P_k) = 4\pi P_k^2 \frac{dN}{dW} (n_\pi, n_k=1, W) \quad (4.1)$$

where P_k is the momentum of the first K , $W = [(2M_N - \sqrt{M_\pi^2 + P_k^2})^2 - P_k^2]^{1/2}$ is the combined energy of the second K and the n_π pions in their center of mass frame, and dN/dW is calculated from the tables in Fialho⁹⁾. The values for $\langle T_k(n_\pi) \rangle$ obtained with $X(n_\pi, P_k)$ are listed in Table III, together with the results of a similar calculation for $\langle E_\pi(n_\pi) \rangle$ in $K\bar{K}$ events. These results were weighted by the probabilities for various n_π in Table II (iv) to obtain the average energies, $\langle T_k \rangle$ and $\langle E_\pi \rangle$, for $K\bar{K}$ events (Table II (vi, vii)).

Table III The average kinetic energy of one K meson, $\langle T_k(n_\pi) \rangle$, and the average total energy of one pion, $\langle E_\pi(n_\pi) \rangle$, for an event in which a $K\bar{K}$ pair and n_π pions are produced.

n_π	$\langle T_k(n_\pi) \rangle$	$\langle E_\pi(n_\pi) \rangle$
0	444	—
1	206	476
2	112	332
3	64	253

Unfortunately, the momentum phase space volume considered by Fermi⁴⁾ and

* This is not quite true for (b) of P , but we still obtain good agreement between (b) and (c) of P in Table II (iv-vii).

accurately evaluated by Fialho⁹⁾ is not covariant, so it is not clear what is the most suitable expression for the K momentum spectrum. In a covariant theory¹⁴⁾ $\int d^3P_i$ is replaced by $\int m_i/E_i \cdot d^3P_i$, and (4.1) can be replaced by the invariant quantity

$$X'(n_\pi, P_k) = \frac{4\pi P_k^2 m_k}{E_k} \frac{dN'}{dW} (n_\pi, n_k=1, W) \quad (4.2)$$

where

$$\frac{dN'(n)}{dW} = \pi \int d^3P_i \frac{m_i}{E_i} \delta(\sum_i \vec{p}_i) \delta(\sum_i E_i - W) \quad (4.3)$$

and W is the same as in (4.1). We have not estimated the change in $\langle n_\pi \rangle$ associated with $K\bar{K}$ events due to the covariant formalism, which may be small because the enhancement of high multiplicities (low kinetic energies) tends to cancel the effect of the smaller interaction volume needed to fit $\langle n_\pi \rangle = 5.36$. However, the factor m_i/E_i clearly favors $\langle T_K \rangle$ at the expense of $\langle E_\pi \rangle$, and we estimate this effect as follows. We can make use of the exact two particle solutions

$$\frac{dN}{dW}(m_1, m_2) = \frac{\pi}{2} \left[(W^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 \right]^{1/2} \left\{ 1 - \left[\frac{m_1^2 - m_2^2}{W^2} \right]^2 \right\} \quad (4.4a)$$

$$\frac{dN'}{dW}(m_1, m_2) = \frac{2\pi m_1 m_2}{W^2} \left[(W^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2 \right]^{1/2} \quad (4.4b)$$

to compare $X(n_\pi=1, P_k)$ and $X'(n_\pi=1, P_k)$ accurately. In this way we find $\langle T_k(n_\pi=1) \rangle = 206$ Mev. using (4.1), and $\langle T'_k(n_\pi=1) \rangle = 221$ Mev. using (4.2). dN'/dW cannot be solved analytically for $n_\pi > 1$ and has not been tabulated, but clearly $(\langle E_\pi(n_\pi) \rangle - \langle E'_\pi(n_\pi) \rangle)$ is less than the 30 Mev. obtained for $n_\pi=1$ because $m_\pi/\langle E_\pi \rangle$ increases with increasing n_π . Rough estimates indicate that the covariant treatment shifts $\langle E_\pi \rangle$ downward by ~ 20 Mev and $\langle T_k \rangle$ upward by ~ 15 to 20 Mev for all models.

We can compare the theoretical values for $\langle T_k \rangle$ with the experimental values¹⁾ 140 Mev for the 3 definite charged K 's observed at Berkely, or 182 Mev for the 10 probable charged K 's seen at Berkeley*. Since slow K 's are more likely to decay or come to rest in the emulsions, the "definite" K 's are biased towards low energies. The 10 probable K 's seem to favor $KT(a)$, but the statistics certainly do not allow a decision as yet.

We have also listed the average total energy of a pion detected in a $K\bar{K}$ event, in Table II (vii). The percentage variation from model to model for this quantity is

* The events in reference (1) had an average total energy of 1927 Mev, including antiproton kinetic energy. To compare with the theory for events at rest in Table II (vi), we subtracted 10 Mev from the 182 Mev experimental result.

somewhat less than for the other quantities discussed in this section.

The physical reason why few pions are associated with $K\bar{K}$ production in $KT(a)$ is that most $K\bar{K}$ pairs are produced from states containing fewer than average cloud pions, and all $K\bar{K}$ pairs are emitted by the small, "high temperature" core which favors high kinetic energies and low core pion multiplicities. On the other hand the large Fermi interaction volume ("low temperature") and the pion-pion interaction build up the pion multiplicity at the expense of K kinetic energy in F and P , and this feature is also expected to hold for other models^{(5),(6)} based on a strong pion interaction.

§ 5. Conclusion

We have found that $KT(b)$ and F , which involve large radii that can only result from strong pion-pion interactions, yield predictions similar to those of P . On the other hand $KT(a)$ implies weak pion-pion interactions and differs significantly from the other models with respect to the details of $K\bar{K}$ events. Therefore a reasonably small number of $K\bar{K}$ events can decide whether antinucleon annihilation is dominated by pion-pion interactions or by the KT cloud-core picture.

Before one compares experimental results with Table II (v-vii), however, it should be remembered that: (a) The numerical predictions obtained for $K\bar{K}$ events depend strongly on the assumed pion multiplicity $\langle n_\pi \rangle = 5.36$ (footnote, § 1.) A trustworthy comparison between statistical theories and experimental $K\bar{K}$ events cannot be made until $\langle n_\pi \rangle$ is determined more accurately.

(b) We have neglected conservation of angular momentum, parity, and charge conjugation, and we have used the usual non-covariant phase space elements, d^3P . In § 4. we estimated that a fully covariant treatment of momentum phase space would increase $\langle T_k \rangle$ by ~ 15 or 20 Mev and decrease $\langle E_\pi \rangle$ by ~ 20 Mev.

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The Σ - Λ Relative Parity and the \bar{K} - N Reaction

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Possible resonances in pion-hyperon scattering at low energies are qualitatively examined and some noticeable differences are found between two cases in which the Σ - Λ relative parity is even and odd. Then, supposing that, in the process $\bar{K} + N \rightarrow \pi + Y$, the strong resonances of pion-hyperon scattering in the final state will dominate over the relatively weak K -meson interaction, we discuss the possibility to conjecture the relative parities of K -meson and baryons from the experiment of this process.

§ 1. Introduction

As was discussed by Gell-Mann¹⁾ the coupling of K -meson to baryons seems to be somewhat weaker than that of pion. Hence in the process $\bar{K} + N \rightarrow \pi + Y^*$, the pion-hyperon scattering is expected to play an important role as a final state interaction. If the pion-baryon interaction has universal form and strength, strong resonances similar to the $(3/2, 3/2)$ -pion-nucleon scattering will occur in the final pion-hyperon scattering¹⁾ and enhance relevant amplitudes of the \bar{K} -absorption. It was further noticed that the condition of universal strength should be somewhat relaxed. Namely, as far as $2 \gtrsim (f_\Sigma/f_\Lambda)^2 \gtrsim 1/2$, the qualitative character of resonances will not be radically altered²⁾, f_Σ and f_Λ being respectively the renormalized $(\Sigma\Sigma\pi)$ - and $(\Sigma\Lambda\pi)$ -coupling constants. In this paper** we mean by universal pion-baryon interaction that the Σ - Λ relative parity $P_{\Sigma\Lambda}$ is even and that f_Σ^2 and f_Λ^2 are of the same order of magnitude with $f_N^2 \approx 0.08$.

The possibility of experimental test of the universal pion-baryon interaction by the \bar{K} -absorption has been discussed by several authors³⁾. It should be noticed, however, that in order to know what are characteristic of the universal pion-baryon interaction or of even $P_{\Sigma\Lambda}$, consequences of other types of interaction must be examined at the same time. Indeed, strong resonances of the pion-hyperon scattering are, as we shall see, not necessarily characteristic of the universal pion-baryon interaction but will occur, in some states, also by another type of pion-hyperon in-

* Here and henceforth Y and the word hyperon will mean either Σ - or Λ -hyperon. This process will simply be called \bar{K} -absorption.

** The spins of hyperons are all assumed to be $1/2$. It is, of course, assumed that the isospin, strangeness and parity are conserved in the strong interactions and that the members of a charge multiplet have same parity relative to others.

teraction ($P_{\Sigma\Lambda}$ is odd.). Therefore, some processes that have been considered to serve as experimental tests of the universal pion-baryon interaction will turn out to be useless.

In Sec. 2 we shall examine the pion-hyperon scattering at low energies in the case of odd $P_{\Sigma\Lambda}$. It will be pointed out that in some states resonance will be found commonly in both cases of even and odd $P_{\Sigma\Lambda}$, while in some states resonance will occur only in one of the two cases. Then implications of these resonances in the \bar{K} -absorption will be discussed in Sec. 3 in relation to the possibility to determine the relative parities of K -meson and baryons. In Sec. 4 some comments will be given on previous investigations on this topics.

§ 2. Pion-hyperon scattering

If the K -meson interaction is sufficiently weak the amplitudes of the \bar{K} -absorption can be approximately expressed by the pion-hyperon scattering amplitudes. The S -wave pion-interaction will, however, be of the same order of strength with that of K -meson, and perhaps cannot be clearly disentangled in the \bar{K} -absorption. Whereas strong P -wave resonances of the pion-hyperon scattering, if exist, will be confirmable by marked enhancement of relevant amplitudes of the \bar{K} -absorption.

We do not intend to make detailed analysis but confine our attention only to strong resonances of the P -wave scattering at low energies. Throughout this section $P_{\Sigma\Lambda}$ is taken to be odd, unless otherwise mentioned. We use the static approximation⁴⁾, reducing the pseudoscalar ($\Sigma\Sigma\pi$)-coupling to the familiar P -wave vertex and the *scalar* ($\Sigma A\pi$)-coupling to the S -wave vertex. The Σ - A mass difference and the virtual effect of K -meson will be ignored.

There are three types of scattering,

$$\begin{aligned} (\Sigma \rightarrow \Sigma) : \quad & \pi + \Sigma \rightarrow \pi + \Sigma, \\ (\Sigma \leftrightarrow A) : \quad & \pi + \Sigma \leftrightarrow \pi + A, \\ (A \rightarrow A) : \quad & \pi + A \rightarrow \pi + A. \end{aligned} \tag{2.1}$$

In $(A \rightarrow A)$, since P -wave is not scattered in the Born approximation, the P -wave resonance will not occur. On the contrary, strong resonance will occur in $(A \rightarrow A)$ if $P_{\Sigma\Lambda}$ is even*. This is the first difference to be noticed. One may conceive here that, if the ($\Sigma A\pi$)-coupling constant is considerably large, a resonance may take place in S -wave scattering in $(A \rightarrow A)$. It will not be the case, however. It is shown that S -wave scattering in $(A \rightarrow A)$ does not occur in the Born approximation,

* Remember that if the pion-baryon interaction is strictly universal the pion-hyperon scattering amplitudes are related to those of pion-nucleon scattering, $T_{1/2}$ and $T_{3/2}$, as follows.

$$\begin{aligned} T(\Sigma \rightarrow \Sigma)_0 &= T_{1/2}, & T(\Sigma \rightarrow \Sigma)_1 &= (2T_{1/2} + T_{3/2})/3, \\ T(\Sigma \rightarrow \Sigma)_2 &= T_{3/2} \text{ and } T(A \rightarrow A)_1 &= (T_{1/2} + 2T_{3/2})/3, \end{aligned}$$

where the subscripts indicate the isospin states.

if the Σ - Λ mass difference is ignored. Though if the mass difference is taken into account the matrix element of the scattering is nonvanishing, its sign indicates that the scattering potential is repulsive.

In ($\Sigma \leftrightarrow \Lambda$), pion's angular momentum changes as $S \leftrightarrow P$; repeating of ($\Sigma \leftrightarrow \Lambda$) contributes to ($\Sigma \rightarrow \Sigma$). We write the T -matrix element of ($\Sigma \rightarrow \Sigma$) with P -wave and that of ($\Sigma \rightarrow \Lambda$) with ($P \rightarrow S$) wave, respectively, as follows.

$$\begin{aligned} (\Sigma \rightarrow \Sigma): & -v_q v_p \cdot 2\pi (\omega_q \omega_p)^{-1/2} P_I P_J h_{IJ}(\omega_p), \\ (\Sigma \rightarrow \Lambda): & -v_q v_p \cdot 2\pi (\omega_q \omega_p)^{-1/2} (\boldsymbol{\sigma} \cdot \mathbf{q}) h'(\omega_p). \end{aligned} \quad (2.2)$$

Here \mathbf{q} and \mathbf{p} are respectively the initial and the final momenta of pion, v_p is a cut-off factor, $\omega_p = (1 + p^2)^{1/2}$, the unit $c = \hbar = m_\pi = 1$ is used. P_I is a projection operator for the $\pi + \Sigma$ state of $I=0, 1$ or 2 . $P_{J=1/2} = (\boldsymbol{\sigma} \mathbf{p})(\boldsymbol{\sigma} \mathbf{q})$, $P_{J=3/2} = 3\mathbf{p}\mathbf{q} - (\boldsymbol{\sigma} \mathbf{p})(\boldsymbol{\sigma} \mathbf{q})$. $\boldsymbol{\sigma}$ is the spin vector of hyperon. The ($\Sigma \rightarrow \Lambda$) belongs to $(I, J) = (1, 1/2)$. The Low equation for h_{IJ} is, in the one meson approximation,

$$\begin{aligned} h_{IJ}(\omega) = & \frac{\lambda_{IJ}}{\omega} + \frac{1}{\pi} \int d\omega_p p v_p^2 \left[\frac{p^2 |h_{IJ}(\omega_p)|^2 + \delta_{I,1} \delta_{J,1/2} |h'(\omega_p)|^2}{\omega_p - \omega - i\epsilon} \right. \\ & \left. + \sum_{I'J'} C_{IJ,I'J'} \frac{p^2 |h_{I'J'}(\omega_p)|^2 + \delta_{I',1} \delta_{J',1/2} |h'(\omega_p)|^2}{\omega_p + \omega} \right]. \end{aligned} \quad (2.3)$$

The coefficient of the Born term is (see Appendix)

$$\lambda_{IJ} = \frac{f_\Sigma^2}{3} \times \begin{cases} 2 \\ -4 \\ -7 \\ 2 \\ -1 \\ 2 \end{cases} \quad \text{for} \quad (I, J) = \begin{cases} (0, 1/2) \\ (0, 3/2) \\ (1, 1/2) \\ (1, 3/2) \\ (2, 1/2) \\ (2, 3/2) \end{cases} \quad (2.4)$$

h' in the Born approximation is given by

$$h'(\omega)_{\text{Born}} = \lambda' / \omega, \quad \lambda' = -2\sqrt{2} f_\Sigma f_\Lambda. \quad (2.5)$$

f_Σ and f_Λ are the renormalized and unrationalized coupling constants of the ($\Sigma \Sigma \pi$)- and the ($\Sigma \Lambda \pi$)-interaction, respectively. The crossing matrix is

$$\begin{aligned} C_{IJ,I'J'} &= C_{II'} \times C_{JJ'}, \\ C_{II'} &= \frac{1}{6} \begin{pmatrix} 2 & -6 & 10 \\ -2 & 3 & 5 \\ 2 & 3 & 1 \end{pmatrix}, \quad C_{JJ'} = \frac{1}{3} \begin{pmatrix} -1 & 4 \\ & & \\ 2 & 1 \end{pmatrix}. \end{aligned} \quad (2.6)$$

In the case of even $P_{\Sigma\Lambda}$, resonances occur in the states with $I=1, 2$, $J=3/2$. Whereas in the case of odd $P_{\Sigma\Lambda}$ the sign of λ_{IJ} of eq. (2.4) suggests that, in addition to these states with $I=1, 2$, $J=3/2$, resonance possibly occurs also when

$I=0$ and $J=1/2$. ($\lambda_{IJ}>0$ means that the scattering potential is attractive. For even $P_{\Sigma\Lambda}$, $\lambda_{0,1/2}=2/3 \cdot (f_{\Sigma}^2 - 5f_{\Lambda}^2)^{1/2}$. Therefore the sign of $\lambda_{0,1/2}$ changes with $P_{\Sigma\Lambda}$.) Indeed, if f_{Λ}^2 is very small ($5f_{\Lambda}^2 \ll f_{\Sigma}^2$) when $P_{\Sigma\Lambda}$ is even, the two cases of even and odd $P_{\Sigma\Lambda}$ will be qualitatively similar to each other, as far as resonance behaviors in the pion-hyperon scattering are concerned. According to Ferrari and Fonda's analysis of the Λ - N force, however, the case of $f_{\Lambda}^2=0$ is ruled out irrespectively of K -meson's parity.⁵⁾ We think their conclusion on this point is reliable, hence we do not consider the case of very small f_{Λ}^2 .

Following Chew and Low⁴⁾ we rewrite eq. (2.3) in the form,

$$h_{\alpha}(\omega) = \frac{\lambda_{\alpha}}{\omega} \cdot \left[1 - \frac{\omega}{\pi} \lambda_{\alpha} \int d\omega_p \frac{p^2 v_p^2}{\omega_p^2} \cdot \left\{ \frac{X_{\alpha}(\omega_p)}{\omega_p - \omega - i\epsilon} + \frac{Y_{\alpha}(\omega_p)}{\omega_p + \omega} \right\} \right]^{-1}, \quad (2.7)$$

$$X_{\alpha}(\omega_p) = p^2 + \delta_{\alpha} |h'(\omega_p)|^2 / |h_{\alpha}(\omega_p)|^2, \quad (2.8)$$

$$Y_{\alpha}(\omega_p) = \sum_{\beta} C_{\alpha\beta} \{ p^2 |h_{\beta}(\omega_p)|^2 + \delta_{\beta} |h'(\omega_p)|^2 \} / \left| \sum_{\beta} C_{\alpha\beta} h_{\beta}(\omega_p) \right|^2,$$

where $\alpha \equiv (I, J)$ and $\delta_{\alpha} \equiv \delta_{I,1} \delta_{J,1/2}$. An approximate solution will be given by substitution for h_{α} and h' in X_{α} and Y_{α} by the corresponding Born terms, $\lambda_{\alpha}/\omega_p$ and λ'/ω_p . Namely,

$$h_{\alpha}(\omega) = \frac{\lambda_{\alpha}}{\omega} \cdot \left[1 - \frac{\omega}{\pi} \left\{ \lambda_{\alpha} F_{-}(\omega) + \delta_{\alpha} \frac{\lambda'^2}{\lambda_{\alpha}} F'_{-}(\omega) + \lambda_{\alpha} F_{+}(\omega) + \lambda'_{\alpha} F'_{+}(\omega) \right\} \right]^{-1}, \quad (2.9)$$

where

$$A_{\alpha} = \sum_{\beta} C_{\alpha\beta} \frac{\lambda_{\beta}^2}{\lambda_{\alpha}} = \frac{f_{\Sigma}^2}{3} \times \begin{cases} 13 \\ 7 \\ 8 \\ 8 \\ -2 \\ 10 \end{cases}, \quad A'_{\alpha} = \sum_{\beta} C_{\alpha\beta} \delta_{\beta} \frac{\lambda'^2}{\lambda_{\alpha}} = 4f_{\Lambda}^2 \times \begin{cases} 1 \\ 1 \\ 1/7 \\ 1 \\ 1 \\ 1 \end{cases}, \quad (2.10)$$

and

$$F_{\pm}(\omega) = \int d\omega_p p^3 v_p^2 / \omega_p^2 (\omega_p \pm \omega - i\epsilon), \quad (2.11)$$

$$F'_{\pm}(\omega) = \int d\omega_p p v_p^2 / \omega_p^2 (\omega_p \pm \omega - i\epsilon).$$

In the $(I, J) \neq (1, 1/2)$ states, the phase shift δ_{α} is given by $\text{Re } h_{\alpha}^{-1}(\omega_p) = p^3 v_p^2 \cot \delta_{\alpha}(p)$. In the $(I, J) = (1, 1/2)$ state, the phase shift is not real because of the

coexistence of two channels, ($\Sigma \rightarrow \Sigma$) and ($\Sigma \rightarrow \Lambda$).

As for $F_{\pm}(\omega)$ and $F'_{\pm}(\omega)$, we simply replace them by their values at $\omega=1$, ignoring their ω -dependence. Using a cut-off energy $\omega_{\max}=6$, we have for $\omega=1$

$$F_+=6.0, \quad F'_+=1.2, \quad \text{Re } F_-=3.0, \quad \text{Re } F'_-=0.4. \quad (2.12)$$

Then eq. (2.9) becomes

$$(\lambda_a/\omega) \text{Re } h_a^{-1}(\omega) \approx 1 - r_a \omega \quad (2.13)$$

where

$$r_a \approx f_{\Sigma}^2 \begin{pmatrix} 5.4 \\ -0.3 \\ -1.9 \\ 3.8 \\ -1.3 \\ 4.5 \end{pmatrix} + f_{\Lambda}^2 \begin{pmatrix} 0.51 \\ 0.51 \\ -1.24 \\ 0.51 \\ 0.51 \\ 0.51 \end{pmatrix}. \quad (2.14)$$

If $f_{\Sigma}^2 \approx 0.08$ and, for example, $(f_{\Lambda}/f_{\Sigma})^2 \approx 5$ are assumed, we have

$$r_a \approx \begin{pmatrix} 0.43+0.21 \\ -0.03+0.21 \\ -0.15-0.50 \\ 0.31+0.21 \\ -0.10+0.21 \\ 0.36+0.21 \end{pmatrix} = \begin{pmatrix} 0.64 \\ 0.18 \\ -0.65 \\ 0.52 \\ 0.11 \\ 0.57 \end{pmatrix}. \quad (2.15)$$

Here the first numbers show the value when $f_{\Lambda}^2=0$. The resonance energy is given by $1/r_a$. Therefore it may be concluded that, if $f_{\Sigma}^2 \approx f_N^2$, and $(f_{\Lambda}/f_{\Sigma})^2 \lesssim 5$, resonances will occur at about $\omega \approx 2 \sim 3$ in the states with $(I, J) = (0, 1/2)$, $(1, 3/2)$ and $(2, 3/2)^*$. If $(f_{\Lambda}/f_{\Sigma})^2 \gg 5$, resonances may occur also in other states, but for the present we tentatively assume that f_{Λ}^2 is not so large. Significant results of this section are summarized in Table I. Contrasts between two cases of even and odd $P_{\Sigma\Lambda}$ are found only in ($\Sigma \rightarrow \Sigma$) of $I=0$ and in ($\Lambda \rightarrow \Lambda$).

Table I. $\bigcirc(\times)$ means presence (absence) of resonance.

$P_{\Sigma\Lambda} \backslash (I, J)$	$(\Sigma \rightarrow \Sigma)$						$(\Lambda \rightarrow \Lambda)$	
	(0, 1/2)	(0, 3/2)	(1, 1/2)	(1, 3/2)	(2, 1/2)	(2, 3/2)	(1, 1/2)	(1, 3/2)
even	\times	\times	\times	\bigcirc	\times	\bigcirc	\times	\bigcirc
odd	\bigcirc	\times	\times	\bigcirc	\times	\bigcirc	\times	\times

* The present approximation applied to the pion-nucleon scattering gives the effective range $r \approx 16f_N^2/\pi = 0.41$ to the $(3/2, 3/2)$ state.

§ 3. The \bar{K} -absorption

First, we present some kinematical considerations. The interaction range R of the K -meson-nucleon reactions would not be much greater than the compton wave length of the K -meson, $1/m_K$. As a result of this we may suppose that a K -meson with angular momentum greater than

$$l = R/\lambda \quad (3.1)$$

will not play an important role. Here $\lambda = 1/p_K$ is the de Broglie wave length of the K -meson with momentum p_K . To illustrate this, in Fig. 1, we plot l as determined from eq. (3.1) with two choices of $R = 1/m_K$ and $2/\bar{m}_K$. The same quantity in case of the pion-nucleon system with $R = 1/m_\pi$ is shown by the dotted line for convenience of comparison. It is seen that the limitation on l is more stringent for the K -meson-nucleon case than for the pion-nucleon case. At very low energies, say, below 50 Mev (K -meson's kinetic energy in the laboratory system), an S -wave K -meson will dominantly interacts with a nucleon.

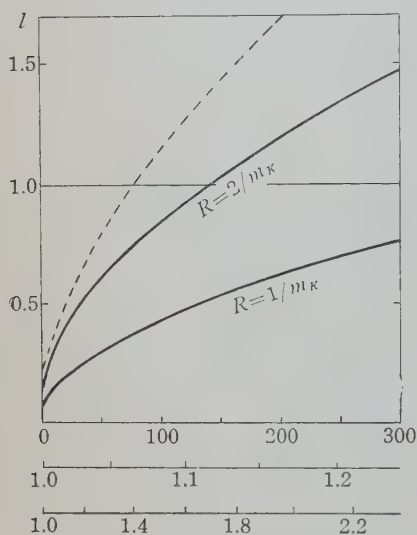


Fig. 1 The first abscissa indicates the lab. kinetic energy in Mev of either a K -meson or a pion. The second and the third ones indicate respectively the c. m. total energy of a K -meson in m_K and that of a pion in m_π .

four possibilities are Σ^+K_+ , Σ^+K_- , Σ^-K_+ and Σ^-K_- , where the subscript indicates parity relative to $N\Lambda$.

If the initial state is $S_{1/2}$, resonance can be involved only in Σ^-K_- . In the Σ^-K_- case the amplitude of the Σ -production related to the resonance ($I=0$, $P_{1/2}$) will be predominant, hence the branching ratio will be $\Sigma^+ : \Sigma^0 : \Sigma^- \approx 1 : 1 : 1$. The superscript indicates the charge. While the Λ -production will be relatively scarce because the final state of the Λ -production is $S_{1/2}$ and is not concerned with resonance. If this is the case, the production ratio of Σ and Λ will rather sensitively depend on energy. The enhancement of the Σ -production alone is especially favorable for the interpretation of the experimental result which shows infrequent Λ -production. Namely, Alvarez et al.⁶⁾ reported that, in flight plus a rest events yield $\Sigma^+ : \Sigma^0 : \Sigma^- : \Lambda$

$=4:4:8:1$. Eisenberg et al. 's analysis⁷⁾ of events in flight showed the rise of the Λ -production with increasing energy. In cases other than Σ^-K_- , this Σ - Λ ratio could not be understood without going into detailed mechanism of pion and K -meson interactions.

If the incident K -meson is P -wave, $P_{3/2}$ resonance will occur not only in the case of Σ^+K_- but also in the case of Σ^-K_+ , and that $I=1$ in both cases*. If we observe only the Σ -production, these two cases could not be distinguished. There will, however, be remarkable difference in the Σ - Λ ratio between two cases. As has been discussed by Ross³⁾ the $I=1$, $P_{3/2}$ resonance in the Σ^+K_- case will result $\Sigma^+ : \Lambda \approx 1 : 2$ (more precisely $\Sigma^+ : \Sigma^0 : \Sigma^- : \Lambda \approx 1 : 0 : 1 : 4$) while in the Σ^-K_+ case Σ will dominate over Λ , because only the Σ -production will be enhanced by the final state resonance. In case of Σ^-K_+ one more resonance is involved ($I=0$, $P_{1/2}$). If we could obtain information on the pure $I=0$ state ($K^- + p \rightarrow \pi^0 + \Sigma^0$), the $P_{1/2}$ resonance could be a clue to distinguish the Σ^-K_+ case from the Σ^+K_- case. The above results are summarized in Table II.

Table II. $\Sigma(1, 3/2)$, for example, means that resonance is involved in the Σ -production of $(I, J) = (1, 3/2)$. The \times means absence of resonance.

initial state \ case	Σ^+K_+	Σ^+K_-	Σ^-K_+	Σ^-K_-
S	\times	\times	\times	$\Sigma(0, 1/2)$
P	\times	$\Sigma(1, 3/2)$ $\Lambda(1, 3/2)$	$\Sigma(0, 1/2)$ $\Sigma(1, 3/2)$	\times

In actual phenomena, the initial state will contain both of S - and P -waves, so the analysis will be rather involved. At sufficiently low energies, however, the S -wave will be dominant in the initial state, and the $P_{1/2}$ resonance in the final state will provide useful clues to determine the parities of the K -meson and the hyperons.

§ 4. Discussion

Though the \bar{K} -absorption will be an efficient tool to determine the relative parities of the K -meson and the baryons, this alone will not suffice to be decisive. If $P_{\Sigma\Lambda}$ is once determined, the dispersion relations applied to the K - N and \bar{K} - N scatterings will be effective for the determination of the K -meson's parity relative to the baryons⁸⁾. As for $P_{\Sigma\Lambda}$ itself, however, the dispersion relations seem to offer rather poor information⁹⁾. Therefore, it is desirable to find clues on $P_{\Sigma\Lambda}$ in other

* These cases could be distinguished from the above mentioned $I=0$, $P_{1/2}$ resonance in Σ^-K_- case by that (i) in $I=0$ state the production ratio $\Sigma^+ : \Sigma^0 : \Sigma^-$ is $1 : 1 : 1$, while it is $1 : 0 : 1$ in the $I=1$ state and (ii) the angular distribution in $P_{3/2}$ state contain $\cos^2\theta$ term while that in $P_{1/2}$ state does not.

processes and/or methods.

Here, confining ourselves to processes wherein the resonances of pion-hyperon scattering at low energies will play important roles, we give some comments on previous investigations. It has been suggested by Ito, Minami and Tanaka¹⁰⁾ that the processes

$$\begin{aligned} (a) \quad & K^- + D \rightarrow \pi^- + p + \Lambda, \\ (b) \quad & \pi^+ + p \rightarrow \pi^+ + K^0 + \Sigma^+, \\ (c) \quad & \pi^+ + p \rightarrow \pi^+ + K^+ + \Lambda, \end{aligned} \quad (4.1)$$

will serve to the test of the universal pion-baryon interaction or of the $P_{\Sigma\Lambda}$. Their discussion on (a) and (c) holds good because the $\pi + \Lambda$ system involves resonance if $P_{\Sigma\Lambda}$ is even and does not if $P_{\Sigma\Lambda}$ is odd. While the process (b) will be useless because the $\pi^+ + \Sigma^+$ system ($I=2$) involves resonance commonly in both cases of even and odd $P_{\Sigma\Lambda}$.

Ross⁹⁾ suggested that the $\pi + \Lambda$ resonance will be found in the processes

$$\Sigma^- + p \rightarrow \begin{cases} \pi^- + p + \Lambda, \\ \pi^0 + n + \Lambda. \end{cases} \quad (4.2)$$

These will be effective as tests by the same reason with the above mentioned processes (a) and (c).

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Appendix

In this appendix we obtain the coefficient λ_α , eq. (2.4). In order to proceed in parallel with the case of pion-nucleon scattering, it is convenient to introduce isospin operator ρ_α ($\alpha=1, 2, 3$) for Σ -hyperon and write the Hamiltonian of the $(\Sigma\Sigma\pi)$ -interaction in the form $\sum_{\alpha=1,2,3} \bar{\Sigma} \rho_\alpha \Sigma \pi_\alpha$ analogous to the $(NN\pi)$ -interaction, $\sum_{\alpha=1,2,3} \bar{N} \tau_\alpha N \pi_\alpha$. Then ρ_α satisfies

$$\rho_1 \rho_2 - \rho_2 \rho_1 = i \rho_3, \quad (\text{cyclic}) \quad \rho^2 = \sum_{\alpha=1,2,3} \rho_\alpha^2 = 1(1+1) = 2. \quad (A.1)$$

An explicit representation is

$$\rho_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (A.2)$$

with $\Sigma = (\Sigma_1, \Sigma_2, \Sigma_3)$ and $\pi = (\pi_1, \pi_2, \pi_3)$.

The method used by Wick¹¹⁾ in pion-nucleon scattering can be applied to this case. Let pion's isospin operator, which has properties similar to ρ , be t and the total isospin operator of the $\pi + \Sigma$ system be I . Then

$$I^2 \equiv (t + \rho)^2 = t^2 + \rho^2 + 2t\rho. \quad (A.3)$$

Remembering that $I^2 = I(I+1)$ and $t^2 = \rho^2 = 2$, we see

$$t\rho = I(I+1)/2 - 2. \quad (A.4)$$

Next, introduce matrices Q and Q' with elements

$$Q_{\mu\lambda} = \rho_{\mu}\rho_{\lambda} \quad \text{and} \quad Q'_{\mu\lambda} = \rho_{\lambda}\rho_{\mu}, \quad (A.5)$$

which are analogues of Wick's Q and Q' . When the index λ (μ) indicates the isospin component of the initial (final) pion, Q and Q' correspond respectively to the uncrossed and the crossed types of scattering diagrams in the Born approximation. The eigenvalues of Q and Q' are obtained by the relations

$$Q = (1 - t\rho)(2 + t\rho) \quad \text{and} \quad Q' = 2 - (t\rho)^2. \quad (A.6)$$

λ_α is obtained by using the result shown in Table A instead of Table II in Wick's paper and replacing f_N by f_Σ .

The projection operator P_I in eq. (2.2) can be expressed in terms of Q and Q' as follows.

$$\begin{aligned} P_0 &= (1 - Q')/3, & P_1 &= Q/2, \\ P_2 &= (2 - 3Q/2 + Q')/3. \end{aligned} \quad (A.7)$$

The crossing matrix C in eq. (2.5) is also obtained by a fashion similar to that explained in Sec. 6 of Wick's paper.

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Effect of Hard Cores on the Binding Energies of H^3 and He^3 —Corrected Values—

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There was a slight error in the kinetic energy formula of the paper I by Ohmura (formerly Kikuta), Morita, and Yamada. The binding energy of H^3 as well as the Coulomb energy of He^3 are recalculated as described in I by using the correct value of the kinetic energy. The results obtained are, however, almost the same as in I and II.

In the previous papers,^{1,2)} (referred to as I, II respectively hereafter), the binding energy of H^3 has been calculated by assuming two-body central forces with hard cores of various sizes. It has been shown that the binding energy is reduced considerably with the increase of the hard core radius, and that the binding energy is not very sensitive to the shape of the potential if the hard core radius is quite large (for example, 0.6×10^{-13} cm). The observed energy difference between He^3 and H^3 (~ 0.76 Mev) is much smaller than the Coulomb energies calculated by many authors³⁾, which are about 1 Mev or more. However, if the two-body nuclear force has a strong repulsive core in it, the two protons of He^3 may be pushed away from one another so that the Coulomb energy can be expected to be reduced. In I and II we found this to be the case. Such an effect of the hard core on the binding energy of H^3 and on the Coulomb energy of He^3 are assumed to be valid even when the two-body potential has the tensor force included in it.

Meanwhile Dr. G. Derrick of the University of Sydney used some numerical values of I as a check on his coding for SILLIAC, and found a small difference (~ 1 Mev) between his kinetic energy values and ours. Then he checked the cumbersome formulae for kinetic energy given in I, and has kindly informed me in the fall of 1957 that the term $\cdots + (D^2 - 16D/x - 8/x^2)C_1(x)$ in the expression on the second line of page 232 of I should be replaced by $\cdots + (D^2 - 14D/x - 8/x^2)C_1(x)$. Thus we decided to recalculate the binding energy of H^3 and the Coulomb energy of He^3 by using the correct values of kinetic energy, though the corrections were expected to be rather small. The same assumptions, the same notation and the same method of calculations as those in I and II are used. We use the same potentials as in I and II; namely, (1), (2), Table I in I and (1), Table I in II. The

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trial function is the same as (4) in I. The binding energy and the values of parameters μ and ν determined with variational method in this way are given in Tables I and II.

Table I. Variationally computed binding energy of H^3 .

Hard core radius D in 10^{-13} cm	B. E. (H^3) in Mev			
	$r_{0s} \doteq 2.7 \times 10^{-13}$ cm		$r_{0s} \doteq 2.4 \times 10^{-13}$ cm	
	Yukawa pot.	Exponential pot.	Yukawa pot.	Exponential pot.
0.0	12.49	10.26	15.97	11.38
0.2	9.15	8.81	10.0	9.98
0.4		7.52		8.85
0.6	6.43	5.79	7.41	7.38
Experimental Value	8.49			

Table II. Adjusted values of μ and ν in 10^{13} cm^{-1}

D in 10^{-13} cm	$r_{0s} \doteq 2.7 \times 10^{-13}$ cm				$r_{0s} \doteq 2.4 \times 10^{-13}$ cm			
	μ		ν		μ		ν	
	Yukawa	Expon.	Yukawa	Expon.	Yukawa	Expon.	Yukawa	Expon.
0.0	0.605	0.479	∞	∞	0.659	0.503	∞	∞
0.2	0.447	0.462	5.93	5.03	0.489	0.479	6.1	5.20
0.4		0.457		4.09		0.500		4.27
0.6	0.452	0.450	4.28	4.20	0.498	0.494	4.47	4.51

Since the adjustment of the potential parameters is somewhat inaccurate in the case of the Yukawa type potential, all the values for the Yukawa well are considered to be about one order of magnitude less precise than those for the exponential well. But if we assume the validity of potential parameters, all the tabulated figures in Table I~X have validity. The exceptions are Tables II and III, where the tabulated values contain errors of the order 1%, or even up to several per cent.

If charge independence of nuclear forces is assumed, the Coulomb energy of He^3 can be calculated by the perturbation method with reasonable precision. The results are given in Table III. The detailed numerical results which appear in the course of our calculation of kinetic energy, potential energy, etc., are given in Table IV~X in the Appendix. By comparing with Tables in references I and II, we see that: 1) The kinetic energies are reduced by an amount of the order of magnitude of 1 Mev in case of $D \neq 0$. Consequently, the binding energies increase by a similar amount. 2) Adjusted values of μ increase somewhat, but ν practically remains unchanged. 3) The results for $D=0$ remain correct, because the corrected term contains a factor D . 4) Some qualitative conclusions stated in I and II are not altered, namely, a) Hard cores push out the wave functions of H^3 and He^3 ,

Table III. Coulomb energy of He^3

Hard core radius D in 10^{-13} cm	C. E. of He^3 in Mev			
	$r_{0s} \doteq 2.7 \times 10^{-13}$ cm		$r_{0s} \doteq 2.4 \times 10^{-13}$ cm	
	Yukawa pot.	Exponential pot.	Yukawa pot.	Exponential pot.
0.0	1.246	0.986	1.358	1.037
0.2	0.830	0.836	0.896	0.861
0.4		0.746		0.798
0.6	0.688	0.686	0.736	0.733
Experimental Value	0.764			

so that the Coulomb energy of He^3 , which is too large without the hard cores, is brought into agreement with the experimental value of the difference between the binding energies of H^3 and He^3 , b) Hard cores reduce the binding energies of H^3 and He^3 considerably, c) The hard core effects are slightly more enhanced for the Yukawa potential than for the exponential, especially near $D=0$, possibly because of the rapid change of the potential form in this region, d) Discussions in I and II about the approximations can be applied also to the present case, and so the values of B.E. (H^3) are expected to be about 1 Mev larger than the values in Table I, while the Coulomb energies may be several per cent smaller than the values in Table III.

If we take the hard core radius D as $0.3 \sim 0.4 \times 10^{-13}$ cm for $r_{0s} = 2.7 \times 10^{-13}$ cm or $D = 0.4 \sim 0.6 \times 10^{-13}$ cm for $r_{0s} = 2.4 \times 10^{-13}$ cm, reasonable fit to the experimental values is obtained.

Since the binding energy thus calculated is rather sensitive to unestablished details of the nuclear force, and since the tensor potential must be taken into account, the absolute value of the binding energies of H^3 should not be considered literally. The inclusion of the tensor force reduces the binding energy³⁾ of H^3 to some extent. On the contrary, we can clearly see from Table III the general trend of the effect of hard cores on the Coulomb energy of He^3 . Because, in the case of the exponential potential, it is inferred from Table II that the diminution of the Coulomb energy for the exponential potential is not due to the behavior of the wave function at large distances, which is mainly specified by the parameter μ , but due to the vanishing of the wave function at short distances.

The inclusion of the tensor potential in the two-body force may decrease the Coulomb energy. This fact will be understood by comparing the calculated values of Pease and Feshbach³⁾ with Table III of the present paper. Using the Yukawa well for both central and tensor parts, Pease and Feshbach obtained 1.01~1.04 Mev for the Coulomb energy, while Table III gives 1.245 Mev (assuming $r_{0s} \doteq 2.7 \times 10^{-13}$ cm) and 1.358 Mev (assuming $r_{0s} \doteq 2.4 \times 10^{-13}$ cm) for the Yukawa well. The reason may be as follows: The tensor potential mixes D -states (and P -states) in H^3

state. These states are pushed away outside by the centrifugal forces, so that the Coulomb energy arising from these mixed parts of the wave function decreases considerably. There is also another reason that the inclusion of tensor force reduces the binding energy so that the nucleus has a larger extension.

Summary. This note reports the corrected values of our results in the previous papers I and II. The correction is done by replacing $(D^2 - 16D/x - 8/x^2)C_1(x)$, the last part of the second line of p. 232 in I, by $(D^2 - 14D/x - 8/x^2)C_1(x)$. Consequently, the kinetic energies decrease about 1 Mev for $D \neq 0$. There is no change for $D = 0$. Otherwise the results obtained are almost the same as in I and II. The absolute value of binding energy is relatively sensitive to unestablished details of the nuclear force, but we can clearly see from Table III the general trend of the effect of hard cores on the Coulomb energy of He^3 . Because it is inferred from Table II that the diminution of the Coulomb energy for the exponential potential is not due to the behavior of the wave function at large distances, which is mainly specified by the parameter μ , but due to the vanishing of the wave function at short distances. The Coulomb energy of He^3 is being calculated by assuming an extended proton. The Coulomb energy may be reduced further.

Table IV. No hard core. Exponential potential. ($\nu = \infty$).

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev	
			$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$		$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$
0.5	∞	33.42	21.19	22.39	44.42	*10.18	*11.38
0.4835	∞	31.67	20.12	21.19	41.54	*10.25	*11.33
0.45	∞	28.15	17.96	18.79	35.98	*10.12	*10.96

Table V. Hard core radius $D=0.2 \times 10^{-13} \text{ cm}$. Exponential potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in ($2 \times 10^{-13} \text{ cm}$) ⁶
			$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$		$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$	
0.55	4.5	41.92	28.12	29.79	62.81	*7.23	*8.90	0.005945
0.5	6.5	40.51	27.04	28.79	60.14	7.41	9.16	0.011327
0.5	5.5	38.77	26.02	27.55	56.51	*8.27	*9.81	0.010975
0.5	5	37.63	25.35	26.74	54.49	*8.49	*9.89	0.010723
0.5	4.5	36.26	24.53	25.77	52.28	*8.51	*9.75	0.010393
0.5	3.5	32.53	22.27	23.11	47.06	7.74	8.58	0.009344
0.45	5.5	32.61	22.09	23.17	45.97	*8.73	*9.81	0.020033
0.45	5	31.70	21.55	22.53	44.49	8.76	9.74	0.019672
0.45	4.5	30.60	20.89	21.74	42.87	*8.62	*9.47	0.019194
0.45	4	29.25	20.07	20.78	41.07	8.26	8.97	0.018546
0.4	4.5	25.01	17.11	17.77	34.19	7.93	8.59	0.037981

Table VI. Hard core radius $D=0.4 \times 10^{-13}$ cm. Exponential Potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in ($2 \times 10^{-13} \text{ cm}$) ⁶
			$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$		$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$	
0.5	5.5	48.04	34.06	36.46	77.36	4.74	7.14	0.017722
0.5	4.5	43.73	31.42	33.21	68.14	*7.01	*8.80	0.016515
0.5	3.5	37.64	27.70	28.75	58.07	*7.26	*8.32	0.014507
0.5	3	33.77	25.25	25.89	52.55	6.46	7.10	0.012975
0.45	3.5	31.65	23.58	24.24	48.07	*7.16	*7.82	0.026619
0.4	4.5	29.72	21.93	22.70	44.39	*7.25	*8.02	0.056483
0.4	4	27.92	20.82	21.38	41.63	*7.10	*7.66	0.05462
0.4	3.5	25.73	19.47	19.80	38.69	*6.52	*6.84	0.05208
0.35	4.5	23.09	17.30	17.70	34.03	6.37	6.77	0.11764

Table VII. Hard core radius $D=0.6 \times 10^{-13}$ cm. Exponential Potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in ($2 \times 10^{-13} \text{ cm}$) ⁶
			$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$		$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$	
0.5	4.5	51.18	39.70	41.89	85.70	*5.18	*7.38	0.025025
0.5	4	46.75	36.98	38.53	78.16	*5.56	*7.12	0.023497
0.5	3.5	42.03	34.03	34.93	70.88	5.17	6.07	0.021423
0.5	3	36.42	30.32	30.57	62.79	3.95	4.21	0.018909
0.45	4.5	42.60	33.56	35.04	70.46	*5.70	*7.18	0.04371
0.4	5.5	39.17	30.43	32.04	64.93	4.68	6.29	0.08576
0.4	5	36.88	29.07	30.33	60.63	*5.32	*6.58	0.08382
0.4	4.5	34.32	27.05	28.38	56.25	*5.57	*6.45	0.08132
0.4	4	31.42	25.67	26.16	51.81	*5.29	*5.77	0.07815
0.35	4.5	26.49	21.62	22.03	43.23	4.87	5.29	0.16429
0.3	5.5	21.88	17.65	18.11	35.76	3.78	4.24	0.38152
0.3	5	20.67	16.92	17.20	33.70	3.88	4.17	0.37685
0.3	4.5	19.30	16.06	16.15	31.62	3.74	3.83	0.37085
0.3	3.5	15.97	13.88	13.58	27.17	2.69	2.38	0.35203

Table VIII. No hard core. Yukawa potential. ($\nu=\infty$)

μ in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev	
		$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$		$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$
0.55	38.92	27.04	29.56	53.75	*12.22	14.73
0.6	45.08	31.38	34.48	63.97	*12.49	*15.58
0.65	51.44	35.89	39.58	75.07	*12.26	*15.96
0.7	57.99	40.52	44.86	87.07	11.45	*15.79

Table IX. Hard core radius $D=0.2 \times 10^{-13}$ cm. Yukawa potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in ($2 \times 10^{-13} \text{ cm}$) ⁶
			$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$		$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$	
0.55	4.5	41.26	28.83	29.90	62.81	7.29	8.36	0.005945
0.5	6.5	40.47	28.25	29.44	60.14	*8.58	**9.77	0.011327
0.5	5.5	38.50	26.87	27.88	56.51	*8.86	**●9.87	0.010975
0.5	5	37.17	25.99	26.90	54.49	8.68	●9.59	0.010723
0.5	4.5	35.67	24.96	25.75	52.28	*8.34	**9.13	0.010393
0.45	6	33.19	23.21	23.98	47.31	9.09	●9.86	0.020312
0.45	5.5	32.33	22.72	23.32	45.97	*9.08	**●9.68	0.020033
0.45	5	31.32	21.93	22.54	44.49	8.76	●9.37	0.019672
0.45	4.5	30.12	21.11	21.63	42.87	*8.35	**8.87	0.019194
0.45	4	28.68	20.12	20.53	41.07	7.73	8.14	0.018546
0.4	4.5	24.67	17.32	17.61	34.19	*7.80	●**8.09	0.037981

[**] gives B. E. =9.94 Mev with adjusted values of $\mu=0.492$, $\nu=5.89 \times 10^{13} \text{ cm}^{-1}$, while [●] gives B. E. =10.07 Mev with $\mu=0.487$, $\nu=6.36 \times 10^{13} \text{ cm}^{-1}$. Since the author does not quite understand such a large difference between values of different two sets, the averaged values are tabulated in Table I and II.

Table X. Hard core radius $D=0.6 \times 10^{-13}$ cm. Yukawa potential.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B. E. (H^3) in Mev		N in ($2 \times 10^{-13} \text{ cm}$) ⁶
			$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$		$r_{0s}=2.7$ $\times 10^{-13} \text{ cm}$	$r_{0s}=2.4$ $\times 10^{-13} \text{ cm}$	
0.5	4.5	51.13	40.50	41.97	85.70	*5.93	*7.41	0.025025
0.5	4	46.68	37.65	38.55	78.16	*6.17	*7.07	0.023497
0.5	3.5	41.95	34.58	34.89	70.88	*5.65	*5.96	0.021423
0.5	3	36.37	30.78	30.31	62.79	4.36	3.89	0.018909
0.45	4.5	42.58	34.24	35.08	70.46	*6.36	*7.20	0.04371
0.4	5.5	39.22	31.16	32.20	64.93	5.45	6.49	0.08576
0.4	5	36.91	29.72	30.44	60.63	6.00	6.72	0.08382
0.4	4.5	34.32	28.07	28.44	56.25	*6.14	*6.52	0.08132
0.4	4	31.42	26.16	26.31	51.81	*5.78	*5.92	0.07815
0.35	4.5	26.53	22.08	22.09	43.23	5.37	5.39	0.16429

Added note after completion of the manuscript.

We have come to know of a paper by J.M. Blatt and G. Derrick⁴⁾ entitled "Repulsive Core Forces in the Triton". They have concluded that the binding energy of H^3 increases with increasing core radius. They use a family of central and tensor potentials for the two-body nuclear force. The 4 adjustable parameters in the potential for the triplet even state can probably be determined so as to give the correct binding energy, the effective range and the electric quadrupole moment

of the deuteron, and also the binding energy of the triton, if the core radius is not too large. Therefore, if we want to argue whether the binding energy of H^3 increases or decreases with increasing core radius, one plausible extra-restriction must be imposed on the 4 potential parameters in order to compare the binding energies of H^3 for the different values of the core radii. It seems that Blatt and Derrick assume that the well depth of the central force should be taken equal for the different core radii when comparing the B.E. of H^3 . The author does not quite understand the reason why they choose the above mentioned criterion. Even if their criterion was most plausible, their conclusion could be doubtful, because the tensor force is completely omitted in their actual calculation of the binding energy of H^3 . The numerical values obtained by them may of course be useful for some specialists as a guide.

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Appendix. Details of the results of calculation

In order to get the minimum of the stationary expression of energy by varying the adjustable parameters μ and ν , we have used the quadratic function approximation as in I and II. In the following tables we give the detailed values which have appeared in the course of our calculation. The values with asterisks are used to obtain Tables I, II and III. Tables IV~VII are revised versions of Table VI in reference I, and Tables XIII~X are revised versions of Table V in reference II. Potential parameters were given in Table I of reference I and Table I of reference II respectively. The values of normalization are given in $(2 \times 10^{-13} \text{cm})^6$.

References

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Superexchange Interaction of MnO*

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The magnitude of superexchange interaction of the MnO crystal is calculated according to the methods proposed previously. Various mechanisms of superexchange interaction are taken into consideration. Overlaps between neighbouring ions are assumed to be small and the energy of the crystal is developed in powers of S , the overlap integral between manganese and oxygen ions. The superexchange interaction of all mechanisms appears from S^4 terms and it is concluded from calculation that the Slater mechanism may play a dominant role in MnO. The numerical value of magnitude of interaction agrees with experiment in the order of magnitude. Several discussions about superexchange interactions of other substances, particularly of MnF_2 and LaMnO_3 , are given.

§ 1. Introduction

The magnetic interactions between magnetic ions in antiferromagnetic ionic crystals were interpreted by Kramers¹⁾ to be due to superexchange interaction and formulated by Anderson²⁾ using the spin operator method. After that, various properties of antiferromagnets were investigated theoretically using the spin Hamiltonian

$$-2J\sum_{i>j}\mathbf{S}_i\mathbf{S}_j \quad (1)$$

and many fruitful results were obtained. The spin operator method, however, was criticized by Slater,³⁾ because it neglected the nonorthogonality of wave functions. Since then any theoretical treatment which overcomes his criticism has not yet been published.

According to Anderson, superexchange interaction was predicted to be negative when the $3d$ shell is more than half filled and positive when this is less than half filled. Because this prediction did not always agree with experimental facts, other mechanisms of superexchange interaction were proposed by Slater,⁴⁾ Anderson-Hasegawa⁵⁾ and Goodenough.⁶⁾ Since these arguments are quite qualitative and are not suited for the quantitative investigation, we cannot say how much each mechanism contributes to the superexchange interaction of a real crystal. Among these mechanisms some give positive contribution to superexchange interaction and some

* This work is based on a thesis submitted in partial fulfilment of the requirements for the degree of Ph. D. at Department of Physics, Faculty of Science, University of Tokyo.

give negative one so that the resultant effect is expected to be different in various crystals. Therefore, in order to make a theoretical prediction as to superexchange interaction in a real crystal, quantitative knowledge of the contribution from each mechanism is indispensable. However, the quantitative investigation of the superexchange interaction of a real crystal must be performed so as to avoid the non-orthogonal difficulty.

J. Yamashita and the author⁷⁾ have proposed a method to evaluate superexchange interaction, assuming the non-orthogonality of wave functions being small compared with unity and a rapid convergence of a power series of the energy of the system with respect to overlap integrals. They formulated superexchange interaction in the case of four electron model. The purpose of the present paper is to extend the above formula to a real crystal and to obtain numerical results about the MnO crystal.

In the actual calculation of superexchange interaction the choice of wave functions of the MnO crystal is the most essential point. For alkali-halides Lowdin⁸⁾ assumed the totally ionic state and expanded the average energy of that state in powers of the overlap integral S , retaining up to the terms of order S^2 which gave repulsive energies between ions, and obtained good agreement with experiment. The author and J. Yamashita⁹⁾ calculated the uniaxial quadrupolar relaxation times in alkali-halides by using the purely ionic model of the crystal without neglecting overlaps between ions. The obtained result agrees fairly well with experiment. Thus the purely ionic model may be regarded as a fairly good approximation for the above phenomena of alkali-halides unless overlaps between ions are neglected. For other phenomena and materials, however, the approximation may have to be refined. The following effects should be taken into account for that purpose. (1) Ions may be deformed from spherical symmetry in a crystal. (2) Some covalent states may be mixed such as $[M^+X^-] + [MX]$.

We shall treat the superexchange interaction of MnO crystal in the same approximation. We have already shown in reference 7) that the totally ionic model gives rise to a spin-dependent energy in the fourth order terms of S . Now, it is true that the refinement of the approximation in the above scheme may be expected to have only a minor effect on the cohesive energy of MnO crystal, but as the magnitude of superexchange interaction is rather small, it might be supposed to give the contribution of the same order of magnitude as the above S^4 term to the superexchange interaction. In fact the deformation of O^{--} ions leads to the Slater mechanism and the Anderson mechanism is led by mixing a covalent state $[Mn^+O^-]$ to the totally ionic state $[Mn^{++}O^{--}]$. Mixing an excited state $[Mn^+O^{--}Mn^{+++}]$ to the totally ionic state also gives to a spin-dependent energy. We shall calculate in the following sections the magnitudes of the superexchange interaction which arise from all the above mechanisms.

§ 2. Totally ionic state

When the totally ionic state of MnO crystal is assumed to be $[\text{Mn}^{++}(3d)^5 {}^6S] + [\text{O}^{--}(2p)^6 {}^1S]$, the orbital degeneracy is removed but the spin direction of each manganese ion may be arbitrary. The ground state of an antiferromagnet is a linear combination of states with various spin configurations, but we assume from neutron diffraction experiment¹⁰⁾ that the energy of the ordered antiparallel spin state is close to that of the ground state and compare its energy with that of the parallel spin state. But for a while we shall consider states of various spin configurations, only restricting z component of spin angular momentum of each manganese ion to $5/2$ or $-5/2$. These states can be described by a single Slater determinant and their energies can be obtained by Lowdin's method⁸⁾. As the atomic orbitals of a manganese ion we took solutions¹¹⁾ of the Hartree-Fock equation of a free ion and as the $2s$ orbital of an oxygen ion the Hartree-Fock wave function¹²⁾ of the free O^- ion. As to the $2p$ orbitals of an O^{--} ion we used the variational solution¹³⁾ determined in MgO crystal.* We denote these atomic orbitals by $\phi_\mu(\mathbf{x})$, where $\mathbf{x}=(\mathbf{r}, \chi)$ means the spatial and the spin coordinates of an electron and μ specifies the position of the ion, the species of the orbital and the spin direction.

Let the total Hamiltonian be

$$H = \sum_i H_1(i) + \sum_{i \neq j} e^2 / 2r_{ij}, \quad (2)$$

where

$$H_1(i) = p_i^2 / 2m - e^2 \sum_g Z_g / r_{ig}. \quad (3)$$

Here, i and j specify electrons and g a nucleus and r_{ig} means the distance between the i -th electron and the g -th nucleus whose atomic number is denoted by Z_g . On the other hand, r_{ij} is the distance between the i -th electron and the j -th electron. The average energy of the states mentioned above is given by the following expression

$$\begin{aligned} E = & \sum_\mu (\phi_\mu | H_1 | \phi_\mu) + (1/2) \sum_{\mu\nu} \{ (\phi_\mu \phi_\nu | \phi_\mu \phi_\nu) - (\phi_\mu \phi_\nu | \phi_\nu \phi_\mu) \} \\ & - \sum_{\mu\nu} P_{\mu\nu} [(\phi_\nu | H_1 | \phi_\mu) + \sum_\kappa \{ (\phi_\nu \phi_\kappa | \phi_\mu \phi_\kappa) - (\phi_\nu \phi_\kappa | \phi_\kappa \phi_\mu) \}] \\ & + (1/2) \sum_{\mu\nu\kappa\lambda} P_{\mu\kappa} P_{\nu\lambda} \{ (\phi_\kappa \phi_\lambda | \phi_\mu \phi_\nu) - (\phi_\kappa \phi_\lambda | \phi_\nu \phi_\mu) \}, \end{aligned} \quad (4)$$

$$\begin{aligned} \text{where } (\phi_\mu | H_1 | \phi_\nu) = & \int \phi_\mu(1) H_1(1) \phi_\nu(1) d\mathbf{x}_1, \\ (\phi_\mu \phi_\nu | \phi_\kappa \phi_\lambda) = & \int \phi_\mu(1) \phi_\nu(2) (e^2 / r_{12}) \phi_\kappa(1) \phi_\lambda(2) d\mathbf{x}_1 d\mathbf{x}_2, \end{aligned} \quad (5)$$

$$P_{\mu\nu} = \delta_{\mu\nu} - (1 + S)_{\mu\nu}^{-1}. \quad (6)$$

The summation in (4) is to be taken over all atomic orbitals in the crystal which

* Watson¹⁴⁾ obtained a Hartree-Fock solution of O^{--} ion, distributing $+e$ (and $+2e$) charge on a sphere of radius 1.4 \AA centred on the nucleus of the ion. His solution agrees fairly well with the above wave functions and the ambiguity of our numerical results which comes from the ambiguity of oxygen wave functions may be small.

are filled by an electron. Here we introduce the same approximation as what Lowdin used, that is, we neglect* all overlap integrals except those between the nearest neighbour ions which we assume to be quite small as compared with unity and expand the energy expression (4) in powers of them.** When the expansion

$$P_{\mu\nu} = S_{\mu\nu} - \sum_{\alpha} S_{\mu\alpha} S_{\alpha\nu} + \sum_{\alpha\beta} S_{\mu\alpha} S_{\alpha\beta} S_{\beta\nu} - \dots \quad (7)$$

is inserted into (4) and terms up to the second order of S are retained, the following expression results:

$$E = \sum_{\mu} (\phi_{\mu} | H_1 | \phi_{\mu}) + \sum_{\mu > \nu} \{ (\phi_{\mu} \phi_{\nu} | \phi_{\mu} \phi_{\nu}) - (\phi_{\mu} \phi_{\nu} | \phi_{\nu} \phi_{\mu}) \} + \sum_{\mu > \nu} B(\mu, \nu), \quad (4)'$$

where

$$\begin{aligned} B(\mu, \nu) = & -2S_{\mu\nu} \{ (\phi_{\nu} | H_1 | \phi_{\mu}) + \sum_{\kappa \neq \mu, \nu} (\phi_{\nu} \phi_{\kappa} | \phi_{\mu} \phi_{\kappa}) \} \\ & + S_{\mu\nu}^2 \{ (\phi_{\mu} | H_1 | \phi_{\mu}) + (\phi_{\nu} | H_1 | \phi_{\nu}) + \sum_{\kappa \neq \mu} (\phi_{\mu} \phi_{\kappa} | \phi_{\mu} \phi_{\kappa}) + \sum_{\kappa \neq \nu} (\phi_{\nu} \phi_{\kappa} | \phi_{\nu} \phi_{\kappa}) - (\phi_{\nu} \phi_{\mu} | \phi_{\nu} \phi_{\mu}) \}. \end{aligned} \quad (8)$$

As was pointed out by Yamashita,¹⁵⁾ the value of (4)' is the same for all the states mentioned above.

A spin dependent energy is contained in the fourth order part of (4). For example, a term of $-\sum_{\mu\nu} P_{\mu\nu} (\phi_{\nu} | H_1 | \phi_{\mu})$ in (4) gives rise to a term

$$-\sum_{\mu\nu\alpha\beta} S_{\mu\alpha} S_{\alpha\beta} S_{\beta\nu} (\phi_{\nu} | H_1 | \phi_{\mu}) \quad (9)$$

when (7) is inserted in it. Let ϕ_i and ϕ_j be two $3d$ orbitals of neighbouring Mn^{++} ions, then the part of (9) which contains ϕ_i and ϕ_j is

$$\begin{aligned} & -\sum_{\alpha\beta} \{ S_{i\alpha} S_{\alpha j} S_{j\beta} (\phi_{\beta} | H_1 | \phi_i) + S_{\alpha i} S_{i\beta} S_{\beta j} (\phi_j | H_1 | \phi_{\alpha}) + S_{j\alpha} S_{\alpha i} S_{i\beta} (\phi_{\beta} | H_1 | \phi_j) \\ & + S_{\alpha j} S_{j\beta} S_{\beta i} (\phi_i | H_1 | \phi_{\alpha}) \}, \end{aligned}$$

which remains only when ϕ_i and ϕ_j have the same spin direction. In this way the part of (4) which remains when ϕ_i and ϕ_j have the same spin direction and vanishes when they have opposite directions is given up to the terms of S^4 by $G_1 + G_2 + G_3 + G_4$, where

$$\begin{aligned} G_1 = & (\sum_{\alpha} S_{i\alpha} S_{\alpha j})^2 \{ (\phi_i | \sum_{\kappa}^4 Q_{\kappa} | \phi_i) + (\phi_j | \sum_{\kappa}^4 Q_{\kappa} | \phi_j) - (\phi_i \phi_j | \phi_i \phi_j) \} \\ & + (\sum_{\alpha} S_{i\alpha} S_{\alpha j}) \sum_{\alpha} [2S_{i\alpha} S_{\alpha j} \{ (\phi_{\alpha} | \sum_{\kappa}^2 Q_{\kappa} | \phi_{\alpha}) - (\phi_{\alpha} \phi_i | \phi_{\alpha} \phi_i) - (\phi_{\alpha} \phi_j | \phi_{\alpha} \phi_j) \} - 2(\phi_j \phi_{\alpha} | \phi_{\alpha} \phi_i) \\ & - S_{i\alpha} \{ (\phi_{\alpha} | \sum_{\kappa \neq \alpha j} Q_{\kappa} + \sum_{\kappa}^2 Q_{\kappa} | \phi_j) - 2(\phi_{\alpha} \phi_i | \phi_j \phi_i) \} \\ & - S_{j\alpha} \{ (\phi_{\alpha} | \sum_{\kappa \neq \alpha i} Q_{\kappa} + \sum_{\kappa}^2 Q_{\kappa} | \phi_i) - 2(\phi_{\alpha} \phi_j | \phi_i \phi_j) \}], \end{aligned} \quad (10)$$

* The largest of overlap integrals between a manganese ion and an oxygen ion which is a next-to-nearest neighbour of it is 0.007 and one tenth of that between nearest neighbour ions.

** We define the order of magnitude of the matrix elements as follows: $(\phi_{\mu} | H_1 | \phi_{\nu})$, $(\phi_{\mu} \phi_{\nu} | \phi_{\mu} \phi_{\mu})$, and $(\phi_{\mu} \phi_{\kappa} | \phi_{\nu} \phi_{\kappa})$ are first-order quantities in S and $(\phi_{\mu} \phi_{\nu} | \phi_{\nu} \phi_{\mu})$ is a second-order quantity in S , when the orbitals ϕ_{μ} and ϕ_{ν} are those of nearest neighbours.

$$G_2 = -\sum_{\alpha\beta} S_{i\alpha} S_{j\beta} (\phi_\alpha \phi_j | \phi_j \phi_\beta) - \sum_{\alpha\beta} S_{j\alpha} S_{i\beta} (\phi_\alpha \phi_i | \phi_i \phi_\beta), \quad (11)$$

$$G_3 = \sum_{\alpha\beta} S_{i\alpha} S_{\alpha j} \{ 2S_{i\beta} (\phi_\beta \phi_\alpha | \phi_j' \phi_\alpha) + 2S_{j\beta} (\phi_\beta \phi_\alpha | \phi_i \phi_\alpha) - S_{j\beta} S_{i\beta} (\phi_\alpha \phi_\beta | \phi_\alpha \phi_\beta) \}, \quad (12)$$

$$G_4 = -2\sum_{\alpha\beta} S_{i\alpha} S_{j\beta} (\phi_\alpha \phi_\beta | \phi_j \phi_i), \quad (13)$$

$$Q_\kappa(1) = -Z_\kappa e^2 / r_{1\kappa} + e^2 \sum_{\mu}^{(\kappa)} \int |\phi_\mu(2)|^2 / r_{12} \cdot d\mathbf{x}_2. \quad (14)$$

Q_κ is the potential energy which is exerted on an electron by the κ -th ion. The summation, unless specified otherwise, is taken over all atomic orbitals in the crystal and is taken over the atomic orbitals of the κ -th ion, when attached a symbol (κ and is taken over all orbitals except those of the i -th ion when attached a symbol i). When we neglect all overlap integrals except those between nearest neighbours, G remains only when ϕ_i and ϕ_j are orbitals either of nearest neighbouring ions (so-called 90° interaction) or of next nearest neighbouring ions (so-called 180° interaction).

Let σ_i be a symbol which takes a value of $+1$ when the i -th ion has α spin direction and -1 when it has β spin direction. Then r_i and s_i are defined by

$$r_i = (1/2)(1 - \sigma_i), \quad s_i = (1/2)(1 + \sigma_i). \quad (15)$$

From the preceding paragraph we can write the interaction between the i -th ion and the j -th ion as

$$(\sigma_i \sigma_j + 1) / 2 \cdot \sum_{(i)} \sum_{(j)} (G_1 + G_2 + G_3 + G_4), \quad (16)$$

where the summation $\sum_{(i)}$ is to be taken over those orbitals of the i -th ion which have unpaired spin (five $3d$ orbitals). Let us denote the coefficient of $\sigma_i \sigma_j$ by J , then from this model J_{overlap} is given by

$$J_{\text{overlap}} = (1/2) \sum_{(i)} \sum_{(j)} (G_1 + G_2 + G_3 + G_4). \quad (17)$$

We give numerical results of 180° interaction. For ϕ_α and ϕ_β in (10), ..., (13) we took $2p\sigma$, $2p\pi$, and $2s$ orbitals of the oxygen ion which lies between ϕ_i and ϕ_j . Various integrals have been calculated by the same approximation as Lowdin used in his computation of cohesive energies of alkali-halides (Chapt. 5 of ref. 8)). $(\phi_i \phi_j | \phi_p \phi_p)$ and $(\phi_i \phi_j | \phi_p \phi_p)$ where replaced by $(e^2/2a) S_{pi} S_{pj}$ and $(e^2/2a) S_{pi}$ respectively, where a is the distance between a manganese ion and an oxygen ion of nearest neighbour. The results are

$$\begin{aligned} \sum_{(i)} \sum_{(j)} G_1 &= 0.0019 \text{ ev}, & \sum_{(i)} \sum_{(j)} G_2 &= -0.0034 \text{ ev}, \\ \sum_{(i)} \sum_{(j)} G_3 &= 0.0004 \text{ ev}, & \sum_{(i)} \sum_{(j)} G_4 &= -0.0001 \text{ ev}, \\ J_{\text{overlap}} &= -0.0007 \text{ ev}. \end{aligned} \quad (18)$$

This favours the parallel spin arrangement. But, as seen in the above table, J_{overlap} is a result of cancellation of large positive and negative quantities and it cannot be decided in the accuracy of the present calculation whether this term is positive or negative. It may be said, however, that this term is not the most important in MnO crystal, although it cannot be neglected completely. The 90° interaction is expected to

be much smaller than 180° interaction, because its expression contains squares of overlap between $2p\pi$ and $3d\epsilon$ orbitals.

§ 3. Kramers-Anderson model

Now we mix to the totally ionic state a covalent state in which one electron from an oxygen ion transfers to a neighbouring manganese ion. We shall see that the energy of the crystal which is described in this way depends on the spin direction. Since it is difficult to perform configuration interaction correctly in an N -body problem, we assume as in reference 7) that each $2p$ electron of all oxygen ions transfers to nearest neighbour manganese ions irrespective of transfer of other $2p$ electrons. A $2p$ electron with α spin which has occupied $\phi_{p\alpha}$ (which shall be denoted as ϕ_p hereafter) in Fig. 1 is now to occupy the following orbital,

$$\phi_p = \frac{\phi_p + r_i \lambda_i \phi_i + r_j \lambda_j \phi_j}{\sqrt{1 + r_i (2\lambda_i S_{pi} + \lambda_i^2) + r_j (2\lambda_j S_{pj} + \lambda_j^2)}}, \quad (19)$$

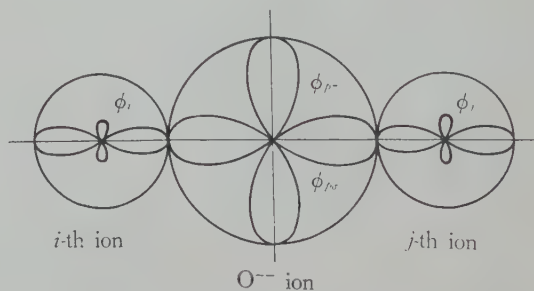
where λ_i and λ_j are variational parameters and $r_i(r_j)$ takes account of the fact that when the i -th (j -th) ion has α spin the $2p$ electron with α spin cannot transfer to that ion. Similarly, a $2p$ electron with β spin which has occupied ϕ_p , now occupies

$$\phi_p' = \frac{\phi_p + s_i \lambda_i' \phi_i + s_j \lambda_j' \phi_j}{\sqrt{1 + s_i (2\lambda_i' S_{pi} + \lambda_i'^2) + s_j (2\lambda_j' S_{pj} + \lambda_j'^2)}}. \quad (20)$$

Even when we transform all $2p$ orbitals in the crystal in this way, the total system can be described by a single Slater determinant, the energy of which can be expressed by (4) where ϕ_p 's are replaced by ϕ_p 's. When we expand the energy in powers of S assuming λ 's to be of the first order of S , we get the following expression,

$$E = E(S^0) + E(S^2) + \sum_i (2A\lambda_i + B\lambda_i^2) + E(S^4) + \lambda E(S^3) + \lambda^2 E(S^2) + \lambda^3 E(S) + \lambda^4 E(S^0), \quad (21)$$

where $E(S^n)$ represents an energy matrix of the order of S^n and A and B are of the first and the zeroth order of S , respectively. The first term is the sum of kinetic energies of all electrons in the crystal and the Coulomb interactions between electrons and between nuclei and electrons. The second term is exchange and overlap interactions between nearest neighbour ions which are of the second order of S . The third term is also of the second order and the rest is of the fourth order.



Here ϕ_i and ϕ_j are those $3d$ orbitals which have zero component of orbital angular momentum with respect to the axis joining the oxygen ion to manganese ions.

Fig. 1 Configuration of Atomic Orbitals

The last three terms contain cross products

of λ_i and λ_j . We determine λ_i so as to minimize the second order part $2A\lambda_i + B\lambda_i^2$, that is, $\lambda_i = -A/B$ and insert this value to the fourth order part. Spin dependent energies are contained in the last five terms of (21), the first of which we have discussed in the preceding section. From the rest of it we obtain as a coefficient of $\sigma_i\sigma_j$ the following expressions (see Appendix I.)

$$J_{Ao} = J_{Ao1} + J_{Ao2} \quad (22)$$

$$J_{Ao1} = t \{ B(\phi_p, \phi_j) - C \} + S_{pj}^2 \Delta E + A_2, \quad (23)$$

$$J_{Ao2} = -\frac{1}{2} A_1 + t \Delta E, \quad (24)$$

where

$$t = 2\lambda_i S_{pi} + \lambda_i^2, \quad (25)$$

$$\lambda_i = -(\phi_p | H_p - E_p | \phi_i) / (E_d - E_p), \quad (26)$$

$$E_d = (\phi_i | H_p | \phi_i), \quad E_p = (\phi_p | H_p | \phi_p). \quad (27)$$

H_p , ΔE , A_1 and A_2 are defined by (A.5), (A.9), (A.12) and (A.17), respectively. The denominator of (26) is the energy necessary to transfer an electron from an oxygen ion to a nearest neighbour manganese ion and the numerator of (26) is a transfer matrix for it. $B(\phi_p, \phi_j)$ is the overlap energy between ϕ_p and ϕ_j ⁸⁾, C is the exchange energy between ϕ_p and the 3d shell of the j -th manganese ion. This result is quite analogous to that of four electron model⁷⁾, in which J_{Ao1} was interpreted to arise from the Anderson mechanism and J_{Ao2} was seen to correspond to the Anderson-Hasegawa mechanism plus the mechanism of mixing two electron transition process. In reality J_{Ao2} can be interpreted to arise from the above mechanisms only when the correct configuration mixing rather than one-electron approximation (19), (20) is carried out.* The sign of J_{Ao2} is seen to be positive in the correct configuration interaction⁷⁾ but that of (24) cannot be determined without numerical calculation. Numerical results given in Appendix I are

$$J_{Ao1} = -0.0015 \text{ eV}, \quad J_{Ao2} = 0.0010 \text{ eV}, \quad J_{Ao} = -0.0004 \text{ eV}. \quad (28)$$

The sign of J_{Ao1} is opposite to what Anderson predicted. It is because $B-C$ is negative in our calculation, while Anderson assumed a positive value (favourable for antiparallel spin arrangement) for an exchange integral which corresponds to $B-C$ in our case. Although the accuracy of our calculation cannot decide the sign of $B-C$ to be negative, it is probable that the exchange and the overlap interactions between outermost orbitals are almost cancelled out as in alkali-halides.** It may be said, therefore, that in MnO the superexchange interaction from Anderson's mechanism is not so great even in the accuracy of our calculation, though its sign is not obvious.

* We may regard $J_{Ao1} + J_{Ao2}$ as arising from transfer mechanism in our approximation, because it contains λ , which measures the degree of transfer.

** According to ref. 8), exchange and overlap energies of NaCl crystal at 5.0 a. u. of lattice constant are given as $R(\text{Cl}3p, \text{Na}2p) = 6.23 (\times 10^{-3} \text{ a.u.})$, $C(\text{Cl}3p | \text{Na}2p) = 6.20$, $B(\text{Cl}3p | \text{Na}2s) = 15.74$, $C(\text{Cl}3p | \text{Na}2s) = 5.66$.

$J_{A\sigma 2}$ is positive, as should be the case in the correct configuration interaction of the four electron model. It may be said also that it is not the most important mechanism of the superexchange interaction in MnO.

A superexchange interaction arises also from mechanisms in which a $2p\pi$ electron transfers to a $3d\epsilon$ orbital and a $2s$ electron to a $3d\gamma$ orbital.* The formulation of these interactions is analogous to the above case and will not be given here. In numerical calculations of its magnitude, it is necessary to know the value of the energy necessary to take a $2s$ electron from an oxygen ion. We used a calculated value of 13 eV for it. Contributions to J are calculated as $J_{A\pi}=0.00007$ eV from $2p\pi$ orbital and $J_{A\sigma}=0.00005$ eV from $2s$ orbital.

In the above calculation of E_d-E_p (and E_d-E_{2s}) the effect of polarization was not taken into account. Although it is not at once obvious whether the effect of polarization should be included in our E_d-E_p and it cannot be taken into account in our formulation, assuming a value of -5 eV for the effect of polarization on E_d-E_p (and E_d-E_{2s}) alters the results to

$$\begin{aligned} J_{A\sigma 1} &= -0.0030 \text{ eV}, & J_{A\sigma 2} &= 0.0034 \text{ eV}, & J_{A\sigma} &= 0.0004 \text{ eV}, \\ J_{A\pi} &= 0.00008 \text{ eV}, & J_{A\sigma} &= 0.00006 \text{ eV}. \end{aligned} \quad (29)$$

From these results the contributions from $2p\pi$ and $2s$ orbitals can be said to be rather small.

Our next problem is 90° interaction which can be formulated in the analogous way to 180° interaction. The results which are not given here show that there always appear in its expression factors of square of overlap between $2p\pi$ and $3d\epsilon$ orbitals, and its magnitude is supposed to be much smaller than 180° interaction. The $2s$ contribution is almost identical with that of 180° interaction. Therefore no large contribution to 90° interaction is expected from the mechanism in this section.

§ 4. Slater mechanism

In a crystal electron orbitals of ions are deformed from those of free ions. We shall take account of this effect by mixing some configurations with different symmetry into original orbitals. Various configuration mixings are possible and a superexchange interaction arises from a configuration mixing. We have already discussed⁷⁾ the physical picture of superexchange interaction which arises from mixing an s function into $2p$ orbital of an oxygen ion. We can consider also the effect of another deformation which is a uniform contraction or expansion of the oxygen ion. When the two manganese ions have the same spin direction α , $2p$ electrons of α spin contract uniformly. When they have opposite spins, $2p$ electrons of both α spin and β spin contract to the less extent than in the parallel case and the gain of the energy by contraction is larger in the parallel case than in the anti-

* The contributions from $2s$ and $2p\sigma$ are not additive. It would not alter, however, the order of magnitude of J to take account of its correction.

parallel case. We represent this mechanism by mixing another p state into $2p$ orbital.

We calculated the magnitude of the superexchange interaction arising from $(2p+s)$ mechanism. We assumed for $2p$ electrons of oxygen ions the following orbital,

$$\begin{aligned}\varphi_p &= (\phi_p + \varepsilon \phi_s) / \sqrt{1 + \varepsilon^2}, \quad \text{for } \alpha \text{ electrons,} \\ \varphi_p' &= (\phi_p - \varepsilon \phi_s) / \sqrt{1 + \varepsilon^2}, \quad \text{for } \beta \text{ electrons.}\end{aligned}\quad (30)$$

Here ϕ_s is an s type orbital. Then the energy average is given by (4)' where ϕ_p 's are replaced by φ_p 's. Assuming ε of order S^2 , the part of energy which contains ε is given, up to fourth order, by

$$4L\varepsilon(\sigma_i - \sigma_j)/2 + 2M\varepsilon^2, \quad (31)$$

where

$$\begin{aligned}M &= \{(\phi_s | H_{p \text{ intra}} | \phi_s) - (\phi_p | H_{p \text{ intra}} | \phi_p) - 2(\phi_s \phi_p | \phi_p \phi_s)\} \\ &\quad + \{\sum_{\mu}^{(p)} (\phi_s \phi_{\mu} | \phi_s \phi_{\mu}) - \sum_{\mu}^{(p)} (\phi_p \phi_{\mu} | \phi_p \phi_{\mu})\}, \\ L &= S(\phi_s \phi_i) B(\phi_p \phi_i) / S_{pi} - \sum_{(i)} (\phi_i \phi_s | \phi_p \phi_i) \\ &\quad + S(\phi_s \phi_i) \{(\phi_p | H_1 | \phi_i) + \sum_{\kappa \neq p, i} (\phi_p \phi_{\kappa} | \phi_i \phi_{\kappa})\} - S_{pi} \{(\phi_s | H_1 | \phi_i) \\ &\quad + \sum_{\kappa \neq p, i} (\phi_s \phi_{\kappa} | \phi_i \phi_{\kappa})\} - S_{pi}^2 (\phi_p \phi_i | \phi_s \phi_i),\end{aligned}\quad (32)$$

$$\begin{aligned}H_{p \text{ intra}}(1) &= p_1^2/2m - Z_0 e^2/r_{1p} + e^2 \sum_{\mu \neq p}^{(p)} [\phi_{\mu}^2(2)/r_{12} \cdot d\mathbf{x}_2 \\ &\quad - \int \phi_{\mu}(2) \phi_{\mu}(1) P_{12}/r_{12} \cdot d\mathbf{x}_2].\end{aligned}$$

Here i and j specify two manganese ions which lie on the both ends of ϕ_p orbital. When the i -th ion and the j th ion have the same spin direction, the linear term of ε vanishes because of symmetry, which is taken into account by the factor $\frac{1}{2}(\sigma_i - \sigma_j)$. Cross products of two parameters ε and ε' which belong to different $2p$ orbitals do not arise in the S^4 approximation. In order to transform the last three terms of (32), we assume that ϕ_i fulfil the equation

$$[p_1^2/2m - Z_i e^2/r_{1i} + e^2 \sum_{\kappa \neq i}^{(i)} \phi_{\kappa}^2(2)/r_{12} \cdot d\mathbf{x}_2] \phi_i(1) = E' \phi_i(1), \quad (33)$$

which is not Hartree-Fock equation but contains no exchange term. Then

$$\begin{aligned}L &= S(\phi_s \phi_i) B(\phi_p \phi_i) / S_{pi} - \sum_{(i)} (\phi_i \phi_s | \phi_p \phi_i) \\ &\quad + S(\phi_s \phi_i) (\phi_p | \sum_{\kappa}^{(i)} Q_{\kappa} | \phi_i) - S_{pi} (\phi_i | \sum_{\kappa}^{(i)} Q_{\kappa} | \phi_i) + S_{pi} (\phi_i \phi_p | \phi_s \phi_p) \\ &\quad - S(\phi_s \phi_i) (\phi_i \phi_p | \phi_p \phi_p)\end{aligned}\quad (34)$$

results. As the last four terms of (34) proves to be much smaller than the first and the second terms, the error in assuming (33) may be small. (31) is a minimum with respect to ε when $\varepsilon = -\frac{1}{2}(\sigma_i - \sigma_j)L/M$, which is of the second order of S , because so is L . Then the minimum value of (31) is given by $-2(\sigma_i - \sigma_j/2)^2 \times L^2/M$, which gives for the contribution to J the following value,

$$J_s = L^2/M. \quad (35)$$

This mechanism gives no contribution to 90° interaction. For ϕ_s we took $\phi_s \propto P_{2p}(r)Y_{00}(\theta, \varphi) + c\phi_{2s}$, where $P_{2p}(r)/r$ is the radial part of $2p$ function of the oxygen ion¹³⁾ and c is to be determined so that ϕ_s is orthogonal to ϕ_{2s} . Then calculation shows $L=0.323$ eV, to which the last four terms of (34) contributes only -0.009 eV. The first term of M is calculated to be 13 eV, which is the intra-atomic energy necessary to excite a $2p$ electron to ϕ_s orbital. The second term of M is the change of the Coulomb interaction with surrounding ions in the excitation, which we neglected because it is supposed to be of order 1 eV. Then

$$J_s = 0.0080 \text{ eV} \quad (36)$$

is obtained.

Next we shall consider the contributions to superexchange interaction arising from various configuration mixings other than $(2p+\phi_s)$ mixing. For a while we replace the effect of the manganese ions on electrons of the oxygen ion which they surround by a crystalline field which depends on the relative orientation of the spin direction of $2p$ or $2s$ electron with that of a manganese ion. This simplification is still expected to give the correct sign to superexchange interaction and it is easy to extend this treatment to that of many-body problem like that of the preceding paragraph, if necessary. When surrounded octahedrally by manganese ions, orbitals of the oxygen ion are not necessarily of s or p type but is expected to have symmetry of Γ_1 or Γ_{15} . In the ground state we assume the orbitals of Γ_1 and Γ_{15} type, which we denote by ϕ_s and ϕ_x, ϕ_y, ϕ_z , are filled. Furthermore we assume that excited orbitals of $\Gamma_1, \Gamma_{15}, \Gamma_{12}$ and $\Gamma_{25'}$ type (which we denote by $\phi'_s; \phi'_x, \phi'_y, \phi'_z; \phi_{T1}, \phi_{T2}; \phi_{E1}, \phi_{E2}, \phi_{E3}$) are known (Fig. 2). We denote by V_i the spin dependent part of the potential which the i -th manganese ion exerts on the electrons of the oxygen ion which have the same spin direction as the manganese ion. Then besides the ordinary crystalline field α electrons experience a potential $V_\alpha = \sum_i s_i V_i$ and β electrons $V_\beta = \sum_i r_i V_i$. The second order perturbed energy, taking the above potentials as perturbations, gives as the coefficient of $\sigma_i \sigma_j$ the following expressions,

$$J_{180} = -\frac{(\phi_s|V_1|\phi'_s)^2}{E_{s'}-E_s} + \frac{(\phi'_s|V_1|\phi_y)^2}{E_{s'}-E_s} - \frac{(\phi_s|V_5|\phi_{T1})^2}{E_T-E_s} + \frac{(\phi_z|V_3|\phi_{E1})^2}{E_E-E_s} \\ - \frac{(\phi_z|V_5|\phi'_z)^2 + 2(\phi_z|V_1|\phi'_z)^2}{E_{s'}-E_s} + \frac{(\phi_z|V_5|\phi'_s)^2}{E_{s'}-E_s} + \frac{(\phi_z|V_5|\phi_{T1})^2}{E_T-E_s}, \quad (37)$$

$$J_{90} = -\frac{(\phi_z|V_1|\phi'_z)^2 + 2(\phi_z|V_5|\phi'_z)(\phi_z|V_1|\phi'_s)}{E_{s'}-E_s} - \frac{(\phi_s|V_1|\phi'_s)^2}{E_{s'}-E_s} + \frac{(\phi_s|V_5|\phi_{T1})^2}{2(E_T-E_s)}. \quad (38)$$

The first term of J_{180} , for example, is the superexchange interaction arising from mixing into ϕ_s an orbital of Γ_1 type. The denominator $E_{s'}-E_s$ is the excitation energy from ϕ_s to $\phi_{s'}$. Each term of (37) and (38) have a definite sign and can

be said to be ferromagnetic or antiferromagnetic. The result is summarized in Table 1. Though no quantitative argument can be made from the above results, we may expect that $(2p+p')$ mechanism might give some contributions to superexchange interaction. We give our estimate of this mechanism. We took for ϕ_x, ϕ_y, ϕ_z $2p$ type orbitals, the radial part of which is

$$P(r) \propto r^2 (\exp(-2.9r) + 0.15 \exp(-\lambda r)), \quad (39)$$

where λ is a variational parameter. It coincides with that of ref. 13), when $\lambda=1.1$. The variational parameter λ for α electrons is not set equal to that for β electrons. Its extreme value should depend on the spin configuration of the surrounding manganese ions. The intra-atomic energy depends on λ , when this is close to 1.1, in the following way,

$$18.6(\lambda-1.1)^2 - 0.122(\lambda-1.1) + 7.0 \text{ eV}. \quad (40)$$

The repulsive energy between the oxygen ion and the manganese ions is roughly estimated to depend on λ as

$$4.73(\lambda-1.1)^2 - 2.43(\lambda-1.1) + 0.5 \text{ eV}. \quad (41)$$

Besides these energies there are the exchange and the overlap energies between a $2p$ electron and the $3d$ shell of a manganese ion of the same spin as the $2p$ electron, which are given by

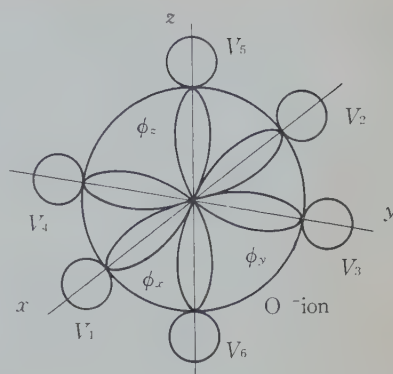
$$0.78(\lambda-1.1)^2 - 0.06(\lambda-1.1) - 0.22 \text{ eV}. \quad (42)$$

Here λ is determined so as to minimize the sum of (40), (41) and (42). When we carry out the process in the ferromagnetic case and then in the antiferromagnetic case there appears a difference in the minimum energy, which corresponds to

$$J = -0.00001 \text{ eV}, \quad (43)$$

which gives a rather small contribution.

Because the energy differences of this section are given in the Hartree-Fock approximation, which takes more account of the correlation between electrons of the parallel spin than between those of the opposite spins, the energy differences might



$$\begin{aligned} \phi_x &\sim x, \quad \phi_y \sim y, \quad \phi_z \sim z \\ \phi_{x'} &\sim x, \quad \phi_{y'} \sim y, \quad \phi_{z'} \sim z \\ \phi_{\tau_1} &\sim 3z^2 - r^2, \quad \phi_{\tau_2} \sim x^2 - y^2 \\ \phi_{\epsilon_1} &\sim yz, \quad \phi_{\epsilon_2} \sim zx, \quad \phi_{\epsilon_3} \sim xy \end{aligned}$$

Fig. 2 Oxygen ion orbitals

Table 1

mechanisms	180°	90°
s+s'	F	F
s+p'	AF	
s+r	F	AF
p+p'	F	F
p+s'	AF	
p+r	AF	
p+ε	AF	

be supposed to be overestimated. If we proceeded to the many-body treatment which takes account of also the correlation between electrons of opposite spins, the energy difference might be reduced to some extent.

We notice the possibility that the effect of Slater mechanism could be detected by the agency of nuclear quadrupolar interaction with p or s electrons of negative ions, because the degree of deformation of negative ions is different in the parallel and the antiparallel states.

§ 5. The effect of triply ionized manganese ions

A spin dependent energy is also obtained by mixing an excited configuration $[\text{Mn}^+\text{O}^{--}\text{Mn}^{++}]$ into the ground configuration $[\text{Mn}^{++}\text{O}^{--}\text{Mn}^{++}]$. Because the above configuration does not mix in the ferromagnetic state if we assume $(3d)^6$ configuration for Mn^+ ion, this mechanism always favours the antiferromagnetic state. We shall take the ordered antiparallel spin state as the starting point of the antiferromagnetic state and assume the rate of mixing of the excited configuration to be small. Instead of carrying out the configuration interaction, $3d$ orbitals of a manganese ion are assumed to be extended to sites of six next-to-nearest neighbouring manganese ions. Thereby only $d\gamma$ orbitals are considered, because $d\varepsilon$ orbitals have small overlaps with $2p\pi$ orbitals of oxygen ions. Of two $d\gamma$ orbitals, we denote that of $3z^2-r^2$ type by ϕ_0 and that of x^2-y^2 type by ϕ_1 . In the antiferromagnetic state the spin direction of the central ion is opposite to that of the neighbouring ions. Now ϕ_0 and ϕ_1 of the manganese ion which lies on the $+x$ axis is denoted by ϕ_0^x and ϕ_1^x etc. The direction of z (in $3z^2-r^2$) is chosen to the line joining the ion and the central ion. On the other hand, ϕ_0 and ϕ_1 of the central ion are denoted simply by ϕ_0 and ϕ_1 . In place of ϕ_0 and ϕ_1 of the central ion we take

$$\phi_0 = \{\phi_0 + \sum_x (\mu_{x0}\phi_0^x + \mu_{x1}\phi_1^x)\} / \sqrt{1 + \sum_x (\mu_{x0}^2 + \mu_{x1}^2)}, \quad (44)$$

$$\phi_1 = \{\phi_1 + \sum_x (\nu_{x0}\phi_0^x + \nu_{x1}\phi_1^x)\} / \sqrt{1 + \sum_x (\nu_{x0}^2 + \nu_{x1}^2)}. \quad (45)$$

Substituting these into (4)', we obtain the following expression of energy which contains μ 's and ν 's

$$2\mu_{x0}(P-Q) + 2(\mu_{x0} + \mu_{-x0} + \mu_{y0} + \mu_{-y0})(-1/2)(P-Q) \\ + 2(\nu_{x0} + \nu_{-x0} + \nu_{y0} + \nu_{-y0})(\sqrt{3}/2)(P-Q) + \sum_x (\mu_{x0}^2 + \mu_{x1}^2 + \nu_{x0}^2 + \nu_{x1}^2)R, \quad (46)$$

where

$$P = (\dot{\lambda}_i + S_{pi})(H_{pi} - S_{pi}E_p) + S_{pi}\{H_{pi} - S_{pi}(\phi_i|H_i|\phi_i) - (\phi_p\phi_j|\phi_i\phi_j) - (\phi_p\phi_i|\phi_i\phi_i)\} \\ + S_{pi}^2(\phi_i\phi_p|\phi_i\phi_p) - \lambda_i(\phi_p\phi_i|\phi_i\phi_i) + \lambda_i S_{pi}\{(\phi_i\phi_i|\phi_i\phi_i) - (\phi_i\phi_j|\phi_i\phi_j)\}, \quad (47)$$

$$Q = (\phi_p\phi_i|\phi_j\phi_p) - \lambda_i(\phi_p\phi_j|\phi_i\phi_j), \quad (48)$$

$$R = (\phi_j|H_i|\phi_j) - (\phi_i|H_i|\phi_i) = \sum_{m=0\pm 1\pm 2}(\phi_{3i0}\phi_{3im}|\phi_{3im}\phi_{3i0}) - (\phi_i\phi_j|\phi_i\phi_j), \quad (49)$$

$$H_i(1) = H_1(1) + e^2 \sum_{\mu \neq i} \int \{\phi_\mu^2(2) - \phi_\mu(2)\phi_\mu(1)P_{12}\} / r_{12} \cdot d\mathbf{x}_2. \quad (50)$$

The configuration of orbitals which are contained in the integrals of (47), (48) and (49) is shown in Fig. 1. In (46) we assumed μ 's and ν 's to be of order S^2 and retained only the fourth order terms. The first term of the right-hand side of (49) is the sum of the intra-atomic exchange integrals of $3d0$ orbital with five $3d$ orbitals. To minimize (46) μ 's and ν 's are determined as, e.g., $\mu_{\pm 0} = -(P-Q)/R$, which is of the second order of S , as assumed. The minimum value of (46) is

$$-6(P-Q)^2/R. \quad (51)$$

Using wave functions of the preceding sections this term is calculated to be $-0.0026 \times 6\text{eV}$.

§ 6. Discussions

From the above results regarding 180° interaction we notice the following points:

(1) The results in the N -body case are N times those of the four electron model.

(2) All terms of superexchange interaction arise from terms of order S^4 .

(3) The assumption of the totally ionic state also leads to a spin dependent energy.

(4) The value of λ is small compared with unity. The energy gain arising from transfer of $2p$ electrons to manganese ions is much smaller than the cohesive energy of the crystal, so the introduction of Anderson's mechanism does not violate the usual theory of cohesive energies of ionic lattices.

(5) The superexchange interaction arising from *transfer* mechanism is rather small.

(6) Slater's mechanism gives the dominant contribution to the superexchange interaction of MnO crystal.

(7) The mechanism of mixing $[\text{Mn}^+\text{O}^{--}\text{Mn}^{++}]$ configuration makes also a considerable contribution.

The sum of all the mechanisms considered in this paper gives

$$J = 0.0097\text{eV} = 112^\circ\text{K} \quad (52)$$

for 180° interaction.* For 90° interaction, although we did not carry out numerical calculations, no mechanism considered in the present paper seemed to give as large contribution as 10°K . Now analyzing the susceptibility data¹⁶⁾ of MnO by the molecular field approximation, we get $J_{180} = 44^\circ\text{K}$, $J_{90} = 87^\circ\text{K}$. Our calculation for 180° interaction proves to give a correct order of magnitude, while for 90° interaction no mechanism is expected to give as large interaction as the experimental value. If we assume the above analysis to be correct, 90° interaction may arise either from direct exchange interaction or from some other mechanisms not con-

* The mechanism of §5 contributes to J an amount of 0.0026 eV .

sidered here. The following fact supports the direct exchange interaction. By analyzing the susceptibility data we determined values of J_{180} and J_{90} of MnS and MnSe. They are 59°K, 88°K for J_{180} , and 65°K, 20°K for J_{90} , respectively. We see that J_{180} increases from MnO to MnSe, while J_{90} decreases. It may be natural to expect that both the interactions showed the same tendency, if they originated from the same kind of interaction. On the other hand, the distance between nearest neighbour manganese ions increases from MnO to MnSe, so that the direct exchange interaction may account for the decrease of J_{90} . However, the 90° interaction may be the problem in future

The theory of superexchange interaction given in the previous sections is quite general and contains no arbitrary assumptions except for four points.

First, the convergence of expansion of $(1+S)^{-1/2}$ in S is assumed. For MnO the largest of overlap integrals, that between $2p0$ and $3d0$ functions, is 0.076 which is much smaller than unity. Furthermore the S^4 part of energy which we calculated here is smaller than the repulsive energy (S^2 part) of the crystal. So this assumption is considered to be well fulfilled in MnO crystal.

Second, the ordered antiparallel spin arrangement was assumed for the antiferromagnetic state. In reality this is not the ground state of an antiferromagnet, which contains besides the ordered antiparallel state states of various spin configurations. Two types of these configurations is to be considered. One is those in which spin directions of manganese ions are declined without change of their valence. The other is those in which manganese ions with valences different from two appear ($\text{Mn}^+ - \text{Mn}^{++}$). Mixing the latter configurations which appear in the usual band theory stabilizes the antiferromagnetic state, which was already pointed out by the author¹⁷⁾. Our estimate of this effect in MnO, given in § 5, shows that this has a contribution to superexchange interaction which cannot be neglected compared with other mechanisms. The importance of this effect is pointed out here for the first time as far as the author knows. To mix the former configurations, it is necessary to obtain matrix elements between two states of different spin configurations. It has not been successful, however, to obtain those in the case of N -body problem without neglecting the non-orthogonalities of wave functions. If these could be obtained the ground state of an antiferromagnet could be represented by a linear combination of various spin states.

Third, instead of carrying out the correct configuration interaction complicated orbitals of the forms (19), (20), (30), (44) and (45) are assumed. In reference 7) and a forthcoming paper the results of the correct configuration interaction for the four electron model are compared with the results of our treatment. It is shown that only J_{A02} term is modified. Numerical estimate shows⁷⁾ that the two methods give nearly equal values for J_{A02} .

Fourth, our development can be carried out only when the orbital degeneracy of magnetic ions is removed. If magnetic ions have orbital degeneracy, the total system cannot be represented by a single Slater determinant, so the same difficulty

as that met with configuration interaction arises. Although our method cannot be applied, therefore, directly to FeO or CoO, etc., such materials as NiO, CaMnO_3 and LaCrO_3 in which orbital degeneracies of magnetic ions are removed completely in a cubic field can be treated by our method. Several discussions for these materials are given in Appendix II.

The above calculation shows that the mechanism of superexchange interaction is rather complicated. Although for MnO Slater's mechanism seems to play a dominant role, other mechanisms seem to have some contributions which cannot be neglected. Furthermore other mechanisms which were not considered here might give some contributions, though small. In view of these points, in the theory of superexchange interaction it is necessary to take all terms of the same order in some quantity which are expected to give any contribution. In our theory the power expansion in S was used. Then superexchange interaction appears from S^4 terms. Since S^0 terms are considered to be neglected as far as $S \sim 0.1$, our method will cover a comprehensive scope of consideration.

Acknowledgment

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Appendix I

Here we show the detailed calculation of superexchange interaction from Anderson's mechanism. We observe the modification of energy expression (4)' when ϕ_p 's are introduced instead of ϕ_p 's. (Because the modification of S^4 terms is higher than S^4 , it is sufficient to deal with (4)' rather than (4).) From (19) the following expression is obtained,

$$\phi_p(2)\phi_p(1) = \phi_p(2)\phi_p(1) + \rho_{p2}(2, 1) + \rho_{p4}(2, 1), \quad (\text{A} \cdot 1)$$

where

$$\begin{aligned} \rho_{p2}(2, 1) = & r_i \{ \lambda_i (\phi_p(2)\phi_i(1) + \phi_i(2)\phi_p(1) - 2S_{pi}\phi_p(2)\phi_p(1)) + \lambda_i^2 (\phi_i(2)\phi_i(1) \\ & - \phi_p(2)\phi_p(1)) \} + r_j \{ \lambda_j (\phi_p(2)\phi_j(1) + \phi_j(2)\phi_p(1) - 2S_{pj}\phi_p(2)\phi_p(1)) \\ & + \lambda_j^2 (\phi_j(2)\phi_j(1) - \phi_p(2)\phi_p(1)) \} \equiv r_i \Gamma_{pi}(2, 1) + r_j \Gamma_{pj}(2, 1), \end{aligned} \quad (\text{A} \cdot 2)$$

$$\rho_{p4}(2, 1) = - \{ r_i (2\lambda_i S_{pi} + \lambda_i^2) + r_j (2\lambda_j S_{pj} + \lambda_j^2) \} \rho_{p2}(2, 1). \quad (\text{A} \cdot 3)$$

From (20) a similar expression for ϕ_p 's can be obtained. We regard the first, the second and the third term of (A.1) to be of the zeroth, the second and the fourth order of S . When these are introduced to the first two terms of (4)', we get up to S^4 terms

$$E_0 + \sum_p (\phi_p | H_p | \phi_p) + \sum_{p>q} \left\{ \int \rho_{p2}(1, 1) \rho_{q2}(2, 2) e^2 / r_{12} \cdot d\mathbf{x}_1 d\mathbf{x}_2 \right. \\ \left. - \int \rho_{p2}(1, 2) \rho_{q2}(2, 1) e^2 / r_{12} d\mathbf{x}_1 \cdot d\mathbf{x}_2 \right\}, \quad (\text{A} \cdot 4)$$

where E_0 does not contain λ 's and H_p is given by

$$H_p(1) = H_1(1) + e^2 \sum_{\mu \neq p} \left\{ \int \phi_\mu^2(2) / r_{12} \cdot d\mathbf{x}_2 - \int \phi_\mu(2) \phi_\mu(1) P_{12} / r_{12} \cdot d\mathbf{x}_2 \right\}. \quad (\text{A} \cdot 5)$$

The second order part of (A·4) which contains λ 's is

$$\sum_i r_i \{ 2\lambda_i (H_{pi} - S_{pi} E_p) + \lambda_i^2 (E_d - E_p) \}, \quad (\text{A} \cdot 6)$$

which is a minimum with respect to λ_i when

$$\lambda_i = - (H_{pi} - S_{pi} E_p) / (E_d - E_p), \quad (\text{A} \cdot 7)$$

where

$$H_{pi} = (\phi_p | H_p | \phi_i) \quad (\text{A} \cdot 8)$$

and E_p and E_d are given by (27). The minimum value of (A·6) is

$$-\Delta E = - (H_{pi} - S_{pi} E_p)^2 / (E_d - E_p). \quad (\text{A} \cdot 9)$$

This is the amount per ion pair by which the energy of the crystal gains by virtue of electron transfer. Introducing (A·7) into (A·4) and after some calculation, we obtain as the interaction energy

$$\{ t(\Delta E - C) - \frac{1}{2} A_1 \} \sigma_i \sigma_j, \quad (\text{A} \cdot 10)$$

where

$$C = \sum_{(i)} (\phi_p \phi_i | \phi_i \phi_p), \quad (\text{A} \cdot 11)$$

$$A_1 = \int \Gamma_{pi}(1, 1) \Gamma_{pj}(2, 2) e^2 / r_{12} \cdot d\mathbf{x}_1 d\mathbf{x}_2 - \int \Gamma_{pi}(1, 2) \Gamma_{pj}(2, 1) e^2 / r_{12} \cdot d\mathbf{x}_1 d\mathbf{x}_2 \quad (\text{A} \cdot 12)$$

and t is given by (25). Here the value of (A·7) is to be taken for λ_i .

Next we observe the change of overlap energies between ions when ϕ_p 's are substituted into the B terms of (4)'. We notice that there appear overlap energies between oxygen ions of nearest and next-to-nearest neighbours because electrons from the two oxygen ions have a probability to reside at the same intervening manganese ion. It can be easily seen, however, that these overlap energies do not depend on spin configuration up to S^4 terms, so we do not consider these terms further. We observe only the change of the overlap energy between $2p$ orbital and $3d$ orbital which depends on spin configurations. Inserting the following expressions,

$$S(\phi_p, \phi_i) = \{ 1 - \frac{1}{2} t(r_i + r_j) \} S(\phi_p, \phi_i) s_i \quad (\text{A} \cdot 13)$$

$$(\phi_p | H_1 | \phi_i) = \{ 1 - \frac{1}{2} t(r_i + r_j) \} (\phi_p | H_1 | \phi_i) s_i \quad (\text{A} \cdot 14)$$

$$(\phi_p | H_1 | \phi_p) + \sum_{\mu \neq p} (\phi_p \phi_\mu | \phi_p \phi_\mu) \simeq (\phi_p | H_1 | \phi_p) + \sum_{\mu \neq p} (\phi_p \phi_\mu | \phi_p \phi_\mu) - \Delta E(r_i + r_j) \quad (\text{A} \cdot 15)$$

into the expression of B between ϕ_i and ϕ_p , we can obtain the following interaction energy from B terms

$$\{tB(\phi_p, \phi_i) + S_{pi}^2 \Delta E + A_2\} \sigma_i \sigma_j, \quad (\text{A} \cdot 16)$$

where
$$A_2 = \int \Gamma'_{pi}(1, 1) \Gamma_{pj}(2, 2) e^2 / r_{12} \cdot d\mathbf{x}_1 d\mathbf{x}_2 \quad (\text{A} \cdot 17)$$

$$\Gamma'_{pi}(1, 1) = -2S_{pi} \phi_p(1) \phi_i(1) + S_{pi}^2 \{ \phi_p(1) \phi_p(1) + \phi_i(1) \phi_i(1) \}. \quad (\text{A} \cdot 18)$$

Adding (A·10) and (A·16), we obtain as the coefficient of $\sigma_i \sigma_j$

$$J_{A\sigma} = J_{A\sigma 1} + J_{A\sigma 2}, \quad (\text{A} \cdot 19)$$

where
$$J_{A\sigma 1} = t \{ B(\phi_p, \phi_i) - C \} + S_{pi}^2 \Delta E + A_2, \quad (\text{A} \cdot 20)$$

$$J_{A\sigma 2} = -\frac{1}{2} A_2 + t \Delta E. \quad (\text{A} \cdot 21)$$

In order to transform these expressions to a suitable form for numerical calculation we assume that $2p$ orbital of an oxygen ion satisfies the Hartree-Fock equation of free ion,

$$H_{p \text{ intra}}(1) \phi_p(1) = -X \phi_p(1), \quad (\text{A} \cdot 22)$$

where X corresponds to electron affinity of O^- ion. Similarly $3d$ orbital ϕ_i of a manganese ion satisfies the Hartree-Fock equation of the free manganese ion

$$[p_1^2/2m - Z_i e^2/r_{1i} + e^2 \sum_{\mu}^{(i)} \{ \phi_{\mu}^2(2)/r_{12} \cdot d\mathbf{x}_2 - \int \phi_{\mu}(2) \phi_{\mu}(1) P_{12}/r_{12} \cdot d\mathbf{x}_2 \}] \phi_i = I_2 \phi_i \quad (\text{A} \cdot 23)$$

where I_2 corresponds to the second ionization energy of manganese. Then it can be easily shown that

$$H_{pi} - S_{pi} E_p = (\phi_p | \sum_{\kappa}^{(i)} Q_{\kappa} | \phi_i) - S_{pi} (\phi_p | \sum_{\kappa}^{(i)} Q_{\kappa} | \phi_p), \quad (\text{A} \cdot 24)$$

$$E_d - E_p = X - I_2 + (\phi_i | \sum_{\kappa}^{(i)} Q_{\kappa} | \phi_i) - (\phi_p | \sum_{\kappa}^{(i)} Q_{\kappa} | \phi_p) - (\phi_i \phi_j | \phi_i \phi_j). \quad (\text{A} \cdot 25)$$

We evaluated these integrals by the same approximation as Lowdin used (Chapter 5 of ref. 8)) except for $(\phi_i \phi_j | \phi_i \phi_j)$, which was evaluated by point charge approximation. For X and I_2 we used experimental values. Then $E_d - E_p = 14$ eV and $H_{pi} - S_{pi} E_p = -0.97$ eV, so λ becomes 0.067. Then $6N\Delta E$, which is the gain of the crystal energy by virtue of transfer, is 9.2 kcal/mol which is much smaller than the repulsive energy of MnO crystal. Integrals $(\phi_p \phi_i | \phi_j \phi_p)$ and $(\phi_p \phi_j | \phi_i \phi_j)$ which appear in A_1 and A_2 were replaced by $S_{pi} S_{pj} e^2/2a$ and $S_{pi} e^2/2a$, respectively. Other integrals A_1 and A_2 can be calculated by Lowdin's approximation. The results are

$$S_{pi} = 0.076, \quad t = 0.015, \quad (\text{A} \cdot 26)$$

$$J_{A\sigma 1} = -0.0015 \text{ eV}, \quad J_{A\sigma 2} = 0.0010 \text{ eV}, \quad (\text{A} \cdot 27)$$

$$J_{A\sigma} = -0.0004 \text{ eV}. \quad (\text{A} \cdot 28)$$

Appendix II

From the knowledges obtained by the calculation of superexchange interaction

of MnO crystal we can make some qualitative discussions about superexchange interactions of other substances.

(1) We consider 180° interaction between ions of configuration $(3d)^3$ and $(3d)^5$ which are surrounded by six negative ions in a regular octahedral arrangement. We suppose that these ions have maximum spin. The contribution from $2p\pi$ orbital may be neglected, because of small overlap with $3d\varepsilon$ orbitals and of orthogonality to $3d\gamma$ orbitals. Then from totally ionic model we obtain from (10), (11), (12) and (13)

$$E_{\text{parallel}} - E_{\text{antiparallel}} = -S^2 \sum_{(j)} (\phi_p \phi_j | \phi_j \phi_p), \quad (\text{A} \cdot 29)$$

where $S = \int \phi_p \phi_{i0} d\mathbf{x}$ and i and j denote $(3d)^5$ and $(3d)^3$ ions, respectively. Here ϕ_{i0} is the $3d\gamma$ orbital of the i -th ion which has maximum overlap with ϕ_p . This term may be interpreted as follows: because of non-orthogonality S between ϕ_p and ϕ_{i0} , the density of ϕ_p electron becomes $(1+S^2)\phi_p^2(\mathbf{r})$ in the region not so close to the i -th ion when this electron has the same spin as the i -th ion and the density is $\phi_p^2(\mathbf{r})$ when it has the opposite spin to the i -th ion. Then the difference $S^2\phi_p^2(\mathbf{r})$ between the density of α electron and that of β electron results, which gives rise to exchange energy with the j -th ion. This is given by (A·29).

Next we introduce Anderson's mechanism. One must recognize that $2p\sigma$ electrons of both spin directions can transfer to the $(3d)^3$ ion because $3d\gamma$ orbital of that ion is completely empty. Of course, Hund's coupling favours the transfer of the electron with the same spin as the $(3d)^3$ ion, but the magnitude of Hund's coupling is supposed to be about 1 eV, while the energy necessary to transfer a $2p$ electron to the metallic ion may amount to 10 eV in the usual case. Then the superexchange interaction from this mechanism may be much reduced. By the same reason superexchange interaction from Anderson-Hasegawa's mechanism and two electron transition process is supposed to be much reduced. On the other hand the mechanism in which a $2p$ electron transfers to the $(3d)^5$ ion gives the following interaction,

$$E_{\text{parallel}} - E_{\text{antiparallel}} = -(2\lambda S + \lambda^2) \sum_{(j)} (\phi_p \phi_j | \phi_j \phi_p). \quad (\text{A} \cdot 30)$$

Adding (A·29) and (A·30), we obtain

$$E_{\text{parallel}} - E_{\text{antiparallel}} = -(\lambda + S)^2 \sum_{(j)} (\phi_p \phi_j | \phi_j \phi_p), \quad (\text{A} \cdot 31)$$

which gives ferromagnetic interaction. We notice that the ferromagnetic interaction results only when both the overlap effect (A·29) and the Anderson term (A·30) are considered. The mechanism which corresponds to that of § 5 is also reduced, because excited configurations can be mixed both in ferromagnetic and in antiferromagnetic case.

Next we consider Slater's mechanism. We suppose the overlap and the exchange interactions, which act between the $(3d)^5$ ion and $2p$ electron of the same spin direction, to be repulsive. On the other hand, since the overlap interaction is

absent between the $(3d)^3$ ion and $2p\sigma$ electron, the $(3d)^3$ ion may attract $2p\sigma$ electron of the same spin direction. Then, contrary to the case of MnO, the oxygen ion can polarize when the two magnetic ions have the same spin direction and Slater's mechanism also acts ferromagnetically.

From above discussions 180° interaction between $(3d)^3$ and $(3d)^6$ ions is inferred to be ferromagnetic*. The same will hold between $(3d)^3$ and $(3d)^4$ ions, when orbital degeneracy of the $(3d)^4$ ion is removed by some crystal distortion from cubic symmetry and as a result the $d\gamma$ orbital of $3z^2-r^2$ type which is directed to the $(3d)^3$ ion is stabilized.

(2) Next we shall discuss 180° interaction between $(3d\epsilon)^3$ ions. Slater's mechanism of $(2p+s)$ configuration mixing is concluded to be antiferromagnetic by the same reason as in MnO, except for one point that $(3d)^3$ ion attracts rather than repels the $2p\sigma$ electron of the same spin as the ion because of the absence of overlap interaction between the $2p\sigma$ electron and the $(3d)^3$ ion. Exchange interaction between $2p\sigma$ orbital and $3d\epsilon$ orbitals is pertinent to the Slater mechanism. *This is the reason why this mechanism is expected to give the dominant contribution to superexchange interaction.* In the other mechanisms the coupling between $2p\sigma$ and $3d\epsilon$ orbitals is absent because of the symmetry of wave functions. There remain only $(2p\pi, 3d\epsilon)$ coupling which is small because of small overlap and $(2p\sigma, 3d\gamma)$ coupling which is reduced because both α and β electrons can transfer to $3d\gamma$ orbital.

(3) We consider 180° interaction of NiO. In this case only $d\gamma$ electrons are relevant to superexchange interaction. It may also be sufficient here to consider only the effect of $2p\sigma$ orbital. The Slater mechanism is caused by the repulsive force of $3d\gamma$ electron which is exerted on $2p\sigma$ electron of the same spin, whereas in the case of MnO, both the repulsive force of $3d\gamma$ electron and the attractive force of $3d\epsilon$ electrons are effective. Then it is supposed that in MnO the superexchange interaction from Slater's mechanism is much reduced by cancellation. The similar argument can be applied to Anderson's mechanism. Although the polarization in NiO cannot be said at once to be larger than that in MnO because of the difference of wave functions, the above situation may be one of the causes that superexchange interaction is larger in NiO than in MnO. Whether $3d\epsilon$ shell is completely filled or half filled does not directly affect Anderson-Hasegawa's mechanism, two electron transition process and configuration mixing $(\text{Ni}^+\text{O}^{--}\text{Ni}^{++})$, which all act antiferromagnetically.

(4) Here we have a criterion for 180° interaction which states that, when one ion exerts a repulsive force and the other an attractive one on $2p\sigma$ electron of the same spin as the ion, the coupling is ferromagnetic, and when both ions exert a repulsive or an attractive force, the coupling is antiferromagnetic. We argue that the attractive case occurs when the ion has no $d\gamma$ electron or when the ion has

* We anticipate, therefore, $\text{La}(\text{Fe}, \text{Cr})\text{O}_3$ and $(\text{Fe}, \text{Cr})\text{F}_3$ to be ferromagnetic.

its $d\gamma$ orbital directed sideways from the negative ion, and that the repulsive case occurs when the ion has at least one $d\gamma$ electron which is directed to the negative ion. We stated the above criterion in terms of the Slater mechanism, but we shall see that other mechanisms often give the same conclusion as Slater's mechanism as to the sign of superexchange interaction.

(5) Now we discuss the superexchange interaction of MnF_3 ¹⁸⁾ and LaMnO_3 ¹⁹⁾, which are antiferromagnetics with the Neel temperatures of 43 and about 140°K, respectively. Their magnetic structures were determined by neutron diffraction methods and it was found that they had the same structure, in which in a c -plane spins are coupled ferromagnetically and spin directions of two adjacent planes are coupled antiferromagnetically. In these substances manganese is considered to be triply ionized. The Mn^{3+} ion has three $d\varepsilon$ electrons and one $d\gamma$ electron, so that its orbital state is doubly degenerate in a cubic crystal. A manner of ordering of single $d\gamma$ orbital in MnF_3 was proposed by E. O. Wollan *et al*¹⁸⁾. Their description is as follows: in a c -plane one of two Mn^{3+} ions which are sited on opposite sides of a fluorine ion has its $d\gamma$ orbital (of $3z^2 - r^2$ type) directed to the fluorine ion and the other sideways from the ion. Then according to the above criterion we have ferromagnetic coupling in c -plane, which agrees with the observed facts.* As for the interaction between c -planes, two Mn^{3+} ions have their $d\gamma$ orbitals directed sideways and we have antiferromagnetic coupling between c -planes, which is that observed. The situation will be similar in the case of LaMnO_3 .

It was found that adding a small amount of BaTiO_3 to LaMnO_3 makes the symmetry of the substance cubic, accompanied by ferromagnetic transition. We imagine the ordering of $d\gamma$ orbitals is removed by addition of BaTiO_3 and each Mn^{3+} ion has no particular direction of its $d\gamma$ orbital. Then, if we assume the interaction between two Mn^{3+} ions averaged over various orbital directions to be ferromagnetic**, the substance becomes ferromagnetic, as is observed.

* Goodenough⁶⁾ assumed *a priori* ferromagnetic coupling in c -plane in the case of LaMnO_3 .

** This may be reasonable, because the ferromagnetic interaction is supposed to be much larger than the antiferromagnetic interaction, and because the appearance of the configurations in which two Mn^{3+} ions have their $d\gamma$ orbitals directed to the intervening oxygen ion and in which the coupling is supposed to be strongly antiferromagnetic may probably be suppressed by short range order because of high energy of that configuration.

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Strong Fermi-Type Interaction and Its Application to the G_A/G_V Ratio in β -Decay and the Anomalous Magnetic Moment of Nucleon

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The G_A/G_V ratio in β -decay and the anomalous nucleon magnetic moment are calculated on the basis of the compound model in which the elementary strong interactions are assumed to be the CP-invariant scalar, pseudoscalar and tensor Fermi type couplings.

In this model the P -invariance of strong interactions and the equality of the vector coupling constant of β -decay to that of μ - e decay are guaranteed automatically, and the experiment proposed by Gell-Mann is found to be useful as a test of the composite model.

§ 1. Introduction

Meson theory has achieved great success in explaining the phenomena in which only the outside region of the nucleon cloud is concerned. In phenomena in which the contribution from the inner nucleon cloud region cannot be neglected, however, meson theory has not yet clarified a number of points, such as:

- (i) The coupling constant ratio G_A/G_V in β -decay becomes smaller than 1, if we apply static or covariant perturbational methods in lowest order to the theory of Gell-Mann et al.^{1,2)}
- (ii) The experimental result for the iso-scalar part of the anomalous nucleon magnetic moment is smaller than the theoretical value.
- (iii) The electric charge distribution of the neutron.
- (iv) S -wave pion nucleon scattering.

On the other hand, Fermi and Yang, Heisenberg, and Sakata et al. considered that the Fermi type interaction is more fundamental than the Yukawa type and proposed the composite model of mesons. Many authors have discussed how one may construct mesons from this point of view. Some authors have applied this model to the problem of S -wave pion-nucleon scattering, but owing to the high degree of divergency they could not get any decisive results. Quite a few people have expressed the opinion that the composite model perhaps gives just the same results as the conventional π -meson theory.

The present author³⁾ has raised the question why parity happens to be conserved in strong and electromagnetic interactions whereas it is not conserved in weak interactions. And we have found that the composite model appears more

beautiful than Yukawa meson theory from the standpoint of transformation properties. In our previous paper we made the following 3 assumptions:

(Assumption i) "The strong, electromagnetic and weak interaction Hamiltonians are invariant under the combined transformation CP, but not necessarily under P - and C -transformation separately".

(Assumption ii) "The electromagnetic interaction is gauge invariant".

(Assumption iii) "The strong interaction is charge independent".*

With these assumptions we attempted to guarantee that P and C are conserved in the strong and electromagnetic interactions by restricting the type of the interaction, and found:

(i) In the electromagnetic interaction,⁴⁾⁵⁾⁶⁾ it is sufficient to restrict the interaction to the usual type.**

(ii) In the strong interaction,

a) When the fundamental interaction is assumed to be the following Fermi type;

$$H = (\bar{N}N)(\bar{N}N) + (\bar{P}P)(\bar{P}P) + (\bar{P}P)(\bar{N}N) + (\bar{P}N)(\bar{N}P) \\ + (\bar{A}A)(\bar{A}A) + (\bar{A}A)(\bar{P}P) + (\bar{A}A)(\bar{N}N) \quad (1)$$

and the idea of Sakata's compound model⁸⁾ is applied, it is sufficient to restrict the interaction type to an arbitrary linear combination of scalar, pseudoscalar and tensor couplings.³⁾

b) When the fundamental interaction is assumed to be of the Yukawa type, π -meson interactions are P -invariant provided that the coupling is non-derivative.⁴⁾⁵⁾⁶⁾ In order to make the K -meson interaction P -invariant, however, we must take another, stronger assumption⁹⁾ than the above three.

Thus if we make use of Sakata's compound model and assume the fundamental interaction to be of the Fermi type, the characteristics of the baryon weak interaction, namely change of strangeness and parity non-conservation, are due to the change of the fundamental particles ($N \leftrightarrow A$), and the strong interaction conserves parity as a natural consequence. So the Fermi type interaction is more beautiful than the Yukawa type interaction from the point of view of transformation properties.

Through these considerations we are led to the viewpoint that the elementary interaction is of Fermi type and the coupling is restricted to scalar, pseudoscalar and tensor couplings. With this viewpoint we may make a fresh attack on the phenomena which cannot be explained by usual meson theory. In this paper we calculate the G_A/G_V ratio and the anomalous nucleon magnetic moment and consider

* We can get the condition that parity non-conserving terms cannot appear in the strong interaction Hamiltonian, even without the charge independence hypothesis. See the previous paper.³⁾

** If we take an unfamiliar electromagnetic interaction, we can get P -noninvariant and gauge invariant electromagnetic interaction as in Zeldovich's letter.⁷⁾

why the results calculated on the grounds of Fermi type interaction are qualitatively different from the results obtained by Yukawa type interaction.

The fundamental interactions assumed in this note have the following form,

$$H_w(\beta\text{-decay}) = \left\{ \bar{\psi} \gamma_\mu (G_V^0 + G_A^0 \gamma_5) \frac{\tau_1 + i\tau_2}{2} \phi \right\} \cdot \left\{ \bar{e} \gamma_\mu \frac{1 + \gamma_5}{\sqrt{2}} \nu \right\} + \text{H.C.} \quad (2)$$

$$H_w(\mu \rightarrow e) = \left\{ \bar{\nu} \gamma_\mu (g_V^0 + g_A^0 \gamma_5) \mu \right\} \left\{ \bar{e} \gamma_\mu \frac{1 + \gamma_5}{\sqrt{2}} \nu \right\} + \text{H.C.} \quad (3)$$

$$H_{\pi N} = i e \bar{\psi} \gamma_\mu \frac{1 + \tau_3}{2} \phi A_\mu \quad (4)$$

$$H_{gt} = \sum_i \sum_k f_{1i} (\bar{\psi} O_i \tau_k \phi) (\bar{\psi} O_i \tau_k \phi) + \sum_i f_{0i} (\bar{\psi} O_i \phi) (\bar{\psi} O_i \phi) \quad (5)$$

where*

$$\begin{aligned} O_S &= 1 \\ O_V &= \gamma_\mu \\ O_T &= (i/2) (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \\ O_A &= i \gamma_\mu \gamma_5 \\ O_P &= \gamma_5 \end{aligned} \quad (6)$$

ϕ represents the wave function of nucleon and

$$\bar{\psi} = \psi^\dagger \gamma_4. \quad (7)$$

We assume the existence of other elementary interactions such as: μ -capture, electromagnetic interaction of particles other than nucleons, strong interactions concerning the Λ -particle, and baryon weak interactions, but we shall not enter into the details of these interactions.

§ 2. Electric charge renormalization and the anomalous nucleon magnetic moment

According to the requirement of gauge invariance the renormalized charge is exactly equal to the bare charge in the lowest order of e . (Appendix A)

Even in the interaction of the 2nd kind we can eliminate divergence by a finite number of renormalization constants, if we restrict the calculation to perturbations of a finite order. However, we must consider not only mass and charge, but also magnetic moment, etc., as the renormalization constants; the calculated value of the anomalous magnetic moment diverges, so we cut off the divergent integral by Feynman's cut-off factor. The calculation is performed in the lowest order and

* Our standpoint is $f_{1V} = f_{0V} = f_{1A} = f_{0A} = 0$. However, we write the expression including V, A coupling for convenience of future reference.

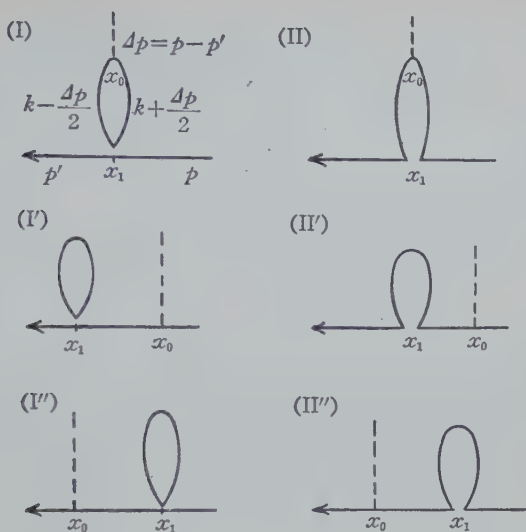


Fig. 1

the relevant diagram is given in Fig. 1. Only the effective Hamiltonian of the form,

$$A_\mu \bar{\psi}(x_0) \Delta P_\nu \sigma_{\mu\nu} \psi(x_0) f[(\Delta P)^2], \quad (8)$$

$$\sigma_{\mu\nu} = (i/2) (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu),$$

contributes to the anomalous magnetic moment, and this can be expressed as follows :

$$\begin{aligned} H_{eff} &= -\frac{e}{4} A_\mu \int_{-\infty}^{\infty} dx_1 \{ \bar{\psi}(x_1) \sigma_{\rho\nu} (F_{1T} \tau_3 + F_{0T}) \psi(x_1) \} \\ &\quad \times S_P \{ S_F(x_0 - x_1) \gamma_\mu S_F(x_1 - x_0) \sigma_{\rho\nu} \} \\ &= 4\kappa^2 J(0) \left[\bar{\psi}(x_0) \left(-\frac{e}{2\kappa} \boldsymbol{\sigma} \cdot \mathbf{H} \right) (F_{1T} \tau_3 + F_{0T}) \psi(x_0) \right] \end{aligned} \quad (9)$$

where

$$F_{1T} = f_{1T} + \frac{3}{4} (f_{1S} - f_{0S}) - \frac{1}{4} (f_{1T} - f_{0T}) + \frac{3}{4} (f_{1P} - f_{0P}) \quad (10)$$

$$F_{0T} = f_{0T} - \frac{3}{4} (3f_{1S} + f_{0S}) + \frac{1}{4} (3f_{1T} + f_{0T}) - \frac{3}{4} (3f_{1P} + f_{0P})$$

$$J((\Delta P)^2) = -i \int_0^1 dx \int d^4k \frac{1}{[k^2 + \kappa^2 + (\Delta P)^2 (\frac{1}{2} - x)^2]^2}. \quad (11)$$

Introducing Feynman's cut off factor,¹⁰⁾ $J(0)$ is rewritten as

$$\begin{aligned} J(0) &= -i \int d^4k \frac{\lambda^2}{[k^2 + \kappa^2]^2 [k^2 + \kappa^2 + \lambda^2]} \\ &= \pi^2 \left[-1 + \frac{\kappa^2 + \lambda^2}{\lambda^2} \ln \frac{\kappa^2 + \lambda^2}{\kappa^2} \right]. \end{aligned} \quad (12)$$

The experimental and the calculated values, based on the π -meson theory, of the anomalous magnetic moment of the nucleon are as follows:

		$\Delta\mu_P$	$\Delta\mu_N$	$\Delta\mu_P + \Delta\mu_N$	$\Delta\mu_P - \Delta\mu_N$
experimental value		1.790	-1.910	-0.120	3.700
previous theoretical value	2nd order perturbation	0.53	-3.93	-3.40	4.46
(π meson theory)	2nd order perturbation (cut off)	0.61	-1.90	-1.29	2.51

Besides the theoretical values tabulated above, many authors have worked out the anomalous magnetic moment by various other methods (static theory, intermediate coupling theory...). In most of them the theoretical value of the iso-vector part coincides with the experimental one, but the usual π -meson theory failed to explain the isoscalar part. The effects of strange particles are also not sufficient⁽¹⁾ to alter these values.

However, if the elementary interaction is of the Fermi type, the contribution from the diagram of Fig. 1 which does not exist in the π -meson theory becomes large and when $F_{0T} > 0$ the large iso-scalar anomalous magnetic moment calculated by the π -meson theory will possibly be canceled out by this diagram.

Table 1

λ^2/κ^2	$F_{0T}\kappa^2$
0.5	0.06
1.0	0.03
2.0	0.02
10	0.008
100	0.003
1000	0.002

Table 2

	sign	F_{1T}/F_{0T}
f_{1P} the other=0	-	$-\frac{1}{3}$
f_{1S} "	-	$-\frac{1}{3}$
f_{0T} "	+	1
f_{0P} "	-	1
f_{0S} "	-	1
f_{0T} "	+	$\frac{1}{5}$
$f_{1P}=f_{0P}=f_P$ "	-	0
$f_{1S}=f_{0S}=f_S$ "	-	0
$f_{1T}=f_{0T}=f_T$ "	+	0

For this purpose, $\Delta\mu_P + \Delta\mu_N$ obtained by Eqs. (9) and (10) must be ~ 1 , and F_{0T} becomes as shown in Table 1, where λ^2 is Feynman's cut-off parameter. If we restrict the type of coupling to one kind, the sign of the coupling constant and F_{1T}/F_{0T} ratio have the values shown in Table 2.

§ 3. G_A/G_F ratio in β -decay interaction

Recently Feynman and Gell-Mann⁽¹⁾⁽²⁾ have paid attention to the following relations between effective weak coupling constants:

$$g_V \simeq G_V$$

$$G_A/G_V \simeq 1.2. \quad (13)$$

They have proposed weak universal $V-A$ interactions ($g_A^0 = g_V^0 = G_A^0 = G_V^0$) and assumed an interaction $(\bar{e}\gamma_\mu(1+\gamma_5)/\sqrt{2}\nu)$ coupled with pions as well as with nucleus

$$\mathcal{J}_{\mu+} \left(\bar{e}\gamma_\mu \frac{1+\gamma_5}{\sqrt{2}} \nu \right) + \text{H.C.}$$

$$\mathcal{J}_\mu = \frac{1}{2} i \bar{\psi} \boldsymbol{\pi} \gamma_\mu \psi + \boldsymbol{\pi} \times \partial \boldsymbol{\pi} / \partial x_\mu \quad (14)$$

in order to explain $g_V = G_V$, by means of a method identical with that of Gershtein and Zeldovich.¹²⁾ However, in the static form of meson theory or in the lowest order covariant perturbation theory, G_A/G_A^0 is always < 1 .

In this note we make use of Sakata's composite model and assume the elementary strong interactions to be of the Fermi type. Then it is not necessary to assume β -decay interaction other than (2), in order to explain $g_V = G_V$.

Namely, if the result of the experiment* proposed by Gell-Mann²⁾ is negative, it means that Sakata's composite model is not true or that the gauge invariance is not completely correct in β -decay. We have calculated G_A/G_A^0 in the lowest order perturbation (Fig. 2), and the result is given by

$$\frac{G_A}{G_A^0} = 1 + \frac{2iF_1}{(2\pi)^4} \int d^4k \, Sp \left[\frac{(i\mathbf{k}-\kappa)\gamma_\mu(1+\gamma_5)(i\mathbf{k}-\kappa)\gamma_\mu\gamma_5}{(k^2+\kappa^2)^2} \right], \quad (15)$$

$$\mathbf{k} = \sum_p \mathbf{k}_p \gamma_p$$

where

$$F_1 = f_{1A} + \frac{1}{2}(f_{1S} - f_{0S}) - \frac{1}{2}(f_{1P} - f_{0P}) + \frac{1}{4}(f_{1V} - f_{0V}) - \frac{1}{4}(f_{1A} - f_{0A}). \quad (16)$$

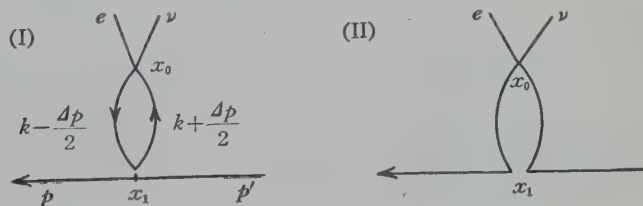


Fig. 2

After the calculation of the integral in Eq. (15), G_A/G_A^0 becomes, (Case 1)

$$\frac{G_A}{G_A^0} = 1 + \frac{4F_1\kappa^2}{(2\pi)^2} \left[-1 + \frac{\kappa^2 + \lambda^2}{\lambda^2} \ln \frac{\kappa^2 + \lambda^2}{\kappa^2} \right] \quad (17)$$

* Gell-Mann has proposed his experiment in order to check the last term of Equation (14): if the result of the experiment is negative, $(\bar{e}\nu)$ does not couple with the pion directly.

(Case 2)

$$\frac{G_A}{G_A^0} = 1 + \frac{2F_1\kappa^2}{(2\pi)^2} \left[1 + \frac{1}{2} \frac{\lambda^2}{\kappa^2} - \frac{\kappa^2 + \lambda^2}{\lambda^2} \ln \frac{\kappa^2 + \lambda^2}{\kappa^2} \right] \quad (18)$$

(Case 3)

$$G_A/G_A^0 = [(G_A/G_A^0)_{\text{case 1}} + (G_A/G_A^0)_{\text{case 2}}] \div 2 \quad (19)$$

depending on the method of calculation.* (Appendix C)

Table 3

λ^2/κ^2	$F_1\kappa^2$ (Case 1)	$F_1\kappa^2$ (Case 2)	$F_1\kappa^2$ (Case 3)
0.5	9	110	17
1.0	5	35	9
2.0	3	11	5
10	1	1	1
10^2	0.5	0.09	0.2
10^3	0.3	0.008	0.02

Table 4

	sign
$f_{1P} \neq 0$, all other f 's = 0	-
f_{1S} "	+
f_{0P} "	+
f_{0S} "	-
$f_{1P} = f_{0P} = f_P$ "	☆
$f_{1S} = f_{0S} = f_S$ "	☆

☆ In these cases $F_1=0$, and the diagrams of Fig. 2 do not contribute to the value of G_A/G_A^0 .

In order to make $G_A/G_V \sim 1.2$, F_1 must be positive and the required order of F_1 is given in Table 3. If we restrict the type of coupling to one kind, S or P , the signs of the coupling constants required are shown in Table 4.

In our approximation (lowest order perturbation theory) the signs of f_{1S} and f_{0P} shown in Table 2 and Table 4 are inconsistent with each other, so these possibilities can be excluded.

§ 4. Comparison of Fermi type interaction with Yukawa type interaction

Now we consider the reason why the Fermi type interaction can explain processes which cannot be explained by usual Yukawa type interaction. The physical quantities calculated in usual meson theory are expressed either by an even or an odd power of coupling constant, never by both even and odd powers. In those quantities which are considered in this note only the even terms appear. However, if the elementary interaction is of the Fermi type, such a quantity is expanded in both even and odd powers of the Fermi coupling constant. Thus in the latter theory the relative signs of the coupling constants are important.

Contributions from Fig. 1 have no counter-parts in meson theory and change their sign if the signs of coupling constants are changed. However, we could establish, if we like, a relation between such diagrams of the Fermi type interaction

* The calculation of these diagrams contain divergences and is ambiguous. However, the above 3 cases give the same restrictions on the sign of f as shown in Table 4.

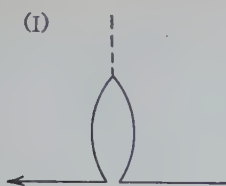


Fig. 3

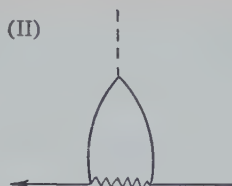


Fig. 4

and diagrams in meson theory; the diagram (I) should correspond to the diagram (II) in Fig. 3 when the mass of the virtual meson μ is extremely large.

The effective Fermi type coupling constant derived from a direct coupling of pseudoscalar or scalar mesons with iso-spin 1 is, in the lowest order perturbation (Fig. 4), given by

$$f_{1P}' = -\frac{g^2}{2} \frac{1}{k^2 + \mu^2} \quad f_{1S}' = -\frac{g^2}{2} \frac{1}{k^2 + \mu^2}. \quad (20)$$

The signs of f_{1P}' and f_{1S}' are determined uniquely, $f_{1P}' > 0$ and $f_{1S}' < 0$, but if the elementary interaction is of the Fermi type they can have arbitrary signs. The sign of f_{1P}' is different from that of f_{1P} in Tables 2 and 4.

§ 5. Conclusion and discussion

We can point out the following qualitatively beautiful properties of the composite model:

- (i) Strong and electromagnetic interactions become P -invariant when we restrict coupling type in a simple way.
- (ii) G_V (the vector coupling constant of β -decay) and g_V (the vector coupling constant of μ - e decay) have, automatically, equal magnitude, and the experiment proposed by Gell-Mann²⁾ is useful as a test of the composite model.
- (iii) If the elementary weak baryon interactions are to be of the Fermi type

$$(\bar{A}\psi)(\bar{\psi}\psi) + (\bar{A}\psi)(\bar{A}\psi),$$

these interactions give the selection rules $\Delta I = \pm \frac{1}{2}$ and $\pm \frac{3}{2}$, and the transition $\Delta I = \pm \frac{5}{2}$ is forbidden. A $\Delta I = \pm \frac{5}{2}$ transition can occur only through electromagnetic corrections and is very small. This selection rule imposes a strict condition* in the branching ratio of K -decay, $(K_1^0 \rightarrow \pi^0 + \pi^0)/(K_1^0 \rightarrow \pi^+ + \pi^-)$.

Quantitatively we can explain G_A/G_V ratio and the scalar part of anomalous magnetic moment of nucleon, which can hardly be accounted for by the usual meson theoretical calculations.

In spite of the divergency of high degree, we think that it is very important to investigate various phenomena by means of the Fermi type interaction and to examine the characteristics of the Fermi type interaction.

* This condition is the same with that of Fig.2 of ref. 13).

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Appendix A. Electric charge renormalization

The correction to the electric charge is calculated to the lowest order, the 1st order Feynman diagrams are shown in Fig. 1. The contribution from the diagram (I') (II') (I'') (II'') are renormalized into mass terms. The effective Hamiltonian derived from the diagram (I) and (II) are written as follows.

$$H_{eff}^I = \frac{i(ie)}{4} \sum_i A_\mu(x_0) \int_{-\infty}^{\infty} \{ \bar{\psi}(x_1) O_i (f_{1i} \tau_3 + f_{0i}) \psi(x_1) \} \\ \times Sp \{ S_F(x_0 - x_1) \gamma_\mu S_F(x_1 - x_0) O_i \} dx_1 \quad (A \cdot 1)$$

$$H_{eff}^{II} = -\frac{i(ie)}{4} \sum_i A_\mu(x_0) \int_{-\infty}^{\infty} dx_1 \left\{ \bar{\psi}(x_1) \left(f_{1i} \frac{3 - \tau_3}{2} + f_{0i} \frac{1 + \tau_3}{2} \right) \right. \\ \left. \times O_i S_F(x_0 - x_1) \gamma_\mu S_F(x_1 - x_0) O_i \psi(x_1) \right\} \quad (A \cdot 2)$$

where

$$S_{F\beta\alpha}(x-y) = 2 \langle P \{ \bar{\psi}_\alpha(x), \psi_\beta(y) \} \rangle_{vac}. \quad (A \cdot 3)$$

H_{eff}^{II} can be rewritten in the form of H_{eff}^I , with the use of the formula in Appendix B. Only the term in which $O_i = \gamma_\mu$ contributes to the charge renormalization, then the required effective Hamiltonian is

$$H_{eff} = -\frac{e}{4} A_\mu(x_0) \int_{-\infty}^{\infty} dx_1 \{ \bar{\psi}(x_1) \gamma_\mu (F_1 \tau_3 + F_0) \psi(x_1) \} Sp \{ S_F(x_0 - x_1) \gamma_\mu S_F(x_1 - x_0) \gamma_\mu \} \quad (A \cdot 4)$$

where

$$F_1 = +\frac{1}{2}(f_{1S} - f_{0S}) - \frac{1}{2}(f_{1P} - f_{0P}) - \frac{1}{4}(f_{1V} - f_{0V}) + \frac{1}{4}(f_{1A} - f_{0A}) + f_{1V} \\ F_0 = -\frac{1}{2}(3f_{1S} + f_{0S}) + \frac{1}{2}(3f_{1P} + f_{0P}) + \frac{1}{4}(3f_{1V} + f_{0V}) - \frac{1}{4}(3f_{1A} + f_{0A}) + f_{0V}. \quad (A \cdot 5)$$

(A·4) can be rewritten as

$$H_{eff} = \frac{e}{(2\pi)^4} A_\mu(x_0) \{ \bar{\psi}(x_0) \gamma_\mu (F_1 \tau_3 + F_0) \psi(x_0) \} \\ \int d^4k Sp \left[\frac{\{ i(\mathbf{k} - \mathbf{A}\mathbf{p}/2) - \kappa \} \gamma_\mu \{ i(\mathbf{k} + \mathbf{A}\mathbf{p}/2) - \kappa \} \gamma_\mu}{\{ (\mathbf{k} - \mathbf{A}\mathbf{p}/2)^2 + \kappa^2 \} \{ (\mathbf{k} + \mathbf{A}\mathbf{p}/2)^2 + \kappa^2 \}} \right] \quad (A \cdot 6)$$

where

$$\mathbf{k} = \sum_\mu k_\mu \gamma_\mu \\ \mathbf{A}P = P - P' \quad (A \cdot 7)$$

$$\bar{\psi}(x_0) = \bar{b}e^{-iP \cdot x_0}, \quad \psi(x_0) = ae^{iP \cdot x_0}$$

$$I_{\mu\nu} = \int d^4k S_P \left[\frac{\{i(\mathbf{k} - \mathbf{A}\mathbf{p}/2) - \kappa\} \gamma_\mu \{i(\mathbf{k} + \mathbf{A}\mathbf{p}/2) - \kappa\} \gamma_\nu}{\{(\mathbf{k} - \mathbf{A}\mathbf{p}/2)^2 + \kappa^2\} \{(\mathbf{k} + \mathbf{A}\mathbf{p}/2)^2 + \kappa^2\}} \right]$$

must be of the form

$$I_{\mu\nu} = (\mathbf{A}P_\mu \mathbf{A}P_\nu - (\mathbf{A}P)^2 \delta_{\mu\nu}) C((\mathbf{A}P)^2), \quad (\text{A} \cdot 8)$$

which follows from the relativistic and gauge invariance of the theory. Then the correction of electric charge must be zero. However, the integration of (A.6) is quadratically divergent, and in order that the above general theory is correct the following condition must be fulfilled,

$$\int d^4k S_P \left[\frac{(i\mathbf{k} - \kappa) \gamma_\mu (i\mathbf{k} - \kappa) \gamma_\nu}{(k^2 + \kappa^2)^2} \right] = 0. \quad (\text{A} \cdot 9)$$

Appendix B

The Hamiltonian of the charge independent, nonderivative Fermi interaction is a linear combination of the following terms

$$\begin{aligned} H_{1i} &= \sum_k (\bar{\psi} O_i \tau_k \psi) (\bar{\psi} O_i \tau_k \psi) \\ H_{0i} &= (\bar{\psi} O_i \psi) (\bar{\psi} O_i \psi) \end{aligned} \quad (\text{A} \cdot 10)$$

where O_i 's are defined in (6).

If we exchange ψ and $\bar{\psi}$ and define H'_{1i} and H'_{0i} by

$$\begin{aligned} H'_{1i} &= \sum_k (\bar{\psi} O_i \tau_k \psi) (\bar{\psi} O_i \tau_k \psi) \\ H'_{0i} &= (\bar{\psi} O_i \psi) (\bar{\psi} O_i \psi), \end{aligned} \quad (\text{A} \cdot 11)$$

then H'_{1i} and H'_{0i} can be written as a linear combination of H_{1i} 's and H_{0i} 's as follows:

$$\begin{aligned} 8H'_{1S} &= + H_{1S} + H_{1V} + H_{1T} + H_{1A} + H_{1P} - H_{0S} - H_{0V} - H_{0T} - H_{0A} - H_{0P} \\ 8H'_{1V} &= + 4H_{1S} - 2H_{1V} + 2H_{1A} - 4H_{1P} - 4H_{0S} + 2H_{0V} - 2H_{0A} + 4H_{0P} \\ 8H'_{1T} &= + 6H_{1S} - 2H_{1T} + 6H_{1P} - 6H_{0S} + 2H_{0T} - 6H_{0P} \\ 8H'_{1A} &= + 4H_{1S} + 2H_{1V} - 2H_{1A} - 4H_{1P} - 4H_{0S} - 2H_{0V} + 2H_{0A} + 4H_{0P} \\ 8H'_{1P} &= + H_{1S} - H_{1V} + H_{1T} - H_{1A} + H_{1P} - H_{0S} + H_{0V} - H_{0T} + H_{0A} - H_{0P} \\ 8H'_{0S} &= - 3H_{1S} - 3H_{1V} - 3H_{1T} - 3H_{1A} - 3H_{1P} - H_{0S} - H_{0V} - H_{0T} - H_{0A} - H_{0P} \\ 8H'_{0V} &= - 12H_{1S} + 6H_{1V} - 6H_{1A} + 12H_{1P} - 4H_{0S} + 2H_{0V} - 2H_{0A} + 4H_{0P} \\ 8H'_{0T} &= - 18H_{1S} + 6H_{1T} - 18H_{1P} - 6H_{0S} + 2H_{0T} - 6H_{0P} \\ 8H'_{0A} &= - 12H_{1S} - 6H_{1V} + 6H_{1A} + 12H_{1P} - 4H_{0S} - 2H_{0V} + 2H_{0A} + 4H_{0P} \\ 8H'_{0P} &= - 3H_{1S} + 3H_{1V} - 3H_{1T} + 3H_{1A} - 3H_{1P} - H_{0S} + H_{0V} - H_{0T} + H_{0A} - H_{0P}. \end{aligned} \quad (\text{A} \cdot 12)$$

Appendix C

The integral

$$J = \int d^4k S p \left[\frac{(i\mathbf{k} - \kappa) \gamma_\mu (1 + \gamma_5) (i\mathbf{k} - \kappa) \gamma_\mu \gamma_5}{(k^2 + \kappa^2)^2} \right] = \int d^4k \frac{4(-k_\mu k_\mu - \kappa^2)}{(k^2 + \kappa^2)^2} \quad (\text{A} \cdot 13)$$

is given, under the condition (A·9), by

$$J' = \int d^4k S p \left[\frac{(i\mathbf{k} - \kappa) \gamma_\mu (i\mathbf{k} - \kappa) \gamma_\mu}{(k^2 + \kappa^2)^2} \right] = \int d^4k \frac{4(-k_\mu k_\mu + \kappa^2)}{(k^2 + \kappa^2)^2} = 0. \quad (\text{A} \cdot 14)$$

(Case 1) If we drop the quadratically divergent term in (A·13) using (A·12),

$$J = -8 \int d^4k \frac{\kappa^2}{(k^2 + \kappa^2)^2}. \quad (\text{A} \cdot 15)$$

This is rewritten, using Feynman's cut-off factor, as

$$\begin{aligned} J &= -8 \int d^4k \frac{\kappa^2}{(k^2 + \kappa^2)^2} \frac{\lambda^2}{(k^2 + \kappa^2 + \lambda^2)} \\ &= -8i\pi^2 \kappa^2 \left[-1 + \frac{\kappa^2 + \lambda^2}{\lambda^2} \ln \frac{\kappa^2 + \lambda^2}{\kappa^2} \right]. \end{aligned} \quad (\text{A} \cdot 16)$$

(Case 2) If we drop the logarithmically divergent term in (A·12), using (A·12), and rewrite it with Feynman's cut-off factor,

$$\begin{aligned} J &= -8 \int d^4k \frac{k_\mu k_\mu}{(k^2 + \kappa^2)^2} \\ &= -8 \int d^4k \frac{k_\mu k_\mu}{(k^2 + \kappa^2)^2} \cdot \frac{\lambda^4}{(k^2 + \kappa^2 + \lambda^2)^2} \\ &= -4i\pi^2 \kappa^2 \left[1 + \frac{1}{2} \left(\frac{\lambda^2}{\kappa^2} \right) - \frac{\kappa^2 + \lambda^2}{\lambda^2} \ln \frac{\lambda^2 + \kappa^2}{\kappa^2} \right]. \end{aligned} \quad (\text{A} \cdot 17)$$

(Case 3) This corresponds to the value obtained by Feynman's cut-off without using the condition (A·14).

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Approximate Solution of the Relativistic Two-Body Equation and its Application to the Nuclear Forces

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The approximate but analytical solutions of Bethe-Salpeter equation¹ are given and the relations between binding energy and coupling constant are obtained. With the use of these solutions the validity of the non-relativistic approximation (i.e. neglect of retardation and recoil) is investigated.

§ 1. Introduction and summary

Since a few years ago, the properties of the nuclear potential have been investigated with the use of meson theory, and many successful results have been obtained in the region where the distance between two nucleons is larger than 0.7 times meson compton wave length¹⁾. However, there are some ambiguities in the definition of “*nuclear potential*” in the inner region. When we construct the nuclear potential using meson theory, we usually neglect the “*recoil*” of nucleon. But the recoil is considered to be important in the inner region. The ambiguity comes from to what extent the recoil is taken into account. We can therefore remove the ambiguity by investigating the recoil effect. The effect, however, has not been examined even in a very simple case (that is one meson exchange interaction).

On the other hand, it has been investigated how to describe the two-body system relativistically on the basis of field theory, and the covariant two-body equations have been presented by many authors.²⁾ The equation contains the recoil without ambiguity. If we solve it, we can therefore see the recoil effect completely and decide the best definition of the nuclear potential. Further, the equations contains another effect, that is, the so-called “*retardation*.” Namely, during a finite (not zero) time a meson propagates from one nucleon to the other. We can examine also the retardation effect by solving the equation. Moreover, we can answer the following questions using the solution. What sort of relation is there between the binding energy and the eigenvalue of coupling constant? Does the equation describe the deuteron state as the most stable one among many possible states?

In this paper we shall start from the relativistic two-body equation in the ladder approximation and try to solve the equation approximately. A simplified

interaction is assumed which is factorizable but takes account of the recoil and the retardation. This simplified equation can be solved analytically. However, the interaction is not realistic in some sense. The validity of this approximation is examined in section 3. We first solve the equation exactly and then solve it by neglecting the retardation and the recoil. Comparing these solutions, we can see the effects of the retardation and of the recoil.

The results obtained are summarized as follows. a) The relations between the binding energy and the coupling constant are obtained for various cases (Figs. 2, 3, 6 and 7). b) For the two boson system the neglect of the retardation is qualitatively a good approximation, but it results in an overestimation of the binding energy. And the neglect of the recoil gives nearly the same effect as the neglect of the retardation (§ 4). c) For the two fermion system coupled with Ps(ps) meson, the most favourable state (that is, the state with the largest binding energy) is the “*deuteron*” (total spin 1, charge singlet and even parity) (§ 6). The wave function of the “*deuteron*” contains P-state besides usual S, D-states (§ 7). It is quite natural since we are dealing with a completely relativistic equation. For this system, the neglect of the retardation is a good approximation qualitatively so far as we confine ourselves to the “*deuteron*” state and another state (i.e. total spin 0, charge triplet and even parity) (§ 6).

In § 2 we shall try to reduce the B-S equation to an approximate nonrelativistic equation and show that the one pion exchange potential can be obtained by this approximation. In § 5 we shall develop a technique which is convenient for the treatment of two fermion system. In the last section (§ 7) we shall consider the meaning of the wave functions in our representation for the two fermion system and classify the relativistic wave functions according to parity, charge state and total angular momentum.

§ 2. Non-relativistic approximation for B-S equation

We can write relativistic two-body equations in ladder approximation as

$$(\square - M^2)^{(1)} (\square - M^2)^{(2)} \phi(x_1, x_2) = g^2 \Delta_F(x_1 - x_2) \phi(x_1, x_2) \quad (2.1)$$

for 2-boson system (2-B. S.), and

$$(\gamma_\mu \partial_\mu + M)^{(1)} (\gamma_\mu \partial_\mu + M)^{(2)} \phi(x_1, x_2) = -g^2 (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \gamma_5^{(1)} \gamma_5^{(2)} \Delta_F(x_1 - x_2) \phi(x_1, x_2) \quad (2.2)$$

for 2-fermion system (2-F. S.), applying p - s coupling and taking account of isospin. We transform the co-ordinate x_1, x_2 to $X = (x_1 + x_2)/2, x = x_1 - x_2$, and separate X by putting $\phi(X, x) = \exp(iP_\mu X_\mu) \psi(x)$ where P is the total energy momentum four vector. Further, we take the center of mass system $P = (\mathbf{0}, iE)$, and transform the equations into momentum representation. Then eqs. (2.1) and (2.2) become

$$[\mathbf{p}^2 + (p_4 + i\epsilon)^2 + 1][\mathbf{p}^2 + (p_4 - i\epsilon)^2 + 1]\Phi(p) = -ig^2/(2\pi)^4 \int [(\mathbf{p}-\mathbf{k})^2 + \mu^2]^{-1} \Phi(k) (dk) \quad \text{for 2-B. S.,} \quad (2.3)$$

$$[i\boldsymbol{\gamma}^{(1)}\mathbf{p} + \gamma_4^{(1)}(ip_4 - \epsilon) + 1][-i\boldsymbol{\gamma}^{(2)}\mathbf{p} - \gamma_4^{(2)}(ip_4 + \epsilon) + 1]\Phi(p) \\ = ig^2/(2\pi)^4 \cdot (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \gamma_5^{(1)} \gamma_5^{(2)} \int [(\mathbf{p}-\mathbf{k})^2 + \mu^2]^{-1} \Phi(k) (dk) \quad \text{for 2-F. S.,} \quad (2.4)$$

where we take $M_1 = M_2 = 1$ and $E/2 \equiv \epsilon$ for simplicity.

Now we shall try to reduce these equations to approximate non-relativistic equations. Our method is somewhat different from Klein's³⁾ and consists of two steps.

(a) *Neglect of retardation*

This means to assume infinitely large velocities of mesons transferred between two particles. In other words it means to neglect the time which it takes for a meson to travel from one particle to the other. This operation drops a degree of freedom for the so-called *relative time* (or *energy*). Mathematically it corresponds to the following operation,

$$A_{\mathbf{r}}(x) \Rightarrow \delta(x_0) \int A_{\mathbf{r}}(x) dx_0, \quad (2.5)$$

or

$$[(\mathbf{p}-\mathbf{k})^2 - (p_0 - k_0)^2 + \mu^2]^{-1} \Rightarrow [(\mathbf{p}-\mathbf{k})^2 + \mu^2]^{-1} \quad (2.5')$$

in p -representation.

Now define $\Phi(\mathbf{p})$ by

$$\Phi(\mathbf{p}) \equiv \int \Phi(p) dp_0, \quad (2.6)$$

then $\Phi(\mathbf{p})$ satisfies the following equations,

$$\sqrt{1 + \mathbf{p}^2} (1 + \mathbf{p}^2 - \epsilon^2) \Phi(\mathbf{p}) = g^2/4 (2\pi)^3 \int [(\mathbf{p}-\mathbf{k})^2 + \mu^2]^{-1} \Phi(\mathbf{k}) d^3k \quad \text{for 2-B. S.,} \quad (2.7)$$

$$\sqrt{1 + \mathbf{p}^2} (1 + \mathbf{p}^2 - \epsilon^2) \Phi(\mathbf{p}) = -g^2/4 (2\pi)^3 \cdot (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \left\{ [i\gamma_5^{(1)}\boldsymbol{\gamma}^{(1)}\mathbf{p} - \gamma_5^{(1)}\gamma_4^{(1)}\epsilon + \gamma_5^{(1)}] \times \right.$$

$$\times [-i\gamma_5^{(2)}\boldsymbol{\gamma}^{(2)}\mathbf{p} - \gamma_5^{(2)}\gamma_4^{(2)}\epsilon + \gamma_5^{(2)}] + (1 + \mathbf{p}^2 - \epsilon^2) \gamma_5^{(1)}\gamma_4^{(1)}\gamma_5^{(2)}\gamma_4^{(2)} \}$$

$$\times \int [(\mathbf{p}-\mathbf{k})^2 + \mu^2]^{-1} \Phi(\mathbf{k}) d^3k \quad \text{for 2-F. S..} \quad (2.8)$$

As these equations are three dimensional, any *spurious solution*⁴⁾ does not appear.

(b) *Neglect of recoil*

What we call the "*recoil*" here is the higher power terms in momentum \mathbf{p} and velocity dependent interaction. In this paragraph we shall put these recoil

terms to be zero.

Now let $\epsilon \equiv 1 + B/2$, then B is non-relativistic energy in center of mass system. The case $B < 0$ corresponds to a bound state, and $B > 0$ to a scattering state. We neglect the higher order terms than B^2 , \mathbf{p}^2 , g^4 in Eqs. (2.7) and (2.8) considering that γ_5 and $\gamma_5 \gamma_4$ are of order of \mathbf{p} , then the terms in curly brackets in eq. (2.8) becomes $-(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p})$. Therefore Eqs. (2.7) and (2.8) become

$$(\mathbf{p}^2 - B)\phi(\mathbf{p}) = g^2/4(2\pi)^3 \int [(\mathbf{p} - \mathbf{k})^2 + \mu^2]^{-1} \phi(\mathbf{k}) d^3k \quad \text{for 2-B. S.,} \quad (2.9)$$

$$(\mathbf{p}^2 - B)\phi(\mathbf{p}) = g^2/4(2\pi)^3 \cdot (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p}) \int [(\mathbf{p} - \mathbf{k})^2 + \mu^2]^{-1} \phi(\mathbf{k}) d^3k$$

for 2-F. S.. (2.10)

The interaction term of eq. (2.10) contains derivatives of $\phi(\mathbf{x})$ in x -representation, that is to say, it is a velocity dependent potential. If these derivatives are removed, Eq. (2.10) becomes

$$(\mathbf{p}^2 - B)\phi(\mathbf{p}) = g^2/4(2\pi)^3 \cdot (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \int (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p} - \mathbf{k})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p} - \mathbf{k}) [(\mathbf{p} - \mathbf{k})^2 + \mu^2]^{-1} \phi(\mathbf{k}) d^3k$$

for 2-F. S.. (2.10')

Eqs. (2.9) and (2.10') are Schrödinger equations of 2 particles interacting through the following potentials:

$$V(r) = -(g^2/16\pi) e^{-\mu r}/r \quad \text{for 2-B. S.,} \quad (2.11)$$

$$V(r) = (g^2 \mu^3/3 \cdot 16\pi) (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \{ (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + S_{12}(1 + 3/\mu r + 3/(\mu r)^2) \} e^{-\mu r}/\mu r$$

for 2-F. S.. (2.12)

These are the well-known second order nuclear forces constructed by meson theory.

§ 3. Approximation for obtaining an analytical solution

Now we try to solve equations (2.3) and (2.4) approximately, since we do not know any solution of these equations except in very special cases.⁶⁾ If we replace the interaction terms of the equations by separable forms, then we can solve the equations analytically. Yamaguchi⁵⁾ has already applied this method to the non-relativistic two nucleon problem and obtained satisfactory agreement with experiment. The interaction term is replaced as follows,

$$\int [(\mathbf{p} - \mathbf{k})^2 + \mu^2]^{-1} \phi(\mathbf{k}) (d\mathbf{k}) \Rightarrow v(\mathbf{p}) \int v(\mathbf{k}) \phi(\mathbf{k}) (d\mathbf{k}) = C v(\mathbf{p}), \quad C \equiv \int v(\mathbf{k}) \phi(\mathbf{k}) (d\mathbf{k}).$$

(3.1)

The approximation in a three dimensional case is quite the same as the above except that p and k are replaced by \mathbf{p} and \mathbf{k} . This procedure corresponds to the replacement $\langle x|V|y \rangle = u(x) \cdot u(y)$ in x -representation. $u(x)$ and $v(\mathbf{p})$ are Lorentz

invariant scalar functions, if we assume that they depend only on x^2 and p^2 respectively. Therefore such an interaction is Lorentz invariant but is not realistic because it is invariant scalar and nonlocal. If we adopt this interaction, the analytical solutions of the equations are easily obtainable by multiplying the right-hand side by the inverse operator of the factor in the left-hand side.

Further, the better the form of $v(p)$ is choosed, the nearer the solution approaches to the exact one. Now it is a question how to select the form of $v(p)$. As we know the exact solution for the 2-boson system in a case $\mu=0$,⁶⁾ $v(p)$ is chosen so that our solution agrees with the exact one, and for the other cases we can find $v(p)$ by analogy to this case. The following $v(p)$'s were obtained in such a manner.

$$v(p) \equiv a[p^2 + \mu^2 - i\sigma]^{-1/2} \quad \text{for 2-B. S.,} \quad (3.2)$$

$$v(p) \equiv a\omega[(p^2 + \mu^2 - i\sigma)(p^2 + \omega^2)]^{-1/2} \quad \text{for 2-F. S..} \quad (3.3)$$

For the case of the 2-fermion system, we have adopted the Feynman cut-off to prevent the divergence with a cut-off parameter ω . The cause of the divergence is that the equation of a free boson is Klein-Gordon equation and bilinear in p but the equation of a free fermion is Dirac equation and linear in p . It has been suggested by Taylor⁷⁾ that the equation for two fermion system does not give discrete eigenvalues of coupling constant. This fact is due to the very cause above mentioned. "a" in Eqs. (3.2) and (3.3) is the only degree of freedom left in $v(p)$. As this interaction is invariant scalar, it should be noted that the method is applicable to the ground state only and spurious solutions⁴⁾ of the equation appear no more. The validity of the approximation will be discussed in the next section.

§ 4. Bound state for 2-boson system and a test of our method

In the first paragraph of this section, we shall answer an interesting question to what extent our method is reliable. Next we shall apply it to the 2-boson system for simplicity, and investigate whether reasonable results can be obtained.

(i) A test of the method

With the use of the approximation in the last section, eq. (2.3) is written as follows:

$$\phi(p) = -i(\lambda/\pi^2)Cv(p) [p^2 + (p_4 + i\epsilon)^2 + 1]^{-1} [p^2 + (p_4 - i\epsilon)^2 + 1]^{-1} \quad (4.1)$$

where we put $g^2/4(2\pi)^2 \equiv \lambda$ so that the symbol coincides with Cutkosky's⁶⁾. Substitute eq. (4.1) into the definition of C , eq. (3.1), then

$$-i(\lambda/\pi^2) \int v^2(p) [p^2 + (p_4 + i\epsilon)^2 + 1]^{-1} [p^2 + (p_4 - i\epsilon)^2 + 1]^{-1} (dp) = 1. \quad (4.2)$$

We find a relation between ϵ and λ for the special case $\mu=0$,

$$\frac{1}{\lambda} = a^2 \left[\frac{2}{\epsilon \sqrt{1 - \epsilon^2}} \tan^{-1} \frac{\epsilon}{\sqrt{1 - \epsilon^2}} + \frac{1}{\epsilon^2} \log(1 - \epsilon^2) \right] \quad (4.3)$$

In order that λ becomes 2 as ϵ approaches zero, a^2 should be $1/2$. The relation between λ and ϵ^2 is shown in Fig. 1 and compared with Cutkosky's which is obtained without the approximation. As no difference can be found graphically between

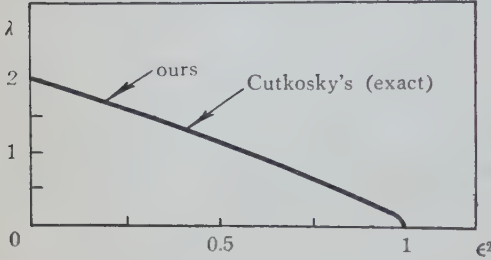


Fig. 1. The relations between the coupling constant λ and the energy ϵ^2 for $\mu=0$. The one which is obtained with the use of our method is compared with that obtained by Wick and Cutkosky without any approximation. There is no difference between these two curves graphically.

(ii) *Relations between the energy and the coupling constant*

From eq. (4.2) we obtain for $\mu \neq 0$, ($b=1-\epsilon^2-\mu^2$)

$$\frac{1}{\lambda} = a^2 \left[\frac{2\sqrt{1-\epsilon^2}}{\epsilon b} \tan^{-1} \frac{\epsilon}{\sqrt{1-\epsilon^2}} + \frac{1}{2\epsilon^2} \log \mu^2 + \frac{\sqrt{b^2-4\mu^2\epsilon^2}}{2\epsilon^2 b} \log \frac{b+2\mu^2+\sqrt{b^2-4\mu^2\epsilon^2}}{b+2\mu^2-\sqrt{b^2-4\mu^2\epsilon^2}} \right]$$

for $0 < \epsilon \leq 1-\mu$,

$$\frac{1}{\lambda} = a^2 \left[\frac{2\sqrt{1-\epsilon^2}}{\epsilon b} \tan^{-1} \frac{\epsilon}{\sqrt{1-\epsilon^2}} + \frac{1}{2\epsilon^2} \log \mu^2 - \frac{\sqrt{4\mu^2\epsilon^2-b^2}}{\epsilon^2 b} \tan^{-1} \frac{\sqrt{4\mu^2\epsilon^2-b^2}}{1+\mu^2-\epsilon^2} \right]$$

for $1-\mu \leq \epsilon < 1$, $b \neq 0$,

$$\frac{1}{\lambda} = a^2 \left[\frac{1}{\epsilon \mu} \tan^{-1} \frac{\epsilon}{\mu} - \frac{1}{2\epsilon^2} \log \frac{\epsilon^2 + \mu^2}{\mu^2} \right] \quad \text{for } b=0, \text{ i.e. } \epsilon^2 = 1-\mu^2,$$

$$\frac{1}{\lambda} = a^2 \left[\frac{1}{1-\mu^2} + \frac{\mu^2}{(1-\mu^2)^2} \log \mu^2 \right] \quad \text{for } \epsilon=0,$$

and

$$\frac{1}{\lambda} = a^2 \left[\log \mu + \frac{\sqrt{4-\mu^2}}{\mu} \tan^{-1} \frac{\sqrt{4-\mu^2}}{\mu} \right] \quad \text{for } \epsilon=1.$$

Similar relations are obtained in the case in which the retardation is neglected.

$$\frac{1}{\lambda} = a^2 \left[\frac{2\sqrt{1-\epsilon^2}}{\epsilon b} \tan^{-1} \frac{\epsilon}{\sqrt{1-\epsilon^2}} - \frac{2\mu}{\sqrt{1-\mu^2} b} \tan^{-1} \frac{\sqrt{1-\mu^2}}{\mu} \right] \quad \text{for } \epsilon \neq 0, 1,$$

$$\frac{1}{\lambda} = a^2 \left[\frac{2}{1-\mu^2} - \frac{2\mu}{(1-\mu^2)^{3/2}} \tan^{-1} \frac{\sqrt{1-\mu^2}}{\mu} \right] \quad \text{for } \epsilon=0,$$

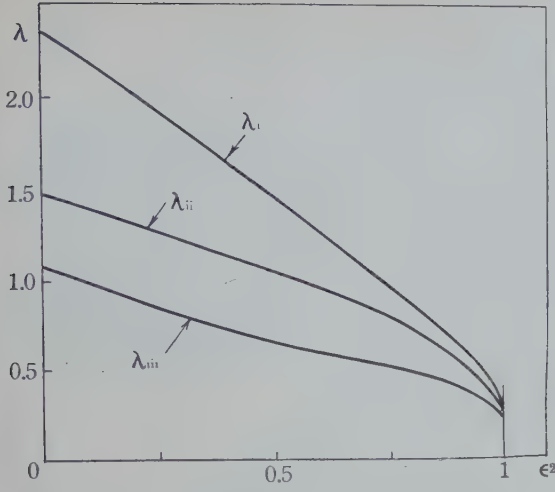


Fig. 2. The relations between the energy and the coupling constant in the case of 2-boson system for $\mu=0.282$.

λ_i : without non-relativistic approximation
 λ_{ii} : retardation is neglected
 λ_{iii} : retardation and recoil are neglected.

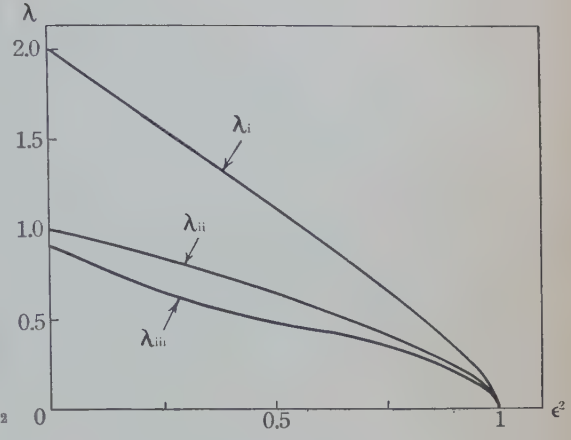


Fig. 3. The same relations as in Fig. 2 for $\mu=0$.

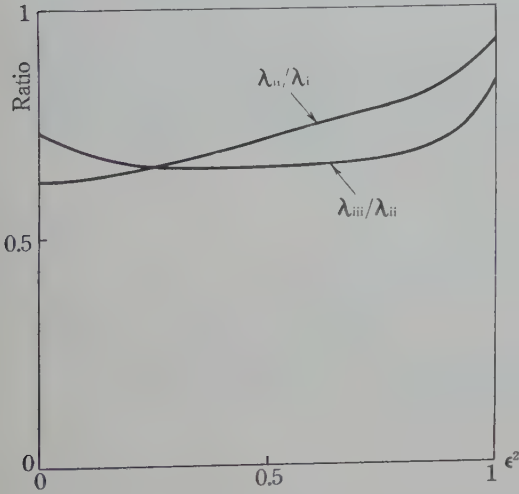


Fig. 4. The ratios of the coupling constants for $\mu=0.282$. As for the meaning of λ_i, λ_{ii} and λ_{iii} , see the caption for Fig. 2. These curves give a measure of the approximations. If the approximations are perfectly good, the ratios become 1 in whole energy region.

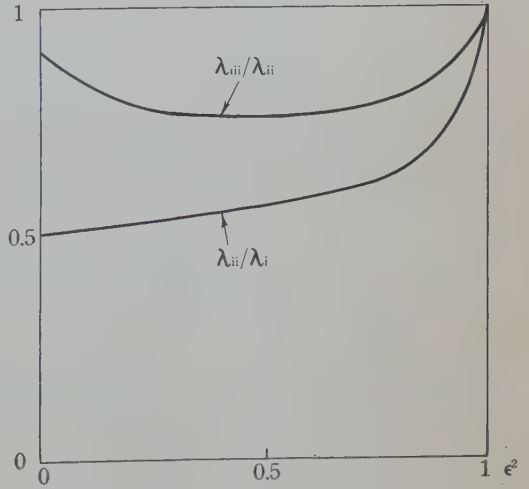


Fig. 5. The ratios of the coupling constants for $\mu=0$, cf. the caption for Fig. 4.

and

$$\frac{1}{\lambda} = a^2 \frac{2}{\mu \sqrt{1-\mu^2}} \tan^{-1} \frac{\sqrt{1-\mu^2}}{\mu} \quad \text{for } \epsilon=1.$$

And also in the case in which the recoil is neglected,

$$\lambda = (\pi a^2)^{-1} (\mu + \sqrt{-B})$$

In the above equation $-B=2(1-\epsilon)$ represents the binding energy of the system.

Now we shall show the above three relations between the energy and the coupling constant in Figs. 2 and 3 where exchanged particles have finite mass ($\mu=0.282$)* and zero mass ($\mu=0$) respectively. In order to give a accuracy of the approximations, we show the ratios of the coupling constants (Figs. 4 and 5). From these Figures we may conclude: a) The neglect of the retardation is qualitatively a good approximation but quantitatively gives the effect that the interaction is strengthened considerably (about 2 times). b) The neglect of the recoil also gives nearly the same effect as the neglect of the retardation.

§ 5. Explanation of technique and representation

To solve the equation for the 2-fermion system is not so simple as that for the 2-boson system. Because the wave function $\Phi(p)$ of the 2-fermion system has $4 \times 4 = 16$ components to represent two spinor states. Further, if we consider their charge states, the number of the independent components increases to $2 \times 2 = 4$ times and becomes $16 \times 4 = 64$. In order to treat $\Phi(p)$, we shall introduce a simple technique.

If we do not consider charge states, Φ has 16 components. Then we write them in 4×4 matrix form $\Phi_{ij}(p)$ ($i, j=1, 2, 3, 4$) and assume that the first suffix i is concerned to the matrix operators of particle 1 and the second j does to those of 2. Hence the operators of particle 2 should be multiplied from the right side of Φ in their transposed forms, i.e.,

$$\Gamma^{(2)}\Phi = \Phi\Gamma^{(2)T} = \Phi\Gamma^T. \quad (5.1)$$

Therefore we consider that the operators which are multiplied from the right in their transposed forms are those of particle 2. Of course the operators from the left are those of 1.

Further, in order to avoid this transposition, we introduce an operator A defined by

$$A\gamma_\mu^T A^{-1} = \gamma_\mu \quad (\mu=1, 2, 3, 4, 5). \quad (5.2)$$

Operating A^{-1} to the last term of eq. (5.1) from the right side, we get

$$\Phi\Gamma^T A^{-1} = \Phi A^{-1} \cdot A\Gamma^T A^{-1}. \quad (5.3)$$

If $\Gamma = \gamma_\mu$, $A\Gamma^T A^{-1} = \Gamma$ and If $\Gamma = \gamma_\mu \gamma_\nu$ ($\mu \neq \nu$),

$$A\Gamma^T A^{-1} = A(\gamma_\mu \gamma_\nu)^T A^{-1} = A\gamma_\nu^T A^{-1} \cdot A\gamma_\mu^T A^{-1} = \gamma_\nu \gamma_\mu = -\Gamma \quad (\mu \neq \nu). \quad (5.4)$$

* Recently $KK\pi$ interaction is considered by many people, and so we take masses of K and π for M and μ respectively. Hence 0.282 is of no serious meaning.

Hence it should be noted that Γ changes its sign in this case. The same is done for charge operator τ_L . We introduce \sum defined by

$$\sum \tau_L^T \sum^{-1} = -\tau_L \quad (L=1,2,3). \quad (5.5)$$

We shall hereafter denote $\phi A^{-1} \sum^{-1}$ newly by ϕ .

Now we shall define our representation of Dirac matrices here. Following Dirac⁸⁾ we define matrices σ, ρ and use the following γ_μ 's

$$\gamma = -\rho_2 \sigma, \quad \gamma_4 = \rho_3, \quad \gamma_5 = \rho_1. \quad (5.6)$$

Then the above operator A is expressed as $\gamma_1 \gamma_3$. Of course, \sum is τ_2 .

At the beginning of this section we saw that ϕ has 64 independent components and we write ϕ for convenience in the following general form ($\tilde{\sigma} \equiv -\rho_1 \sigma$)

$$\begin{aligned} \phi = & \phi^S + \gamma_\mu \phi_\mu^V + \sigma_i \phi_i^T + \tilde{\sigma}_i \tilde{\phi}_i^T + \gamma_5 \gamma_\mu \phi_\mu^A + \gamma_5 \phi^P \\ & + (\phi_L^S + \gamma_\mu \phi_{\mu L}^V + \sigma_i \phi_{iL}^T + \tilde{\sigma}_i \tilde{\phi}_{iL}^T + \gamma_5 \gamma_\mu \phi_{\mu L}^A + \gamma_5 \phi_L^P) \tau_L, \end{aligned} \quad (5.7)$$

where summations are taken over $i, L (=1, 2, 3)$ and $\mu (=1, 2, 3, 4)$. We shall hereafter represent these 64 ϕ^S, ϕ_μ^V , etc., as ϕ^a *en bloc* and the corresponding matrices $1, \gamma_\mu$, etc., as Γ^a . Then ϕ^a 's are scalar functions of p and commute with Γ^a, τ_L .

Before we close the section, we shall express various operators using our notation. Space reflection operator H :

$$H\phi(\mathbf{p}, p_0) = \gamma_4 \phi(-\mathbf{p}, p_0) \gamma_4, \quad (5.8)$$

Spin operator S and isospin operator I :

$$S = \frac{1}{2}(\sigma^{(1)} + \sigma^{(2)}), \quad I = \frac{1}{2}(\tau^{(1)} + \tau^{(2)}), \quad (5.9)$$

$$\text{or} \quad S\phi = \frac{1}{2}(\sigma\phi - \phi\sigma), \quad I\phi = \frac{1}{2}(\tau\phi - \phi\tau), \quad (5.9')$$

$$S^2\phi = \frac{1}{2}(3\phi - \sigma\phi\sigma), \quad I^2\phi = \frac{1}{2}(3\phi - \tau\phi\tau). \quad (5.10)$$

Spin exchange operator P^B :

$$\begin{aligned} P^B = & \frac{1}{4}(1 + \sigma^{(1)} \cdot \sigma^{(2)})(1 + \rho^{(1)} \cdot \rho^{(2)}) \\ = & \frac{1}{4}(1 + \gamma_\mu^{(1)} \gamma_\mu^{(2)} + \sigma^{(1)} \cdot \sigma^{(2)} + \tilde{\sigma}^{(1)} \cdot \tilde{\sigma}^{(2)} - \gamma_5^{(1)} \gamma_\mu^{(1)} \gamma_5^{(2)} \gamma_\mu^{(2)} + \gamma_5^{(1)} \gamma_5^{(2)}), \end{aligned} \quad (5.11)$$

Charge exchange operator P^T :

$$P^T = \frac{1}{2}(1 + \tau^{(1)} \cdot \tau^{(2)}). \quad (5.12)$$

If coordinates exchange operator ($p_\mu \rightarrow -p_\mu$) is represented by P^M , the exchange operator of two fermions is expressed by

$$P = P^M P^B P^T. \quad (5.13)$$

As nucleons obey the Fermi statistics,

$$P\phi = -\phi. \quad (5.14)$$

Therefore if we substitute the general form of Φ into the above eq. (5.14), we find the conditions satisfied by Φ^a .

$$\begin{aligned}\Phi^S(-p) &= -\Phi^S(p), & \Phi_\mu^V(-p) &= -\Phi_\mu^V(p), & \Phi_i^T(-p) &= \Phi_i^T(p), \\ \tilde{\Phi}_i^T(-p) &= \tilde{\Phi}_i^T(p), & \Phi_\mu^A(-p) &= \Phi_\mu^A(p), & \Phi^P(-p) &= -\Phi^P(p), \\ \Phi_L^S(-p) &= \Phi_L^S(p), & \Phi_{\mu L}^V(-p) &= \Phi_{\mu L}^V(p), & \Phi_{iL}^T(-p) &= -\Phi_{iL}^T(p), \\ \tilde{\Phi}_{iL}^T(-p) &= -\tilde{\Phi}_{iL}^T(p), & \Phi_{\mu L}^A(-p) &= -\Phi_{\mu L}^A(p), & \Phi_L^P(-p) &= \Phi_L^P(p),\end{aligned}\quad (5.15)$$

As we see from the above equations, these 64 Φ 's separate into some groups according to the eigenvalues of these operators.

§ 6. Bound state of 2-fermion system

The treatment of the two fermions' wave function becomes very easy when we use the technique developed in the preceding section. The original equation (2.4) can be rewritten as follows:

$$[i\gamma p + \gamma_4(ip_4 - \epsilon) + 1]\Phi(p)[-i\gamma p - \gamma_4(ip_4 + \epsilon) + 1] = -ig^2/(2\pi)^4 \cdot v(p)\tau_L\gamma_5 C\gamma_5\tau_L. \quad (6.1)$$

C is a constant matrix with 4 rows and 4 columns. Then the explicit form of Φ is

$$\Phi(p) = -\frac{ig^2}{(2\pi)^4} v(p) \frac{[-i\gamma p - \gamma_4(ip_4 - \epsilon) + 1]\tau_L\gamma_5 C\gamma_5\tau_L[i\gamma p + \gamma_4(ip_4 + \epsilon) + 1]}{[p^2 - (ip_4 - \epsilon)^2 + 1][p^2 - (ip_4 + \epsilon)^2 + 1]}. \quad (6.2)$$

Substitute eq. (6.2) into eq. (2.1), then

$$-\frac{ig^2}{(2\pi)^4} \int v^2(p) \frac{[-i\gamma p - \gamma_4(ip_4 - \epsilon) + 1]\tau_L\gamma_5 C\gamma_5\tau_L[i\gamma p + \gamma_4(ip_4 + \epsilon) + 1]}{[p^2 - (ip_4 - \epsilon)^2 + 1][p^2 - (ip_4 + \epsilon)^2 + 1]}(dp) = C. \quad (6.3)$$

This is simultaneous homogeneous linear equations of 64 unknown C 's. The condition that they have a non-trivial solution determines the relation between g^2 and ϵ . In order to simplify eq. (6.3), we put C like eq. (5.7), i.e.

$$\begin{aligned}C &= C^S + \gamma_\mu C_\mu^V + \sigma_i C_i^T + \tilde{\sigma}_i \tilde{C}_i^T + \gamma_5 \gamma_\mu C_\mu^A + \gamma_5 C^P \\ &+ (C_L^S + \gamma_\mu C_{\mu L}^V + \sigma_i C_{iL}^T + \tilde{\sigma}_i \tilde{C}_{iL}^T + \gamma_5 \gamma_\mu C_{\mu L}^A + \gamma_5 C_L^P)\tau_L.\end{aligned}\quad (6.4)$$

Inserting it into eq. (6.3), and comparing two coefficients of Γ^a 's in both sides, we find 12 sets of equations which are separated from one another:

$$\left. \begin{aligned} &\left(-\frac{3ig^2 a^2 \omega^2}{(2\pi)^4}\right) \int \frac{(p^2 - p_0^2 + 1 + \epsilon^2)C^S - 2\epsilon C_4^V}{(p^2 + \omega^2)(p^2 + \mu^2)[p^2 - (p_0 + \epsilon)^2 + 1][p^2 - (p_0 - \epsilon)^2 + 1]}(dp) = C^S, \\ &\left(\begin{array}{c} \text{''} \\ \text{''} \end{array}\right) \int \frac{(p^2 + p_0^2 - 1 - \epsilon^2)C_4^V + 2\epsilon C^S}{(\text{''})(\text{''})[\text{''}][\text{''}]}(dp) = C_4^V, \end{aligned} \right\} \quad (6.5a)$$

$$\left(\begin{array}{c} \text{''} \\ \text{''} \end{array} \right) \int \frac{(\mathbf{p}^2 - p_0^2 - 1 + \epsilon^2) \mathbf{C}^V - 2\mathbf{p}(\mathbf{p} \cdot \mathbf{C}^V)}{(\text{''})(\text{''})[\text{''}][\text{''}]} (d\mathbf{p}) = \mathbf{C}^V, \quad (6.5b)$$

$$\left(\begin{array}{c} \text{''} \\ \text{''} \end{array} \right) \int \frac{(-\mathbf{p}^2 - p_0^2 + 1 + \epsilon^2) \mathbf{C}^T + 2\mathbf{p}(\mathbf{p} \cdot \mathbf{C}^T) + 2i\epsilon \mathbf{C}^A}{(\text{''})(\text{''})[\text{''}][\text{''}]} (d\mathbf{p}) = \mathbf{C}^T, \\ \left(\begin{array}{c} \text{''} \\ \text{''} \end{array} \right) \int \frac{-(\mathbf{p}^2 - p_0^2 + 1 + \epsilon^2) \mathbf{C}^A + 2\mathbf{p}(\mathbf{p} \cdot \mathbf{C}^A) + 2i\epsilon \mathbf{C}^T}{(\text{''})(\text{''})[\text{''}][\text{''}]} (d\mathbf{p}) = \mathbf{C}^A \quad (6.5c)$$

$$\left(\begin{array}{c} \text{''} \\ \text{''} \end{array} \right) \int \frac{(\mathbf{p}^2 + p_0^2 + 1 - \epsilon^2) \tilde{\mathbf{C}}^T - 2\mathbf{p}(\mathbf{p} \cdot \tilde{\mathbf{C}}^T)}{(\text{''})(\text{''})[\text{''}][\text{''}]} (d\mathbf{p}) = \tilde{\mathbf{C}}^T, \quad (6.5d)$$

$$\left(\begin{array}{c} \text{''} \\ \text{''} \end{array} \right) \int \frac{-(\mathbf{p}^2 + p_0^2 + 1 - \epsilon^2) \mathbf{C}_4^A}{(\text{''})(\text{''})[\text{''}][\text{''}]} (d\mathbf{p}) = \mathbf{C}_4^A, \quad (6.5e)$$

$$\left(\begin{array}{c} \text{''} \\ \text{''} \end{array} \right) \int \frac{-(\mathbf{p}^2 - p_0^2 - 1 + \epsilon^2) \mathbf{C}^P}{(\text{''})(\text{''})[\text{''}][\text{''}]} (d\mathbf{p}) = \mathbf{C}^P. \quad (6.5f)$$

The equations for C_L^a are quite the same as the above equations except that (-3) in the coefficients of the left side is replaced by 1. By these equations 12 coupling constants are determined for a given energy ϵ .

The integrations of Eqs. (6.5a) – (6.5f) are elementary but somewhat tedious. If we define A_1, A_2 , and A_3 as follows:

$$\int \frac{\mathbf{p}^2}{(p + \omega^2)(p^2 + \mu^2)[\mathbf{p}^2 - (p_0 + \epsilon)^2 + 1][p^2 - (p_0 - \epsilon)^2 + 1]} (d\mathbf{p}) \equiv \frac{i\pi^2}{4} A_1, \quad (6.6a)$$

$$\int \frac{1}{(\text{''})(\text{''})[\text{''}][\text{''}]} (d\mathbf{p}) \equiv \frac{i\pi^2}{6} A_2, \quad (6.6b)$$

and

$$\int \frac{p_0^2}{(\text{''})(\text{''})[\text{''}][\text{''}]} (d\mathbf{p}) \equiv \frac{i\pi^2}{12} (A_1 - 2A_3), \quad (6.6c)$$

then the above equations (6.5a) – (6.5f) become simple to some extent. The explicit forms of these integrals are shown in appendix. Setting

$$G^2 \equiv g^2 a^2 \omega^2 (2^5 \cdot 3\pi^2)^{-1} \quad (6.7)$$

for simplicity and describing 12 G^2 's which correspond to 12 coupling constants as $G_1^2, \dots, G_6^2, G_{1L}^2, \dots, G_{6L}^2$, we find from eqs. (6.5a) – (6.5f)

$$\left. \begin{aligned} [3G_1^2\{A_1+A_3+(1+\epsilon^2)A_2\}-1]C^S-6G_1^2\epsilon A_2C_4^V=0, \\ [3G_1^2\{2A_1-A_3-(1+\epsilon^2)A_2\}-1]C_4^V+6G_1^2\epsilon A_2C^S=0, \end{aligned} \right\} \quad (6.8a)$$

$$[3G_2^2\{A_3-(1-\epsilon^2)A_2\}-1]C^V=0, \quad (6.8b)$$

$$\left. \begin{aligned} [3G_3^2\{A_3-A_1+(1+\epsilon^2)A_2\}-1]C^T+6iG_3^2\epsilon A_2C^A=0, \\ [3G_3^2\{-A_3-(1+\epsilon^2)A_2\}-1]C^A+6iG_3^2\epsilon A_2C^T=0, \end{aligned} \right\} \quad (6.8c)$$

$$[3G_4^2\{A_1-A_3+(1-\epsilon^2)A_2\}-1]\tilde{C}^T=0, \quad (6.8d)$$

$$[3G_5^2\{A_3-2A_1-(1-\epsilon^2)A_2\}-1]C_4^A=0, \quad (6.8e)$$

$$\text{and} \quad [3G_6^2\{-A_1-A_3+(1-\epsilon^2)A_2\}-1]C^P=0. \quad (6.8f)$$

The 8 equations for C_L^a are obtained by replacing $C^a \rightarrow C_L^a$, $3G_i^2 \rightarrow -G_{iL}^2$ in the above equations.

Now these 12 coupling constants correspond to 12 different states respectively and they are designated by the eigenvalues of four operators, i.e. charge state, parity, particle exchange and total spin. Table 1 shows these relations.

Table 1. The relations between coupling constants and the characters of corresponding states.

Coupling constant	g_1^2	g_2^2	g_3^2	g_4^2	g_5^2	g_6^2	g_{1L}^2	g_{2L}^2	g_{3L}^2	g_{4L}^2	g_{5L}^2	g_{6L}^2
Charge state	S	S	S	S	S	S	T	T	T	T	T	T
Parity	+	-	+	-	-	-	+	-	+	-	-	-
Particle exchange	+	+	-	-	-	+	-	-	+	+	+	-
Total spin	0	1	1	1	0	0	0	1	1	1	0	0

As we are treating the 2-fermion system, the wave function ϕ should be anti-symmetric in respect of two particles. Then the states which are symmetric in respect of two particles are physically meaningless as we see in Table 1.

We shall show the relations between ϵ^2 and g_i^2 obtained from eqs. (6.8a) — (6.8f) in Fig. 6 where we take $\omega=1$,* $\mu=0.1486$. The squared coupling constants which have negative values are omitted in Fig. 6, because $g_i^2 < 0$ means that the interaction gives repulsive force for the corresponding states and makes no bound state. In Fig. 6 g_3^2 is the smallest coupling constant in the region $\epsilon^2 \approx 1$ in which the energy of the deuteron is contained. Therefore the most favourable state is the state of the “deuteron,” that is, the state of total spin 1, charge singlet, parity+ and particle exchange — according to Table 1.

For the case in which the retardation is neglected we find the similar relations in the same manner. We do not follow the calculation here but show that result of this case in Fig. 7.

Comparing Fig 7 with Fig. 6, we notice first of all that the graph is simplified

* $\omega=\mu$ may be better than $\omega=1$.

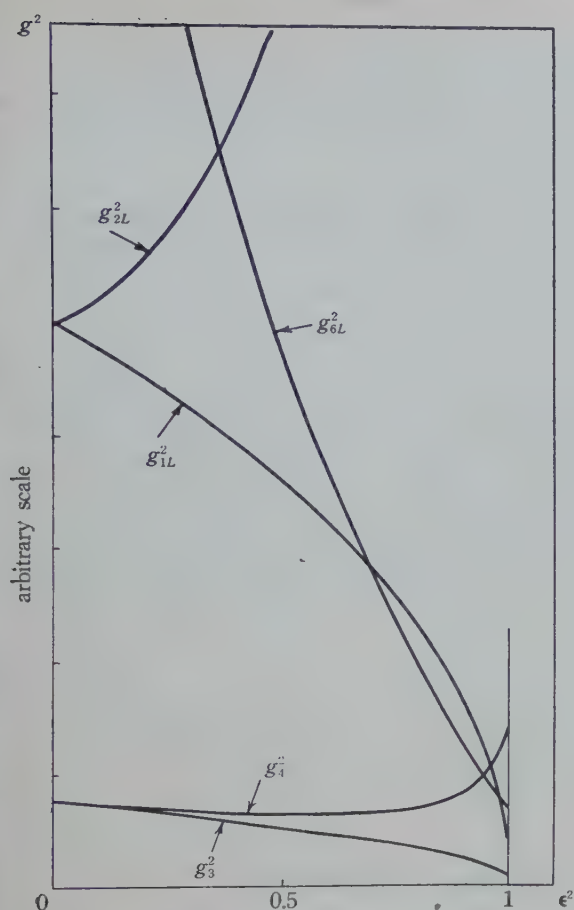


Fig. 6 The relations between the coupling constants and the energy in the case of two fermion system for $\mu=0.1486$. Different curves correspond to different states (cf. Table 1).

good approximation so far as we consider g_{1L}^2 (charge triplet, spin 0, even parity) and g_3^2 (charge singlet, spin 1, even parity), but it is not a good approximation for the other states even qualitatively. The latter conclusion, however, cannot be accepted in its literal sense, because the conclusion is in close relation with the validity of our method.

§ 7. Interpretation of the wave function for 2-fermion system

In § 5 we wrote ϕ in a general form Eq. (5.7) but we did not mention there what Γ^a and τ_L mean. However, we can easily interpret that they express spin and isospin respectively. Their magnitudes can be found out if the operators introduced in § 5 are operated to them. For example, substitute $\sigma_1 + i\sigma_2$ for ϕ in eqs. (5.10) and (5.9'), then

very much by the neglect of the retardation. In more detail g_{2L}^2 and g_4^2 disappear from Fig. 5. Although g_{1L}^2 and g_3^2 have nearly the same tendency in the both graphs, the others are quite different. Therefore we may conclude as follows. The neglect of the retardation is qualitatively a

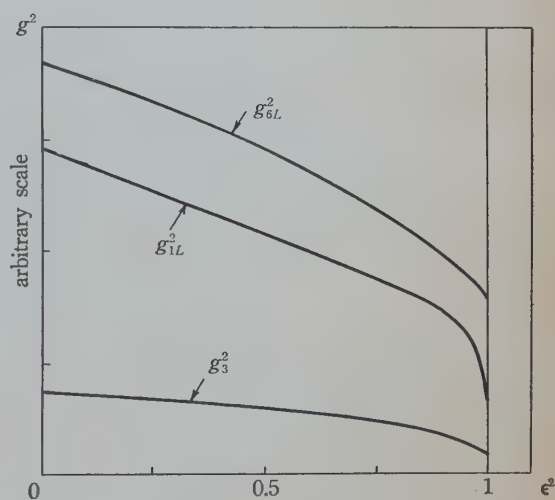


Fig. 7. The same relations as in Fig. 4 when the retardation is neglected.

$$S^2(\sigma_1 + i\sigma_2) = 2(\sigma_1 + i\sigma_2), \quad S_3(\sigma_1 + i\sigma_2) = 1(\sigma_1 + i\sigma_2).$$

This means that $\sigma_1 + i\sigma_2$ expresses a spin of magnitude 1 and third component 1. In this way we can make a table of I^a and τ_L where they are classified according to their magnitudes and third components (Table 2).

Table 2. The magnitude and third component of I^a and τ_L .

Magnitude	Third component	I^a (=spin)				τ_L (=isospin)
0	0	1_3	γ_3	$\gamma_5\gamma_3$	γ_5	1
	1	$\gamma_1 + i\gamma_2$	$\sigma_1 + i\sigma_2$	$\tilde{\sigma}_1 + i\tilde{\sigma}_2$	$\gamma_5\gamma_1 + i\gamma_5\gamma_2$	$\tau_1 + i\tau_2$
1	0	γ_3	σ_3	$\tilde{\sigma}_3$	$\gamma_5\gamma_3$	τ_3
	-1	$\gamma_1 - i\gamma_2$	$\sigma_1 - i\sigma_2$	$\tilde{\sigma}_1 - i\tilde{\sigma}_2$	$\gamma_5\gamma_1 - i\gamma_5\gamma_2$	$\tau_1 - i\tau_2$

Though we know that the solution of eq. (2.4) is expressed as eq. (6.2), it is difficult to interpret such an equation in its form. Then we shall make it an intelligible form. We shall consider the deuteron state which is one of the most interesting states for us. Suppose the wave function is an eigenfunction of total angular momentum (with an eigenvalue 1) and of its third component (eigenvalue 1), then

$$C = (\sigma_1 + i\sigma_2)C^T + (\gamma_5\gamma_1 + i\gamma_5\gamma_2)C^A. \quad (7.1)$$

Substitute eq. (7.1) into eq. (6.2), and we find

$$\begin{aligned} \phi(p) = & K(\mathbf{p}^2, p_0^2) \left[\left\{ (-\mathbf{p}^2 - p_0^2 + \epsilon^2 + 1)C^T + 2i\epsilon C^A \right\} (\sigma_1 + i\sigma_2) + \right. \\ & + \left\{ (\mathbf{p}^2 - p_0^2 + \epsilon^2 + 1)C^A - i\epsilon C^T \right\} (\gamma_5\gamma_1 + i\gamma_5\gamma_2) + 2C^T(p_1 + ip_2)(\mathbf{p} \cdot \boldsymbol{\sigma}) \\ & + 2C^A(p_1 + ip_2)(\mathbf{p} \cdot \boldsymbol{\gamma}_5) - 2(\epsilon C^T + iC^A)(p_1 + ip_2)\gamma_5 - 2iC^A p_0(p_1 + ip_2)\gamma_5\gamma_3 \\ & \left. - 2(C^T + i\epsilon C^A) \left([\mathbf{p} \times \boldsymbol{\gamma}]_1 + i[\mathbf{p} \times \boldsymbol{\gamma}]_2 \right) + 2iC^T p_0 \left([\mathbf{p} \times \tilde{\boldsymbol{\sigma}}]_1 + i[\mathbf{p} \times \tilde{\boldsymbol{\sigma}}]_2 \right) \right], \quad (7.2) \end{aligned}$$

where

$$K(\mathbf{p}^2, p_0^2) = -\frac{3ig^2}{(2\pi)^4} \frac{a}{\sqrt{(\mathbf{p}^2 + 1)(\mathbf{p}^2 + \mu^2)[\mathbf{p}^2 - (p_0 + \epsilon)^2 + 1][\mathbf{p}^2 - (p_0 - \epsilon)^2 + 1]}}. \quad (7.3)$$

Now we shall consider the meaning of eq. (7.2). According to Table 2 it represents a state of angular momentum 1. The first four terms remain in non-relativistic limit, and the others are entirely relativistic effects. The first and the second terms represent the triplet S -state, and the third and fourth terms contain the D -state. That is,

$$(p_1 + ip_2)(\mathbf{p} \cdot \boldsymbol{\sigma}) = \sqrt{\frac{4\pi}{45}} \mathbf{p}^2 [\sqrt{6} Y_2^2(\theta, \phi)(\sigma_1 - i\sigma_2) + \sqrt{6} Y_2^1(\theta, \phi)\sigma_3]$$

$$-Y_2^0(\theta, \phi)(\sigma_1 + i\sigma_2)] + \frac{\sqrt{4\pi}}{3} \mathbf{p}^2 Y_0^0(\theta, \phi)(\sigma_1 + i\sigma_2) \quad (7.4)$$

where $Y_l^m(\theta, \phi)$ is a spherical harmonic function and (θ, ϕ) represents the direction \mathbf{p} . $(p_1 + ip_2)(\mathbf{p} \cdot \boldsymbol{\gamma}_5 \boldsymbol{\gamma})$ can be rewritten quite similarly. The first term of eq. (7.4) is the form that a state of orbital angular momentum 2, i.e. the D -state couples with a state of spin 1 to make a state of total angular momentum 1. The second term of eq. (7.4) is 3S_1 . Among the relativistic terms of Eq. (7.2), the first two terms represent that the orbital angular momentum 1, i.e. the P -state couples with the state of spin zero, and the last two terms represent that the P -state couples with spin 1 to make a state of total angular momentum 1. Therefore it is an interesting relativistic effect that the P -state may appear in the deuteron wave function being accompanied by $\boldsymbol{\gamma}_5$.

We shall classify the relativistic state of the two-nucleon system by three good quantum numbers, i.e. total angular momentum J , charge state T and parity Π . Spin is not a good quantum number in the relativistic case while it is a good quantum number in the non-relativistic case. The result is summarized in Table 3. For $J \geq 2$ we can classify the states in quite the same manner. The states in brackets depend linearly upon the relative time (or energy) of the system.

Table 3. Classification of two-nucleon system.

Π	T	J	non-relativistic	relativistic	Symbol
+	0	1	$^3S_1 + ^3D_1$	$^3P_1 + ^1P_1$	3E
		0		$(^1S_0) + ^3P_0$	
	1	1		$(^3S_1) + ^1P_1 + ^3P_1 + ^3D_1$	
		0	1S_0	3P_0	1E
	0	1	1P_1	$^3S_1 + ^3P_1 + ^3D_1$	1O
		0		$^1S_0 + (^3P_0)$	
-	1	1	3P_1	$^3S_1 + ^1P_1 + ^3D_1$	3O
		0	3P_0	1S_0	3O

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Appendix

The explicit forms of the integrals $A_1 A_2$ and A_3 defined by Eqs. (6.6a), (6.6b) and (6.6c) where we take $\omega=1$ are as follows:

$$A_1 = \frac{1}{\epsilon^2} - \frac{1}{\epsilon^2(1-\mu^2)} \left(3\mu^2 - \frac{\alpha^2}{2\epsilon^2} \right) \log \mu^2 + \frac{8(1-\epsilon^2)^{3/2}}{\epsilon^3 \alpha} \tan^{-1} \frac{\epsilon}{\sqrt{1-\epsilon^2}} +$$

$$+ \frac{2(4-\epsilon^2)^{3/2}}{\epsilon^3(1-\mu^2)} \tan^{-1} \frac{\epsilon}{\sqrt{4-\epsilon^2}} +$$

$$\begin{aligned}
& \left\{ -\frac{R^3}{\epsilon^4(1-\mu^2)\alpha} \tan^{-1} \frac{R}{1-\epsilon^2+\mu^2} \quad \text{for } \epsilon > 1-\mu \text{ i.e. } R^2 > 0, \right. \\
& \left. + \frac{R^2\sqrt{-R^2}}{2\epsilon^4(-\mu^2)\alpha} \log \frac{1-\epsilon^2+\mu^2+\sqrt{-R^2}}{1-\epsilon^2+\mu^2-\sqrt{-R^2}} \quad \text{for } \epsilon < 1-\mu \text{ i.e. } R^2 < 0, \right. \\
A_2 = & \frac{3}{\epsilon^2(1-\mu^2)} \log \mu^2 - \frac{12\sqrt{1-\epsilon^2}}{\epsilon^3\alpha} \tan^{-1} \frac{\epsilon}{\sqrt{1-\epsilon^2}} - \frac{12\sqrt{4-\epsilon^2}}{\epsilon^3(1-\mu^2)} \tan^{-1} \frac{\epsilon}{\sqrt{4-\epsilon^2}} + \\
& \left\{ + \frac{6R}{\epsilon^2(1-\mu^2)\alpha} \tan^{-1} \frac{R}{1-\epsilon^2+\mu^2} \quad \text{for } \epsilon > 1-\mu, \right. \\
& \left. - \frac{3\sqrt{-R^2}}{\epsilon^2(1-\mu^2)\alpha} \log \frac{1-\epsilon^2+\mu^2+\sqrt{-R^2}}{1-\epsilon^2+\mu^2-\sqrt{-R^2}} \quad \text{for } \epsilon < 1-\mu, \right. \\
A_3 = & -\frac{1}{\epsilon^2} - \frac{\alpha^2}{2\epsilon^4(1-\mu^2)} \log \mu^2 + \frac{4(1-\epsilon^2)^{3/2}}{\epsilon^3\alpha} \tan^{-1} \frac{\epsilon}{\sqrt{1-\epsilon^2}} \\
& - \frac{2\sqrt{4-\epsilon^2}(\epsilon^2+2)}{\epsilon^3(1-\mu^2)} \tan^{-1} \frac{\epsilon}{\sqrt{4-\epsilon^2}} + \\
& \left\{ -\frac{(\alpha^2+2\epsilon^2\mu^2)R}{\epsilon^4(1-\mu^2)\alpha} \tan^{-1} \frac{R}{1-\epsilon^2+\mu^2} \quad \text{for } \epsilon > 1-\mu, \right. \\
& \left. + \frac{(\alpha^2+2\epsilon^2\mu^2)\sqrt{-R^2}}{2\epsilon^4(1-\mu^2)\alpha} \log \frac{1-\epsilon^2+\mu^2+\sqrt{-R^2}}{1-\epsilon^2+\mu^2-\sqrt{-R^2}} \quad \text{for } \epsilon < 1-\mu, \right.
\end{aligned}$$

where $\alpha \equiv \epsilon^2 + \mu^2 - 1$, $R \equiv \sqrt{4\epsilon^2\mu^2 - (\epsilon^2 + \mu^2 - 1)^2}$, and

for $\epsilon = 0$

$$\begin{aligned}
A_1 = A_3 = & \frac{3(1-3\mu^2)}{2(1-\mu^2)^2} - \frac{3\mu^4}{(1-\mu^2)^3} \log \mu^2, \\
A_2 = & \frac{3(1+\mu^2)}{(1-\mu^2)^2} + \frac{6\mu^2}{(1-\mu^2)^3} \log \mu^2.
\end{aligned}$$

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On the Relations between Spin Polarizations and Distorted Wave Theory of the Direct Reactions

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The spin polarization in the distorted wave theory of stripping is presented in a general form, and the interesting problems are discussed, such as, what polarization comes from what types of the distortion of the wave functions. Under certain conditions, the exact relations are derived between the signs of polarization and the distortion of particle wave functions. It is pointed out that the similar relations are also present in other direct reactions. As an example, polarization in the inelastic scattering is studied briefly.

§ 1. Introduction

Many pieces of work have so far been carried out to improve on the simple theories of the direct reactions^{1),2)} by using distorted waves for the incident and emergent particles.³⁾⁴⁾⁵⁾ This is important for the polarization of the emergent particles. The simple theories are essentially based upon the "plane-wave" Born approximation, so that they predict no polarization. If either the incident, or outgoing particle wave function, or both, are distorted by the interaction with nucleus, then the spin polarization can result. Now, the interesting question arises what polarization comes from what types of distortions.

The first aim of this paper is to show elementarily, from the quantum mechanical point of view, that polarization takes a different sign according to whether only the incident particle wave is distorted or only the distortion of an emergent particle wave is considered. In cases of the stripping reactions, these were also predicted by Newns⁶⁾ and Tobocman,⁴⁾ with the use of the semi-classical model of two dimension. However, this is valid not only to the stripping reactions but also to all reactions, the theories of which are treated by the distorted-wave Born approximation. In view of these facts, polarization effect will be very interesting as to showing what distortion is produced by a nucleus, and it throws light on the mechanism of the direct reactions.

Other types of the direct reactions are well known, viz., (p, p') , (p, n) reactions, etc. As an example, the inelastic scattering of nucleon is considered in § 3.

It is our second aim to show that, in the inelastic scattering, the polarization may occur on the same ground as that of the stripping reactions: the use of the distorted waves for the incident and emergent particle states. Angular distributions

of the both types of the direct reactions are very similar, and strongly forward peaked⁷⁾ in remarkable contrast to the predictions made by the statistical theory of nuclear reactions. Such forward peaked angular distributions, in general, show that there are important interferences of different sub-channel wave functions and different compound nuclear states if the compound nucleus formation is concerned. As polarization phenomena are essentially interference effects,⁸⁾ similar relations are expected to hold with respect to the spin polarization in both types of direct interactions. There is, however, only one difference between them. In the stripping reactions, polarization is due to the spin triplet state interaction and mainly due to the singlet state interaction in the simple inelastic scattering as discussed here for the short range character of the inter-nucleonic force.

In this paper, the theory is not developed from the standpoint of general theory of polarization, but in each respective reactions in order to clarify how polarization comes from distortions of particle wave functions.

§ 2. Polarization in the stripping reaction

The polarization of emitted protons from an $A(d, p)B$ reaction is defined as

$$P(\mathbf{k}_d, \mathbf{k}_p) = \frac{\sum_{\mu_p, \mu_d} I_{\mu_p M_B M_A \mu_d} \cdot I_{\mu_p M_B M_A \mu_d}^* (\mu_p | \mathbf{i}_p | \mu_p)}{i_p \cdot \sum_{\mu_p, \mu_d} |I_{\mu_p M_B M_A \mu_d}|^2} \quad (2.1)$$

where, $\mathbf{i}_p = \frac{1}{2} \hbar \boldsymbol{\sigma}_p : \boldsymbol{\sigma}_p$ is the Pauli spin of the emergent proton. $I_{\mu_p M_B M_A \mu_d}$ is an amplitude for an $A(d, p)B$ reaction, and given by a matrix element of the form

$$I_{\mu_p M_B M_A \mu_d} = \int \psi_p^{(-)}(\mathbf{k}_p, \mathbf{r}_p) \chi_{\mu_p}^* (s_p) \chi_{I_B M_B}^* (\xi; \mathbf{r}_n, s_n) V_{np}(|\mathbf{r}_n - \mathbf{r}_p|) \psi_d^{(+)}(\mathbf{k}_d, \mathbf{R}_d) \\ \times \chi_{\mu_d}(\mathbf{r}, s_n, s_p) \chi_{I_A M_A}(\xi) d\tau. \quad (2.2)$$

Here, $\chi_{I_A M_A}$ and $\chi_{I_B M_B}$ are the wave functions of the target nucleus A and residual nucleus B . $\psi_d^{(+)}$ and $\psi_p^{(-)}$ are the orbital wave functions of the deuteron and proton channel, distorted by some suitable potentials due to the nucleus under the outgoing and incoming radiation conditions, respectively. χ_{μ_p} is the proton spin wave function, and χ_{μ_d} is the deuteron triplet state wave function. V_{np} is a triplet state central potential. The coordinates are as follows: ξ are internal coordinates of A . \mathbf{r}_n and \mathbf{r}_p are neutron and proton coordinates relative to the mass centre of A (assumed to be infinitely heavy). \mathbf{R}_d and \mathbf{r} are given by $\frac{1}{2}(\mathbf{r}_n + \mathbf{r}_p)$ and $\mathbf{r}_n - \mathbf{r}_p$ respectively. s_n and s_p are spin coordinates of the neutron and proton. (2.2) involves certain approximation of its own, which will be discussed later in detail.

We start with (2.2) which makes us possible to carry out the integration over ξ , and write it into the nuclear overlap integral⁹⁾

$$\int \chi_{I_B M_B}^* (\xi, \mathbf{r}_n, s_n) \chi_{I_A M_A}(\xi) d\xi$$

$$= \sum_{j\mu_j l m \mu_n} (I_A j M_A \mu_j | I_B M_B) (l 1/2 m \mu_n | j \mu_j) \cdot \theta_{jl}^* \cdot u_l^*(r_n) \cdot (i^l Y_{lm}(\Omega_n))^* \cdot \chi_{\mu_n}^*(s_n). \quad (2.3)$$

Here, j and l are the total and orbital angular momenta with which the neutron is captured by A . χ_{μ_n} and $u_l(r_n)$ are a spin and a normalized orbital (real) wave function of the captured neutron. θ_{jl} is the reduced width amplitude, and made real by taking a suitable phase.

Substitution of (2.3) into (2.2) yields

$$I_{\mu_p M_B M_A \mu_d} = \sum_{j\mu_j l m \mu_n} (I_A j M_A \mu_j | I_B M_B) (l 1/2 m \mu_n | j \mu_j) (1/2 1/2 \mu_p \mu_n | 1 \mu_d) \cdot \hat{l} \cdot \theta_{jl} \cdot B_{lm}, \quad (2.4)$$

where,

$$B_{lm} = \frac{1}{\hat{l}} \cdot \int \psi_p^{(-)}(\mathbf{k}_p, \mathbf{r}_p) \cdot u_l(r_n) \left(i^l Y_{lm}(\Omega_n) \right)^* V_{np} \chi(\mathbf{r}) \psi_d(\mathbf{k}_d, \mathbf{R}_d) d\mathbf{r}_n d\mathbf{r}_p \quad (2.5)$$

and $\hat{l} = \sqrt{2l+1}$. The matrix element B_{lm} gives the amplitude for the capture of the neutron into the state (l, m) .

For calculation of $(\mu_p' | i_p | \mu_p)$, it is convenient to introduce and evaluate the irreducible tensor of rank one $T_{\kappa}^{(1)}$ where

$$T_0^{(1)} = \frac{i_{pz}}{[i_p(i_p+1)]^{1/2}}, \quad T_{\pm 1}^{(1)} = \mp \frac{i_{pz} \pm i \cdot i_{py}}{[2i_p(i_p+1)]^{1/2}}. \quad (2.6)$$

By using (2, 1) and (2.6), we can write the polarization intensity as

$$T_{\kappa}^{(1)}(\mathbf{k}_d, \mathbf{k}_p) = \frac{\sum_{\mu_p' \mu_p} (\mu_p' | i_p | \mu_p) I_{\mu_p M_B M_A \mu_d} \cdot I_{\mu_p' M_B M_A \mu_d}^*}{i_p \cdot \sum_{\mu_p M_B M_A \mu_d} |I_{\mu_p M_B M_A \mu_d}|^2}. \quad (2.7)$$

(2.7) involves the matrix element quadratically, and therefore we define a density matrix

$$\rho_{lm; l'm'} = B_{lm} \cdot B_{l'm'}^*, \quad (2.8)$$

and a statistical tensor

$$\rho_{kl}(l, l') = \sum_{m, m'} (-)^{l'-m'} \cdot (ll'm-m'|kq) \rho_{lm; l'm'}. \quad (2.9)$$

Substitution of (2.4) into (2.7) results, using (2.8) and (2.9),

$$T_{\kappa}^{(1)}(\mathbf{k}_d, \mathbf{k}_p) = \frac{4}{9} \cdot \frac{\sum_{j l} (-)^{j-l-1/2} \cdot [l(l+1)(2l+1)]^{1/2} \cdot \frac{\theta_{jl}^2}{2j+1} \cdot (-)^{\kappa} \cdot \rho_{1-\kappa}(l, l)}{\sum_{j l} \hat{l} \cdot \theta_{jl}^2 \cdot \rho_{00}(l, l)}. \quad (2.10)$$

In order to investigate what polarization comes from what types of distortions, we must go further. $\rho_{1\kappa}(l, l)$ must be related explicitly to the physical factors which

are radial parts of the B_{lm} 's.

The particle wave functions $\psi_d^{(+)}$ and $\psi_p^{(-)}$ are normalized like $\exp(i\mathbf{k}\cdot\mathbf{r})$ and as the quantization axis the direction of \mathbf{k}_d is chosen. Then, they are expressed as

$$\psi_d^{(+)}(\mathbf{k}_d, \mathbf{r}) = (\pi)^{1/2} \cdot \sum_{l_1} i^{l_1} \cdot \hat{l}_1 \cdot w_{l_1}(k_d r) Y_{l_1 0}(\Omega), \quad (2.11)$$

and

$$\psi_p^{(-)}(\mathbf{k}_p, \mathbf{r}) = 2\pi \cdot \sum_{l_2 m_2} i^{l_2} \cdot v_{l_2}(k_p r) Y_{l_2 m_2}(\Omega) Y_{l_2 m_2}^*(\Omega_{k_p}). \quad (2.12)$$

w_l and v_l at large distances have the asymptotic forms

$$w_l(x) = h_l^{(2)}(x) + \eta_l \cdot h_l^{(1)}(x), \quad (2.11')$$

and

$$v_l(x) = h_l^{(1)}(x) + \hat{\eta}_l^* \cdot h_l^{(2)}(x), \quad (2.12')$$

where $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$ are the first and second kind of spherical Hankel functions of order l . The notation $\hat{\eta}_l$, different from η_l , is used corresponding to the distortions by different potentials. For the internal wave function of deuteron, the Yukawa type is used. This makes the calculation of the matrix element very easy, since

$$V_{np} \chi(\mathbf{r}) = -(\hbar^2/M) \sqrt{8\pi\alpha} \cdot \delta(\mathbf{r}), \quad (2.13)$$

where $\hbar^2\alpha^2/M$ is the deuteron binding energy.

Substitution of (2.11), (2.12), and (2.13) into (2.5) results in

$$B_{lm} = -\frac{2\pi\hbar^2}{M} \cdot \sqrt{2\pi\alpha} \cdot \sum_{l_1 l_2 m_2} (\hat{l}_1/\hat{l})^2 \cdot \hat{l}_2 \cdot (l_1 l_2 00 | l 0) \cdot i^{l_1-l_2-l} \cdot \mathcal{R}(l_2 l l_1) \\ \times (l_1 l_2 0 m_2 | l m) Y_{l_2 m_2}^*(\Omega), \quad (2.14)$$

where,

$$\mathcal{R}(l_2 l l_1) = \int v_{l_2}^*(k_p r) u_l(r) w_{l_1}(k_d r) r^2 dr. \quad (2.15)$$

Thus, the statistical tensor $\rho_{1\kappa}(l, l)$ can be expressed in terms of $\mathcal{R}(l_2 l l_1)$'s as follows

$$\rho_{1\kappa}(l, l) = \left(\frac{3}{4\pi}\right)^{1/2} \cdot 2\pi\alpha \cdot \left(\frac{2\pi\hbar^2}{M}\right)^2 \cdot \sum_{l_1 l_2 l_1' l_2' l} \left(\frac{\hat{l}_1 \cdot \hat{l}_2 \cdot \hat{l}_1' \cdot \hat{l}_2'}{\hat{l}}\right)^2 \cdot i^{l_1+l_2+l_1'+l_2'} \\ \times (l_1 l_2 00 | l 0) (l_1' l_2' 00 | l 0) \\ \times (l_1 l_1' 00 | L 0) (l_2 l_2' 00 | L 0) \cdot \mathcal{R}(l_2 l l_1) \mathcal{R}(l_2' l_1')^* \\ X \begin{pmatrix} l_1 & l_1' & L \\ l_2 & l_2' & L \\ l & l & 1 \end{pmatrix} \cdot (L 1 - \kappa \kappa | L 0) Y_{L-\kappa}(\Omega). \quad (2.16)$$

Derivation of (2.16) will be given in the Appendix. Combination of (2.16) with

(2.6) and (2.10) shows that polarization is normal to the plane formed by the directions of the incoming deuteron and outgoing proton. In terms of unit vector $\mathbf{n} = \mathbf{k}_d \times \mathbf{k}_p / |\mathbf{k}_d \times \mathbf{k}_p|$, we have a final result

$$P(\mathbf{k}_d, \mathbf{k}_p) = \mathbf{n} \cdot \frac{\text{const.}}{\sum_{j,l} \hat{l} \cdot \theta_{jl}^2 \cdot \rho_{00}(l, l)} \cdot \sum_{j,l} (-)^{j-l-1/2} \cdot \left[\frac{l(l+1)}{2l+1} \right]^{1/2} \cdot \frac{\theta_{jl}^2}{2j+1} \cdot \sum_L \phi_L(l) \cdot P_L^1(\cos\theta). \quad (2.17)$$

Here,

$$\begin{aligned} \phi_L(l) = & \sum_{l_1 l_2 l_1' l_2'} i^{l_1+l_2+l_1'+l_2'} \cdot (\hat{l}_1 \cdot \hat{l}_2 \cdot \hat{l}_1' \cdot \hat{l}_2')^2 \cdot (l_1 l_2 00 | l 0) (l_1' l_2' 00 | l 0) \\ & \times (l_1 l_1' 00 | L 0) (l_2 l_2' 00 | L 0) \text{Re}[i \mathcal{R}(l_2 l_1) \mathcal{R}(l_2' l_1')^*] \cdot X \begin{pmatrix} l_1 & l_1' & L \\ l_2 & l_2' & L \\ l & l & 1 \end{pmatrix}. \end{aligned} \quad (2.18)$$

When only one l value is important, (2.17) indicates that the polarizations take a sign opposite to one another for the total angular momentum $j = l \pm 1/2$ of the captured neutron. For the polarization resulted from the distortion of particle waves, all information is involved in the quantity $\phi_L(l)$, and will be analysed here in detail.

First, it is a trivial result that the "plane-wave" Born approximation predicts no polarization. This is due to the fact that $\mathcal{R}(l_2 l_1)$ is always real in this approximation.

Secondly, polarization can occur whenever a real potential is chosen for the distortions. This comes from the fact that the radial part of a wave function distorted by real potential under incoming or outgoing radiation condition cannot be made real by any means.

Next, we consider the general cases where the particle waves are distorted by optical (complex) potentials. Theoretically there may be two cases: One is the case (a), where the incident wave is only distorted, and another the case (b), where only distortion of the emergent wave is considered. It seems interesting to study how different effects would be brought on the polarization by the alternative of case (a) or (b) for distortions. Let us assume the polarization for case (a) to be given by (2.18), then that for case (b) is obtained by exchanging suffix 1 and 2, and replacing $\hat{\mathcal{R}}(l_1 l_2)$ for $\mathcal{R}(l_1 l_2)$ in (2.18). Here

$$\mathcal{R}(l_1 l_2) = 2 \cdot \int j_{l_1}(k_p r) u_l(r) w_{l_2}(k_d r) r^2 dr,$$

and

$$\hat{\mathcal{R}}(l_1 l_2) = 2 \cdot \int v_{l_1}^*(k_p r) u_l(r) j_{l_2}(k_d r) r^2 dr.$$

By the above procedure $\phi_L(l)$ becomes $\hat{\phi}_L(l)$. The kinematical factors involved

in (2.18) have well-defined symmetry properties. By exchanging suffix 1 and 2, X -coefficient changes its sign and the other factors remain unchanged. Therefore, we obtain exactly the following relation,

$$\hat{\phi}_L(l) = -\phi_L(l), \quad \text{if} \quad \hat{\mathcal{R}}(l_1 l_2) = \mathcal{R}(l_2 l_1). \quad (2.19)$$

Under this condition, cases (a) and (b) lead to the completely symmetric polarization to the scattering plane: their magnitudes are the same, but signs opposite to one another. For our purposes, it is convenient to divide a condition in (2.19) into two parts as follows,

$$\left. \begin{array}{l} (1) \text{ The wave number of the incident particle is equal to} \\ \text{that of the outgoing particle. That is to say, } k_d = k_p. \\ (2) \text{ The distorting potentials for both channels are the same.} \\ \text{In this case, } v_i^*(x) = w_i(x). \end{array} \right\} \quad (2.20)$$

Finally, both the deuteron and proton interactions with the nucleus should be considered. We consider here the cases where the conditions (2.20) hold in general. In this case, the condition (2.19) is replaced by

$$\mathcal{R}(l_1 l_2) = \mathcal{R}(l_2 l_1). \quad (2.21)$$

By using the condition above, it is easy to show that the net polarization vanishes identically.

In general, the condition (1) in (2.20) is satisfied approximately, since this is the optimum condition in the angular distribution to obtain the characteristic dependence of the orbital angular momentum l of the captured neutron. The condition (2) is supposed not to be valid in the stripping reaction, because of the interactions of different particles with nucleus. However, this condition may be approximately satisfied in the inelastic processes leaving the residual nucleus in low-lying states. If this is true, polarization in these will be worthwhile to study, as the condition (2) can be realized to the better approximation by using a larger incident energy. Leaving the detailed discussions on the conditions (2.20) to a later section, the polarization in inelastic scattering is briefly given in the next section.

§ 3. The polarization in inelastic scattering

We consider here an inelastic scattering process as simple as possible. It is assumed that the target nucleus consists of a core of spin zero together with an extra neutron of orbital angular momentum l , which is coupled with the intrinsic spin to give spin j . We take the interaction of the incident particle (assumed to be neutron) with the extra nucleon only, the core being regarded as inert.

Along the same line as in the previous section, the polarization intensity is expressed as

$$T_{\kappa}^{(1)}(\mathbf{k}_0, \mathbf{k}_0') = \frac{\sum_{\mu, \mu', M, M_\mu} (i1\mu'\kappa|i\mu'') I_{\mu' M' M_\mu} \cdot I_{\mu'' M'' M_\mu}^*}{i \cdot \sum_{\mu, M' M_\mu} |I_{\mu' M' M_\mu}|^2}. \quad (3.1)$$

Here, $I_{\mu' M' M_\mu}$ is the transition matrix element, and given by

$$I_{\mu' M' M_\mu} = \int \psi^{(-)}(\mathbf{k}_0', \mathbf{r}_0) \chi_{\mu'}^*(s_0) \varphi(j' M') V_{10} \varphi(j M) \chi_{\mu}(s_0) \psi^{(+)}(\mathbf{k}_0, \mathbf{r}_0) d\tau \quad (3.2)$$

where $\varphi(j M)$ is the ground state wave function of the target nucleus, which is expressed as the form

$$\varphi(j M) = \sum_{m\nu} (j M | l 1/2 m \nu) u_l(r_1) (i^l Y_{lm}(\Omega_1)) \chi_\nu(s_1) \Phi_0(\xi),$$

and the excited state of the residual nucleus is obtained by replacing the corresponding primed quantities for each quantum numbers. The core state wave function $\Phi_0(\xi)$ remains unaffected by the assumptions. \mathbf{r}_1 and \mathbf{r}_0 are the coordinates of the extra nucleon and the incident nucleon (scattered by the collision) relative to the mass center of the core (assumed to be infinitely heavy). \mathbf{k}_0 and \mathbf{k}_0' are the propagation vectors of the incident and the scattered particle. V_{01} is taken as the three-dimensional delta function, for the short range character of the inter-nucleonic force, and for simplicity. Then, the triplet state interaction gives no contributions to the direct process of inelastic scattering, and $I_{\mu' M' M_\mu}$ results in

$$I_{\mu' M' M_\mu} = \frac{1}{2} \sum_{m' m'} (l' 1/2 m' - \mu' | j' M') (l 1/2 m - \mu | j M) (-)^{\mu' - \mu} B_{l' m'; l m} \quad (3.3)$$

where

$$B_{l' m'; l m} = V_s \cdot \int \psi^{(-)}(\mathbf{k}_0', \mathbf{r}_0) u_{l'}(r_1) (i^{l'} Y_{l' m'}(\Omega_1))^* \delta(\mathbf{r}_1 - \mathbf{r}_0) u_l(r_1) \times (i^l Y_{lm}(\Omega_1)) \psi^{(+)}(\mathbf{k}_0, \mathbf{r}_0) d\tau. \quad (3.4)$$

Here, V_s is the strength of singlet interaction.

The density matrix and the statistical tensor are slightly different from the stripping cases, and defined by

$$\rho_{lm; l' m'}(L) = \sum_M B_{lm; LM} \cdot B_{l' m'; LM} \quad (3.5)$$

and

$$\rho_{kq}(L; l, l') = \sum_{m, m'} (-)^{l' - m'} (l' m - m' | k q) \rho_{lm; l' m'}(L). \quad (3.6)$$

Using these definitions, the polarization intensity is

$$T_{\kappa}^{(1)}(\mathbf{k}_0, \mathbf{k}_0') = \frac{4}{3} \cdot \frac{(-)^{j' - l' + 1/2} \cdot [l' (l' + 1)]^{1/2} \cdot (-)^{\kappa} \rho_{1-\kappa}(l; l', l')}{(2j' + 1) \cdot \rho_{00}(l; l', l')}. \quad (3.7)$$

When the polarization vector is taken as the quantization axis, the component of polarization of the proton spin in the direction of the axis is

$$\begin{aligned}
 P(\mathbf{k}_0, \mathbf{k}_0') &= \frac{2}{\sqrt{3}} \cdot \frac{(-)^{j'-l'+1/2} \cdot [l'(l'+1)]^{1/2} \cdot \rho_{10}(l; l', l')}{(2j'+1) \cdot \rho_{00}(l; l', l')} \\
 &= \frac{(-)^{j'-l'+1/2} \cdot \sum m' (\sum |B_{l'l'm'; l'm}|^2)}{(j'+1/2) \cdot \sum |B_{l'l'm'; l'm}|^2}.
 \end{aligned} \tag{3.8}$$

If $\langle m \rangle$ denotes the mean value of the component of orbital angular momentum along the quantization axis:

$$\langle m \rangle = \frac{\sum m' (\sum |B_{l'l'm'; l'm}|^2)}{\sum |B_{l'l'm'; l'm}|^2}, \tag{3.9}$$

then,

$$P(\mathbf{k}_0, \mathbf{k}_0') = \frac{(-)^{j'-l'+1/2}}{j'+1/2} \cdot \langle m \rangle. \tag{3.10}$$

Now, the similar expression in stripping reaction results from (2.6), (2.8), (2.9), and (2.10)

$$P(\mathbf{k}_d, \mathbf{k}_p) = \frac{(-)^{j-l-1/2}}{3(j+1/2)} \cdot \langle m \rangle_{dp}, \tag{2.10'}$$

where $P(\mathbf{k}_d, \mathbf{k}_p)$ is the component of the polarization of proton from the stripping in the direction of which is taken as the quantization axis. $\langle m \rangle_{dp}$ is the mean value of the component of orbital angular momentum of the captured neutron along the axis:

$$\langle m \rangle_{dp} = (\sum m |B_{lm}|^2) / \sum |B_{lm}|^2.$$

Comparison of (3.9) with (2.10') shows that the sign of the polarization from the stripping reaction is opposite to that from inelastic scattering, if $\langle m \rangle$ and $\langle m \rangle_{dp}$ have the same sign. This is owing to the different types of interactions used: The triplet state interaction in the deuteron stripping reaction and the singlet state one in the inelastic scattering. If the triplet state interaction should be considered in the inelastic scattering, this enables one to infer that the polarization in the magnitude somewhat reduces.

The final expression of the emitted protons from inelastic scattering can be obtained in the same way as in the previous section, but is not given here, because the conclusions obtained are the same as those in the stripping cases.

It is emphasized here that the condition (2) in (2.20) is approximately satisfied by raising the incident energy high enough compared with the excited state energy. Further, if the distorting potentials are taken as the same in the zero-th approximation for both channels of inelastic scattering, the net polarization vanishes. If the incident energy is comparable to, or, slightly larger than the excited state energy, however, the polarization may appear and will be observed.

§ 4. Discussions

The polarization of the product particle spin has been studied by starting with the formulas (2.2) and (3.2) which involve certain approximations. These are as follows:

(1) The distorting potentials are spin-independent. The spin-orbit coupling is undoubtedly present for a nucleon and probably so for a deuteron, and is possible to produce the polarization in the scattered particles. However, it gives only a minor correction in magnitude compared with that produced by the much stronger spherical symmetric potential. Newns and Refai¹⁰⁾ have shown that this is the case in the stripping reaction.

(2) The effects arising from the tensor force of the two-body interaction and the small D -state of the deuteron wave function are neglected. The estimate by Weidenmüller¹¹⁾ supports this approximation. It is pointed that these tensor effects cannot produce the polarization when the plane waves are used for the particle wave functions.

(3) The Coulomb interactions of charged particles with nucleus have been ignored in this paper. In his work on stripping reaction at very low energy, Yoccoz¹²⁾ has shown that if the effects of the Coulomb field on the deuteron and proton are taken into account, the polarization can result. Since the Coulomb field cannot be responsible for the results at higher energies, and makes the direct process at low energies insignificant, we could ignore its effects altogether in the cases with which we are concerned.

The exact relations obtained in § 2, are derived under rather severe conditions. What must be discussed is to what extent the conditions can be relaxed. For this purpose, we need knowledges about dependence of the dynamical factors \mathcal{R} 's on the distorting potential and particle energy. We must investigate how sensitively the dynamical factors \mathcal{R} 's depend on these. Now, the distorted wave procedure is only appropriate if the incident energy covers several states of the compound nucleus and is far from the single particle resonances (size resonances).¹³⁾ Under these conditions, the factors \mathcal{R} 's may not be expected to be very sensitive to the choice of the distorting potential and incident energy. Otherwise, polarization in the stripping reactions and the others should lose the qualification as powerful tools for the nuclear spectroscopy.

In this paper, any numerical investigation is not attempted for determining the degree of sensitivity, since our primary concerns are to find the qualitative relation between the polarizations and the distortion of waves.

Fortunately, in their works on the polarization of the proton from $C^{12}(d, p)C^{13}$, Newns and Refai,¹⁰⁾ and Weidenmüller¹¹⁾ have calculated it for the various assumptions of the distorting potentials. Their results show that the dependence on the distorting potential is not serious and suggest the qualitative relations obtained here to be valid, at least, at small scattering angles.

In practice, interactions of both incoming and outgoing particles with nucleus should be considered. We failed to find any relations between the polarizations and the distortions except in (2.21). In the cases under (2.21), it may be suggested that the distortions for both particles tend to reduce the magnitude of the polarization in accordance with the predictions of the semi-classical model. This predicts the algebraic sum of effects due to the distortions on the polarization. However, interferences between these distortions are present in quantum mechanical theory, and the relation in (2.21) is nothing but to show the completely destructive interference between them.

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Appendix

The density matrix is defined by (2.8) and expressed with the aid of (2.14) as

$$\begin{aligned} \rho_{lm;l'm'} = & \left(\frac{2\pi\hbar^2}{M} \right)^2 \cdot 2\pi\alpha \cdot \sum_{l_1/2m_2} \sum_{l_1'/2'm_2'} \frac{(\hat{l}_1 \cdot \hat{l}_1')^2 \cdot \hat{l}_2 \cdot \hat{l}_2'}{(\hat{l})^4} \cdot i^{l_1-l_2-l_1'+l_2'} \cdot \mathcal{R}(l_2 l_1) \mathcal{R}(l_2' l_1')^* \\ & \times (l_1 l_2 00 | l 0) (l_1' l_2' 00 | l' 0) (l_1 l_2 0 m_2 | l m) (l_1' l_2' 0 m_2' | l' m') Y_{l_2 m_2}^*(\varrho) Y_{l_2' m_2'}(\varrho). \end{aligned} \quad (\text{A} \cdot 1)$$

The product of two spherical harmonics can be expressed as a linear superposition of the spherical harmonics.

Introducing (A.1) into its definition (2.9), we can write the statistical tensor as

$$\begin{aligned} \rho_{1\kappa}(l, l') = & \left(\frac{2\pi\hbar^2}{M} \right)^2 \cdot 2\pi\alpha \cdot \frac{1}{\sqrt{4\pi}} \cdot \sum_{l_1/2m_2} \sum_{l_1'/2'm_2'} i^{l_1+l_2+l_1'+l_2'} \cdot \frac{(\hat{l}_1 \cdot \hat{l}_2 \cdot \hat{l}_1' \cdot \hat{l}_2')^2}{\hat{L}(\hat{l})^4} \cdot \mathcal{R}(l_2 l_1) \mathcal{R}(l_2' l_1')^* \\ & \times (l_1 l_2 00 | l 0) (l_1' l_2' 00 | l' 0) (l_2 l_2' 00 | L 0) M_\kappa(LM) \cdot Y_{LM}(\varrho), \end{aligned} \quad (\text{A} \cdot 2)$$

where, $M_\kappa(LM)$ is a purely geometrical quantity and defined as follows,

$$\begin{aligned} M_\kappa(LM) = & (-)^{l+l'+l_2} \cdot \sum_{m_2'(m_2 m_2')} [(-)^{m_2-m'} (l l m - m' | 1 \kappa) (l_1 l_2 0 m_2 | l m) (l_1' l_2' 0 m_2' | l' m')] \\ & \times (l_2 l_2' - m_2 m_2' | LM) \end{aligned} \quad (\text{A} \cdot 3)$$

$M_\kappa(LM)$ can be reduced by the Racah formalism to an expression involving no sums of the magnetic quantum numbers. In the reduction, it is convenient to use the following formulas:

$$(l_2 l_2' - m_2 m_2' | LM) (l_1 l_2 0 m_2 | l m) = (-)^{l_2+l_2'+m_2'-m_2} \cdot \frac{\hat{l} \cdot \hat{L}}{\hat{l}_1} \cdot \sum_f \hat{f} \cdot (l_2' l m_2' - m | f M).$$

$$\times (Lf - MM|l_1 0) W(LL_2' l_1 l; l_2 f),$$

$$\sum_{m_2'} (l_1' l_2' 0 m_2' | l m') (l l m - m' | 1 \kappa) (l_2' l m_2' - m | f M) = \hat{l} \cdot \hat{f} \cdot (l_1' f 0 M | 1 - \kappa) W(l_1' l_2' 1 l; l_2 f),$$

and

$$(Lf - MM|l_1 0) (l_1' f 0 M | 1 - \kappa) = \hat{l} \cdot \hat{l}_1 \cdot \sum_g (l_1 l_1' 0 0 | g 0) (L 1 - M - \kappa | g 0) \cdot W(LL_1 1 l_1'; f g).$$

By the use of these formulas, we find that

$$M_\kappa(LM) = \hat{l} \cdot \hat{L} \cdot (\hat{l})^2 \cdot \sum_g \sum_f (\hat{f})^2 \cdot W(LL_2' l_1 l; l_2 f) W(l_1' l_2' 1 l; l f) W(LL_1 1 l_1'; f g) \\ \times (l_1 l_1' 0 0 | g 0) (L 1 - M - \kappa | g 0).$$

The sum over f can be written in terms of X -coefficients[†] as

$$M_\kappa(LM) = \hat{l} \cdot \hat{L} \cdot (\hat{l})^2 \cdot \sum (-)^{L-g} \cdot (l_1 l_1' 0 0 | g 0) (L 1 M \kappa | g 0) \cdot X \left(\begin{array}{ccc} l_1 & l_1' & g \\ l_2 & l_2' & L \\ l & l & 1 \end{array} \right). \quad (\text{A} \cdot 3)$$

In (A·3), the sum over g reduces to only a single term which is $g=L$.⁸⁾ This is shown as in the following ways:

(1) By the conservation of parity, $l_1 + l_2 + l_1' + l_2' = \text{even integer}$,

(2) by the properties which Clebsch-Gordan coefficient $(ab00|c0)$ vanishes unless $a+b+c = \text{even integer}$, $L+g = \text{even integer}$, and

(3) by the property of $(L 1 M \kappa | g 0)$, $g=L$, or $L \pm 1$.

Hence, there follows the result $g=L$.

Substitution of (A·3) into (A·2) gives the final expression of the statistical tensor

$$\rho_{1\kappa}(l, l) = \left(\frac{2\pi\hbar^2}{M} \right)^2 \cdot 2\pi\alpha \cdot \left(\frac{3}{4\pi} \right)^{1/2} \sum_{l_1 l_2 l_1' l_2' L} i^{l_1 - l_2 + l_1' + l_2'} \left(\frac{\hat{l}_1 \cdot \hat{l}_2 \cdot \hat{l}_1' \cdot \hat{l}_2'}{\hat{l}} \right)^2 \cdot \mathcal{R}(l_2 l l_1) \mathcal{R}(l_2' l l_1')^* \\ \times (l_1 l_2 0 0 | l 0) (l_1' l_2' 0 0 | l 0) (l_1 l_1' 0 0 | L 0) (l_2 l_2' 0 0 | L 0) \cdot X \left(\begin{array}{ccc} l_1 & l_1' & L \\ l_2 & l_2' & L \\ l & l & 1 \end{array} \right) \quad (2 \cdot 16) \\ \times (L 1 - \kappa \kappa | L 0) \cdot Y_{L-\kappa}(\Omega).$$

This is nothing but the (2·16) given in § 2. The components of the polarization are derived, by using the relation (2·6), from (2·10)

$$P_x(\mathbf{k}_a, \mathbf{k}_p) = \frac{1}{3} \cdot \left(\frac{2}{3} \right)^{1/2}.$$

$$\frac{\sum_{j l} (-)^{j-l-1/2} \cdot [l(l+1)(2l+1)]^{1/2} \cdot (\theta_{jl}^2/2j+1) \cdot [\rho_{1,-1}(l, l) - \rho_{1,1}(l, l)]}{\sum_{j l} \hat{l} \cdot \theta_{jl}^2 \cdot \rho_{00}(l, l)},$$

$$P_y(\mathbf{k}_a, \mathbf{k}_p) = (i/3) \cdot \left(\frac{2}{3} \right)^{1/2}.$$

[†] See, for example, Arima, Horie, and Tanabe, Prog. Theor. Phys. **11** (1954), 143.

$$P_z(\mathbf{k}_d, \mathbf{k}_p) = \frac{2}{3\sqrt{3}} \cdot \frac{\sum_{j'l'} (-)^{j-l-1/2} \cdot [l(l+1)(2l+1)]^{1/2} \cdot (\theta_{jl}^2/2j+1) \cdot [-\rho_{1,-1}(l,l) - \rho_{1,1}(l,l)]}{\sum_{j'l'} \hat{l} \cdot \theta_{jl}^2 \cdot \rho_{00}(l,l)}, \quad (\text{A} \cdot 4)$$

where the quantization axis has been taken in the direction of the incident deuteron beam.

Substitution of (2·16) into (A·4) gives the components of the polarization in terms of \mathcal{R} 's. As the dependence on κ of $\rho_{1\kappa}(l_d, l_p)$ is contained only in factors $(L1 - \kappa\kappa|L0) \cdot Y_{L-\kappa}(\mathcal{Q})$, the dependences on the scattering angles are given for each component of the polarization as follows,

$$\begin{aligned} P_x &\sim -i \sin \varphi \cdot P_L^1(\cos \theta) \\ P_y &\sim i \cos \varphi \cdot P_L^1(\cos \theta) \\ P_z &\equiv 0 \end{aligned}$$

where $P_L^1(\cos \theta)$ is the normalized associated Legendre polynomial.

This result shows that the polarization is always normal to the scattering plane. In terms of the unit vector $\mathbf{n} = \mathbf{k}_d \times \mathbf{k}_p / |\mathbf{k}_d \times \mathbf{k}_p|$, we can obtain the final expression by evaluating $[P_y(\mathbf{k}_d, \mathbf{k}_p)]_{\varphi=0}$. Further, this can be expressed entirely in terms of real quantities by the use of the symmetry properties of the X-coefficient for interchange of two rows.

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Stellar Synthesis of the α -Particle Nuclei Heavier than Ne^{20} *

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The rates of (α, γ) and (γ, α) reactions at temperatures above 5×10^8 °K for α -particle nuclei, Ne^{20} , Mg^{24} , Si^{28} , S^{32} , A^{36} , and Ca^{40} , are calculated as well as heavy ion reactions such as $\text{C}^{12} + \text{C}^{12} \rightarrow \text{Ne}^{20} + \alpha$, etc.

The stellar condition, under which the cosmical abundances of the α -particle nuclei can be explained in terms of the above nuclear reactions, is investigated by using these results. At high temperature $T = 2 \sim 3 \times 10^9$ °K the cosmical abundances of α -particle nuclei can be reproduced by taking the matter density ρ as $1 \sim 10^2$ g cm $^{-3}$. Such a stellar condition may be realized in the supernova explosion. It is shown that, to explain the cosmical abundances in concern by the equilibrium theory, $T > 5 \times 10^9$ °K and $\rho \sim 10^6$ g cm $^{-3}$ are required.

§ 1. Introduction

According to the current theory of stellar evolution,¹⁾ the development of a helium core at a star's center due to hydrogen burning by pp -chain or CN -cycle causes a star to move off the main sequence in the direction to the red-giant region. The fusion processes of helium are believed to occur at a later stage of this red-giant evolution in which the hydrogen in the central core has been largely converted into helium and in which gravitational contraction has raised the central temperature to $\sim 10^8$ °K and the density to $\sim 10^5$ g cm $^{-3}$. Once C^{12} has been produced through the $3\alpha \rightarrow \text{C}^{12}$ reaction, nuclei heavier than C^{12} can be built up by successive (α, γ) reactions. Results obtained by Nakagawa *et al.*,²⁾ Hayakawa *et al.*,³⁾ and Cameron⁴⁾ show that these helium capture processes occur at temperatures between 1×10^8 °K and 2×10^8 °K and synthesize C^{12} , O^{16} , Ne^{20} but perhaps little Mg^{24} . The inner core of C^{12} , O^{16} and Ne^{20} develops in the star and eventually undergoes gravitational contraction, which will raise the temperature in the core to the point where further thermonuclear reactions begin to occur.

Recently, Burbidge, Burbidge, Fowler and Hoyle⁵⁾ proposed the following " α -process" as reactions occurring in this stage. At the central temperature near 1×10^9 °K, γ -rays present in the thermal assembly become sufficiently energetic to promote $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$. This is the first and most important (γ, α) reaction to

* An outline of this paper was published in Prog. Theor. Phys. **20** (1958), 110, and reported in the Joint Discussion on Nucleogenesis in Stars, held during the Tenth General Assembly of the International Astronomical Union, Moscow, U.S.S.R., August, 1958.

occur since the α -particle binding energy in Ne^{20} is only 4.75 MeV, while it is 7.37 MeV in C^{12} and 7.15 MeV in O^{16} . The proton and neutron binding energies in these nuclei are 12 to 19 MeV, so that the (γ, α) reactions precede (γ, p) or (γ, n) . The α -particle thus released can now penetrate the Coulomb barrier of the remaining Ne^{20} quite readily at the temperature of the order of 10^9 °K, thereby forming Mg^{24} . Once Mg^{24} is produced we can also expect $\text{Mg}^{24}(\alpha, \gamma)\text{Si}^{28}$ and $\text{Si}^{28}(\alpha, \gamma)\text{S}^{32}$ to occur in some degree. In this way the α -particle nuclei, Mg^{24} , Si^{28} , S^{32} , A^{36} and Ca^{40} , will be built up in decreasing proportion.

The above α -process is, of course, oversimplified, because in a very hot and dense stellar core consisting of C^{12} , O^{16} and Ne^{20} heavy ion reactions such as $\text{C}^{12} + \text{C}^{12} \rightarrow \text{Ne}^{20} + \alpha$ and $\text{C}^{12} + \text{O}^{16} \rightarrow \text{Mg}^{24} + \alpha$ must also be taken into account as possible α -particle sources. The fastest of these reactions is $\text{C}^{12} + \text{C}^{12}$, which competes with $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ at temperature below 2×10^9 °K and at density 10^5 g cm^{-3} as shown later.*

The relative abundances of these α -particle nuclei in the solar neighborhood, as compiled by Suess and Urey,⁶⁾ are shown in Table 1. These values will be called the normal abundances in a sense that they represent the chemical composition of the normal Population I stars.

Table 1. Binding Energy and relative abundances of the α -particle nuclei.

	Ne^{20}	Mg^{24}	Si^{28}	S^{32}	A^{36}	Ca^{40}
α -particle binding energy (in MeV)	4.75	9.31	10.00	6.94	6.66	7.04
Abundance (in number)	8.4	0.78	1.00	0.39	0.14	0.052

The abundances in other celestial objects such as planetary nebulae and early type stars⁷⁾ agree with these values within the factor of 5, as seen in Fig. 1.

The purpose of this article is to find the stellar conditions, that is, the temperature, the density and the time of their duration, which are necessary to explain the normal abundances of α -particle nuclei built-up by the α -process as well as heavy ion reactions. Before doing this, various nuclear reaction rates are calculated by the well-known procedures.⁵⁾ Reaction times for (α, γ) reactions, as shown in Fig. 2, have a strong Z -dependence due to the Coulomb barrier effect at temperature below 2×10^9 °K. This means that nuclei with large Z are scarcely built up at such a low temperature.

At a higher temperature, $T = (2 \sim 3) \times 10^9$ °K, the differences among (α, γ) reaction rates become so small that it seems to be possible to explain the normal abundances in terms of the α -process. Calculations given in § 3 show that, at

* The same result was obtained independently by H. Reeves and E. E. Salpeter, Bull. Am. Phys. Soc. Ser. II, 3 (1958), 227.

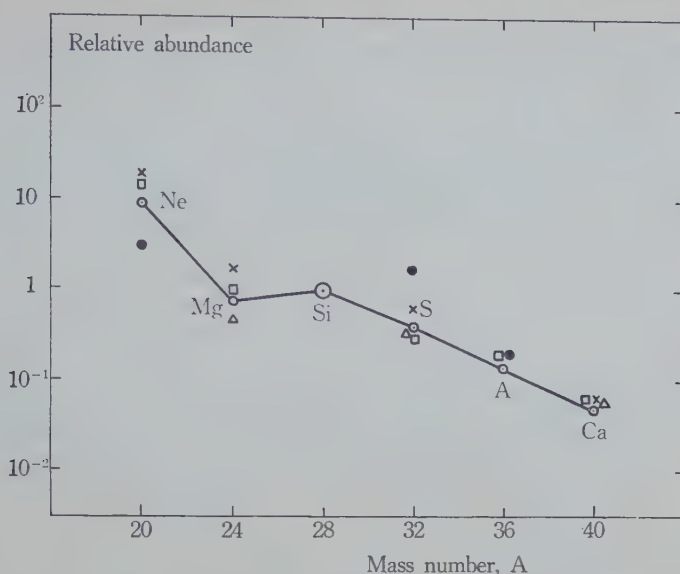


Fig. 1. Relative cosmic abundances of the α -particle nuclei.

○: the normal abundances

△: data for the sun given by Goldberg, Müller and Aller.⁷⁾

●: data for planetary nebulae ($\text{Si}^{28}/\text{O}^{16}$ is assumed to be 0.0214).

×: data for early type stars.

□: mean data for Population I stars.⁷⁾

These relative abundances distributions are normalized at Si^{28} .

density 10^5 g cm^{-3} where (α, γ) reaction rates are much greater than (γ, α) ones, the emitted α -particles are captured instantaneously by C^{12} , O^{16} or Ne^{20} , but they are hardly used for the formation of nuclei heavier than Si^{28} . It is shown, however, that the normal abundances can be reproduced by taking the density as low as $\sim 10 \text{ g cm}^{-3}$, where (α, γ) reaction rates are comparable to that of $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$. At this temperature the reaction time for $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ is $10^{-2} \sim 1$ sec, while $10^3 \sim 10^7$ sec for the other (γ, α) reactions. Since the latter gives the time scale for the establishment of the statistical equilibrium, in which the abundance of Mg^{24} is too low to explain the normal abundances, it is necessary for such a temperature to continue for a rather short time, $1 \sim 10^2$ sec, with rapid heating-up and subsequent cooling. Such a stellar condition will be realized in the envelope in the supernovae explosion, as the time scale of implosion preceding the supernovae explosion is estimated to be of the order of 10^{-1} sec⁵⁾ and such a high temperature will continue for $1 \sim 10^2$ sec.

It is shown in § 4 that, to explain the normal abundances of α -particle nuclei heavier than Mg^{24} by the equilibrium theory,⁸⁾ temperature above 5×10^9 °K and density of the order of 10^5 g cm^{-3} are required. Under this condition the nuclear reaction times are of the order of 10^{-5} sec and it seems rather unlikely that we

can find stellar phenomena that are rapid enough to assure the freezing-in of this established equilibrium.

In § 5, the results obtained in § 3 and § 4 are summarized. Also, discussions are given on another possibility that there initially exists a considerable amount of hydrogen as well as He^4 and Ne^{20} at the temperature above 1×10^9 °K. In this case, α -particle nuclei heavier than Ne^{20} can be built up through the (p, γ) and subsequent (α, p) reactions, such as $\text{Ne}^{20}(p, \gamma)\text{Na}^{21}(\alpha, p)\text{Mg}^{24}$, $\text{Mg}^{24}(p, \gamma)\text{Al}^{25}(\alpha, p)\text{Si}^{28}$, etc.

§ 2. Reaction rates

Let us assume a gas composed of two kinds of nuclei 1 and 2, each of which is uniform in space and obeys the Maxwellian distribution of velocities at given temperature T . Then the reaction rate per unit time per unit volume is expressed as

$$P_{1+2} = p_{1+2} n_1 n_2, \\ p_{1+2} = 4(2\pi m_H A)^{-1/2} (kT)^{-3/2} \int_0^\infty \sigma(E) \exp(-E/kT) E dE, \quad (2.1)$$

where $A = A_1 A_2 / (A_1 + A_2)$ and n_1, n_2 are the densities in number. Using the well-known resonance formula for $\sigma(E)$, p_{1+2} is written as

$$p_{1+2} = \frac{(2\pi)^{1/2} \hbar^2}{(m_H A kT)^{3/2} (2S_1 + 1)(2S_2 + 1)} \sum_n (2J_n + 1) \int_0^\infty \frac{\Gamma_{an} \Gamma_{bn}}{(E - E_n)^2 + \Gamma_n^2/4} e^{-E/kT} dE, \quad (2.2)$$

where S_1 and S_2 are the spins of nuclei 1 and 2, respectively, and $E_n, J_n, \Gamma_{an}, \Gamma_{bn}$ and Γ_n are the energy, the spin, the absorption width, the emission width and the total width of resonance level of compound nuclei, respectively. Γ_{an} is given by

$$\Gamma_{an} = 6\theta_{an}^2 (EE_R)^{1/2} P_l, \quad (2.3)$$

where θ_{an}^2 is the dimensionless reduced width which tends to unity for Teichmann-Wigner limit,⁹⁾ P_l is the penetration factor for orbital angular momentum l , and $E_R = \hbar^2/2m_H AR^2$. For channel radius R , we adopt $1.4 \times 10^{-13} (A_1^{1/3} + A_2^{1/3})$ cm throughout the following. In the case of the particle emission the formula (2.3) must be used also for Γ_{bn} . According to Blatt and Weisskopf¹⁰⁾, θ_{an}^2 is given approximately as

$$\theta_{an}^2 = (\lambda_0/3\pi R) (D/E_R), \quad (2.4)$$

where $\lambda_0 \sim 10^{-13}$ cm is the characteristic wave length of nucleons inside the nucleus and D is the mean distance of resonance levels with the same spin and parity.

The factor $\Gamma_{an} \exp(-E/kT)$ in the integrand of (2.2) has the Gamow peak,

which can be approximated by the error function centered at $E_0 = (\pi e^2 Z_1 Z_2 \sqrt{2m_H A} kT/2)^{1/3}$ with width $\Delta E_0 = (16E_0 kT/3)^{1/2}$. Regarding Γ_m as independent of energy of colliding nuclei, (2.2) is reduced to³⁾

$$p_{1+2} = \frac{(2\pi)^{1/2} \hbar^2}{(m_H A kT)^{3/2} (2S_1+1)(2S_2+1)} \left[\sum_n (2J_n+1) \sqrt{\pi} \frac{\Gamma_a(E_0) \Gamma_m}{(E_0 - E_n)^2} \frac{\Delta E_0}{2} e^{-E_0/kT} \right. \\ \left. + \sum_n (2J_n+1) \frac{2\pi \Gamma_a(E_n) \Gamma_m}{\Gamma_n(E_n)} e^{-E_n/kT} \right]. \quad (2.5)$$

The first summation in the square bracket refers to the nonresonance absorption and the second one refers to the resonance absorption.

If the Gamow peak is so broad that many resonance levels lie within it, we may take only the resonance terms into account. Then, taking the average value of Γ_a , Γ_m and Γ_n at energy E , which are denoted as $\bar{\Gamma}_a(E)$, $\bar{\Gamma}_b(E)$ and $\bar{\Gamma}(E)$ respectively, and substituting $\int_0^\infty \bar{D}^{-1} dE$ for \sum_n , (2.5) is rewritten as

$$p_{1+2} = \frac{(2\pi)^{3/2} \hbar^2}{(m_H A kT)^{3/2} (2S_1+1)(2S_2+1)} \sum_{J,\pi} (2J+1) \int_0^\infty \frac{\bar{\Gamma}_a(E) \bar{\Gamma}_b(E)}{\bar{\Gamma}(E)} e^{-E/kT} \frac{dE}{\bar{D}}, \quad (2.6)$$

in which \bar{D} is the average level distance.

In the following, reaction rates of helium capture and production at temperatures above 5×10^8 °K will be evaluated by using the above formulas.

(A) Helium capturing reaction rates.

i) $\text{C}^{12} + \alpha \rightarrow \text{O}^{16} + \gamma + 7.15 \text{ MeV}$.

Below 1.5×10^9 °K the Gamow peak lies between 7.12 MeV (1^-) level and 9.58 MeV (1^-) level and so the reaction rate will be determined by the nonresonance absorption of these levels. As for the contribution from 7.12 MeV level, calculations have been made by Nakagawa *et al.*,²⁾ Hayakawa *et al.*,³⁾ and Salpeter,¹¹⁾ whose results coincide with each other within the limit of uncertainties, while the contribution from 9.58 MeV level ($\Gamma_\gamma = 6 \times 10^{-3} \text{ eV}$,⁴⁾ $\theta_\alpha^2 = 0.85^{12)}$ is estimated to be at most of the same magnitude as that from 7.12 MeV level. So we may take only the contribution from 7.12 eV level into account at $T < 1.5 \times 10^9$ °K.

Above 1.5×10^9 °K the resonance absorption through 9.58 MeV level and higher levels contributes to the reaction rate mainly, and of these levels the most important ones are the 9.58 MeV level and the 9.84 MeV (2^+) level. For the 9.58 MeV level $\Gamma_\alpha(E_n) = 860 \text{ keV}^{12)}$ and $\Gamma_\gamma = 6 \times 10^{-3} \text{ eV}$, and for the 9.84 MeV level $\Gamma_\alpha(E_n) = 1 \text{ keV}^{12)}$ but Γ_γ is experimentally unknown. Assuming the $E2$ transition from the 9.84 MeV level to the ground level, we get $\Gamma_\gamma = 0.54 \text{ eV}$ theoretically. Since we may put $\Gamma_\alpha \Gamma_\gamma / \Gamma = \Gamma_\gamma$ with sufficient accuracy for both levels, it is obvious that the contribution from the 9.84 MeV level is much larger than that from the 9.58 MeV level because of the larger value of Γ_γ . Contributions from levels higher than 9.84 MeV will not exceed those from the 9.84 MeV level due to the effect of

the exponential factor in (2.5). Thus, taking only the resonance absorption through the 9.84 MeV level into account, we have ($T_0 \equiv T/10^9 \text{ }^\circ\text{K}$)

$$p_{\text{C}+\alpha} = 1.1 \times 10^{-9} T_0^{-3/2} \exp(-30.74 T_0^{-1}) \text{cm}^3 \text{sec}^{-1} \quad (T_0 \geq 1.5). \quad (2.7)$$

Since the error in the estimate of Γ_γ may probably be limited within a factor of 10, the above result is accurate only within the same uncertainty.

ii) $\text{O}^{16} + \alpha \rightarrow \text{Ne}^{20} + \gamma + 4.75 \text{ MeV}$

Above $5 \times 10^8 \text{ }^\circ\text{K}$, the resonance absorption through the 4.95 MeV level is insignificant, due to the very small Γ_α of the level, in comparison with those through higher levels. Then, we shall first evaluate resonance absorption through 5.62 MeV level. Because neither spin nor parity of the level is known experimentally, we estimate Γ_α and Γ_γ theoretically, assuming that spin and parity are 0^+ , 1^- , 2^+ , etc. From (2.4) with $D \sim 1 \text{ MeV}$, we get $\theta_\alpha^2 \sim 0.1$, while from the elastic scattering experiment of α -particle at higher energies¹²⁾ we found $\theta_\alpha^2 = 1 \sim 0.01$; accordingly we adopt here $\theta_\alpha^2 = 0.1$ within uncertainty of factor of 10. Then, we get $\Gamma_\alpha(E_n) = 0.16 \text{ eV}$ for $l=2$, and $\Gamma_\alpha(E_n) = 2.3 \text{ eV}$ for $l=0$. Since Γ_γ is estimated to lie between 1 and 10^{-3} eV , we can put $(2l+1)\Gamma_\alpha\Gamma_\gamma/\Gamma = 0.1 \text{ eV}^*$ within the uncertainty of factor of 30 and finally we get

$$p_{\text{O}+\alpha} = 4.5 \times 10^{-21} T_0^{-3/2} \exp(-10.09 T_0^{-1}) \text{cm}^3 \text{sec}^{-1}. \quad (2.8)$$

If the resonance absorption through the 5.62 MeV level is forbidden, we must take the resonance absorption through 6.74 MeV (0^+) level into account. As for the 6.74 MeV level $\Gamma_\alpha(E_n) = 24 \text{ keV}^{12)}$ but Γ_α is unknown. A theoretical estimate of Γ_γ , assuming the E2 transition to the second excited level, gives $\Gamma_\gamma = 2 \times 10^{-2} \text{ eV}$ and then putting $\Gamma_\alpha\Gamma_\gamma/\Gamma = 2 \times 10^{-2} \text{ eV}$, we get

$$p_{\text{O}+\alpha} = 9.0 \times 10^{-22} T_0^{-3/2} \exp(-23.09 T_0^{-1}) \text{cm}^3 \text{sec}^{-1}. \quad (2.9)$$

This equation gives a value about 1/100 times smaller than (2.8). In the following calculations we shall adopt the value given by (2.8), but it must be kept in mind that there is no assurance for the validity of adopting (2.8) and true reaction rate may be given by (2.9).

iii) $\text{Ne}^{20} + \alpha \rightarrow \text{Mg}^{24} + \gamma + 9.31 \text{ MeV}$

$$\text{Mg}^{24} + \alpha \rightarrow \text{Si}^{28} + \gamma + 10.00 \text{ MeV}$$

$$\text{Si}^{28} + \alpha \rightarrow \text{S}^{32} + \gamma + 6.94 \text{ MeV}$$

$$\text{S}^{32} + \alpha \rightarrow \text{A}^{36} + \gamma + 6.66 \text{ MeV}$$

$$\text{A}^{36} + \alpha \rightarrow \text{Ca}^{40} + \gamma + 7.04 \text{ MeV}$$

$$\text{Ca}^{40} + \alpha \rightarrow \text{Ti}^{44} + \gamma + 5.28 \text{ MeV}$$

Details of the level schemes of Mg^{24} , Si^{28} , etc., has been scarcely known but it is expected that widths of the Gamow peak are much larger than level distances at temperature above $10^8 \text{ }^\circ\text{K}$. Therefore, we evaluate reaction rates with (2.6).

* Salpeter¹¹⁾ put $(2l+1)\Gamma_\alpha\Gamma_\gamma/\Gamma = 0.02 \text{ eV}^{11)}$, so his result is 1/5 times smaller than ours.

The quantity $\bar{\Gamma}_\alpha \bar{\Gamma}_\tau / \bar{\Gamma}$ in the integrand of (2.6) can be reduced to $\bar{\Gamma}_\alpha$ or $\bar{\Gamma}_\tau$ according to $\bar{\Gamma}_\alpha < \bar{\Gamma}_\tau$ or to $\bar{\Gamma}_\tau < \bar{\Gamma}_\alpha$, respectively. If $\bar{\Gamma}_\tau$ is equal to $\bar{\Gamma}_\alpha$ at the energy E' , we can separate the integral into two parts as follows,

$$\int_0^\infty \frac{\bar{\Gamma}_\alpha \bar{\Gamma}_\tau}{\bar{\Gamma} D} e^{-E/kT} dE \cong \int_0^{E'} \frac{\bar{\Gamma}_\alpha}{D} e^{-E/kT} dE + \int_{E'}^\infty \frac{\bar{\Gamma}_\tau}{D} e^{-E/kT} dE. \quad (2.10)$$

From (2.4) with $D \sim 0.4$ MeV, we get $\theta_\alpha^2 \sim 0.05$, while θ_α^2 is expected to lie between 1 and 0.001 from the extrapolation of the experimental data for lighter nuclei. So we fix for all nuclei $\theta_\alpha^2 = 0.05$, within uncertainty of factor 30. Theoretically, $\bar{\Gamma}_\tau$ is expected to lie between several eV and 10^{-2} eV, so we adopt for all nuclei $\bar{\Gamma}_\tau = 0.3$ eV within uncertainty of factor 30. Values of E' for $J=0$ calculated in this way are given in Table 2.

Table 2

Reaction	$\text{Ne}^{20} + \alpha$	$\text{Mg}^{24} + \alpha$	$\text{Si}^{28} + \alpha$	$\text{S}^{32} + \alpha$	$\text{A}^{36} + \alpha$	$\text{Ca}^{40} + \alpha$
$E'(\text{MeV})$	1.08	1.37	1.66	1.96	2.25	2.53

Since $\bar{\Gamma}_\alpha$ is a very rapidly increasing function of E , the value E' is determined very sharply, *i.e.*, even if $\bar{\Gamma}_\tau$ becomes larger or smaller by a factor of 10, its effect to E' is less than 10 per cent. Therefore, for levels of $J=1$ and 2, E' 's do not vary greatly from the values given in Table 2, and contributions from $J=1$ and $J=2$ levels in (2.6) are as large as that from $J=0$ levels. So, for the reaction rate, we adopt approximately the value of four times the contribution from $J=0$ levels. Uncertainties in the results are estimated to be as large as a factor of 100. The results of the calculations are illustrated in Fig. 2.

(B) Helium producing reaction rates

- i) $\text{C}^{12} + \text{C}^{12} \longrightarrow \text{Ne}^{20} + \alpha + 4.62 \text{ MeV}$
 $\text{C}^{12} + \text{O}^{16} \longrightarrow \text{Mg}^{24} + \alpha + 6.80 \text{ MeV}$
 $\text{O}^{16} + \text{O}^{16} \longrightarrow \text{Si}^{28} + \alpha + 9.7 \text{ MeV}$

Above 5×10^8 °K, the width of the Gamow peak is expected to be much larger than level distance, and the reaction rate, in these cases, can be evaluated also from (2.6). Though $\bar{\Gamma}_\alpha$ must be much larger than $\bar{\Gamma}_{\text{C}+\text{C}}$, $\bar{\Gamma}_{\text{C}+\text{O}}$ and $\bar{\Gamma}_{\text{O}+\text{O}}$, it is not accurate to put $\bar{\Gamma}_\alpha \bar{\Gamma}_{\text{C}+\text{C}} / \bar{\Gamma} = \bar{\Gamma}_{\text{C}+\text{C}}$, etc., because proton emission would occur as well as α -particle emission. In the following, however, we put $\bar{\Gamma}_\alpha / \bar{\Gamma}$ equal to unity.

From (2.4) we get $\theta^2 / \bar{D} \sim 0.2 \text{ MeV}^{-1}$. Though we have no experimental information about it, the above value may be accurate within uncertainty of a factor 100. Taking only levels with $J=0$ into account, reaction rates are expressed as

$$p_{\text{C}+\text{C}} = 2.7 \times 10^3 T_9^{-2/3} \exp(-84.25 T_9^{-1/3}) \text{ cm}^3 \text{ sec}^{-1}, \quad (2.11)$$

$$p_{\text{C}+\text{O}} = 4.4 \times 10^7 T_9^{-2/3} \exp(-106.6 T_9^{-1/3}) \text{ cm}^3 \text{ sec}^{-1}, \quad (2.12)$$

$$p_{\text{O}+\text{O}} = 1.8 \times 10^{13} T_9^{-2/3} \exp(-136.0 T_9^{-1/3}) \text{ cm}^3 \text{ sec}^{-1}, \quad (2.13)$$

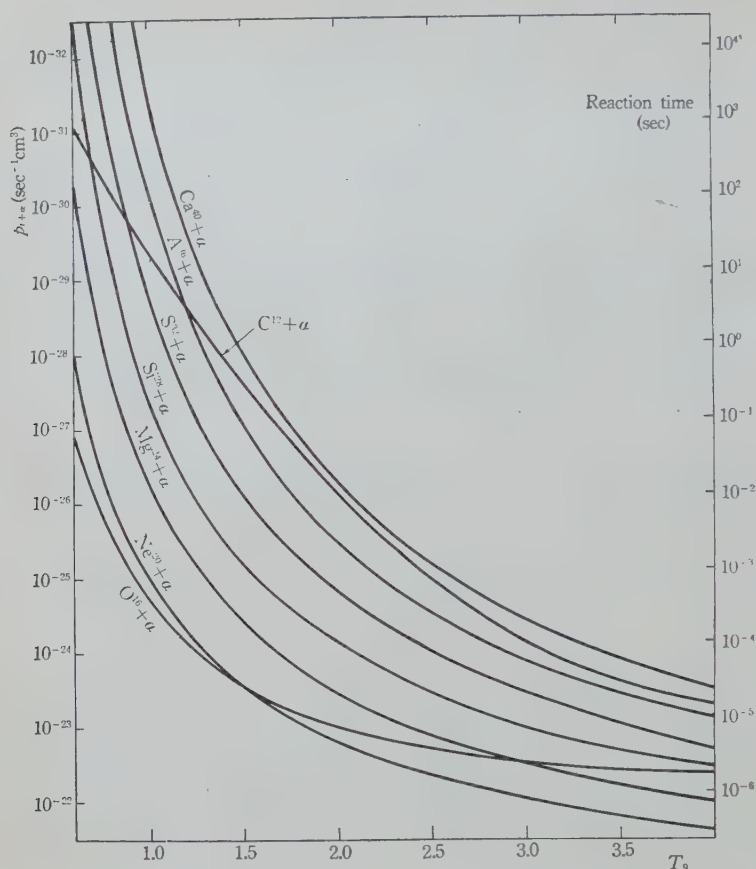


Fig. 2. α -capture reaction rate $p_{1+\alpha}$. Reaction times are evaluated for $\rho_\alpha = 10^5 \text{ g cm}^{-3}$.

ii) The rate of (γ, α) reactions.

Suppose a photo-reaction $A + \gamma \rightarrow B + C$ occurs at temperature T , where energy of photons obeys the Planck distribution, then the rate of reaction λ_A per one nucleus A per unit time is given in terms of the rate p_{B+C} of the reverse reaction as

$$\lambda_A = p_{B+C} \cdot \frac{g_A}{g_B g_C} \left(\frac{A m_H k T}{2\pi \hbar^2} \right)^{3/2} e^{-Q/kT}, \quad (2.14)$$

where g_A , g_B and g_C are the statistical factors given by

$$g = \sum_n (2J_n + 1) e^{-E_n/kT}, \quad (2.15)$$

in which E_n is the energy of the excited state measured from the ground level. Values of λ for α -particle emission, calculated with $p_{1+\alpha}$ given in subsection (A), are illustrated in Fig. 3.

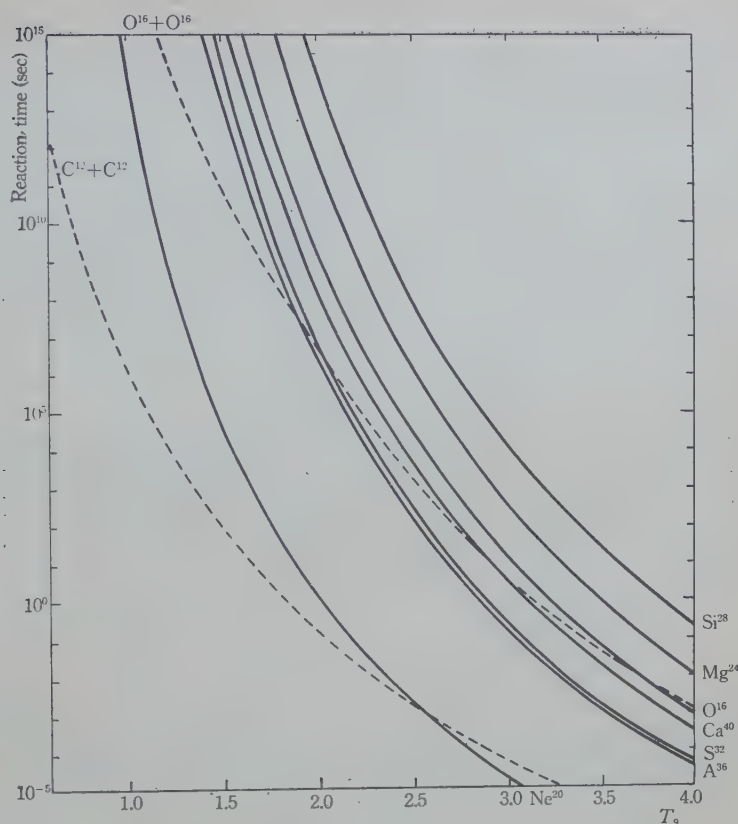


Fig. 3. The reaction times of heavy ion reactions and (γ, α) reactions. Solid curves denote (γ, α) reactions and dotted curves denote heavy ion reactions at density of 10^5 g cm^{-3} .

§ 3. Building-up process of α -particle nuclei

(A) Sources of α -particles

It is seen in Fig. 3 that the $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ reaction rate is exceedingly larger than the other (γ, α) reaction rates because of the small α -particle binding energy of Ne^{20} . For relatively high density, however, heavy ion reactions such as $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$, etc., whose reaction times are inversely proportional to density, may become faster than $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$. Therefore we shall examine which reaction is faster for the production of α -particle under various conditions.

i) Competition between $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ and $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$

Corresponding to $\lambda_{\text{Ne}} n_{\text{Ne}} \cong p_{\text{C}+\text{C}} n_{\text{C}}^2$, we have

$$\log \left(\frac{\rho x_{\text{C}}^2}{x_{\text{Ne}}} \right) \cong -11.16 + \frac{2}{3} \log T_9 - \frac{28.30}{T_9} + \frac{36.56}{T_9^{1/3}}, \quad (3.1)^*$$

* \log means \log_{10} , which is distinguished from natural logarithm, \ln .

where ρ is density in g cm^{-3} and x 's are the concentrations by weight. This relation is illustrated in Fig. 4.

Let the concentration of C^{12} and Ne^{20} be about equal and those of the other element be negligible, then under the condition of relatively high density and low temperature such as $\rho x_{\text{C}} \gtrsim 10^5$ and $T_9 \lesssim 1.5$, we can see from Fig. 4 that $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ cannot become the main source of α -particles before C^{12} is almost exhausted by $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$, while for relatively low density and high temperature such as $\rho \lesssim 10^3$ and $T_9 \gtrsim 2$, $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ occurs more rapidly than $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$.

ii) Competition between $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ and $\text{O}^{16}(\text{C}^{12}, \alpha)\text{Mg}^{24}$

Corresponding to $p_{\text{C}+\text{C}} n_{\text{C}}^2 \gtrless p_{\text{C}+\text{O}} n_{\text{C}} n_{\text{O}}$, we have

$$\log(x_{\text{C}}/x_{\text{O}}) \gtrless 5.09 - 9.75 T_9^{-1/3}. \quad (3.2)$$

This equation shows that for $T_9 < 2.5$, $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ is always faster than $\text{O}^{16}(\text{C}^{12}, \alpha)\text{Mg}^{24}$ until the ratio $x_{\text{C}}/x_{\text{O}}$ becomes smaller than $1/100$. Therefore we can conclude that $\text{O}^{16}(\text{C}^{12}, \alpha)\text{Mg}^{24}$ is of little importance for the production of α -particles.

iii) Competition between $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ and $\text{O}^{16}(\text{O}^{16}, \alpha)\text{Si}^{28}$

The inequality $\lambda_{\text{Ne}} n_{\text{Ne}} \gtrless p_{\text{O}+\text{O}} n_{\text{O}}^2$ corresponds to

$$\log\left(\frac{\rho x_{\text{O}}^2}{x_{\text{Ne}}}\right) \gtrless 20.44 + \frac{2}{3} \log T_9 - \frac{28.30}{T_9} + \frac{59.03}{T_9^{1/3}}. \quad (3.3)$$

This relation is illustrated graphically in Fig. 4, which shows that, if $x_{\text{O}} \sim x_{\text{Ne}}$, $\text{O}^{16}(\text{O}^{16}, \alpha)\text{Si}^{28}$ becomes important only in the case of extremely high density $\rho \gtrsim 10^{10}$ at $T_9 \lesssim 1$. At such a low temperature, however, the reaction time is too long to supply sufficient amount of α -particles necessary for the building-up of the heavier α -particle nuclei. Thus, we can conclude that $\text{O}^{16}(\text{O}^{16}, \alpha)\text{Si}^{28}$ reaction is of no importance for the α -particle source if C^{12} and Ne^{20} are contained even in small fractions.

From the above considerations and also from the results on the reaction rates in § 2, we can draw out the following conclusions. Let the central region of a star, whose density is about 10^5 g cm^{-3} , be made up of mixture of C^{12} , O^{16} and Ne^{20} , and its temperature rises slowly to start helium producing reactions, then the first reaction to occur is $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$, and after C^{12} nuclei are almost exhausted by the reaction and the temperature rises higher, $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ occurs next and $\text{O}^{16}(\text{O}^{16}, \alpha)\text{Si}^{28}$ starts after the complete exhaustion of Ne^{20} . Contrary to the above case of slow heating, if the temperature rises very rapidly to $2 \times 10^9 \text{ K}$ followed by the very rapid cooling within $1 \sim 10$ sec in the region of relatively low density 10^2 g cm^{-3} , —the situation of stellar envelope at the stage of supernova explosion, for instance— the only helium producing reaction to occur is $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$.

(B) α -process as stellar energy sources

Next we shall examine the condition of the central region of a star with luminosity of $100 \sim 10000$ times solar ones, which is supplied by the helium producing reaction such as $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$, $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ or $\text{O}^{16}(\text{O}^{16}, \alpha)\text{Si}^{28}$ followed by the helium capture reactions occurring in the central region. Since, from the consider-

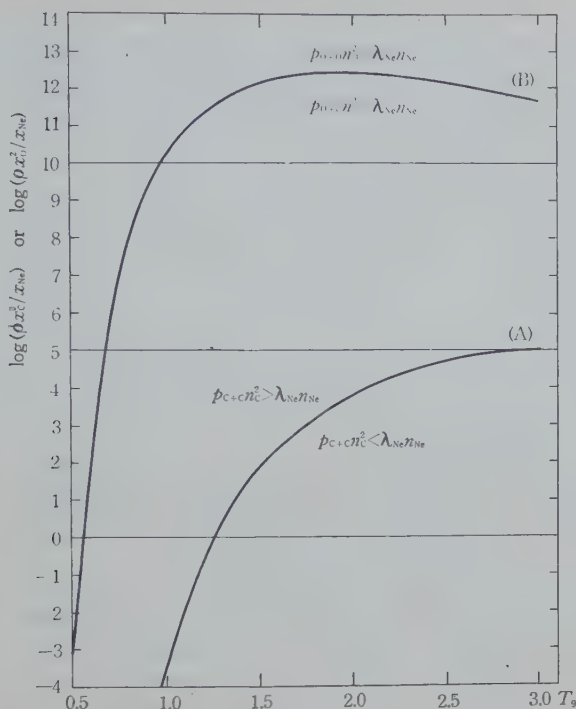


Fig. 4. The curve (A) illustrates the relation $\lambda_{\text{Ne}} n_{\text{Ne}} = p_{\text{C}+\alpha} n_{\text{C}}^2$, and the curve (B) illustrates the relation $\lambda_{\text{Ne}} n_{\text{Ne}} = p_{\text{O}+\alpha} n_{\text{O}}^2$.

ation so far made, the first reaction to occur is found to be $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$, we shall first deal with the case in which the stellar energy is generated by this reactions.

As the reaction rate has steep temperature dependence, the reaction can occur only in a small central region. Let the rate of energy generation per unit mass per unit time near the center be expressed approximately as

$$\varepsilon = \varepsilon_0 x_{\text{C}}^2 \rho T^s \quad (3.4)$$

and also let us assume for simplicity that the structure of central region can be described by the polytrope of index n , and that in this region the radiation pressure is negligible compared with the gas pressure, and electron degeneracy is also negligible. Then, the total energy release is given by³⁾

$$L = \int \varepsilon \rho \cdot 4\pi r^2 dr \cong \varepsilon_0 x_{\text{C}}^2 \left(\frac{3n+3}{2n+s} \right)^{3/2} \left(\frac{k}{2G\mu m_{\text{H}}} \right)^{3/2} \rho_c^{1/2} T_c^{s+3/2}, \quad (3.5)$$

where G is the gravitation constant, μ the mean molecular weight of gas, T_c and ρ_c the central temperature and density respectively. If T_c is about 6×10^8 K and each $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ reaction followed by the α -capture reaction generates energy of 13 MeV, we get, with the aid of (2.11),

$$\varepsilon = 1.0 \times 10^7 \rho x_c^2 T_9^{33.3} \text{ erg sec}^{-1} \text{ g}^{-1}. \quad (3.4')$$

Considering that L is rather insensitive to the value of n so long as $s \gg n$, we may take $n=1.5$, which corresponds to the polytropic index for the convective equilibrium, and consequently we obtain

$$L/L_\odot = 3.2 \times 10^3 x_c^2 \rho_c^{1/2} (T_c/10^9)^{34.8}. \quad (3.5')$$

Let $x_c=1$, $\rho_c=10^5 \text{ g cm}^{-3}$ and $L/L_\odot=1000$, then we have $T_c=6 \times 10^8 \text{ }^\circ\text{K}$. For a star with $10^2 \sim 10^4 L_\odot$ we have $T_c=5.5 \sim 6.5 \times 10^8 \text{ }^\circ\text{K}$. The value of T_c is rather insensitive to the variation of L , x_c and ρ_c .

Next we shall consider the case in which C^{12} has been exhausted and $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ is a main source of α -particles. In the same way as above, the energy generation and the luminosity are given by

$$\varepsilon = \varepsilon_0 x_{\text{Ne}} T^s, \quad (3.6)$$

and

$$L = \varepsilon_0 x_{\text{Ne}} \left(\frac{3n+3}{n+s} \right)^{3/2} \left(\frac{k}{2G\mu m_H} \right)^{3/2} \rho^{-1/2} T_c^{s+3/2}. \quad (3.7)$$

Let the central temperature T_c be about $1 \times 10^9 \text{ }^\circ\text{K}$ and let each $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ reaction followed by the α -capture reaction generate energy of about 3.3 MeV, then we have

$$\varepsilon = 6.0 \times 10^3 x_{\text{Ne}} T_9^{65.2} \text{ erg sec}^{-1} \text{ g}^{-1}, \quad (3.6')$$

and

$$L/L_\odot = 1.0 \times 10^4 x_{\text{Ne}} \rho_c^{-1/2} (T_c/10^9)^{66.7}. \quad (3.7')$$

Putting $x_{\text{Ne}}=1$ and $\rho_c=10^5 \text{ g cm}^{-3}$, we have $T_c \simeq 1.0 \times 10^9 \text{ }^\circ\text{K}$ for stars with $10^2 \sim 10^4 L_\odot$. It should be noted that, on account of very steep temperature dependence of the reaction rate, T_c is determined within a rather narrow range even if L is varied in a wide range.

Finally, we shall deal with the case in which both C^{12} and Ne^{20} have been exhausted in the stellar core so that α -particles are produced mainly by $\text{O}^{16}(\text{O}^{16}, \alpha)\text{Si}^{28}$. In this case, we have

$$\varepsilon = 5.3 \times 10^{-8} \rho x_o^2 T_9^{41.5} \text{ erg sec}^{-1} \text{ g}^{-1}, \quad (3.8)$$

and

$$L/L_\odot = 4.6 \times 10^{-4} \rho_c^{1/2} x_o^2 (T_c/10^9)^{43.0}, \quad (3.9)$$

in which the central temperature is assumed to be about $1.25 \times 10^9 \text{ }^\circ\text{K}$ and each $\text{O}^{16}(\text{O}^{16}, \alpha)\text{Si}^{28}$ reaction followed by the α -capture reaction is assumed to generate energy of 13 MeV. Let $x_o=1$ and $\rho_c=10^5 \text{ g cm}^{-3}$, then we have $T_c=1.25 \times 10^9 \text{ }^\circ\text{K}$ for $L/L_\odot=10^3$ and $T_c=1.2 \sim 1.3 \times 10^9 \text{ }^\circ\text{K}$ for $L/L_\odot=10^3 \sim 10^4$.

The mean reaction time in the above three cases is about 10^5 years in common if we assume that $L \sim 1000 L_\odot$ and $\rho_c \sim 10^5 \text{ g cm}^{-3}$.

(C) *Building-up process of α -particle nuclei*

In this subsection we shall investigate how the α -particle nuclei heavier than Ne^{20} are built up by the successive helium capture reactions starting from the end products of helium burning reactions, namely C^{12} , O^{16} and Ne^{20} . Since $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ and $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ have been found to be the most probable reactions for the source of α -particles, we shall first treat the case in which $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ is the main source of α -particles and secondly the case in which $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ is the main source of α -particles. Finally, we shall treat the case in which α -particles are mixed, by some way or other, with the products of helium burning reaction.

 i) *The case in which α -particles are supplied by $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ reaction*

In this case calculations are made under a very simplified condition that there occurs only $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ and following α -capture reactions, though there remain considerable possibilities that $\text{C}^{12}(\text{C}^{12}, p)\text{Na}^{23}$ and the subsequent proton capture reactions may occur to synthesize various nuclei other than α -particle nuclei. Then, the time variation of abundances of each α -particle nuclei is expressed by a set of equations

$$\begin{aligned} \frac{dn_\alpha}{dt} &= p_{\text{C+C}} n_{\text{C}}^2 - (p_{\text{C}+\alpha} n_{\text{C}} + p_{\text{O}+\alpha} n_{\text{O}} + p_{\text{Ne}+\alpha} n_{\text{Ne}} + p_{\text{Mg}+\alpha} n_{\text{Mg}} + \dots) n_\alpha, \\ \frac{dn_{\text{C}}}{dt} &= -2p_{\text{C+C}} n_{\text{C}}^2 - p_{\text{C}+\alpha} n_{\text{C}} n_\alpha, \\ \frac{dn_{\text{O}}}{dt} &= (p_{\text{C}+\alpha} n_{\text{C}} - p_{\text{O}+\alpha} n_{\text{O}}) n_\alpha, \\ \frac{dn_{\text{Ne}}}{dt} &= p_{\text{C+C}} n_{\text{C}}^2 + (p_{\text{O}+\alpha} n_{\text{O}} - p_{\text{Ne}+\alpha} n_{\text{Ne}}) n_\alpha \\ \frac{dn_{\text{Mg}}}{dt} &= (p_{\text{Ne}+\alpha} n_{\text{Ne}} - p_{\text{Mg}+\alpha} n_{\text{Mg}}) n_\alpha, \\ \frac{dn_{\text{Si}}}{dt} &= (p_{\text{Mg}+\alpha} n_{\text{Mg}} - p_{\text{Si}+\alpha} n_{\text{Si}}) n_\alpha, \text{ etc.} \end{aligned} \tag{3.10}$$

Since the above equations cannot be solved analytically, we calculated them with a method of numerical integration and found out that it was impossible to reproduce the normal abundance ratio, whatever initial chemical composition and temperature we might choose. More precisely, even if we assume $p_{\text{C}+\alpha} = p_{\text{O}+\alpha} = p_{\text{Ne}+\alpha} = p_{\text{Mg}+\alpha} = \dots$, which is the most favourable case to get large abundance of heavier elements, the main products of this process are O^{16} , Ne^{20} and Mg^{24} and the relative abundances of elements heavier than Mg^{24} remain far smaller than the normal abundance even at the stage of complete exhaustion of C^{12} nuclei. If we adopt the value of α -capture reaction rate given in § 2, the deviation from the normal abundance would become much larger. When $\text{C}^{12}(\text{C}^{12}, p)\text{Na}^{23}$ occurs simultaneously, the synthesizing of α -particle nuclei would become very complicated, but this would not improve the result.

 ii) *The case in which α -particles are supplied by $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ reaction*

At relatively high density, the helium producing reaction occurring next to $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ is $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$. When density is relatively low, and temperature is relatively high, $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ is likely to occur more rapidly than $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ as has been shown in subsection (A).

According to Hayakawa *et al.*³⁾ and Nakagawa *et al.*²⁾, the abundance of Ne^{20} may become much larger than those of C^{12} and O^{16} in helium burning reaction under the condition corresponding to the center of the giant stars. So we deal here with the very simplified case in which there initially exists only Ne^{20} , and $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ is the only reaction to supply α -particles. Small contaminations of C^{12} and O^{16} will not greatly change the result.

The time variations of abundances of each α -particle nuclei are expressed as

$$\begin{aligned} dn_\alpha/dt &= \lambda_{\text{Ne}} n_{\text{Ne}} - (p_{\text{O}+\alpha} n_{\text{O}} + p_{\text{Ne}+\alpha} n_{\text{Ne}} + p_{\text{Mg}+\alpha} n_{\text{Mg}} + \cdots) n_\alpha, \\ dn_{\text{O}}/dt &= \lambda_{\text{Ne}} n_{\text{Ne}} - p_{\text{O}+\alpha} n_{\text{O}} n_\alpha, \\ dn_{\text{Ne}}/dt &= -\lambda_{\text{Ne}} n_{\text{Ne}} + (p_{\text{O}+\alpha} n_{\text{O}} - p_{\text{Ne}+\alpha} n_{\text{Ne}}) n_\alpha, \\ dn_{\text{Mg}}/dt &= (p_{\text{Ne}+\alpha} n_{\text{Ne}} - p_{\text{Mg}+\alpha} n_{\text{Mg}}) n_\alpha, \\ dn_{\text{Si}}/dt &= (p_{\text{Mg}+\alpha} n_{\text{Mg}} - p_{\text{Si}+\alpha} n_{\text{Si}}) n_\alpha, \text{ etc.} \end{aligned} \quad (3.11)$$

The solutions of the above equations depend not only on temperature but also on density, contrary to the above case i), where the solutions do not depend on density if we suitably change the scale of time. To show the effect of temperature and density more explicitly, let us rewrite (3.11) in terms of the following dimensionless variables :

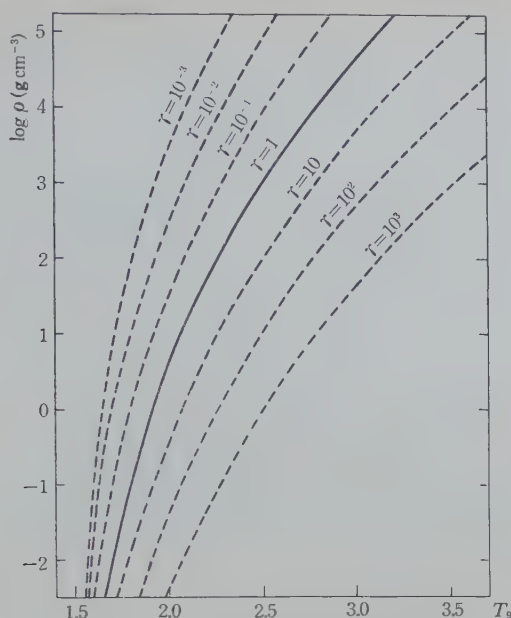
$$\begin{aligned} n_\alpha/N &= y_\alpha, \quad n_{\text{O}}/N = y_0, \quad n_{\text{Ne}}/N = y_1, \quad n_{\text{Mg}}/N = y_2 \text{ etc.} \\ p_{\text{O}+\alpha} n_\alpha dt &= dx, \\ p_{\text{Ne}+\alpha}/p_{\text{O}+\alpha} &= p_1, \quad p_{\text{Mg}+\alpha}/p_{\text{O}+\alpha} = p_2, \quad p_{\text{Si}+\alpha}/p_{\text{O}+\alpha} = p_3 \text{ etc.} \\ \lambda_{\text{Ne}}/p_{\text{O}+\alpha} N &= \gamma \end{aligned} \quad (3.12)$$

where $N = n_{\text{O}} + n_{\text{Ne}} + n_{\text{Mg}} + \cdots$. Then (3.11) is written

$$\begin{aligned} dy_\alpha/dx &= \gamma y_1/y_\alpha - (y_0 + p_1 y_1 + p_2 y_2 + \cdots), \\ dy_0/dx &= \gamma y_1/y_\alpha - y_0, \\ dy_1/dx &= -\gamma y_1/y_\alpha + y_0 - p_1 y_1, \\ dy_2/dx &= p_1 y_1 - p_2 y_2, \\ dy_3/dx &= p_2 y_2 - p_3 y_3, \text{ etc.} \end{aligned} \quad (3.13)$$

with initial condition that $y_1=1$, $y=y_0=y_2=y_3=\cdots=0$ at $x=0$. In the above equations $p_1, p_2, p_3 \cdots$ are functions of temperature only, while γ is a function of both temperature and density as illustrated in Fig. 5. We shall consider how the building up-process depends on the value of γ .

Assuming that $p_{\text{Ne}+\alpha}$ is not so greatly different from $p_{\text{O}+\alpha}$, the value of γ indicates the ratio of the rate of α -particle emission to that of α -particle absorption. If γ is large compared to unity, α -particles emitted from Ne^{20} survive for long time until they are absorbed again and, therefore, the larger the value of γ is, the larger be-


 Fig. 5. The values of $\gamma = \lambda_{\text{Ne}} / p_{\text{O} + \alpha} N$.

comes the concentration of α -particle compared with that of Ne^{20} at the early stage of the reaction, and the ratio $n_{\alpha} / n_{\text{Ne}^{20}}$ approaches the value in the thermal equilibrium of reactions $\text{Ne}^{20} + \gamma \rightleftharpoons \text{O}^{16} + \alpha$. This means that the number of Ne^{20} nuclei used as the source of α -particles relative to that which is converted into Mg^{24} by capturing α -particles increases as γ becomes larger. On the contrary, if γ is small compared with unity, α -particles emitted are instantaneously absorbed by Ne^{20} to form Mg^{24} at the early stage of the reaction and so the considerable amount of Ne^{20} is consumed without being used as the source of α -particles.

Now, we shall show the solution of (3.13). In the idealized case that p_1, p_2, p_3, \dots are put equal to unity and γ is either much larger or much smaller than unity, analytical solution can be obtained approximately. The detail of the calculation is given in the Appendix. Here we show only the final results.

For $\gamma \ll 1$, we have

$$\begin{aligned}
 y_{\alpha} &= \gamma(1-x)e^{-x}, \\
 y_0 &= 1 - e^{-x}, \\
 y_1 &= (1-x)e^{-x}, \\
 y_2 &= \left(x - \frac{x^2}{2}\right)e^{-x}, \\
 &\vdots \\
 y_n &= \left[\frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!}\right]e^{-x},
 \end{aligned} \tag{3.14}$$

and

$$\lambda_{Ne} t = \int_0^x \frac{e^x}{1-x} dx.$$

It should be noted that the above solution does not depend on the value of γ , except y_α . As x approaches unity, t becomes infinity and both y_α and y_1 vanish, namely, the reaction terminates at $x=1$. The value of y_n at $x=1$ is expressed as

$$y_n = (n-1)/n! e \quad (3.15)$$

and the abundance ratio of the neighbouring nuclei is expressed as

$$y_n/y_{n+1} = n-1/n \quad (3.16)$$

which reproduces the normal abundance ratio very well. But it is doubtful, though not impossible, that such a favourable condition adopted here, $p_1=p_2=p_3=\dots$, can be realized, in spite of the considerable uncertainty in the evaluation of the reaction rates. The time variation of abundances is illustrated in Fig. 6a.

For $\gamma \gg 1$, we have

$$\begin{aligned} y_\alpha &= e^{-x/\tau} \cos(x/\sqrt{\gamma}), \\ y_0 &= 1 - (1/\sqrt{\gamma}) e^{-x/\tau} \sin(x/\sqrt{\gamma}), \\ y_1 &= (1/\gamma) e^{-x/\tau} \cos(x/\sqrt{\gamma}), \\ &\vdots \\ y_n &= (1/\gamma) e^{-x/\tau} \left(\cos \frac{x}{\sqrt{\gamma}} + \frac{n-1}{\sqrt{\gamma}} \sin \frac{x}{\sqrt{\gamma}} - e^{-x} \sum_{k=0}^{n-2} \frac{x^k}{k!} \right), \end{aligned} \quad (3.17)$$

and

$$\lambda_{Ne} t = \frac{\gamma^{3/2}}{2} e^{x/\tau} \ln \frac{1 + \sin(x/\sqrt{\gamma})}{1 - \sin(x/\sqrt{\gamma})}.$$

As x approaches $\pi\sqrt{\gamma}/2$, t becomes infinity and both y_α and y_1 vanish, namely, the reaction terminates at $x=\pi\sqrt{\gamma}/2$. For $n \ll \sqrt{\gamma}$, the value of y_n at $x=\pi\sqrt{\gamma}/2$ is expressed as

$$y_n = (n-1)\gamma^{-3/2}, \quad (3.18)$$

and the abundance ratio of the neighbouring nuclei is expressed as

$$y_n/y_{n+1} = 1-1/n. \quad (3.19)$$

The time variation of abundances for $\gamma=10^2$ is illustrated in Fig. 6b.

If we adopt the value of p 's shown in Fig. 2, it is quite difficult to get analytical solutions even approximately. Therefore we solve (3.11') by numerical integration. The solutions for three different cases, namely, $T=2.5 \times 10^9$ K, $\rho=10^5$ g cm $^{-3}$ ($\gamma=10^{-2}$); $T=2.5 \times 10^9$ K, $\rho=10^2$ g cm $^{-3}$ ($\gamma=10$); $T=2.5 \times 10^9$ K, $\rho=1$ g cm $^{-3}$ ($\gamma=10^3$), are illustrated in Figs. 7a, 7b, and 7c, respectively.

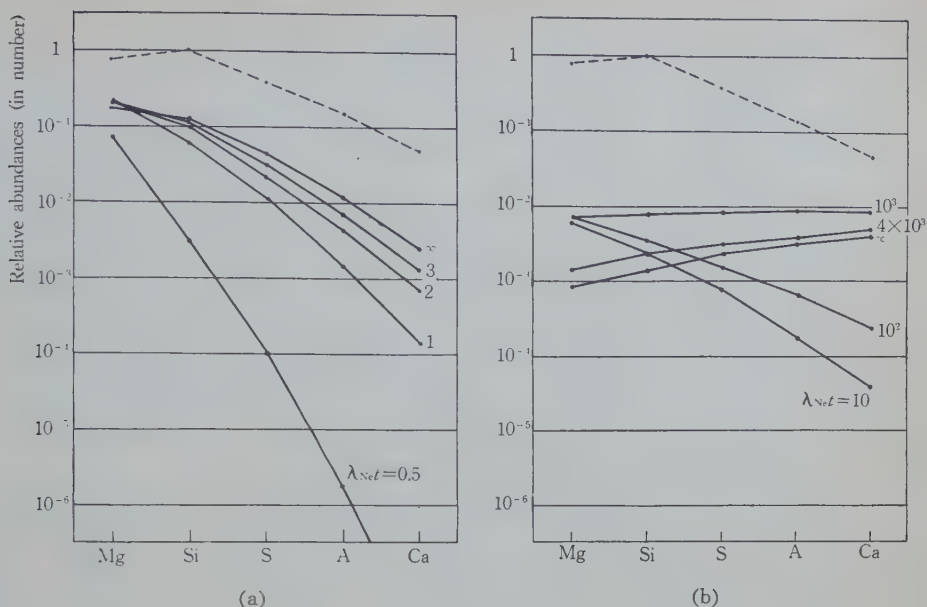


Fig. 6. Abundances of α -particle nuclei relative to that of initially existing Ne^{20} , calculated under the assumption that $p_1 = p_2 = p_3 = \dots = 1$. Fig. 6a illustrates the abundances for $\gamma \ll 1$, and Fig. 6b illustrates those for $\gamma = 100$. The dotted curve denotes the normal abundances given by Suess and Urey, in which abundance of Si^{28} is normalized to be unity.

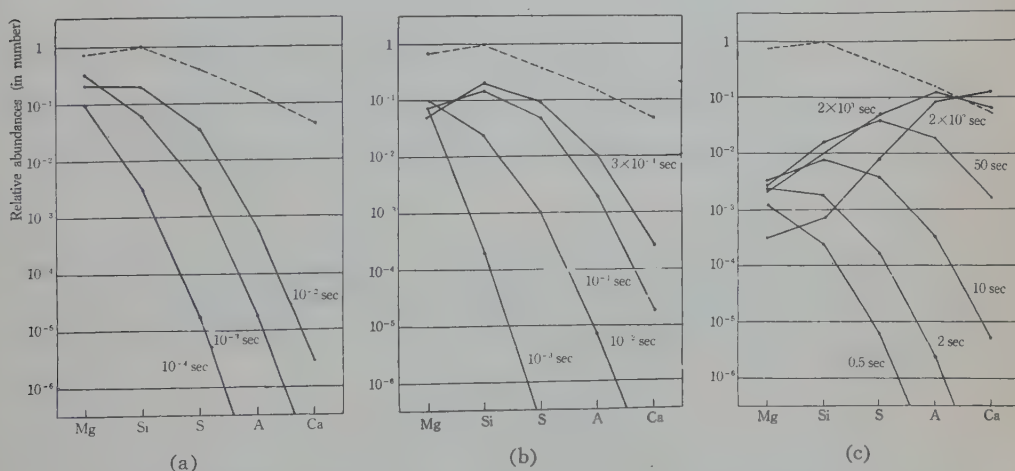


Fig. 7. Abundances of α -particle nuclei relative to that of initially existing Ne^{20} , calculated under the conditions (a) $T = 2.5 \times 10^9$ °K, $\rho = 10^5$ g cm⁻³; (b) $T = 2.5 \times 10^9$ °K, $\rho = 10^2$ g cm⁻³; (c) $T = 2.5 \times 10^9$ °K, $\rho = 1$ g cm⁻³. The dotted curve denotes the same as that in Fig. 6.

From the results obtained above, we can draw the following conclusions.

In the case of small γ , it is impossible to reproduce the normal abundance distribution except in the case in which all p 's are put equal to unity. At slow

synthesizing process which requires much lower temperature than 2.5×10^9 °K, the deviation from the normal abundance will be much larger.

In the case of large γ , contrary to the case of small γ , it is not impossible to reproduce the normal abundances if we suitably choose the value of γ and the time scale of reaction. As is seen from Fig. 7a, 7b and 7c, the abundance curve has a peak at a particular element and this peak shifts in the direction of heavier elements as γ becomes larger. The steepness of the peak is determined by the values of α -capture reaction rate, and the smaller the differences between these reaction rates, *i.e.*, the higher the temperature is, the broader the peak becomes, hence the smaller the deviation from the abundance curve.

Thus, we can conclude that both high temperature, at least higher than 2×10^9 °K, and relatively low density are required in order to reproduce the normal abundance. It is also required that matter must be heated very rapidly to such temperature and cooled again rapidly, because in the case of slow heating $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ reaction would occur before sufficiently high temperature is attained, and in the case of slow cooling various (γ, α) reactions other than $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ would destroy the established abundance distribution. These conditions imposed on temperature, density and time scale of reaction will be realized at the stage of supernova explosion.

At temperature above 3×10^9 °K, various (γ, α) reactions occur within a second so that the assumption here adopted cannot be applied unless we think of an extremely short time scale reaction.

iii) *The case in which α -particles are supplied from outside*

Let us consider the simple case in which α -particles are supplied to Ne^{20} from outside. To simplify the calculation, the $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ reaction is neglected. Then the abundances of heavier elements synthesized depend only on the quantity of α -particles captured, but not on how they are supplied. The variation of abundance is given by the equation as

$$dn_{k+1}/dn_{\alpha}' = \{p_{k+\alpha}n_k - p_{(k+1)+\alpha}n_{k+1}\} / \sum_k p_{k+\alpha}n_k, \quad (3.20)$$

where n_{α}' is not the density of α -particles, but the number of them already captured.

When α -capture reaction rate $p_{k+\alpha}$'s are equal for all nuclei, the solution of (3.20) is expressed as

$$n_k = N_0 (x^k/k!) e^{-x} \quad (3.21)$$

where n_0, n_1, n_2 , etc., denote the number of $\text{Ne}^{20}, \text{Mg}^{24}, \text{Si}^{28}$ etc., respectively, N_0 is the number of Ne^{20} initially existing which is equal to the total number of α -particle nuclei, and $x = n_{\alpha}'/N_0$. Variation of abundance is illustrated in Fig. 8a.

If $p_{k+\alpha}$ differs from each other, the abundance of each nuclide is given by the recurrence formula,

$$n_{k+1} = \exp(-p_{(k+1)+\alpha}x) \int_0^x p_{k+\alpha}n_k(x') \exp(p_{(k+1)+\alpha}x') dx', \quad (3.22)$$

with

$$n_0 = N_0 \exp(-p_{0+\alpha} x),$$

N_0 and n_0, n_1, n_2 , etc., denote the same as before and $dx = dn'_\alpha / \sum_k p_{k+\alpha} n_k$. Fig. 8b illustrates the result of computation in which the values of $p_{k+\alpha}$'s for $T = 2.5 \times 10^9$ °K are adopted.

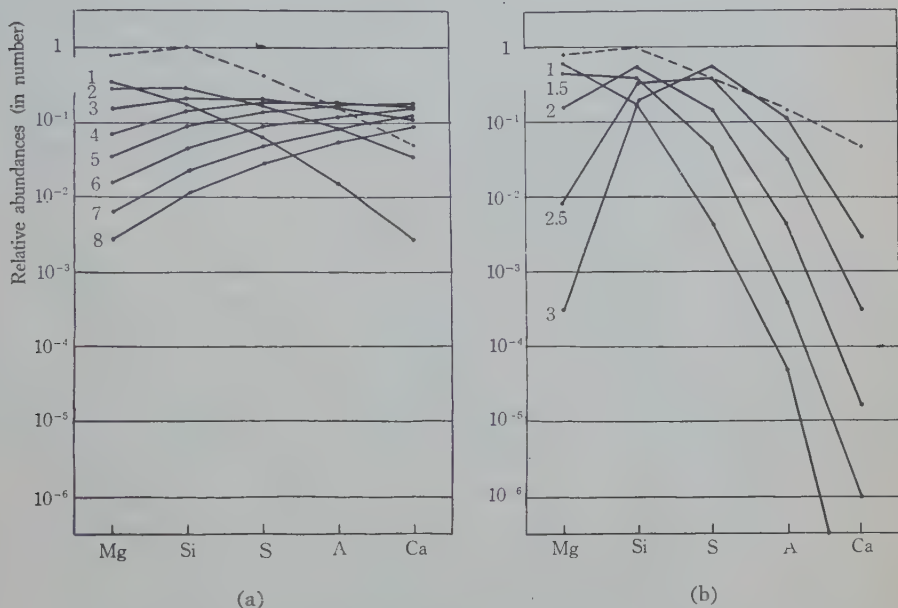


Fig. 8. Abundances of α -particles nuclei relative to that of initially existing Ne^{20} calculated under the condition that (a) $p_{\text{Ne}+\alpha} = p_{\text{Mg}+\alpha} = p_{\text{Si}+\alpha} = \dots$, and (b) $T = 2.5 \times 10^9$ °K. The dotted curve denotes the same as that in Fig. 6. The numbers at the left-ends of the curves represent the ratio n_α/N_0 .

Comparing the results given in Figs. 8a and 8b with those given in the former case, we cannot find any essential difference between them. The restrictions to be imposed on temperature and on the time scale of reaction, are also the same in both case. Even in the case where $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ cannot be neglected, the situation does not essentially differ from the case in which the number of α -particles per one initial Ne^{20} nucleus is somewhat increased.

Finally, we shall briefly give additional remarks on the evaluation of building-up process, in which we neglected the effects of (α, p) reactions throughout because these reactions are endothermic. Preliminary calculations show that, at temperature above 2.5×10^9 °K, the rates of some (α, p) reactions may become larger than those of (α, γ) reactions. In particular, the $\text{A}^{36}(\alpha, p)\text{K}^{39}$ reaction rate is considerably larger than the $\text{A}^{36}(\alpha, \gamma)\text{Ca}^{40}$ reaction rate even at $T \gtrsim 1.5 \times 10^9$ °K. (The threshold energy for $\text{A}^{36}(\alpha, p)$ reaction is 1.29 MeV, while those for other (α, p) reactions are larger than 1.6 MeV.) Therefore, our results on the α -process so far obtained

must be somewhat corrected either when Ca^{40} becomes the main product of this process or when temperature is higher than $2.5 \times 10^9 \text{K}$. Cases of large contamination of hydrogen will be discussed in § 5.

§ 4. Equilibrium theory

In this section, we shall study the extreme case in which the time of duration of temperature and density is longer than any reaction time of α -capture processes and their reverse ones. In this case the statistical equilibrium is finally established among the α -particle nuclei, each (α, γ) reaction and its reverse (γ, α) reaction being in a state of detailed balance.

The equilibrium abundances are given by⁸⁾

$$\frac{n_A n_\alpha}{n_{A+4}} = \left(\frac{g_A g_\alpha}{g_{A+4}} \right) \left(\frac{4A}{A+4} \cdot \frac{2m_H kT}{h^2} \right)^{3/2} e^{-Q/kT}. \quad (4.1)$$

The contribution of excited levels is included in the statistical factors g_α and g_A , which are given by (2.15). At temperature $T=10^{10} \text{K}$, we find $g_{\text{Ne}}=1.75$, $g_{\text{Mg}}=2.03$, $g_{\text{Si}}=1.63$, $g_{\text{S}}=1.37$, $g_{\text{A}}=1.52$, $g_{\text{Ca}}=1.10$, etc., using the nuclear level data.¹²⁾ Since the contributions from excited levels are rather small at temperature below 10^{10}K , we shall put all g 's equal to unity in the following calculations. Then, (4.1) is expressed as

$$\log(n_{A+4}/n_A) = -X_A - \frac{3}{2} \log T_9 + Y_A/T_9 + \log n_\alpha, \quad (4.2)$$

where X_A and Y_A are constants given in Table 3. For binding energies Q we have taken the values given by Wapstra¹³⁾ except for Ti^{44} , Cr^{48} , Ni^{56} and Zn^{60} , for which we have used the semiempirical mass formula given by Cameron.¹⁴⁾

From (4.2) we first find the temperature and the density of α -particles that give the normal abundance ratio of Ca^{40} to Si^{28} , i.e., $\log(n_{\text{Ca}}/n_{\text{Si}}) = -1.28$. Then,

Table 3

A	X	Y	A	X	Y	A	X	Y
4	34.225	-0.4839	24	34.577	50.384	44	34.620	39.305
8	34.413	37.168	28	34.590	35.001	48	34.625	40.384
12	34.489	36.034	32	34.600	33.549	52	34.629	35.691
16	34.531	23.922	35	34.608	35.505	56	34.632	13.1
20	34.558	47.037	40	34.615	26.573			

abundances of all the α -particle nuclei, including unstable Ti^{44} , Cr^{48} , Fe^{52} , Ni^{56} , and Zn^{60} , at such density and temperature are calculated. The results are shown in Fig. 9. Fig. 10 shows the relations between temperature and the total density, $\rho = 4m_H(n_\alpha + 2n_{\text{Be}} + 3n_{\text{C}} + \dots + 15n_{\text{Zn}})$, which give the above ratio $\log(n_{\text{Ca}}/n_{\text{Si}}) = -1.28$ and a larger value $\log(n_{\text{Ca}}/n_{\text{Si}}) = 0$, respectively.

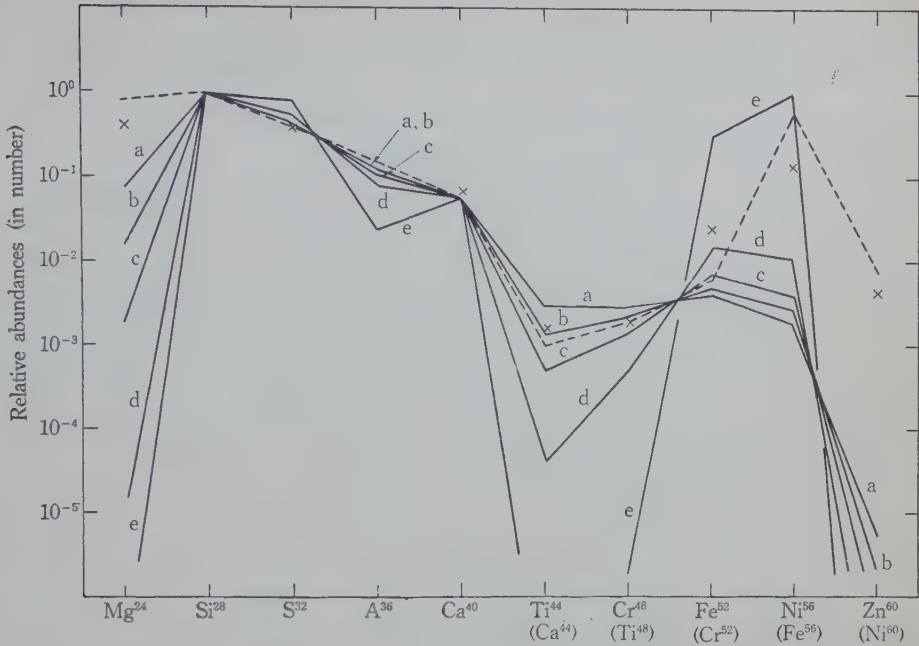


Fig. 9. Equilibrium abundances of α -particle nuclei. The dotted line represents the value adopted by Suess and Urey.⁶⁾ The points marked by across represent the value observed by Goldberg, Müller and Aller.⁷⁾ The solid lines represent the calculated values by the equilibrium theory.

- (a) $T_9=10$, $\log n_\alpha=32.21$, $\rho=1.07 \times 10^9$
- (b) $T_9=7$, $\log n_\alpha=30.49$, $\rho=2.03 \times 10^7$
- (c) $T_9=5$, $\log n_\alpha=28.29$, $\rho=1.29 \times 10^5$
- (d) $T_9=3$, $\log n_\alpha=23.33$, $\rho=3.78$
- (e) $T_9=1$, $\log n_\alpha=1.49$, $\rho=2.04 \times 10^{-24}$

The above results show that at $T \gtrsim 5 \times 10^9 \text{ K}$ and $\rho \gtrsim 10^5 \text{ g cm}^{-3}$ the normal abundances of the α -particle nuclei, except Mg^{24} , together with those of Ca^{44} , Ti^{48} and Cr^{52} are explained fairly well by the equilibrium theory, if the last three nuclei are considered as the decay products of Ti^{44} , Cr^{48} and Fe^{52} , respectively, (in the freezing-in process). At these temperature and density, however, time of (γ, α) reactions are as short as $\lesssim 10^{-5} \text{ sec}$, as seen in Fig. 3. It seems rather unlikely at present that there exist explosive stellar phenomena of such a short time scale as yielding the freezing-in of the equilibrium abundances.

To explain the abundance of Mg^{24} in the equilibrium theory, we must take $T \sim 10^{10} \text{ K}$ and $\rho \sim 5 \cdot 10^8 \text{ g cm}^{-3}$. In this case the reaction time is much shorter. On the other hand, if Fe^{56} is considered as the decay product of Ni^{56} , its abundance does not agree with the normal value unless the temperature is as lower as $\sim 10^9 \text{ K}$.

Next, we shall consider the case in which the time of duration of the temperature and density is not long enough to establish the statistical equilibrium among all

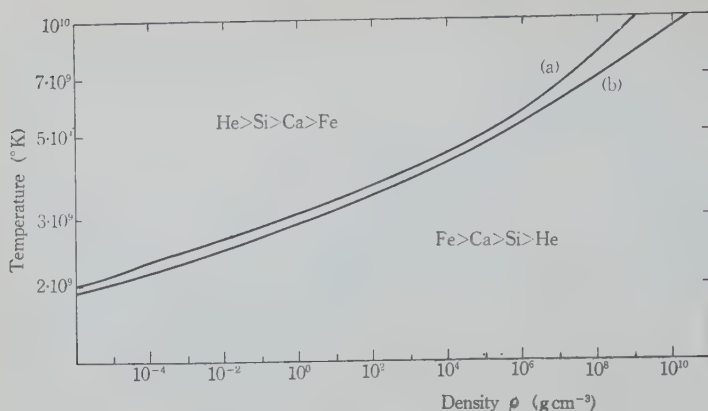


Fig. 10. Temperature and density corresponding to the equilibrium abundance ratios of Ca^{40} to Si^{28} ,

(a) $\log(n_{\text{Ca}}/n_{\text{Si}}) = -1.28$ (normal abundances)

and

(b) $\log(n_{\text{Ca}}/n_{\text{Si}}) = 0$.

α -particle nuclei, but is longer than the case studied in § 4 so that (γ, α) reactions other than $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ proceed in some degree.

For example, we assume $T = 3 \times 10^9$ °K, $\rho = 10^5$ g cm $^{-3}$ and $x_{\text{Mg}} = x_{\text{Si}} = 1/2$ as initial composition, since Mg^{24} and Si^{28} are the main products of the slow α -process caused by $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ or $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$. Time variation of the abundances of all the α -particle nuclei are numerically calculated taking account of all the (α, γ) and (γ, α) reactions. The result shows, as illustrated in Fig. 11, that Mg^{24} is first destroyed by a (γ, α) reaction, and nuclei lighter than Mg^{24} are successively produced by (γ, α) reactions. Then, a statistical equilibrium is approximately established among α -particles and α -particle nuclei heavier than Si^{28} , and the abundances of these nuclei gradually change with the increase of α -particles to approach the complete thermal equilibrium abundance. The normal abundances from Si^{28} to Ca^{40} are obtained if the time of duration is taken to be 10^5 sec. If it is longer than 10^5 sec, abundance of Ca^{40} exceeds that of Si^{28} .

Almost the same result is obtained if we

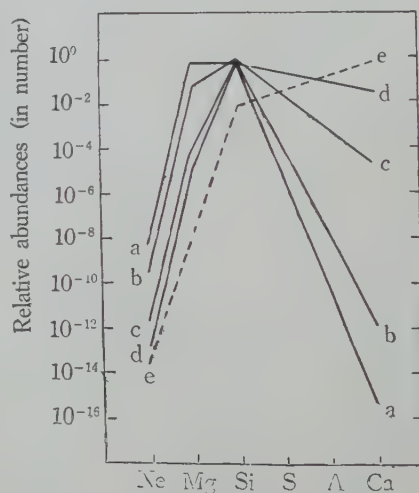


Fig. 11. Variation of relative abundances of α -particle nuclei approaching equilibrium at $T = 3 \cdot 10^9$ °K, $\rho = 10^5$ g cm $^{-3}$ for the initial composition $x_{\text{Mg}} = x_{\text{Si}} = 1/2$.

(a) $t = 1$ sec,

(b) $t = 2 \cdot 10^4$ sec,

(c) $t = 8 \cdot 10^4$ sec,

(d) $t = 10^5$ sec,

(e) equilibrium abundances.

assume higher temperature and density $T=3.5 \times 10^9 \text{ K}$, $\rho=10^8 \text{ g cm}^{-3}$, or lower temperature and density $T=2.5 \times 10^9 \text{ K}$, $\rho=10^2 \text{ g cm}^{-3}$, with a change of time scale of factors 10^{-4} or 10^4 , respectively.

§ 5. Summary and discussions

We summarize in Table 4 our results obtained so far.

Table 4. Summary of our results

Temperature (in 10^9 K)	Density (gcm^{-3})	Initial Composition	Main α -particle source	Our results	Time scale	Location
0.8	$\sim 10^5$	$x_{\text{Ne}} \approx x_{\text{O}} \approx x_{\text{C}}$	$\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$	Abundances of nuclei heavier than Si^{28} are too low.	$\sim 10^4$ years	In the cores of the stars with $10^2 \sim 10^4$ times solar luminos- ities which generate their energy by α - process.
1~1.5		i) Ne^{20} only ii) $x_{\text{Ne}} \approx x_{\text{O}} \approx x_{\text{C}}$	$\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$ $\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$		~ 1 year	
~ 2.5	$\sim 10^5$	Ne^{20} only	$\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$	Abundances of nuclei heavier than Si^{28} are too low. (See Fig. 7a)	$10^{-1} \sim 10$ sec.	In supernovae explosions
		$x_{\text{Ne}} \approx x_{\text{O}} \approx x_{\text{C}}$	$\text{C}^{12}(\text{C}^{12}, \alpha)\text{Ne}^{20}$ and $\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$			
	$\sim 10^2$	i) Ne^{20} only ii) $x_{\text{Ne}} \approx x_{\text{O}} \approx x_{\text{C}}$	$\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$	In fair agree- ment with the normal abun- dance in the range from Mg^{24} to Ca^{40} . (See Fig. 7b)	$1 \sim 10^2$ sec.	
		He^4 and $x_{\text{Ne}} \approx x_{\text{O}} \approx x_{\text{C}}$	He^4 existing initially			
	~ 1	Ne^{20} only	$\text{Ne}^{20}(\gamma, \alpha)\text{O}^{16}$	The peak of the abundance distribution moves forward the larger A with time. (See Fig. 7c)	$10^2 \sim 10^4$ sec.	

We shall briefly discuss another possible case in which there exists a considerable amount of hydrogen as well as He^4 and Ne^{20} in the expanding envelope of the supernovae explosion. In this case the α -particle nuclei are built up mainly through the (p, γ) and subsequent (α, p) reactions, such as $\text{Ne}^{20}(p, \gamma)\text{Na}^{21}(\alpha, p)\text{Mg}^{24}$, $\text{Mg}^{24}(p, \gamma)\text{Al}^{25}(\alpha, p)\text{Si}^{28}$, etc. Preliminary calculations of these nuclear reaction rates give

$$p(\text{Ne}^{20}(p, \gamma)\text{Na}^{21})/p(\text{Na}^{21}(\alpha, p)\text{Mg}^{24}) = 7 \times 10^{-1}, 1 \times 10^{-1}, 3 \times 10^{-2},$$

$$p(\text{Na}^{21}(\alpha, p)\text{Mg}^{24})/p(\text{K}^{35}(\alpha, p)\text{Ca}^{40}) = 5 \times 10^3, 1 \times 10^3, 5 \times 10^2$$

for $T=2.0, 2.5, 3.0 \times 10^9$ °K, respectively, and

$$p(\text{A}^{36}(p, \gamma)\text{K}^{37})/p(\text{K}^{37}(\alpha, p)\text{Ca}^{40}) \gtrsim 1$$

for 3.0×10^9 °K $> T > 1.0 \times 10^9$ °K. In estimating this ratio, we assume that $\text{A}^{36}(p, \gamma)\text{K}^{37}$ has a thermal resonance, since it seems likely that K^{37} has one resonant level in the region of excitation energy near 1 MeV. Thus, in the route from Ne^{20} to Mg^{24} , the (p, γ) reaction is slower than the (α, p) reaction, while in the route from A^{36} to Ca^{40} , the (p, γ) reaction is more rapid than the (α, p) reaction. Therefore, $p(\text{Ne}^{20}(p, \gamma)\text{Na}^{21})/p(\text{K}^{37}(\alpha, p)\text{Ca}^{40})$ plays the most important role in determining the Z -dependence of the relative abundances of α -particle nuclei. For this ratio, we have

$$p(\text{Ne}^{20}(p, \gamma)\text{Na}^{21})/p(\text{K}^{37}(\alpha, p)\text{Ca}^{40}) = 3 \times 10^3, 1 \times 10^2, 1 \times 10^1$$

for $T=2.0, 2.5, 3.0 \times 10^9$ °K, respectively. These values are smaller than the corresponding values of $p(\text{Ne}^{20}(\alpha, \gamma)\text{Mg}^{24})/p(\text{A}^{36}(\alpha, \gamma)\text{Ca}^{40})$, which plays the same role in the absence of hydrogen. The presence of hydrogen, therefore, seems to be more preferable to reproduce the normal abundance distribution.

So far, we have considered the building-up processes of the α -particle nuclei under the stellar condition specified by a given temperature and density. There remains another possibility as follows. The abundance distributions, which are formed through the slow α -process at various depths of the stellar interior, may show different shapes with peaks at different atomic numbers corresponding to the different temperatures and densities of these depths. For the possibility that the superposition of these distributions may give the normal one, we cannot draw a definite conclusion from our present limited knowledges about the stellar structures at such advanced evolutionary stages. It seems, however, more likely that such superpositions show sharp peaks at particular atomic numbers or give different distributions in different celestial objects, rather than they give always the normal abundances as shown in Fig. 1.

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Appendix

Here we consider the solution of equations (3.13) in detail, especially the solution in which γ is either much larger or much smaller than unity.

Let us first investigate the solution where x is so small that an inequality $y_1/y_2 \gg y_0 + p_1 y_1 + p_2 y_2 + \dots$ holds. Neglecting the terms other than the first one in the first three equations of (3.13), we obtain the solution

$$y = y_0 = 1 - y_1, \quad (\text{A} \cdot 1)$$

$$x = -\{\ln(1 - y_n) + y_\alpha\}/\gamma. \quad (A.2)$$

Assuming $y_0 + p_1 y_1 + p_2 y_2 + \dots = 1$ and extrapolating the solution (A.1) to $x = x_1$, where $dy_\alpha/dx = 0$ at $x = x_1$, we get approximately

$$y_0 \cong y_\alpha \cong \gamma/(1 + \gamma), \quad y_1 = 1/(1 + \gamma) \quad \text{at } x = x_1 \quad (A.3)$$

and

$$x_1 = \begin{cases} \gamma/2 & \text{for } \gamma \ll 1 \\ \ln \gamma / \gamma & \text{for } \gamma \gg 1. \end{cases} \quad (A.4)$$

Hence, both for $\gamma \gg 1$ and for $\gamma \ll 1$, x_1 is far smaller than unity and y_2, y_3 , etc., are negligibly small at $x = x_1$. Therefore, the initial condition is approximately taken as

for $\gamma \ll 1$

$$y_0 = y_2 = y_3 = \dots = 0, \quad y_1 = 1, \quad y_\alpha = \gamma \quad \text{at } x = 0,$$

and

for $\gamma \gg 1$

$$y_0 = y_\alpha = 1, \quad y_1 = 1/\gamma, \quad y_2 = y_3 = y_4 = \dots = 0 \quad \text{at } x = 0.$$

For $x > x_1$ (3.13) can be written approximately as follows.

1) $\gamma \ll 1$: Putting $\gamma y_1 = y_\alpha \hat{\xi}$ as the first approximation and, with the method of iteration, expanding $\gamma y_1/y_\alpha$ in power series of γ , we get

$$\gamma y_1/y_\alpha = \hat{\xi} - (\gamma/\hat{\xi})(\hat{\xi} - y_0 + p_1 y_1) - (y_\alpha/\hat{\xi})(d\hat{\xi}/dx) + O(\gamma^2) \quad (A.5)$$

where $\hat{\xi} = y_0 + p_1 y_1 + p_2 y_2 + \dots$. Since the inequality, $-y_\alpha < y_\alpha \cdot d\hat{\xi}/dx < \gamma y_1$, is found to hold from (3.13), the magnitude of $y_\alpha \cdot d\hat{\xi}/dx$ is at most of the order of γ . Then, neglecting the higher order term in γ , the first three equations of (3.13) are reduced to

$$\begin{aligned} \gamma y_1/y_\alpha &= \hat{\xi} \\ dy_0/dx &= \hat{\xi} - y_0 \\ dy_1/dx &= -\hat{\xi} + y_0 - p_1 y_1. \end{aligned} \quad (A.6)$$

When $p_1 = p_2 = p_3 = \dots = 1$, $\hat{\xi}$ is reduced to unity, and so an analytical solution can be easily obtained. The solution is given by (3.14). If values of p 's are different from each other, it is difficult to obtain an analytical solution because $\hat{\xi}$ is not constant.

2) $\gamma \gg 1$: Putting $\gamma y_1 = y_\alpha y_0$ as the first approximation and expanding $\gamma y_1/y_\alpha$ in power series of $1/\gamma$ with the method of iteration, we get

$$\gamma y_1/y_\alpha = y_0 \{1 + (\hat{\xi} - y_0 - p_1 y_\alpha)/\gamma\} + O(1/\gamma^2). \quad (A.7)$$

Neglecting the terms of higher order and inserting (A.7) into (3.13), we get

$$\begin{aligned} dy/dx &= -(\hat{\xi} - y_0) + y_0(\hat{\xi} - y_0 - p_1 y_\alpha)/\gamma, \\ dy_0/dx &= y_0(\hat{\xi} - y_0 - p_1 y_\alpha)/\gamma, \end{aligned} \quad (A.8)$$

$$dy_1/dx = -y_0(\xi - y_0 - p_1 y_\alpha)/\gamma - y_1.$$

To see the characteristic behaviour of the solution, we draw the locus of $dy_0/dy_\alpha = \text{const.} = m$ on the y_0 vers. y_α plane. Putting $\xi=1$, the locus of constant m is expressed as

$$y = \left(1 - \frac{\gamma m}{1-m}\right) - y_0 + \frac{\gamma m}{1-m} \frac{1}{y_0}.$$

Fig. A.1 illustrates the locus corresponding to various values of m . From this we can see that y_0 is nearly constant so that it may be put equal to unity for the solution which starts at $y_\alpha \sim y_0 \sim 1$. Then, (A.8) is approximately written as

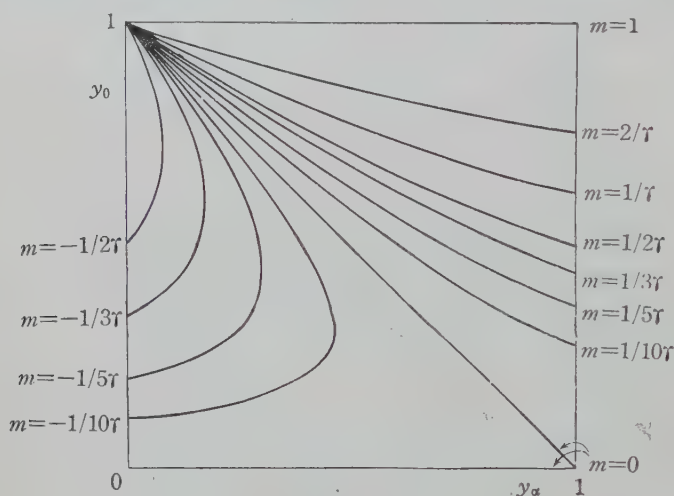


Fig. A.1. The locus for constant value of $dy_0/dy_\alpha = m$.

$$dy_\alpha/dx = -(\xi - y_0) + (\xi - y_0 - p_1 y_\alpha)/\gamma$$

$$dy_0/dx = (\xi - y_0 - p_1 y_\alpha)/\gamma \quad (\text{A.8}')$$

$$dy_1/dx = -(\xi - y_0 - p_1 y_\alpha)/\gamma - y_1.$$

When $p_1 = p_2 = p_3 = \dots = 1$, analytical solution can be easily obtained. Neglecting the higher order small terms, we get the results given by (3.17). If p 's differ from each other, it is difficult to obtain an analytical solution in this case, too.

If the magnitude of γ is of the order of unity, simplification so far made cannot be applied.

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Ordinary and Anomalous Thresholds in Perturbation Theory

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Ordinary and anomalous thresholds of the matrix element corresponding to the general Feynman graph are rigorously investigated in detail. Necessary conditions for the ordinary threshold are obtained.

§ 1. Introduction

It has been believed for a long time that the energy spectrum of the absorptive part of the S -matrix element would be determined by the energies of the possible intermediate states. Though the Feynman-Dyson theory brought a considerable progress to the perturbational calculation, such an important problem was not rigorously investigated. The above conjecture is quite natural from the "physical consideration", but "physical consideration" cannot be applied to the unphysical regions.

In relation to the inability of the rigorous proof of the dispersion relations¹⁾, it has recently been pointed out that in some cases there appear the anomalous energy spectra which cannot be expected from the possible intermediate states, even in perturbation theory²⁾. We call the threshold in such case "the anomalous threshold", and the physically expected threshold "the ordinary threshold." It is highly desirable that we have a simple criterion for judging whether the matrix element corresponding to an arbitrarily given Feynman graph has the ordinary threshold or the anomalous one. Two methods are so far proposed as sufficient conditions for the ordinary threshold; those are Nambu and Symanzik's majorization method³⁾ and the method previously proposed by the present author⁴⁾. But necessary conditions have so far been obtained only for the lowest-order graphs of vertex function and elastic scattering.

In this paper, we mathematically investigate the general, fundamental properties on the ordinary and the anomalous thresholds, by making use of the general integral formula of perturbation term given by the present author⁵⁾, which is summarized in § 2. In § 3, we give the very fundamental theorem on the continuity of the important function $V(x)$. A physical application of this theorem is found in a work on the nucleon structure⁶⁾. In § 4, necessary conditions for the ordinary threshold are obtained, and it is proved that the total rest energy of a possible intermediate state always corresponds to a singular point (a branch point). Some illustrative examples of the general theory in § 4 are presented in § 5. The general lowest-order graphs are studied in detail in § 6.

§ 2. Summary of the previous results

An arbitrary Feynman graph G consists of α internal lines, which are named as 1, 2, ..., α . Without loss of generality, we may assume that all particles are scalar and all couplings are direct, since we are interested only in energy spectra. The each line i of G corresponds to a propagator

$$1/\{(p_i+q_i)^2+m_i^2-i\epsilon\} \quad (i=1, 2, \dots, \alpha) \quad (2.1)$$

where p_i , q_i , and m_i stand for an integration momentum, a constant momentum, and a mass, respectively. It should be understood that $p_i \equiv 0$ for the propagator which contains no integration variable. Not all of integration momenta p_i are independent. We denote the number of independent ones by n . Constant momentum q_i is expressible by external momenta, but q_i is not uniquely determined, namely q_1, q_2, \dots, q_n are completely arbitrary if p_1, p_2, \dots, p_n are independent. This arbitrariness is a powerful tool in the following arguments.

Theorem 1. The matrix element corresponding to G is given by the following parametric integral. The integration domain is $x_i \geq 0$ ($i=1, 2, \dots, \alpha$).

$$\text{const.} \int \frac{\delta(1-\sum_{i=1}^{\alpha} x_i) \prod_{i=1}^{\alpha} dx_i}{U^2(x) \{V(x)-i\epsilon\}^{\alpha-2n}}. \quad (2.2)$$

Here we assumed $\alpha > 2n$. If not so, only a trivial modification is necessary (cf. N(4.1)).* No essential change is brought into the following arguments.

The notations used in (2.2) are as follows.

$$U(x) \equiv \sum x_{\nu_1} x_{\nu_2} \dots x_{\nu_n}. \quad (2.3)$$

The summation goes over all possible sets $(\nu_1, \nu_2, \dots, \nu_n)$ such that $p_{\nu_1}, p_{\nu_2}, \dots, p_{\nu_n}$ are independent.

$$V(x) \equiv \sum_{i=1}^{\alpha} x_i (m_i^2 + q_i^2) - \sum_C Q_C(x) / U(x), \quad (2.4)$$

$$Q_C(x) \equiv U_C(x) (\pm x_{c_1} q_{c_1} \pm x_{c_2} q_{c_2} \pm \dots \pm x_{c_l} q_{c_l})^2. \quad (2.5)$$

C is a closed circuit belonging to G , and c_1, c_2, \dots, c_l are the lines belonging to C . The double signs correspond to the relative directions of $q_{c_1}, q_{c_2}, \dots, q_{c_l}$ on the circuit C .

$$U_C(x) \equiv \sum x_{\nu_1} x_{\nu_2} \dots x_{\nu_{n-1}}. \quad (2.6)$$

The summation goes over all possible sets $(\nu_1, \nu_2, \dots, \nu_{n-1})$ such that none of $\nu_1, \nu_2, \dots, \nu_{n-1}$ coincides with any one of c_1, c_2, \dots, c_l and $p_{\nu_1}, p_{\nu_2}, \dots, p_{\nu_{n-1}}$ are independent. The summation \sum_C in (2.4) goes over all possible circuits belonging to

* N stands for reference 5).

G . But we may exclude the circuit which can be divided into two circuits, because $U_c(x)$ corresponding to the such circuit identically vanishes.

The physical properties of the matrix element are almost represented by the function $V(x)$. The matrix element has no singularity if $V(x) > 0$ for all x , but there appears an imaginary part if $V(x) < 0$ for some x .

It is obvious from (2.3) that $U(x) \geq 0$. But $U(x)$ can vanish when all x_i 's belonging to a circuit vanish. In general, $U(x)$ has an m -th order zero point at all $x_i = 0$ belonging to m independent circuits.* (The number of independent circuits is defined by the number of the independent integration momenta p_i which can flow in these.) When $U(x) = 0$, $V(x)$ is not well defined by (2.4). Therefore the limiting behaviour of $V(x)$ in such case is important.

For convenience, we introduce the following functions:

$$Y_i(x) \equiv (1/2x_i) \partial V(x) / \partial q_i = q_i - (1/U) \sum_{C(i)} U_{C(i)}(x_i q_i \pm \dots), \quad (2.7)$$

where $C(i)$'s denote the circuits containing the line i .

§ 3. Continuity of $V(x)$

Theorem 2. Let A be a set of lines contained in m independent circuits, and B be a set of the other lines of G . Then we can write

$$U = U^{(A)} U^{(B)} + U', \quad (3.1)$$

and

$$V = V^{(B)} + V', \quad (3.2)$$

where $U^{(A)}$ is the U defined as (2.3) with respect to the subgraph A (an m -th order polynomial of $x_i \in A$),** $U^{(B)}$ is the U of the graph B^* which is obtained from G by contracting A to one point (an $(n-m)$ -th order polynomial of $x_j \in B$), and U' stands for a sum over the terms that are at least of $(m+1)$ -th order with respect to $x_i \in A$; $V^{(B)}$ denotes the V of the graph B^* , and V' is at least of first order with respect to $x_i \in A$.

Therefore, when $\sum_{i \in A} x_i \rightarrow 0$, we have uniformly

$$U/U^{(A)} \rightarrow U^{(B)}, \quad (3.3)$$

and

$$V \rightarrow V^{(B)}. \quad (3.4)$$

* Of course, in general, they mutually overlap.

** $a \in \mathfrak{A}$ means that a belongs to \mathfrak{A} . For convenience, the set of lines i and the set of the corresponding parameters x_i are denoted by the same letter.

Further, if $j \in B$,

$$Y_j \rightarrow Y_j^{(B)} \equiv (1/2x_j) \partial V^{(B)} / \partial q_j, \quad (3.5)$$

because the operation $(1/2x_j) \partial / \partial q_j$ is irrelevant to the limiting procedure. But it is not the case for Y_i , ($i \in A$).

(Proof). Since there appear only summations in the both sides of (3.1), it is convenient to regard U as a set of terms $x_{v_1} x_{v_2} \cdots x_{v_n}$, $U^{(A)} \cdot U^{(B)}$ as a direct product of two sets $U^{(A)}$ and $U^{(B)}$, and (3.1) as an equality relation between sets. As A consists of m independent circuits, each term of U must be of at least m -th order with respect to $x_i \in A$. The terms of $(m+1)$ -th order or more with respect to $x_i \in A$ belong to U' . For the term of just m -th order, the product of $x_i \in A$ must be contained in $U^{(A)}$ by definition, and so $n-m$ remaining x_j 's must belong to B . Since the corresponding p_j 's are independent, the product of x_j must belong to $U^{(B)}$. Therefore*

$$U \subset U^{(A)} U^{(B)} + U'.$$

Conversely, since the $n-m$ p_j 's corresponding to the term $\prod x_j \in U^{(B)}$ and the m p_i 's corresponding to the term $\prod x_i \in U^{(A)}$ are independent (if not so, all integration momenta could be represented by $n-1$ p_v 's.), we have

$$U \supset U^{(A)} U^{(B)}.$$

Thus (3.1) holds for an appropriate U' .

Next, we prove (3.2). (3.2) obviously holds for the term $\sum x_i (m_i^2 + q_i^2)$. Hence we consider only the part $\sum_C Q_C / U$. The following three cases are possible for the circuit C . We count the order of $x_i \in A$ of the lowest order terms.

(i) $C \subset A$.

U : m -th order, U_C : $(m-1)$ -th order,
 $(x_{c_1} q_{c_1} \pm \cdots)^2$: 2nd order; therefore Q_C / U is 1st order.

(ii) $C \not\subset A$.

U_C : m -th order, $(x_{c_1} q_{c_1} \pm \cdots)^2$: 0-th order; therefore Q_C / U is 0-th order.

(ii-a) $C \subset B$.

Quite analogously to (3.1), we have

$$U_C = U^{(A)} U_C^{(B)} + U'_C,$$

because U_C is nothing but the U of the graph in which C is contracted to one point. Therefore the part of 0-th order with respect to $x_i \in A$ among Q_C / U is $(U_C^{(B)} / U^{(B)}) (x_{c_1} q_{c_1} \pm \cdots)^2$.

(ii-b) $C \not\subset A$ and $C \not\subset B$.

We consider the set $\{C^k\}$ of the circuits C^k such that $C^{(B)} \equiv C^k \cap B$ is common*

* $\mathfrak{B} \subset \mathfrak{A}$ means that \mathfrak{B} is a subset of \mathfrak{A}

** $\mathfrak{B} \cap \mathfrak{A}$ stands for the common part of \mathfrak{A} and \mathfrak{B} .

($C^{(R)}$ is not a circuit in G). Since $(x_{e_1}q_{e_1} \pm \cdots)^2$ is common for C^k when $A^2x_i=0$, its coefficient becomes a sum $\sum_k U_{C^k}$. We then have

$$\sum_k U_{C^k} = U^{(A)} U_{C^{(R)}}^{(R)} + U_{C^{(B)}}^{(B)},$$

which is proved as follows. $\tilde{A} = A + C^{(R)}$ contains $m+1$ independent circuits. Hence $B - C^{(R)}$ can contain only $n-m-1$ independent quantities. So, each term of U_{C^k} , which is of $(n-1)$ -th order, must contain at least m x_i 's ($x_i \in A$). Consider an arbitrary term $\prod_{i=1}^m x_{v_i} \in U_{C^k}^{(A)}$. When we denote by p' the momentum corresponding to a line $e \in C^{(R)}$, $p_{v_1}, p_{v_2}, \dots, p_{v_m}, p'$ form basic momenta in \tilde{A} , and so the circuit along which p' flows is determined uniquely. Denoting it by C^1 , we have

$$U_{C^1} \supset \prod_{i=1}^m x_{v_i} U_{C^{(R)}}^{(R)}, \quad C^1 \in \{C^k\}.$$

Conversely, an arbitrary term $e \in U_{C^k}$ of m -th order with respect to $x_i \in A$ is, of course, contained in $U_{C^k}^{(A)} U_{C^{(R)}}^{(R)}$. Finally, we must show that $\sum_k U_{C^k}$ has no overlaps. If there were $\prod_{j=1}^{n-1} x_{v_j} \in U_{C^k} \cap U_{C^l}$ for $k \neq l$, then all integration momenta except for those corresponding to the lines $e \in C^k \cap C^l$ could be represented by $p_{v_1}, p_{v_2}, \dots, p_{v_{n-1}}$. However, since $C^k \cap C^l$ does not compose a circuit, another independent p does not exist. This means that G has not n independent integration momenta. This is a contradiction. Thus (3.2) is established.

A simple example of this theorem is found in the calculation of 3π -state contribution of nucleon structure.⁶⁾

Theorem 3. $V(x)$ is continuous for $x_i \geq 0$.

(Proof). This is self-evident for $x_i > 0$. The problem occurs when some x_i 's vanish. $V(x)$ exhibits no singularity when the part of $x_i=0$ does not contain any circuit. If the part of $x_i=0$ contain some circuits, from theorem 2 $V(x)$ has a definite limiting value irrelevant to limiting methods. Therefore if we define by it the value of $V(x)$ for $x_i=0$, then $V(x)$ becomes continuous.

Theorem 4. $V(x)$ has its smallest value in the domain $\{x_i \geq 0, \sum_{i=1}^n x_i = 1\}$.

Because $V(x)$ is continuous in this closed domain.

§ 4. Conditions for the ordinary threshold

$V(x)$ is a linear function of the squares of some linear combinations of external momenta. We denote one of them by P^2 , and regard $V(x)$ as a function of P^2 . If at $-P^2=M^2$ it holds that $V(x) \geq 0$ for all x satisfying $x_i \geq 0$ & $\sum_{i=1}^n x_i = 1$ and $V(x)=0$ for some particular x , then we call $-P^2=M^2$ the "lowest threshold" or simply "threshold." In this case, for an infinitesimal $\varepsilon (\neq 0)$ we have $V(x) > 0$ for all x at $-P^2=M^2-\varepsilon$ and $V(x) < 0$ for some x at $-P^2=M^2+\varepsilon$, because $V(x)$ is linear with regard to P^2 . Namely, $-P^2=M^2$ is an end point of the energy spectrum of the absorptive part. It is usually expected

that ε is positive, but that is not essential in the present investigation.

From the above definition, it is necessary for the threshold $-P^2=M^2$ that " $V(x)=0$ " is the minimum of $V(x)$ with respect to x . We call such a minimum "threshold minimum."

Theorem 5. The necessary condition for a threshold minimum is

$$\begin{aligned}\partial V/\partial x_i &= 0 \quad \text{for } i=1, 2, \dots, r, \\ x_j &= 0 \quad \text{for } j=r+1, \dots, \alpha,\end{aligned}\tag{4.1}$$

where, of course, numbering is arbitrary.

(*Proof*). The minimum of $V(x)$ requires that some x_j 's ($j=r+1, \dots, \alpha$) are the boundary value 0, and that the other x_i 's ($i=1, 2, \dots, r$) satisfy the usual minimum conditions under the restriction $\sum_{i=1}^r x_i=1$. By means of Lagrange's method of an indeterminate coefficient (which is denoted by λ), we obtain

$$\partial V/\partial x_i - \lambda = 0 \quad (i=1, 2, \dots, r).$$

Now, being a first-order homogeneous function of x , $V(x)$ satisfies Euler's differential equation

$$V = \sum_{i=1}^{\alpha} x_i \partial V/\partial x_i.$$

Hence we have

$$V = \sum_{i=1}^r x_i \partial V/\partial x_i = \lambda \sum_{i=1}^r x_i = \lambda.$$

The condition $V(x)=0$ therefore means $\lambda=0$. Thus (4.1) is obtained.

Lemma.

$$\partial V/\partial x_i = m_i^2 + Y_i^2 \quad (i=1, 2, \dots, \alpha).\tag{4.2}$$

This identity is already given in N(A.3). We present its detailed proof in the Appendix.

For simplicity we hereafter assume $m_i > 0$ ($i=1, 2, \dots, \alpha$).

Theorem 6. The conditions (4.1) cannot be satisfied if G is not divided into disconnected parts by opening the lines 1, 2, ..., r .

(*Proof*). Since G is not divided by opening 1, 2, ..., r , we can choose

$$q_i = 0 \quad (i=1, 2, \dots, r).$$

From the assumption all x_i 's ($i=1, 2, \dots, r$) do not vanish, namely at least one is non-vanishing, say, $x_1 > 0$. Then by (3.5) Y_1 has a definite value, which is $Y_1=0$ on account of $q_i=0$ (cf. (2.7)). Hence we have from (4.2)

$$\partial V/\partial x_1 = m_1^2 \neq 0,$$

which is inconsistent with (4.1).

We call a set of lines 1, 2, ..., r satisfying the following conditions "the in-

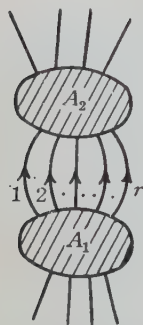


Fig. 1

intermediate-state set": G is divided into two subgraphs A_1 and A_2 by opening the lines 1, 2, \dots , r , but not by opening any true subset of $\{1, 2, \dots, r\}$.

Theorem 7. When the lines 1, 2, \dots , r form an intermediate-state set, (4.1) is satisfied if and only if

$$\left. \begin{aligned} -P^2 &= \left(\sum_{i=1}^r m_i \right)^2, \\ \text{and} \quad x_i &= (1/m_i) / \left(\sum_{k=1}^r 1/m_k \right), \end{aligned} \right\} (i=1, 2, \dots, r) \quad (4.3)$$

where P stands for the total momentum which flows from A_1 to A_2 .

(*Proof*). We take the direction of q_i as $A_1 \rightarrow A_2$ (see Fig. 1). We can then choose

$$q_i = (m_i / \sum_{k=1}^r m_k) \cdot P. \quad (4.4)$$

In the terminology of theorem 2, we may write $A = A_1 + A_2$ and $B = \{1, 2, \dots, r\}$, and graph B^* is shown in Fig. 2. Therefore, when $x_j = 0$ ($j = r+1, \dots, \alpha$), we have



Fig. 2

$$\begin{aligned} Y_i &= q_i - (1/U^{(B)}) \sum_{k \neq i} U_{C(i,k)}^{(B)} (x_i q_i - x_k q_k) \\ &= (1/\sum_{k=1}^r m_k) (m_i - b_i) P, \end{aligned} \quad (4.5)$$

where

$$b_i \equiv (1/U^{(B)}) \sum_{k \neq i} U_{C(i,k)}^{(B)} (x_i m_i - x_k m_k), \quad (4.6)$$

and $C(i, j)$ denotes the circuit containing both the lines i and j , or more explicitly

$$U_{C(i,j)}^{(B)} = \prod_{k=1}^r x_k / x_i x_j \quad (4.7)$$

and

$$U^{(B)} = \sum_{i=1}^r \left(\prod_{k=1}^r x_k / x_i \right). \quad (4.8)$$

Now, from (4.1) and (4.2) we have

$$m_i^2 + Y_i^2 = 0, \quad (i=1, 2, \dots, r)$$

$$\text{i.e.} \quad m_i^2 - c^2 (m_i - b_i)^2 = 0, \quad (4.9)$$

where

$$c^2 \equiv -P^2 / \left(\sum_{k=1}^r m_k \right)^2. \quad (4.10)$$

(4.9) cannot be satisfied when $c^2 \leq 0$. When $c^2 > 0$ (we take $c > 0$), the solutions of (4.9) are

$$m_i - cm_i + cb_i = 0, \quad (4.11)$$

or

$$m_i + cm_i - cb_i = 0. \quad (4.12)$$

But (4.12) is impossible because of

$$b_i \leq (1/U^{(B)}) \sum_{k \neq i} U_{O(i,k)}^{(B)} x_i \cdot m_i \leq m_i.$$

From (4.11) we obtain

$$(1-c) \sum_{i=1}^r m_i + c \sum_{i=1}^r b_i = 0, \quad (4.13)$$

and from (4.6)

$$\sum_{i=1}^r b_i = 0. \quad (4.14)$$

Therefore

$$c = 1 \quad (4.15)$$

and

$$b_i = 0 \quad (4.16)$$

from (4.11). (4.15) with (4.10) is the first equation of (4.3). All of equations (4.16) cannot be independent because of the identity (4.14). Rewriting

$$x_i m_i - x_k m_k = (x_i m_i - x_r m_r) - (x_k m_k - x_r m_r),$$

we regard

$$b_i = 0 \quad (i=1, 2, \dots, r-1) \quad (4.17)$$

as $r-1$ simultaneous equations with respect to $x_i m_i - x_r m_r$ ($i=1, 2, \dots, r-1$).

Then the determinant of coefficients is

$$D \equiv 1/[U^{(B)}]^{r-1} \cdot \begin{vmatrix} \sum_{k \neq 1} U_{O(1,k)}^{(B)} & -U_{O(1,2)}^{(B)} & \dots & -U_{O(1,r-1)}^{(B)} \\ -U_{O(2,1)}^{(B)} & \sum_{k \neq 2} U_{O(2,k)}^{(B)} & \dots & -U_{O(2,r-1)}^{(B)} \\ \dots & \dots & \dots & \dots \\ -U_{O(r-1,1)}^{(B)} & -U_{O(r-1,2)}^{(B)} & \dots & \sum_{k \neq r-1} U_{O(r-1,k)}^{(B)} \end{vmatrix}. \quad (4.18)$$

From (4.7) and (4.8) it turns out that $[U^{(B)}]^{r-1} \cdot D$ is exactly the reciprocal determinant of

$$U^{(R)} = \begin{vmatrix} x_1 + x_r & x_r & \cdots & x_r \\ x_r & x_2 + x_r & \cdots & x_r \\ \cdots & \cdots & \cdots & \cdots \\ x_r & x_r & \cdots & x_{r-1} + x_r \end{vmatrix}. \quad (4.19)$$

We therefore obtain

$$D = 1/U^{(R)} \neq 0 \quad (4.20)$$

on account of the well-known property of the reciprocal determinant. Thus equations (4.17) have the unique solution

$$x_i m_i - x_r m_r = 0 \quad (i = 1, 2, \dots, r-1), \quad (4.21)$$

i. e.

$$x_1 : x_2 : \cdots : x_r = 1/m_1 : 1/m_2 : \cdots : 1/m_r \quad (4.22)$$

with $\sum_{i=1}^r x_i = 1$.

Theorem 8. (4.4) always satisfies the minimum condition

$$\sum_{i,k=1}^r \frac{\partial^2 V}{\partial x_k \partial x_i} \Delta x_k \Delta x_i > 0. \quad (4.23)$$

(*Proof*). From (4.2), (4.5), (4.3) and (4.16) we have

$$\begin{aligned} \partial^2 V / \partial x_k \partial x_i &= 2(m_i - b_i) \partial b_i / \partial x_k \\ &= \begin{cases} -2m_i m_k U_{O(i,k)}^{(R)} / U^{(R)} & \text{for } k \neq i, \\ 2m_i^2 \sum_{j \neq i} U_{O(i,j)}^{(R)} / U^{(R)} & \text{for } k = i, \end{cases} \end{aligned}$$

i.e.

$$\partial^2 V / \partial x_k \partial x_i = \begin{cases} -2 \left(\sum_{j=1}^r m_j \right)^{-1} \left(\sum_{j=1}^r 1/m_j \right) m_i^2 m_k^2 & \text{for } k \neq i, \\ 2 \left(\sum_{j=1}^r m_j \right)^{-1} \left(\sum_{j=1}^r 1/m_j \right) m_i^3 \sum_{j \neq i} m_j & \text{for } k = i. \end{cases} \quad (4.24)$$

Therefore it suffices to prove that

$$\begin{aligned} J &\equiv \sum_{i=1}^r m_i^3 \left(\sum_{j \neq i} m_j \right) (\Delta x_i)^2 - \sum_{i \neq k} m_i^2 m_k^2 \Delta x_i \Delta x_k \\ &= \left(\sum_{i=1}^r m_i \right) \left\{ \sum_{i=1}^r m_i^3 (\Delta x_i)^2 \right\} - \left(\sum_{i=1}^r m_i^2 \Delta x_i \right)^2 \end{aligned} \quad (4.25)$$

is positive definite. Putting

$$\alpha_i \equiv \sqrt{m_i}, \quad \beta_i \equiv \sqrt{m_i^3} \Delta x_i, \quad (4.26)$$

we have

$$J = (\sum_i \alpha_i^2) (\sum_i \beta_i^2) - (\sum_i \alpha_i \beta_i)^2 \quad (4.27)$$

and therefore

$$J \geq 0 \quad (4.28)$$

because of Cauchy's inequality. $J=0$ is possible only when β_i is proportional to α_i , i.e.

$$m_1 \Delta x_1 = m_2 \Delta x_2 = \dots = m_r \Delta x_r. \quad (4.29)$$

So, under the restriction $\sum_{i=1}^r \Delta x_i = 0$, it must be

$$J > 0 \quad (4.30)$$

except for all $\Delta x_i = 0$. Thus (4.23) was proved.

We may call (4.3) "a physical threshold." Theorem 8 then tells us that a physical threshold generally corresponds to a branch point. Because: we consider the parametric integration of (2.2) in the neighbourhood of this point x_i . Carrying out the integrations over x_{r+1}, \dots, x_α , we notice only the term coming out from the end point $x_{r+1} = \dots = x_\alpha = 0$. In this term V takes the positive definite form (4.23) near this point. Therefore the singularity at $\Delta x_i = 0$ cannot be eliminated by the integrations over x_1, x_2, \dots, x_r .

At $x_i = (1/m_i) / (\sum_{k=1}^r 1/m_k)$ ($i=1, 2, \dots, r$) and $x_j = 0$ ($j=r+1, \dots, \alpha$), V can be written as

$$V = \{ (\sum_{k=1}^r m_k)^2 + P^2 \} / (\sum_{k=1}^r m_k) (\sum_{k=1}^r 1/m_k). \quad (4.31)$$

Hence for $-P^2 > (\sum_{k=1}^r m_k)^2$ V becomes negative, namely a branch-cut appears in the side of $-P^2 > (\sum_{k=1}^r m_k)^2$, as is expected.

The minimum condition (4.23) is not complete. We must further take into account the variations of x_{r+1}, \dots, x_α . That is to say, it should be

$$V(x) \geq 0 \quad (4.32)$$

for $\sum_{j=r+1}^\alpha x_j = \varepsilon > 0$ (infinitesimal) irrelevantly to the ratios of $x_j \geq 0$ ($j=r+1, \dots, \alpha$).

This condition (4.32) is, of course, not always satisfied.

We count the orders of various factors in $V(x)$ with respect to ε . First, we have $U \sim \varepsilon^s$, where s is the number of independent circuits contained in A_1 or A_2 (i.e. $s=n-r+1$). Next, for the circuit C which contains some of the lines 1, 2, \dots , r

(of course an even number), we find $U_C \sim \varepsilon^s$, and under the choice (4.4)

$$(x_{c_1} q_{c_1} \pm \cdots)^2 \sim \varepsilon^2$$

owing to the cancellation of main terms (cf. $x_i = (1/m_i) / (\sum_k 1/m_k) + O(\varepsilon)$ ($i=1, 2, \dots, r$)). Therefore Q_C/U is a higher order term of ε .

For $C \subset A_1$ or $C \subset A_2$, we have $U_C \sim \varepsilon^{s-1}$ and

$$(x_{c_1} q_{c_1} \pm \cdots)^2 \sim \varepsilon^2.$$

We thus obtain

$$V(x) = \varepsilon_1 V_1(y) + \varepsilon_2 V_2(y) + O(\varepsilon^2) \geq 0, \quad (4.33)$$

where

$$x_j = \varepsilon_1 y_j \quad (j \in A_1), \quad \sum_{j \in A_1} y_j = 1,$$

$$x_j = \varepsilon_2 y_j \quad (j \in A_2), \quad \sum_{j \in A_2} y_j = 1,$$

and

$$\varepsilon_1 + \varepsilon_2 = \varepsilon, \quad \varepsilon_1 \geq 0, \quad \varepsilon_2 \geq 0.$$

V_1 and V_2 respectively are the V defined relating to the subgraphs A_1 and A_2 . This is easily verified by noticing (for $C \subset A_1$)

$$\left. \begin{aligned} U &= U^{(A_1)} U^{(A_2)} \cdot U^{(B)} + O(\varepsilon^{s+1}), \\ U_C &= U_C^{(A_1)} U^{(A_2)} U^{(B)} + O(\varepsilon^s) \end{aligned} \right\} \quad (4.34)$$

from theorem 2, and therefore

$$U_C/U = U_C^{(A_1)}/U^{(A_1)} + O(\varepsilon). \quad (4.35)$$

(4.33) can be equivalently written as

$$\left. \begin{aligned} V_1(x) &\geq 0 \quad \left(\sum_{j \in A_1} x_j = 1, \quad x_j \geq 0 \right), \\ V_2(x) &\geq 0 \quad \left(\sum_{j \in A_2} x_j = 1, \quad x_j \geq 0 \right). \end{aligned} \right\} \quad (4.36)$$

The condition (4.36) is the necessary and sufficient condition for that (4.3) gives a minimum of V . The implications of (4.36) are illustrated in terms of some simple examples in the next two sections.

Now, (4.36) is, of course, the minimum condition in the local region of x . The lowest threshold defined in the beginning requires $V(x) \geq 0$ in the whole domain. Hence (4.36) is merely a necessary condition for the lowest threshold, but very crucial one. When (4.3) gives the lowest threshold, we call it "the ordinary threshold." Thus

Theorem 9. A necessary condition for the ordinary threshold is the inequalities (4.36) where q_i ($i=1, 2, \dots, r$) is defined by (4.4).

We call the lowest threshold other than the ordinary one "the anomalous threshold." But we shall use also, in a wide sense, the terminology of "anomalous threshold" for the branch point other than physical thresholds. From theorems 6 and 7, an anomalous threshold can appear only in the case in which the set $\{1, 2, \dots, r\}$ stated in theorem 5 contains an intermediate-state set as its true subset. Examples of the anomalous threshold are presented by Karplus et al.^{2,7)}

§ 5. Elastic scattering

In this section we illustrate the general results obtained in the last section by familiar examples.

The elastic scattering is characterized by the following scalar quantities:

$$\begin{aligned} p^2 = p'^2 = -m^2, \quad k^2 = k'^2 = -\mu^2, \\ (k-k')^2 = 4\mathcal{A}^2 \quad (\text{momentum transfer}), \\ (p+k)^2, \quad (p-k')^2, \end{aligned} \quad (5.1)$$

where

$$p+k = p'+k'. \quad (5.2)$$

There is an identity between them,

$$(p+k)^2 + (p-k')^2 + (k-k')^2 = p^2 + k^2 + p'^2 + k'^2 \quad (5.3)$$

i. e.

$$(p+k)^2 + (p-k')^2 + 4\mathcal{A}^2 + 2(m^2 + \mu^2) = 0 \quad (5.4)$$

on account of (5.2). Therefore, in the consideration of the spectrum of $-(p+k)^2$, we must take into account also that of $-(p-k')^2$ at the same time. If M^2 is the ordinary threshold of $-(p+k)^2$, there is a branch-cut $-(p+k)^2 \geq M^2$ (infinitesimally lower than the real axis). Likewise, if M'^2 is that of $-(p-k')^2$, there is a branch cut $-(p-k')^2 \geq M'^2$. The latter can be rewritten as

$$-(p+k)^2 \leq 2(m^2 + \mu^2) + 4\mathcal{A}^2 - M'^2 \quad (5.5)$$

because of (5.4) (infinitesimally upper than the real axis). Hence the two branch cuts partially overlap when

$$4\mathcal{A}^2 > M^2 + M'^2 - 2(m^2 + \mu^2). \quad (5.6)$$

Now, the condition that $-(p+k)^2 = M^2$ is the lowest threshold is $V(x) \geq 0$ with $-(p+k)^2 = M^2$. When x_i 's take the value (4.3) where the lines i belong to the lowest-energy intermediate-state set with respect to $p-k'$, $V(x) \geq 0$ is reduced to

$$M'^2 + (p-k')^2 \geq 0 \quad (5.7)$$

from (4.31), namely (5.7) gives a necessary condition for $V(x) \geq 0$. Therefore

in order that $-(p+k)^2=M^2$ is the lowest threshold, it is necessary that

$$4\mathcal{A}^2 \leq M^2 + M'^2 - 2(m^2 + \mu^2) \quad (5.8)$$

from (5.7) and (5.4). Comparing it with (5.6), we find that $-(p+k)^2=M^2$ cannot be the lowest threshold in our definition, if the two branch-cuts overlap. This is naturally a self-evident fact. But the above method is useful for extracting simple (but rather weak) necessary conditions from the very complicated condition such as $V(x) \geq 0$.

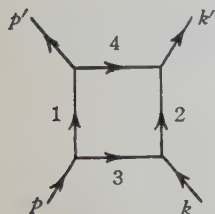


Fig. 3

Example 1. The lowest order graph shown in Fig. 3 is very simple. An extensive investigation has been carried out by Karplus et al.⁷⁾ But our purpose is merely to see the direct consequences from the general results. Necessary conditions for that $-(p+k)^2=(m_1+m_2)^2$ is the lowest threshold are

$$\begin{aligned} (m_1+m_3)^2+p^2 &\geq 0, & (m_2+m_3)^2+k^2 &\geq 0, \\ (m_1+m_4)^2+p'^2 &\geq 0, & (m_2+m_4)^2+k'^2 &\geq 0, \\ (m_3+m_4)^2+(k-k')^2 &\geq 0, \end{aligned} \quad (5.9)$$

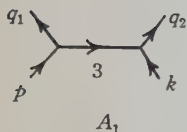
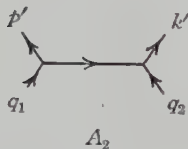
 A_1  A_2

Fig. 4

which are obtained from $V(x) \geq 0$ by using the above-mentioned method. The first four are stability conditions, and the last is trivial.

Next we consider the necessary condition (4.36) for the ordinary threshold.

The subgraphs A_1 and A_2 are shown in Fig. 4. From (4.4) q_1 and q_2 must be taken as

$$\left. \begin{aligned} q_1 &= m_1(p+k)/(m_1+m_2), \\ q_2 &= m_2(p+k)/(m_1+m_2), \end{aligned} \right\} \quad (5.10)$$

Then q_3 and q_4 are uniquely determined:

$$\left. \begin{aligned} q_3 &= p - q_1 = (m_2 p - m_1 k)/(m_1 + m_2), \\ q_4 &= q_1 - p' = (-m_2 p' + m_1 k')/(m_1 + m_2), \end{aligned} \right\} \quad (5.11)$$

The conditions (4.36) are

$$\left. \begin{aligned} V_1(x) &= x_3(m_3^2 + q_3^2) \geq 0 & (x_3=1), \\ V_2(x) &= x_4(m_4^2 + q_4^2) \geq 0 & (x_4=1), \end{aligned} \right\} \quad (5.12)$$

that is to say,

$$\left. \begin{aligned} m_3^2 &\geq \frac{m_2 m^2 + m_1 \mu^2}{m_1 + m_2} - m_1 m_2, \\ m_4^2 &\geq \frac{m_2 m^2 + m_1 \mu^2}{m_1 + m_2} - m_1 m_2, \end{aligned} \right\} \quad (5.13)$$

which are the conditions obtained by many authors.^{2)~4)}

Example 2. As an example of higher order term we consider Fig. 5, whose subgraph A_1 (when $B = \{1, 2\}$) is shown in Fig. 6. q_1 and q_2 are given by (5.10). The condition (4.36) for A_1 is

$$V_1(x) = \sum_{i=3}^6 x_i (m_i^2 + q_i^2) - (x_3 q_3 - x_4 q_4 - x_5 q_5 + x_6 q_6)^2 \geq 0, \quad (5.14)$$

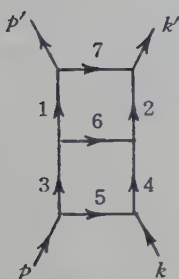


Fig. 5

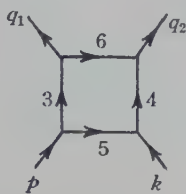


Fig. 6

for which the knowledge for the lowest order can be applied as it is. But we present here only the simple necessary conditions for (5.14), which are obtained analogously to (5.9). These are four stability conditions and

$$\left. \begin{aligned} (m_3 + m_4)^2 + (p + k)^2 &\geq 0, \\ (m_5 + m_6)^2 + (p - q_1)^2 &\geq 0, \end{aligned} \right\} \quad (5.15)$$

namely,

$$\left. \begin{aligned} m_3 + m_4 &\geq m_1 + m_2, \\ (m_5 + m_6)^2 &\geq \frac{m_2 m^2 + m_1 \mu^2}{m_1 + m_2} - m_1 m_2. \end{aligned} \right\} \quad (5.16)$$

The former inequality means that the physical threshold $-(p + k)^2 = (m_3 + m_4)^2$ cannot be lower than $(m_1 + m_2)^2$ if the latter is the lowest threshold (This is a very natural matter). The latter inequality of (5.16), which corresponds to (5.13), is a new necessary condition, which is simple but weaker than (5.14).

Now, in the axiomatic proof of dispersion relation,¹⁾ we know that there is usually some restriction for the momentum transfer \mathcal{A}^2 . But in perturbation theory, the minimum condition (4.36) never gives any limitation on \mathcal{A}^2 . The restriction for \mathcal{A}^2 which can so far be obtained are rather trivial ones, such as (5.8). If there appears the limitation on \mathcal{A}^2 for assuring the ordinary threshold in perturbation theory, it must come out from others than the minimum condition in the local region.

§ 6. Lowest order graphs

In this section we study the graph which consists of only one circuit. An example is shown in Fig. 7. For convenience, we define q_i 's such that they have

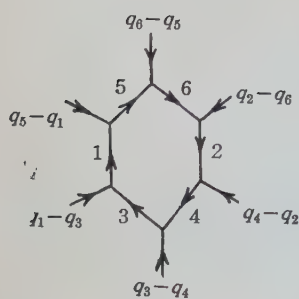


Fig. 7

the same direction. Then we have

$$\begin{aligned} V(x) &= \sum_i x_i (m_i^2 + q_i^2) - (1/\sum_k x_k) (\sum_i x_i q_i)^2 \\ &= (1/2 \sum_k x_k) \sum_{i,k} \{m_i^2 + m_k^2 + (q_i - q_k)^2\} x_i x_k, \end{aligned} \quad (6.1)$$

where $q_i - q_j$ is expressed by a sum of external momenta.

The condition for the ordinary threshold, (4.36), becomes

$$m_i^2 + q_j^2 \geq 0 \quad (j=3, 4, \dots), \quad (6.2)$$

where

$$\left. \begin{aligned} m_2 q_1 + m_1 q_2 &= 0 \\ -(q_1 - q_2)^2 &= (m_1 + m_2)^2 \end{aligned} \right\} \quad (6.3)$$

and

By means of (6.3) q_j is expressed in terms of external momenta:

$$q_j = -\{m_2(q_1 - q_j) + m_1(q_2 - q_j)\} / (m_1 + m_2). \quad (6.4)$$

Then the condition (6.2) is rewritten as

$$m_2 \{m_1^2 + m_j^2 + (q_1 - q_j)^2\} + m_1 \{m_2^2 + m_j^2 + (q_2 - q_j)^2\} \geq 0, \quad (6.5)$$

which is a generalization of (5.13).

An anomalous threshold can appear when three or more parameters x_i do not vanish, as was stated in § 4. We investigate threshold minimums other than the physical threshold $-(q_1 - q_2)^2 = (m_1 + m_2)^2$.

i) $x_1 \neq 0$ and $x_2 \neq 0$.

$V'(x) \equiv \sum_k x_k \cdot V(x)$ is also a homogeneous function of x , therefore theorem 5 is applicable. Namely we have

$$\begin{aligned} \partial V' / \partial x_1 &= 2m_1^2 x_1 + \{m_1^2 + m_2^2 + (q_1 - q_2)^2\} x_2 + \sum_{j \geq 3} \{m_1^2 + m_j^2 + (q_1 - q_j)^2\} x_j = 0 \\ \partial V' / \partial x_2 &= \{m_1^2 + m_2^2 + (q_1 - q_2)^2\} x_1 + 2m_2^2 x_2 + \sum_{j \geq 3} \{m_2^2 + m_j^2 + (q_2 - q_j)^2\} x_j = 0. \end{aligned} \quad (6.6)$$

Calculating $m_2 \partial V' / \partial x_1 + m_1 \partial V' / \partial x_2$, we obtain

$$\begin{aligned} &(m_1 x_1 + m_2 x_2) \{ (m_1 + m_2)^2 + (q_1 - q_2)^2 \} \\ &= - \sum_{j \geq 3} [m_2 \{m_1^2 + m_j^2 + (q_1 - q_j)^2\} + m_1 \{m_2^2 + m_j^2 + (q_2 - q_j)^2\}] x_j \end{aligned} \quad (6.7)$$

Hence, if (6.5) is satisfied, we get

$$-(q_1 - q_2)^2 \geq (m_1 + m_2)^2. \quad (6.8)$$

Thus the anomalous threshold (in a wide sense), even if it exists, cannot be lower than the physical threshold $-(q_1 - q_2)^2 = (m_1 + m_2)^2$, if the condition (6.5) is satisfied.

ii) $x_1 = 0$ and/or $x_2 = 0$.

$V(x)$ then is independent of $(q_1 - q_2)^2$, provided that we adopt $(q_i - q_j)^2$ as the independent variables. If $V(x) < 0$ happens for some special x , the whole real axis becomes the branch-cut on the complex plane of $-(q_1 - q_2)^2$. If not so, this case is trivial.

Therefore, if (6.5) is satisfied, then *either* $-(q_1 - q_2)^2 = (m_1 + m_2)^2$ is the lowest threshold *or* the whole real axis is the branch-cut.

It is an open question whether or not there is an anomalous threshold (in a wide sense) satisfying (6.8). We can definitely exclude such a possibility in the case in which the number of the non-vanishing x_i 's is three (i.e. $B = \{1, 2, 3\}$), without assuming stability conditions.

For simplicity, we write

$$\alpha_{ik} = \alpha_{ki} \equiv m_i^2 + m_k^2 + (q_i - q_k)^2. \quad (6.9)$$

The condition for the threshold minimum is

$$\alpha_{1k}x_1 + \alpha_{2k}x_2 + \alpha_{3k}x_3 = 0 \quad (k=1, 2, 3). \quad (6.10)$$

The simultaneous equations (6.10) have the non-trivial solution only when

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = 0. \quad (6.11)$$

The solution then is

$$x_1 : x_2 : x_3 = \beta_{11} : \beta_{12} : \beta_{13} = \beta_{21} : \beta_{22} : \beta_{23} = \beta_{31} : \beta_{32} : \beta_{33}, \quad (6.12)$$

where $\beta_{ik} = \beta_{ki}$ is the cofactor of α_{ik} . Since $x_i > 0$, all β_{ik} 's must have the same sign because of (6.12) and $\beta_{ik} = \beta_{ki}$.

Now, we have

$$\begin{aligned} \beta_{33} &= \alpha_{11}\alpha_{22} - \alpha_{12}^2 \\ &= -\{(m_1 - m_2)^2 + (q_1 - q_2)^2\} \{(m_1 + m_2)^2 + (q_1 - q_2)^2\} < 0 \end{aligned} \quad (6.13)$$

on account of (6.8). We thus obtain

$$\beta_{ik} < 0. \quad (6.14)$$

The minimum condition (4.23) is

$$\sum_{i,k=1}^3 \alpha_{ik} \Delta x_i \Delta x_k > 0 \quad \left(\sum_{i=1}^3 \Delta x_i = 0 \right). \quad (6.15)$$

The necessary condition for (6.15) is obtained by the well-known elementary method:

$$\sum_{i,k} \beta_{ik} > 0, \quad (6.16)$$

which contradicts (6·14). Thus such an anomalous threshold is impossible.

Appendix. Proof of lemma (4·2)

The notations in N §§ 2~3 are used. From N(3·5), i.e.

$$V = \sum_i x_i m_i^2 + \det B/U, \quad (\text{A} \cdot 1)$$

we obtain

$$\frac{\partial V}{\partial x_i} = m_i^2 + \frac{1}{U} \frac{\partial}{\partial x_i} \det B - \frac{1}{U^2} \frac{\partial U}{\partial x_i} \cdot \det B. \quad (\text{A} \cdot 2)$$

We choose p_i as one of the basic momenta $p_{v_1}, p_{v_2}, \dots, p_{v_n}$ (i.e. $p_{v_i} \equiv p_i$). Then from N(3·4) we have

$$(\partial/\partial x_i) \det B = q_i^2 U + 2q_i B_{01} + B_{11}, \quad (U = \det A = B_{00}), \quad (\text{A} \cdot 3)$$

where B_{01} and B_{11} respectively are the cofactors of b_1 and $a_{11} (\equiv x_i)$. On the other hand, from (2·7) and (A·1) we can write

$$Y_i = \frac{1}{2x_i} \frac{1}{U} \frac{\partial}{\partial q_i} \det B = q_i + B_{01}/U, \quad (\text{A} \cdot 4)$$

and therefore

$$m_i^2 + Y_i^2 - \partial V/\partial x_i = (1/U^2) \{B_{01}^2 - UB_{11} + (\partial U/\partial x_i) \det B\}. \quad (\text{A} \cdot 5)$$

Since

$$\partial U/\partial x_i = \mathfrak{A}_{11} = B_{00,11}$$

is an $(n-1)$ -th order minor determinant of B , Jacobi's theorem gives an identity

$$\begin{vmatrix} U & B_{01} \\ B_{01} & B_{11} \end{vmatrix} = \frac{\partial U}{\partial x_i} \det B. \quad (\text{A} \cdot 6)$$

Thus by (A·5) and (A·6) we have proved

$$m_i^2 + Y_i^2 - \partial V/\partial x_i = 0. \quad (4 \cdot 2)$$

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Notes on Magneto-Hydrodynamic Equilibrium

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A proper coordinate system and a suitable form of equilibrium equations for a successive approximation method are reported in this letter. By these means we can solve equilibrium equations in the case of a rotational transformed field.^{1,2)} But we could not apply these methods to the case in which lines of force are closed and have no symmetric property.

As is well known, lines of electric current and lines of magnetic force lie on equi-pressure surfaces in the M. H. equilibria. We have proved³⁾ that it is possible to construct a global coordinate system on each equi-pressure surface by taking these two kinds of lines as coordinate lines and defining coordinate x_i on these lines according to

$$dx_i = ds_i / |X_i| \quad (i=1,2) \quad (1)$$

In Eq. (1), X_1 and ds_1 are the magnetic field strength and its line-element, X_2 and ds_2 , the current density and its line-element, respectively.

We have studied the structure of

these coordinate systems and got three main results.³⁾ The first is that any closed equi-pressure surface without singular points is topologically torus, as was said by many other authors.²⁾ The second is that there must exist simple closed loops which lie on each equi-pressure surface with non-vanishing pressure-gradient and have the following property, that is,

$$\int_A^B \frac{ds_1}{|X_1|} = \text{const. on the surface} \quad (2)$$

where the integrals are carried out along lines of force from an arbitrary starting point A on the loop to the final point B on it. Generally, there are one or more loops passing through any given point on the surface, turning arbitrary given times along and around the magnetic axis in an equilibrium configuration.

The third result is that by taking these loops belonging to two different homology classes as coordinate lines and defining coordinates ξ_i on these lines in proportion to magnetic flux through an infinitesimally narrow ribbon between the surface and its neighbouring one, limited by $\xi_i=0$ and ξ_i itself, we can construct a many valued periodic coordinate system on each equi-pressure surface, and in this system, lines of current and lines of force are represented as parallel straight lines, respectively.

But when the rotational transform ratio is rational, such a surface may be called a rational one and in the other case the surface may be called an irrational one; this result will fail unless the loops are carefully selected. The reason is the following. In almost all cases there exists one and only one loop through a given point, belonging to a given homology class for an irrational surface. But there exists an infinite number of such loops for a rational surface when the loops exist. Therefore, a selection rule is required for rational cases, but, at present, we cannot get it except in the case with symmetric property.

Equation (2) can be used as a necessary condition for an external field to be a magnetic trap, because Eq. (2) must hold as the fluid pressure tends to 0. Equation (2) is the condition for no charge separation. By means of the second and the third results we can get useful knowledge of current density in the first order of pressure when a rotational transformed external field is given.

We have transformed ordinary equilibrium equations into suitable forms for the successive approximation method. That is,

$$\nabla \chi = \mathbf{H} + 4\pi p \nabla x_1, \quad \nabla \cdot \mathbf{H} = 0, \quad \mathbf{H} \nabla p = 0 \quad (3)$$

where p and \mathbf{H} are the fluid pressure and the magnetic field strength, x_1 is defined in Eq. (1), and χ is a many valued scalar function which is obtained by solving the Poisson equation successively together with suitable boundary conditions and periods.

When p is sufficiently small and the

external field is rotationally transformed, p and x_1 are given by the data in lower order and we can make the iteration to higher order of p .

Equation (2) suggests the possibility of a magnetic trap without rotational transform. In this case Eq. (2) gives magnetic surfaces. B. B. Kadomcev has already discussed this possibility in detail.

Finally, we have studied an effect of rational surfaces in a stellarator with varying rotational transform ratio. Generally speaking, almost all rational surfaces are incompatible with Eq. (2). Thus $|\nabla p|$ is zero on the surfaces densely distributed in the device, or else charge separation should occur. By means of the second and the third results, it can be shown that lines of current describe rapidly oscillating curves in the neighbourhood of such surfaces. Therefore, the diffusion velocity of fluid across magnetic lines increases due to the finite conductivity of fluid and the long paths of current lines in that region. Probably, a vortex flow may also occur under some conditions. But according to our semi-quantitative treatments, the diffusion effect will be small in practical cases.

Detailed discussions will be published elsewhere.

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Ambiguity in the Mixing Parameter for Nucleon-Nucleon Scattering*

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April 30, 1959

In their recent work Zastavenko et al.¹⁾ have discussed a unitary transformation for nucleon-nucleon scattering. This is the generalization of the transformation which gives rise to Minami's ambiguity²⁾ for pion-nucleon scattering.

We would like to add just a few remarks to their results. First of all, suppose the S matrix of nucleon-nucleon scattering for a given total angular momentum J , parity $(-1)^{J+1}$, and triplet state, is given in terms of two eigen phase shifts δ_1 and δ_2 , and a mixing parameter ε , in the form³⁾

$$S \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} e^{2i\delta_1} \cos^2 \varepsilon + e^{2i\delta_2} \sin^2 \varepsilon & \\ \cos \varepsilon \sin \varepsilon (e^{2i\delta_1} - e^{2i\delta_2}) & \\ e^{2i\delta_1} \sin^2 \varepsilon + e^{2i\delta_2} \cos^2 \varepsilon & \end{pmatrix},$$

where the subscripts 1 and 2 correspond to the orbital angular momentum $L = J-1$ and $J+1$, respectively. Now, their transformation is given by

$$U = \frac{(\sigma_1 \mathbf{p})}{p} \cdot \frac{(\sigma_2 \mathbf{p})}{p}, \quad (2)$$

where \mathbf{p} is the relative momentum

operator of two nucleons in the center-of-mass system, and σ_1 and σ_2 are the spin matrices of the first and second nucleons. This transforms S in the following way:

$$S' = U^{-1} S U = \frac{1}{(2J+1)^2} \begin{pmatrix} S_{11} - 4\sqrt{J(J+1)} S_{12} + 4(J+1) S_{22} \\ 2\sqrt{J(J+1)} (S_{22} - S_{11}) + (4J(J+1) - 1) S_{12} \\ 2\sqrt{J(J+1)} (S_{22} - S_{11}) + (4J(J+1) - 1) S_{12} \\ S_{22} + 4\sqrt{J(J+1)} S_{12} + 4J(J+1) S_{11} \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} e^{2i\delta_1} \cos^2 \varepsilon' + e^{2i\delta_2} \sin^2 \varepsilon' \\ \cos \varepsilon' \sin \varepsilon' (e^{2i\delta_1} - e^{2i\delta_2}) \\ \cos \varepsilon' \sin \varepsilon' (e^{2i\delta_1} - e^{2i\delta_2}) \\ e^{2i\delta_1} \sin^2 \varepsilon' + e^{2i\delta_2} \cos^2 \varepsilon' \end{pmatrix}, \quad (4)$$

where

$$\varepsilon' = -\varepsilon - \cos^{-1} \frac{1}{2J+1}. \quad (5)$$

This result shows that there always exists another set of mixing parameters ε' , which gives exactly the same angular distribution for the scattering.

It is interesting to note that this change of the mixing parameter is almost equivalent to the exchange of δ_1 and δ_2 for large J .

The differential cross section for the triplet state is written in terms of the Racah coefficients as³⁾

$$\frac{d\sigma}{d\Omega} = \frac{1}{4k^2} \frac{1}{3} \sum R^*(1) R(2) Z(l_1 J_1 l_2 J_2, 1 \lambda) \times Z(l'_1 J'_1 l'_2 J'_2, 1 \lambda) P_\lambda(\cos \theta). \quad (6)$$

The above ambiguity in the mixing

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parameter can be seen by using the following relations of Z -coefficients:

$$\begin{aligned}
 & (2J_1+1)(2J_2+1)Z(J_1\pm 1 J_1 J_2\pm 1 J_2, 1\lambda) \\
 & = Z(J_1\pm 1 J_1 J_2\pm 1 J_2, 1\lambda) \\
 & + 4\sqrt{J_1(J_1+1)J_2(J_2+1)} \\
 & \quad \times Z(J_1\mp 1 J_1 J_2\mp 1 J_2, 1\lambda) \\
 & \pm 2\sqrt{J_2(J_2+1)}Z(J_1\mp 1 J_1 J_2\pm 1 J_2, 1\lambda) \\
 & \pm 2\sqrt{J_2(J_2+1)}Z(J_1\pm 1 J_1 J_2\mp 1 J_2, 1\lambda), \quad (7)
 \end{aligned}$$

and

$$\begin{aligned}
 & \sqrt{J_2}Z(J_1 J_1 J_2-1 J_2, 1\lambda) = \\
 & -\sqrt{J_2+1}Z(J_1 J_1 J_2+1 J_2, 1\lambda). \quad (8)
 \end{aligned}$$

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Effect of the Finite Size of the Proton on the Coulomb Energy of He^3

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The Coulomb energy of He^3 is calculated by assuming an extended proton. The Coulomb energy is shown to be reduced by a large amount.

The recent electron scattering¹⁾ experiments at Stanford University have es-

tablished the finite size of the proton. If a finite charge distribution is assumed for the proton, the Coulomb energy (referred to as C. E.) arising from the small inter-proton distance in the He^3 nucleus would be reduced. We shall estimate this effect. It seems to be enough to assume that r_i ($i=1, 2, 3$, see Fig. 1 of reference 2) are the mutual distances of the centres of nucleons, and that the proton charge is distributed in the spherically symmetric way around the centre of the proton, because we are interested in evaluating only the order of magnitude of the effect of this on the Coulomb energy. A further simplification will be made; we first calculate C. E. between one extended proton and the other point proton. In the case of two extended protons, the effect should be larger, but cannot become twice that for a single extended proton. The neutron is assumed to be completely neutral.

The experimental data on the electron scattering from protons is well reproduced by assuming an exponential type charge distribution $\rho(r)$ for the proton, so that one may take $\rho(r)$ for the present purpose as

$$\rho_1(r) = \frac{e\beta_1^3}{8\pi} e^{-\beta_1 r}, \quad \langle r^2 \rangle = 12/\beta_1^2. \quad (1)$$

We also choose another function for $\rho(r)$, in order to see shape dependence of the effect,

$$\rho_2(r) = \frac{e\beta_2}{4\pi} \frac{e^{-\beta_2 r}}{r^2}, \quad \langle r^2 \rangle = 2/\beta_2^2. \quad (2)$$

although this shape cannot always be fitted to experimental data over the wider

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range of electron energies. Charge distributions (1) and (2) modify the Coulomb potential e^2/r for one extended proton and the other point proton, respectively, as follows,

$$V_1(r) = \frac{e^2}{r} \left\{ 1 - [1 + \beta_1 r + (\beta_1 r)^2/2] e_1^{-\beta_1 r} \right\} \quad ; \text{ for (1)}$$

$$V_2(r) = \frac{e^2}{r} \left\{ 1 - e^{-\beta_2 r} \right\}, \quad ; \text{ for (2)}$$

The mean square radius of the charge distribution is assumed to be 0.8×10^{-13} cm as indicated by low energy electron scattering data. The Coulomb energy of He^3 is evaluated in a usual way. The only difference is that we now use $V_1(r)$ or $V_2(r)$ instead of the usual Coulomb potential e^2/r . We use the wave function of He^3 as given in reference 3). In the case* of $D=0$, the He^3 nucleus has the largest radius (namely,** $\mu = 0.479 \times 10^{13}$ cm) for $r_{0s}^{***} = 2.7 \times 10^{-13}$ cm with exponential, and the smallest ($\mu = 0.659 \times 10^{-13}$ cm) for $r_{0s} = 2.4 \times 10^{-13}$ cm with Yukawa potential. Coulomb energy values in such extreme cases are summarized in Table I, where we see C. E. of He^3 is

Table I. Coulomb energies of He^3 for $D=0$. One proton is assumed to have charge distribution (1) or (2). The values of point protons are also included for comparison. Observed difference of binding energies between H^3 and He^3 is 0.764 Mev.

Hard core radius $D=0$	C. E. of He^3 in Mev		
	$V_1(r)$	$V_2(r)$	e^2/r
Expon. pot., $r_{0s}=2.7$	0.761	0.802	0.986
Yukawa pot., $r_{0s}=2.4$	0.900	1.000	1.358
Experimental value	0.764		

much reduced, especially for the Yukawa well assuming $r_{0s}=2.4$.

Thus the Coulomb energy of He^3 , which is too large without the hard cores, may be brought into agreement with the experimental value of the difference between the binding energies of H^3 and He^3 without hard cores, if the finite size effect of the proton is taken into consideration. It is desirable to evaluate C. E. for two extended protons in the near future.

Since the presence of a hard core prevents the approach of two protons very near each other, the effect should be smaller for $D \neq 0$. The evaluation

Table II. Coulomb energies of He^3 for $D=0.2 \times 10^{-13}$ cm. One proton has charge distribution (2). The values for point protons are also included.

Hard core radius $L=0.2 \times 10^{-13}$ cm	C. E. of He^3 in Mev	
	$V_2(r)$	e^2/r
Expon. pot., $r_{0s}=2.7$	0.717	0.836
Yukawa pot., $r_{0s}=2.4$	0.753	0.896
Experimental value	0.764	

Table III. Coulomb energies of He^3 for $D=0.6 \times 10^{-13}$ cm

Hard core radius $D=0.6 \times 10^{-13}$ cm	C. E. of He^3 in Mev	
	$V_2(r)$	e^2/r
Expon. pot., $r_{0s}=2.7$	0.631	0.686
Yukawa pot., $r_{0s}=2.4$	0.670	0.736
Experimental value	0.764	

* D ; the hard core radius of the nuclear force.

** μ ; the assumed wave function is of the form $\prod_{i=1,2,3} [\exp(-\nu r'_i) - \exp(-\mu r'_i)]$, $r'_i = r_i - D$, for $r_1, r_2, r_3 \geq D$. ($\nu > \mu$).

*** r_{0s} ; the singlet effective range.

of the Coulomb energy is carried out only for $V_2(r)$, because of the complexity of actual computation for $V_1(r)$. Table II, III show the results obtained. Calculated values are always smaller than the experimental value.

The precise values of the Coulomb energy are expected to be several per cent smaller than the calculated ones in Table I owing to the simplicity of our trial function. This seems to indicate that one extended proton is even consistent with the experimental data without hard core. The inclusion of the tensor potential in the two-body force may not change this statement, because the tensor force may decrease, more or less, the Coulomb energy.³⁾ However, since it is quite probable that there exists a rather strong repulsive potential in the two-body nuclear force, and also as it is very natural to assume the protons to be extended rather than point charges, we are inclined to speculate that a slight violation of charge-symmetry hypothesis, which guarantees the equality of nuclear forces for proton-proton and neutron-neutron systems may occur. The actual nuclear force is apparently not charge-independent. The same order of magnitude of the violation of the charge-symmetry law is enough to account for the observed value of the difference between the binding energies of H^3 and He^3 . If it is true, the neutron-neutron force is somewhat stronger than the proton-proton force. This seems to be quite possible, but we dare not insist on this possibility only.

The author wishes to thank Dr. H. Miyazawa for discussion.

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Spin-Orbit Splitting and Tensor Force

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We have computed with sufficient accuracy the second order effect of the tensor force on the spin-orbit splitting in the light nuclei through the perturbation theoretic treatment, using the meson-theoretic potential and Serber one.

It has been found that, at least, a half of the experimental value of the P -state doublet splitting in He^5 or N^{15} is explained through this calculation. The previous, approximate calculations of this effect by several authors¹⁾ have shown to lead to the splitting of wrong sign or of the too small magnitude, while Feingold's²⁾ variational calculation has given small splitting of right sign, compared with the experiment.

The comparison of this calculation with Feingold's one is discussed in some detail.

Assuming the harmonic oscillator wave function, the second order perturbation energy has been calculated with sufficient accuracy in the following way. The matrix elements for many particles are

reduced to the matrix elements for two particles by the usual method, which are dealt with through introducing the relative and the centre of mass coordinate system.³⁾ Using the orthogonality of the transformation coefficients between these coordinate systems, the summation with respect to many degenerate intermediate states with the same excitation energy can be carried through, resulting into the single particle matrix element, which becomes a simple function of N if the energy of these intermediate states is $E_0 + 2N\hbar\omega$. E_0 is the zeroth order energy, ω the angular frequency of the harmonic oscillator, and N a positive integer. Therefore, the second order energy can be easily calculated with sufficient accuracy.

Using the mentioned procedure, at first, the P -state double splitting of He^5 is calculated. The second order energies of the $P_{3/2}$ and $P_{1/2}$ states are expressed as the sum of the two kinds of contributions. The one contribution comes from the matrix elements of the two particles in the triplet even states, and the other, from those in the triplet odd states. By the former contribution the $P_{3/2}$ state is always more depressed than the $P_{1/2}$ state, whereas in the latter energies of these two states depend on the shape of the two-body potential. The inverted splitting ($P_{3/2} < P_{1/2}$), therefore, will be expected since the tensor force, stronger in the triplet even states than in the triplet odd states, is consistent with the experimental data of the two-body system. The following tensor forces have actually been adopted in this calculation.

Meson potential⁴⁾: for the triplet odd

state,

$$V_T = \begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 1.0) \\ 0 & (\kappa r < 1.0) \end{cases}$$

and for the triplet even state,

$$V_T = \begin{cases} V_T^{(1\pi)}(\kappa r) & (\kappa r \geq 0.7) \\ \text{or (I) } 3V_T^{(1\pi)}(0.7) & (\kappa r < 0.7) \\ \text{or (II) } V_T^{(1\pi)}(0.7) & \end{cases}$$

$$V_T^{(1\pi)}(\kappa r) = (g_e^2/4\pi) \cdot \mu C^2 \cdot (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2/3) \\ \times S_{12} \cdot (1 + 3/(\kappa r) + 3/\kappa^2 r^2) \exp(-\kappa r) / \kappa r.$$

Serber potential⁵⁾:

$$V_T = V_0 \cdot \{ (1 - \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) / 4 \} \cdot S_{12} \cdot \exp(-\alpha^2 r^2)$$

$$V_0 = -25.8 \text{ Mev}, \alpha^{-1} = 2.41 \times 10^{-13} \text{ cm},$$

where $V_T^{(1\pi)}$ is the one pion exchange potential, $\boldsymbol{\tau}$ the isotopic spin operator, S_{12} the tensor force operator, $\kappa^{-1} = \hbar/\mu c$ the Compton wave length of the pion, and 0.08 is used as the coupling constant $g_e^2/4\pi$.

It would be interesting to examine what kind of the intermediate state is important in getting the inverted splitting. For He^5 , the most important states are $[(1S)^2(010)(1P)^3(\frac{1}{2}SL)](\frac{1}{2}S'L)J$, and $[(1S)^2(010)\{(1P)^2(TSL)n\}(\frac{1}{2}S'L')]\cdot(\frac{1}{2}S''L')J$, where (TSL) expresses the isotopic spin T , the intrinsic spin S , and the orbital angular momentum L . In these intermediate states, one or two nucleons in the $1S$ core jump up into the $1P$ -orbit. Since the most outside nucleon already occupies the $1P$ -orbit the jumping nucleons should be correlated with the most outside one, so as to satisfy the Pauli principle. This correlation effect plays a decisive role in getting the inverted splitting. Furthermore, this effect is large only in the inter-

mediate states of the lower excitation energy. In the variational calculation by Feingold,²⁾ however, the contributions from the lower intermediate states have been considerably underestimated, which fact results into the small splitting. It should be also noted that these intermediate states have not been taken into account in the calculations¹⁾ using the Brueckner method.

The similar situation also holds for the *P*-state doublet splitting of N^{15} . The calculated typical values are listed in the table, where we have used the nuclear radius $R=r_0A^{1/3}=1.3A^{1/3}$ which corresponds to the $1S$ -orbit⁶⁾ for He^5 and to the $1P$ -orbit⁷⁾ for N^{15} , because of the importance of the interaction of the nucleons in these orbits.

Table The splitting energy (Mev)

Element	He^5	N^{15}
Meson I	4.0	3.1
Meson II	2.2	1.9
Serber	2.1	6.4
Experiment ⁸⁾	2.5~6.0	6.3

The detailed results will soon be reported in this journal and also, the effects of the tensor force on O^{17} and Ca^{41} are investigated in cooperation with Dr. A. Arima.

The author wishes to express his thanks to Prof. T. Muto, Prof. H. Horie, Drs. A. Arima, M. Kawai, H. Ui and R. Tamagaki for their valuable suggestions and discussions.

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Note on the Low Energy Electron-Hydrogen Scattering

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In our previous paper,¹⁾ the low energy electron-hydrogen scattering was discussed on the effective range approximation. In this note, we shall examine the question of energy range of the incident electron to which this method can be applied. An error in the normalization of the ground state wave function of the negative hydrogen ion is also corrected here.

The effective range and the scattering length for the singlet (symmetric) state

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are given by the formulae (12) and (15) of ref. 1). Namely, the effective range is calculated by

$$\rho = \frac{1}{\gamma} - \frac{1}{4\pi} \int_0^\infty \Psi^2 dr_1 dr_2 \quad (1)$$

where $\psi(r_1, r_2)$ is the wave function of H^- ion, normalized as

$$\begin{aligned} \Psi(r_1, r_2) &\rightarrow \phi_\rho(r_1) \exp(-\gamma r_2) / r_2 \text{ for } r_2 \rightarrow \infty \\ &\rightarrow \phi_\rho(r_2) \exp(-\gamma r_1) / r_1 \text{ for } r_1 \rightarrow \infty \end{aligned} \quad (2)$$

and $\phi_\rho(r) = e^{-r} / \sqrt{\pi}$ is that of the hydrogen atom in the ground state, $\gamma^2/2$ being the affinity of H^- ion. The phase shift δ_+ for the singlet s wave scattering is discussed by the expansion,

$$\begin{aligned} k \cot \delta_+ &= -\gamma + (\rho/2)(\gamma^2 + k^2) \\ &\quad + O((\gamma^2 + k^2)^2), \end{aligned}$$

and the scattering length a_+ is related to γ and ρ through the relation,

$$-1/a_+ \equiv (k \cot \delta_+)_{k=0} = -\gamma + \rho\gamma^2/2. \quad (3)$$

Now, in ref. 1) we used the function of H^- calculated by Hart and Herzberg.²⁾ The definition of the normalization of these authors is the same as that given by the formula (9) of Chandrasekhar and Herzberg,³⁾ in which, however, the factor 2 is unnecessary⁴⁾ so that our previous value for $(1/4\pi) \int \psi^2 dr_1 dr_2$ is to be doubled.⁵⁾ With this correction and $\gamma^2/2 = 0.02764$ ($\gamma = 0.2351$) obtained by Hart and Herzberg, we find

$$\rho = 2.44 \sim 2.60$$

and $a_+ = 6.04$ for $\rho = 2.52$, if the effective range approximation is a good one. To investigate the range of energy for which this is the case, however, the 3rd term

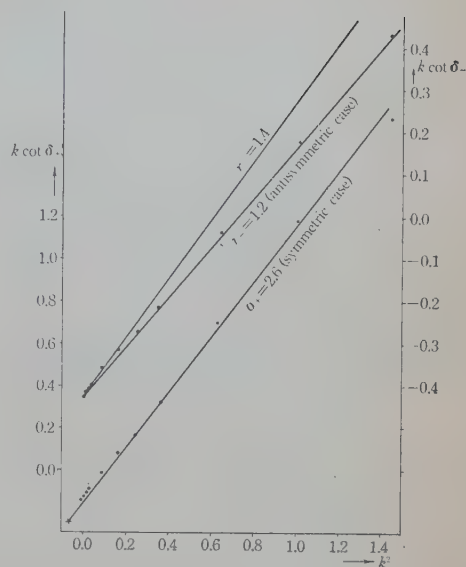


Fig.1 $k \cot \delta$ as the function of k^2 . The dotted points are the variationally computed values adopted by Bransden et al. The solid lines represent the effective range approximation.

in $k \cot \delta$ must be estimated. This is not an easy task. Here, we shall compare the results obtained by the variational method with the prediction by the effective range theory. In Fig. 1, $k \cot \delta$ from the s wave phase shifts adopted by Bransden et al.⁷⁾ are plotted. The good agreement would indicate the validity of the effective range approximation to fairly high energy ($k^2 \leq 0.75$, elastic region).

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ref. 3), so that a factor $1/2$ must be added to (3.2), or the numerical values of L , M , N of the Table I of ref. 6) must be doubled. This was confirmed by the correspondence with Prof. Kinoshita. (We were also informed that the doubling was not necessary in the case of 10 parameters.)

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Note on the Spin-Orbit Coupling and the Tensor Forces

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The origin of the spin-orbit coupling in the nuclear shell model has not been clarified unambiguously. Some people assert that the spin-orbit coupling in nucleon-nucleon interaction is responsible for that in a nucleus. Recent investigation of nuclear forces developed by Japanese group¹⁾ shows, however, that the spin-orbit force with a rather long range, such as proposed by Gammel and Thaler and Signell and Marshak,²⁾ is unnecessary for explaining the two nucleon data with energies lower than about 100 Mev.

Several authors³⁾ attempted to explain

the spin-orbit coupling in terms of the tensor part of nucleon-nucleon interaction but they got negative results. Jancovici⁴⁾ put the tensor force between two nucleons and calculated the spin-orbit coupling of a nucleon with nuclei using the perturbation theoretical approximation. He obtained the favorable result when the nucleon, the spin-orbit coupling of which with the nucleus being considered, has a rather high energy, but when the nucleon has a momentum near the Fermi momentum k_F he has found that the sign of spin-orbit coupling is opposite to that derived from experiments.

His calculation corresponds to the process in which the target nucleus is polarized in the intermediate state by the incident nucleon (the so-called "induced target polarization" term). There is another process which has important contribution to the spin-orbit coupling: The incident nucleon can be exchanged by a nucleon in the target nucleus which is already deformed by the tensor interaction between nucleons in it (the so-called "target exchange" term).⁵⁾ Using the same approximation as Jancovici's we can show that this process can be written in terms of the potential working on the incident nucleon, and this potential contains the spin-orbit coupling if we take the tensor interaction between nucleons. This process can be visualized in Fig. 1 by the Bethe-Goldstone diagram.

The importance of this exchange term in the case where the incident nucleon has a momentum near k_F can be easily seen because in this case the overlap of the wave function of the incident nucleon

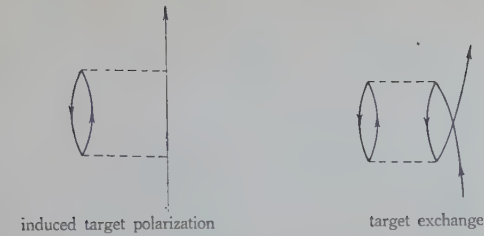


Fig. 1 Bethe-Goldstone diagrams for two processes.

with that of the target nucleus becomes large. Nagata et al.⁶⁾ analyzed the p - α scattering experiments and showed that the splitting of the phase shifts of $p_{3/2}$ and $p_{1/2}$ waves of the proton scattered from He^4 can be explained by this effect. T. Terasawa also estimated the similar terms in the energy-level splitting of He^5 and Ni^{15} .⁷⁾

We used the tensor force $V_{ij} = S_{ij} \cdot v(r_{ij})$ as the interaction between the i -th and j -th nucleon and calculated the term above mentioned in the approximation of second order perturbation with respect to V . We approximated the undeformed nucleus by the Fermi gas model. The result obtained is as follows: The spin-orbit coupling term coming from this process can be written as

$$a \frac{1}{r} \frac{\partial \rho}{\partial r} (\mathbf{l} \cdot \mathbf{s})$$

and the main part of a has the form

$$a \propto \int \Phi(x) [w(x)]^2 dx$$

Here $w(x)$ is the Bessel transform of the potential $v(r)$, x being the momentum measured in unit of the Fermi momentum. The function $\Phi(x)$ is plotted in Fig. 2 in the case where the momentum of the incident nucleon is

k_F . The corresponding function used by Jancovici⁴⁾ is also plotted for the sake of comparison. The numerical value of a depends on the shape and magnitude of $v(r)$. (See Table I.) But the sign of a is positive due to the nature of positive definiteness of $\Phi(x)$.

Table I Value of a for typical potentials

Potential	target exchange	induced polarization	exp.
Meson	63	-46	Several tens Mev (10^{-13}cm) ⁵
Gartenhaus	13	7	

$$\text{Meson: } V(r) = V_0 \left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} \right) \frac{e^{-\mu r}}{\mu r}$$

(μ : reciprocal Compton wave length of pion)

$$\text{Gartenhaus: } V(r) = V_0 (\mu r + 1) (e^{-\mu r} / \mu r)$$

$$V_0 = 13 \text{ Mev.}$$

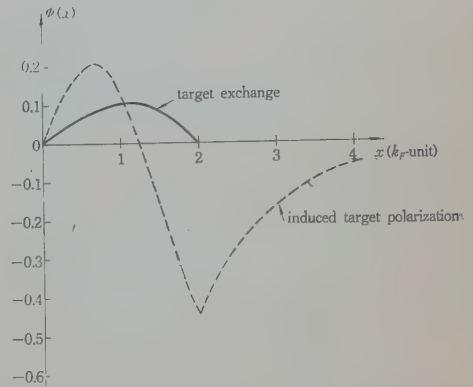


Fig. 2 The function $\Phi(r)$ for two processes

This value of a should be added to that coming from the induced polarization process. As can be seen from the graph of $\Phi(x)$, the parts of Bessel transform of $v(r)$ which give the contribution to a correspond to the region $r \gtrsim 1/3\mu$. As to the induced polarization term, negative contributions come from the region $r \lesssim 1/2\mu$, as can be seen from

Fig. 2. If we used our potential literally in this case the negative contribution to the spin-orbit coupling coming from this region would be appreciable. But the potentials used in Table I should not be extended into such an inner region. It would be reasonable to cut the potential in this region. In momentum space this region corresponds to $k \gtrsim 1.5k_F$. (We took $k_F = 1.09 \times 10^{13} \text{ cm}^{-1}$). The magnitude of induced polarization term derived by Jancovici should be reinvestigated in this sense. Even in the region $1/\mu > r > 1/2\mu$ there are some ambiguities concerning the shape and magnitude of $v(r)$. But we suppose that those in region $r > 1/\mu$ would not be essentially altered even if we take into account the fact that the nucleons are influenced by others in the nucleus. Thus we can conclude that our process gives the spin-orbit coupling in the nucleus which is correct in sign and reasonable in order of magnitude.

The detailed account will be published in near future. The authors would like to express their thanks to Dr. R. Tamagaki for valuable discussions.

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Spin-Spin Interaction in Superconductors

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Recently, Matthias, Suhl and Corenzwit¹⁾ have reported that the superconducting transition temperature of lanthanum is lowered, when small quantities of gadolinium are added, by an amount proportional to the concentration of dissolving gadolinium, and 2.5 at. % or more of gadolinium in lanthanum causes this solid solution to become ferromagnetic, whose Curie points are an approximately linear function of the concentration of gadolinium. This experimental result can be explained by the conduction electron-magnetic ion interaction ($s-d$ exchange interaction²⁾) and the BCS theory.³⁾

The Hamiltonian of the system is taken as follows,

$$H = H_{BCS} + H', \quad (1)$$

where

$$H' = - \sum_{k, k'} \sum_n J_{kk'} e^{i(k-k')R_n} \times \{ (C_{k'\uparrow}^* C_{k\uparrow} - C_{k'\downarrow}^* C_{k\downarrow}) S_n^z + C_{k'\downarrow}^* C_{k\uparrow} S_n^+ + C_{k'\uparrow}^* C_{k\downarrow} S_n^- \}, \quad (2)$$

and H_{BCS} is the Hamiltonian given in

BCS. S_n the spin operator of the n -th magnetic ion; R_n its position vector; and $J_{k,k'}$ depends solely on $|k-k'|$. With the aid of Tables II and III in BCS, the second order perturbed energy of the superconducting state can be written as

$$E_2(\epsilon_0) = \sum_{n,m} A_{nm} (S_n \cdot S_m) \quad (3)$$

where

$$A_{nm} = -\frac{1}{2} \sum_{k,k'} |J_{k,k'}|^2 \left(1 - \frac{\epsilon\epsilon' + \epsilon_0^2}{EE'} \right) \times \frac{e^{i(k-k')(Rn-Rm)}}{E+E'} \quad (4)$$

Eq. (3) expresses the indirect ion-ion interaction via the conduction electrons, which corresponds to the interaction investigated by Ruderman and Kittel⁽⁴⁾ in the case of nuclear magnetic moments. If the magnetic ions are distributed at random, we get from Eqs. (3) and (4)

$$E_2(\epsilon_0) = S(S+1)N_a A f / 100, \quad (5)$$

where

$$A = -\frac{1}{2} \sum_{k,k'} \frac{|J_{k,k'}|^2}{E+E'} \left(1 - \frac{\epsilon\epsilon' + \epsilon_0^2}{EE'} \right), \quad (6)$$

and N_a is the number of atoms in volume \mathcal{Q} ; f the atomic per cent of magnetic impurities. From Eqs. (5) and (6), the energy difference between the superconducting state and the normal state can be written as follows,

$$W_0^* = W_0 + E_2(0) - E_2(\epsilon_0). \quad (7)$$

To the order of $(\epsilon_0/\hbar\omega)$, we get

$$W_0^* = W_0 \left\{ 1 - \frac{3\pi^2}{8} \times 1.14 \times \frac{N_e^2 \langle |J_{k,k'}|^2 \rangle_f}{Z^* E_f} \cdot \frac{S(S+1)}{kT_c} \cdot \frac{f}{100} \right\}, \quad (8)$$

where $\langle \rangle_f$ denotes an average value

over the Fermi surface, N_e the number of conduction electrons in volume \mathcal{Q} , and Z^* the number of conduction electrons per atom. From Eq. (8), we find that the effect of ordinary impurity-scattering can be neglected as compared with that of s - d interaction, because the correction of ordinary impurity-scattering to the energy difference W_0 is of the order of $(\epsilon_0/\hbar\omega)^2$, as shown in the previous paper.⁽⁵⁾ Using Eq. (8) and the empirical relation that $N(0) (kT_c)^2$ is proportional to W_0 , we get for the change of the transition temperature

$$k\Delta T_c = \frac{3\pi^2}{16} \times 1.14 \times \frac{N_e^2 \langle |J_{k,k'}|^2 \rangle_f}{Z^* E_f} \cdot S(S+1) \cdot \frac{f}{100}. \quad (9)$$

From Eq. (9), we find that the change ΔT_c is proportional to the concentration of impurities f and $S(S+1)$, as suggested by Herring.⁽⁶⁾ This result coincides with the experimental fact. It is easily seen that the change ΔT_c depends on the Curie points of the ferromagnetic solid solution. By the use of the molecular field approximation,⁽⁷⁾ the Curie point can be expressed as follows,

$$k\theta_c = \frac{N_e^2 J(0)^2}{2Z^* E_f} \cdot S(S+1) \cdot \frac{f}{100}. \quad (10)$$

Eq. (10) explains the result observed by Matthias et al. From Eqs. (9) and (10), we get

$$\Delta T_c = \frac{3\pi^2}{8} \times 1.14 \times \frac{\langle |J_{k,k'}|^2 \rangle_f}{J(0)^2} \cdot \theta_c. \quad (11)$$

Since $J_{k,k'}$ is nearly equal to $J(0)$, Eq. (11) can be rewritten as

$$\Delta T_c = \frac{3\pi^2}{4} \times 1.14 \times \theta_c. \quad (12)$$

Eq. (12) corresponds to the experimental fact that the change of the transition temperature is proportional to the residual resistivity in the case of non-magnetic impurity.⁸⁾ For the purpose of numerical comparison with the experimental result, we apply Eq. (12) to the case of lanthanum. From Matthias, Suhl and Corenzwit's experiment, we can write as follows,

$$\theta_0 = 0.5 f. \quad (13)$$

Substituting Eq. (13) in Eq. (12), we obtain

$$\Delta T_0 = 2.1 f. \quad (14)$$

The numerical factor of f in Eq. (14) is about one-half of the experimental value. Our calculated value is, therefore, in qualitative agreement with the experimental result. Matthias et al. suggest that there seems to be a correlation between the Curie points in certain alloys and compounds which include the rare earth elements and the superconducting transition temperatures of other closely related alloys and compounds. Moreover, from Eq. (12), it is likely that there is a correlation between the Curie points and the superconducting transition temperatures even

in the same alloy. Furthermore, they suggest that an exchange over conduction electrons leading to ferromagnetism is easily brought in an element which by itself is a superconductor. It is very interesting to investigate the solidity of their suggestion in the light of the BCS theory. The details of this work will be published in the near future.

The author would like to express his sincere gratitude to Professor H. Ichimura for his helpful discussions. Also he is particularly grateful to Dr. Y. Ishikawa, chief of the fundamental research section, for his kind encouragement.

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Errata

σ - π Interaction and Proton Hyperfine Interaction

Shigeyuki Aono

Prog. Theor. Phys. **21** (1959), 779

After the equation (42) on page 786 the following should be inserted.

Summing up about \tilde{J} and considering the property of the delta function, the cross term, e. g. between $\delta(r_{\mu N})$ and $1/r_{\tau\mu}P_{\tau\mu}$ can be dropped, and we obtain

$$\begin{aligned} \delta_N = 1/(E_{AV}S_Z) \{ & \langle \tilde{O} | (1/r_{\tau\mu}) P_{\tau\mu} \delta(r_{\nu N}) S_{\nu Z} (1 - P_{\mu\nu}) | \tilde{O} \rangle \\ & + \langle \tilde{O} | (1/r_{\tau\nu}) P_{\tau\nu} \delta(r_{\mu N}) S_{\mu Z} (1 - P_{\mu\nu}) | \tilde{O} \rangle \}. \end{aligned} \quad (43)$$

Multipole Model of Elementary Particles. I**—Pion-Baryon Interactions—**

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A simple model is presented for a unified understanding of elementary particles phenomena. In the case of the pion-baryon system the strong interaction is interpreted as the monopole interaction and the weak interaction as the dipole interaction between pions and structured baryons of order 10^{-20}cm . The nature of the weak interaction is discussed on the basis of the multipole model.

§ 1 Introduction

In the recent progress in elementary particle theory, concerning the fundamental problems such as those of the divergences of field theory or the curious behaviours of the weak interactions and so on, the several speculations have been presented by many authors in order to get into the detailed knowledge of the micro world. Among others, the following seem to us rather promising in the future analysis of the problems. They are:

- 1) For the removal of the divergence difficulties in field theory the introduction of the fundamental length which in the end concerns the size of the elementary particle is of essential importance¹⁾.
- 2) The coupling length appearing in the second kind interactions might be just the length expected in the above thought. In fact, however, instead of such presumption the singularity of the second kind interactions makes impossible even the renormalization procedure²⁾. Probably, the length in the second kind interactions might indicate some limit beyond which the present field theory should be modified appreciably¹⁾.
- 3) The well-known second kind interaction is only that of β decay, but all the weak interaction may be of the second kind. If really so, the coupling constants can be related with the small length of order of $10^{-17}\sim 10^{-20}\text{cm}$ ³⁾.
- 4) As an example of the remarkable change of the current laws in the very small region, the so-called weak interactions may become compatible with or stronger

than what is called the strong interactions⁴⁾.

These speculations have been given case by case without any unified point of view. However, if each description reproduces a partial aspect of the reality the common picture behind each special view-point will be an essential key to the future unified theory. As an attempt towards such unification, a simple model of the particles is given in the present note where the strong interactions are interpreted as monopole interactions and the weak interactions as multipole interactions between small extended particles. Then, the above speculations will be understood in a more or less concrete and unified picture of the particles. With help of such model one may imagine a possible violation of parity conservation in the weak interaction. The latter problem will be discussed together with the problem of universal Fermi coupling in the next paper.

What is presented here is one of the simplest models, a narrow and specific representation of the reality; this model should not be taken too literally. In order to arrive at a unified description of the world the general character of the internal space of the particles must be analysed in detail. This will be one of the main tasks of our forthcoming work.

§ 2 Pion-baryon interactions

In the research of the time-space structure of elementary particles the most important will be the dimension of length probably coupled intimately to the size of particles. For simplicity and reality of the problem the discussion will be confined first to the pion-baryon interactions, because the structure of baryons is now of real character as it appeared in the electron-nucleon or pion-nucleon scattering.* In addition, both the weak and strong interactions appear in the pion-baryon system.

For the strong interactions we suppose the global symmetry⁶⁾ so far as it goes.** For the moment then, the strong interactions are described by the Lagrangean

$$-L_{strong}(x) = if \sum \bar{B}^{nk}(x) \tau_a \gamma_5 B^{nk}(x) \phi_a(x) \quad (1)$$

where[†]

* Can the nucleon radius of order 10^{-14} cm appearing in $e-N$ scattering be explained by field theory essentially on the basis of the point model? Or, does the change of natural laws, as suggested by Heisenberg, take place in such a region? Our previous analysis could not derive a definite answer. However, according to the preliminary result, the so-called acausality, if existed, might appear in the region of order smaller than 10^{-15} cm.⁵⁾

** We have not any intention to stick to the global symmetry, but only for the simplicity of argument such forms are used hereafter.

† To explain experimental results the assignment of n and k may be changed. However, such modifications are of no importance in our present discussion, because our main concerns are in a unified understanding of the various phenomena by the extended particle model but not yet their detailed description.

$$\begin{array}{cccc}
 n=0 & 1 & 1 & 2 \\
 k=0 & 0 & 1 & 1 \\
 B^{nk} \equiv \left(\begin{array}{c} p \\ n \end{array} \right) & \left(\begin{array}{c} \Sigma^+ \\ Y^0 \end{array} \right) & \left(\begin{array}{c} Z^0 \\ \Sigma^- \end{array} \right) & \left(\begin{array}{c} \Xi^0 \\ \Xi^- \end{array} \right).
 \end{array} \quad (2)$$

For the weak interactions the non conservation of parity have been confirmed experimentally in the Λ and Σ decay processes.⁷⁾ Umezawa showed⁸⁾ then that in the decay interactions if the parity conserving part is of equal amount to that of non conserving part (one to one rule) the Pv interactions are preferred to the Ps interactions in order to explain the experimental asymmetry parameter. So we suppose the following Lagrangean for the weak interactions:*

$$\begin{aligned}
 -L_{weak}(x) &= ig \sum \bar{B}^{nk}(x) (1 + \gamma_5) \gamma_\mu \tau_\alpha B^{n\pm 1k}(x) \frac{d\phi_\alpha(x)}{dx_\mu} \\
 g &\approx 10^{-20} \text{cm.} \quad (\hbar = c = 1)
 \end{aligned} \quad (3)$$

Therefore, combining the above weak and strong Lagrangeans we obtain the following general Lagrangean,

$$\begin{aligned}
 -L(x) &= if \bar{B}^{nk}(x) \gamma_5 \tau_\alpha B^{nk}(x) \phi_\alpha(x) \\
 &\quad + if l \bar{B}^{nk}(x) (1 + \gamma_5) \gamma_\mu \tau_\alpha B^{n\pm 1k}(x) \frac{d\phi_\alpha(x)}{dx_\mu} \\
 l &\approx 10^{-20} \text{cm.} \quad (\hbar = c = 1)
 \end{aligned} \quad (4)$$

One may call it a *semi-phenomenological and semi-speculative* Lagrangean of the pion-baryon interactions. In fact, apart from the details the general character of the pion-baryon interactions is well reproduced in this form. Our present task is to search the particle picture or particle structure behind the above speculative but also phenomenological representation.

§ 3. Multipole model of pion-baryon system

As baryon number is well conserved in the interactions the baryon will be a real substance. Naively speaking, such substance may have some time-space structure as do the atoms or molecules. Indeed, the second kind interaction in the derivative form implies an extended source picture in the space.⁹⁾ If the pion source density at y of baryon (the center of mass at x) is written as $\rho_\alpha(x, y-x) \equiv \rho_\alpha(x, z)$, the pion-baryon interactions become:

* Such straight extension of the global symmetry to the weak interactions is not so reasonable because the direct process $\Sigma^- \rightarrow n + \pi^-$ is not contained. See also the footnote** and † above. The present description of the charge space may be extensively modified in the forthcoming papers.

$$-L(x) = \int \rho_\alpha(x, y-x) \phi_\alpha(y) dy \equiv \int \rho_\alpha(x, z) \phi_\alpha(x+z) dz \quad (5)$$

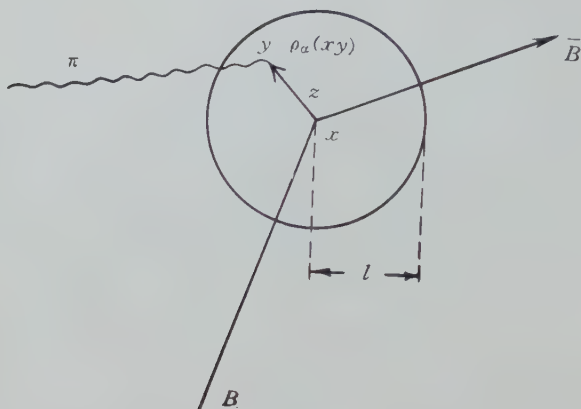


Fig. 1 The pion source density at y of a baryon whose center of mass is at x . $\rho_\alpha(xy) \equiv \rho(x, x-z) \equiv \rho_\alpha(xz)$

where ϕ_α is the pion field. It is supposed hereafter that the density $\rho_\alpha(x, z)$ is non locally spread in the space of order l (Fig. 1). Inside the region smaller than l nothing is known, hence baryon should be represented by the nonlocal field¹⁰⁾:

$$\rho_\alpha(x, z) = if \bar{\psi}(x, z) \tau_\alpha \psi(x, z) \quad (6)$$

Analogously to the relativistic two-body problems¹¹⁾ the internal state of baryon will be assigned by the four numbers $(n, k, j; m)$ as $U_{jm}^{nk}(z)$. If the global symmetry holds, $j = \frac{1}{2}$ and $m \equiv \rho = \pm \frac{1}{2}$.

$$\begin{aligned} \psi(x, z) &= \sum U_{\rho}^{nk}(z) B^{nk}(x) \\ \bar{\psi}(x, z) &= \sum \bar{B}^{n'k'}(x) \bar{U}_{\rho'}^{n'k'}(z) \end{aligned} \quad (7)$$

As l is very small the pion field $\phi(x+z)$ is expanded in the multipole form:

$$\phi_\alpha(x+z) = \phi_\alpha(x) + z_\mu \frac{d\phi_\alpha(x)}{dx_\mu} + \dots \quad (8)$$

Then, supposing that the internal structure has not any preferred direction the Lagrangean is reduced to the following form:

$$\begin{aligned} -L(x) &= if [\langle n'k' | A | nk \rangle \bar{B}^{n'k'}(x) \tau_\alpha B^{nk}(x) \\ &\quad + \langle n'k' | B | nk \rangle \bar{B}^{n'k'}(x) \tau_\alpha \gamma_5 B^{nk}(x)] \phi_\alpha(x) \\ &+ if l [\langle n'k' | C | nk \rangle \bar{B}^{n'k'}(x) \tau_\alpha \gamma_\mu B^{nk}(x) \\ &\quad + \langle n'k' | C | nk \rangle \bar{B}^{n'k'}(x) \tau_\alpha \gamma_5 \gamma_\mu B^{nk}(x)] \frac{d\phi_\alpha(x)}{dx_\mu} + \dots \end{aligned} \quad (9)$$

because the integrals of the internal functions $\int \bar{U} U dz$ or $\int \bar{U} z U dz$ can be expanded in γ -algebra with respect to spin suffix:

$$\begin{aligned} \int \bar{U}_{\rho'}^{n'k'}(z) U_{\rho''}^{n''k''}(z) dz &= \delta_{\rho'\rho''} \langle n'k' | A | n''k'' \rangle + (\gamma_5)_{\rho'\rho''} \langle n'k' | B | n''k'' \rangle \\ \frac{1}{l} \int \bar{U}_{\rho'}^{n'k'}(z) z_{\mu} U_{\rho''}^{n''k''}(z) dz &= (\gamma_{\mu})_{\rho'\rho''} \langle n'k' | C' | n''k'' \rangle + (\gamma_5 \gamma_{\mu})_{\rho'\rho''} \langle n'k' | C | n''k'' \rangle. \end{aligned} \quad (10)$$

Now, regarding the diverse internal states characterized by n and k as different baryons, $B^{nk}(x)$ fields above can be interpreted as the baryon fields just given in (1). Then, the conservation of parity in the strong interactions is guaranteed if $\langle n'k' | A | nk \rangle = 0$. Non conservation of parity in the weak interactions is described by the relation $\langle n'k' | C' | nk \rangle = \alpha \delta_{n,n \pm 1} \delta_{k,k'}$ together with the requirement $\langle n'k' | C | nk \rangle = \beta \delta_{n,n \pm 1} \delta_{k,k'}$. In other words, the selection rules or conservation laws in the baryon interactions are interpreted in our model as those in the transitions between the internal baryon states.

$$\langle n'k' | A | nk \rangle = (1/4) \int \bar{U}_{\rho}^{n'k'}(z) U_{\rho}^{nk}(z) dz = 0$$

(Conservation of parity in the strong interactions)

$$\langle n'k' | B | nk \rangle = (1/4) \int \bar{U}_{\rho}^{n'k'}(z) (\gamma_5)_{\rho\sigma} U_{\sigma}^{nk}(z) dz = \delta_{nn'} \delta_{kk'}. \quad (11)$$

(Conservation of strangeness in the strong interactions)

$$\langle n'k' | C' | nk \rangle = (1/16l) \int \bar{U}_{\rho}^{n'k'}(z) (\gamma \cdot z)_{\rho\sigma} U_{\sigma}^{nk}(z) dz = \alpha \delta_{n,n \pm 1} \delta_{kk'}^*.$$

(Non conservation of parity and strangeness in the weak interactions)

$$\langle n'k' | C | nk \rangle = (1/16l) \int \bar{U}_{\rho}^{n'k'}(z) (\gamma_5 \gamma \cdot z)_{\rho\sigma} U_{\sigma}^{nk}(z) dz = \beta \delta_{n,n \pm 1} \delta_{kk'}^*.$$

$$\alpha \neq 0, \quad \beta \neq 0.$$

Such selection rules reduce the Lagrangean to the simple form as expected:

$$\begin{aligned} -L(x) &= i f \bar{B}_{nk}(x) \tau_{\alpha} \gamma_5 B^{nk}(x) \phi_{\alpha}(x) \\ &+ i f l \bar{B}^{nk}(x) \tau_{\alpha} (\alpha + \beta \gamma_5) \gamma_{\mu} B^{n \pm 1 k}(x) \frac{d\phi_{\alpha}(x)}{dx_{\mu}} + \dots \end{aligned} \quad (12)$$

This is just the Lagrangean (4) proposed in Section 2. That is, if the transition components of baryons obey the selection rules (11), without any detail of the internal mechanics our extended interaction (5) transforms to the semi-

* See the footnote[†] in the next page.

phenomenological form (4) which well reproduces the experimental fact. Therefore, corresponding to the length l appearing in the second kind interaction (4), the spread of the pion source $\rho(x, z)$ in our model is of order of $l \sim 10^{-20}$ cm. In the inner region of $l \sim 10^{-20}$ cm the unknown mechanics may be predominant as expected by Heisenberg.

In the above argument the pion structure is not considered because of lack of its evidence. However, if the pion had a structure of order of 10^{-20} cm our treatment should be so to say 'classical'. That is, the above picture would correspond to 'Correspondentzmassige Strahlungstheorie'. In such terminology the relations (11) may be called 'Baryon's transition components'. There, the unknown mechanics is represented in terms of the transition components of the internal structure.

§ 4. Discussion of the various speculations

Basing on the above simple multipole model, the meaning or relations of the various speculations mentioned in Section 1 are discussed.

1) Weak interactions

As was discussed in the previous sections the second kind interactions are represented in our model by the dipole interactions between the pion and the structured baryons. The length l appearing in the second kind coupling is then interpreted as a dipole arm in the dipole moment of baryon. Consequently, the reason why the interactions should be weak is answered in our terminology by the smallness of the particle size $l \sim 10^{-20}$ cm.

The semi-empirical results of Umezawa that all the weak interactions are of the second kind can be interpreted as the dipole interactions due to the extended structure of baryons of order of 10^{-20} cm.* The monopole interactions between pions and baryons are no other than the strong interactions independent of the baryon structure.*)

[†] Here α becomes scalar, β pseudoscalar. In our picture the coupling constant should be derived, by nature, from the internal laws in the unknown region, and it will not be unaccountable to have a pseudoscalar coupling constant. In fact, the same situation appears in the magnetism where the strength of the magnetic pole m defined by $mH=F$ (H : magnetic field, F : force) is pseudoscalar. Further, in the derivation of this Lagrangean through the relations (10) the appearance of the axial vector $C'\gamma_\mu + C\gamma_5\gamma_\mu$ from the average of the polar vector $\int \bar{U}z_\mu U dz$ is well expected in favour of the non conservation of parity. Because the same is true in the case of optical activity where the electric field E induces the response $p_i = C_{ij}E_j$ in an asymmetric molecule of $C_{ij} \neq C_{ji}$. (See M. Born: Optik pp. 403, 413) There, in the average of the polar vector $\bar{\psi}p_i\psi = \frac{1}{2}(C_{ij} + C_{ji})E_j + (\mathbf{d} \times \mathbf{E})$ the axial vector appears which gives rise to the so-called optical activity. (\mathbf{d} is a vector depending on $(C_{ij} - C_{ji})$). This problem of parity will be discussed in the next paper.

* If there exists a short range interaction $\bar{B}(x)B(x)A(x-x')\bar{L}(x')L(x')$ between baryon B and lepton L just corresponding to the shift from the Coulomb force in $e-N$ scattering, the universal Fermi $V-A$ interaction may be derived as a Van der Waals effect for the core interaction. This problem is discussed in the next paper.

2) Blokhintsev-Umezawa effect

The 'classical' pion field at x by a baryon whose center of mass at x' is: (See Fig. 2)

$$\begin{aligned}
 \phi_\alpha(x) &= \int \Delta(x-x'-z) \rho_\alpha(x'z) dz \\
 &= \Delta(x-x') \int \rho_\alpha(x'z) dz \\
 &\quad + \frac{d\Delta(x-x')}{dx_\mu} \int z_\mu \rho_\alpha(x'z) dz + \dots \quad |x-x'| \gg l \quad (13) \\
 &= \Delta(x-x') i f \bar{B}^{nk}(x') \tau_\alpha \gamma_5 B^{nk}(x') \\
 &\quad + \frac{d\Delta(x-x')}{dx_\mu} i f l \bar{B}^{nk}(x') (\alpha + \beta \gamma_5) \gamma_\mu \tau_\alpha B^{n\pm 1k}(x') \quad (14) \\
 &\quad + \dots
 \end{aligned}$$

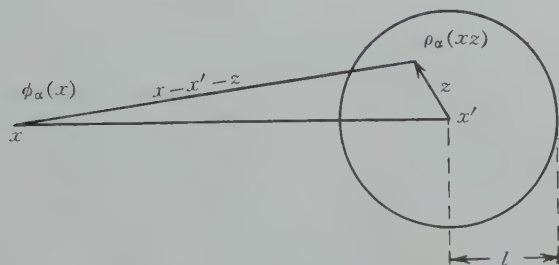


Fig. 2 The classical pionfield at x produced by a baryon whose center of mass is at x' . $\phi_\alpha(x) = \Delta_\pi(x-x'-z) \rho_\alpha(x'z) dz = \Delta_\pi(x-x') \rho_\alpha(x'z) dz + d\Delta_\pi(x-x')/dx_\mu \int z_\mu \rho_\alpha(x'z) dz + \dots$

Evidently such multipole expansion is allowed only for $|x-x'| \gg l$, and if $|x-x'| \approx l$ all multipoles contributes equally. In other words, the distinction between the 'strong' and 'weak' interactions will be possible only for $|x-x'| \gg l \approx 10^{-20}$ cm, and for $|x-x'| \lesssim l$ the 'weak' interactions become so effective as 'strong' interactions losing the meaning of distinction between the two interactions. This is an interpretation by our model of Blokhintsev-Umezawa's proposition.

3) Divergences of the second kind interactions

As seen above, if the second kind interactions are really multipole interactions between pions and structured baryons the phenomenological interaction (4) is nothing but an approximation of the second order in the multipole expansion. Even if such approximation were a good one for $|x-x'| \gg l$, a straight extrapolation to $|x-x'| \sim 0$ should give rise to a strong singularity which makes its renormalization impossible. By origin, the length dimension in the second kind interaction was introduced to give a finite size to particles in hope of eliminating the divergences.

On the contrary, the renormalization was proved to be impossible due to the strong singularity caused by the length dimension. In our language, the unreasonable approximation of multipole expansion answers for the unrenormalizable divergences.

4) Change of laws in the small region

Speaking analogously to the electrostatics, the 'inner part of particles' $|x-x'| < l$ should be expanded in inverse powers of z instead of powers of z . Then, in the inner region the integrals of type $\int \bar{U}U/z \, dz$ would give rise to new selection rules which might be completely different from familiar ones. Such inner transitions would not give any longer the strangeness rules well-known for the 'outer' interactions of baryons. That is, in the inner part of baryons all transitions equally contribute to the pion field, and the so-called "change of current laws" may take place where even the meaning of strangeness is lost.

Such situation intimately concerns the divergence problems. For example, the second order selfenergy of baryon due to the pion field can be written in the form: (See Fig. 3)

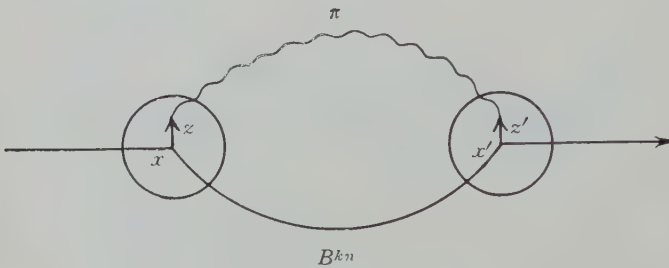


Fig. 3 The second order self energy of a baryon due to pion field.

$$E = - \int dx dx' U(x, x') \quad (15)$$

where U could be written explicitly. This is the selfenergy of baryon taking into account up to the weak interactions. However, our multipole expansion loses its meaning in the inner region $|x-x'| \leq l$. Therefore, E_{self} might have been of the form:

$$E_{self} = - \int_{|x-x'| > l} dx dx' U(x, x') + \int_{|x-x'| > l} dx dx' V(x, x') \quad (16)$$

In the second part V , all baryon fields contributing equally, the selfenergy of a nucleon only due to the pion-nucleon system will be meaningless. By nature, we should calculate the inner part $|x-x'| < l$ leaving nonlocal field as it is. Accordingly, the calculations by the phenomenological interaction (4) corresponds to an extrapolation of the first term U up to the region $|x-x'| < l$ beyond its applying limit, thus resulting a violation of the condition for the primitive divergence. In this sense Heisenberg's opinion is very acceptable that even for the point model the

momentum integral should be cut off at a certain value.

5. Concluding remarks

Taking account of small size of elementary particles we have presented a simple model (multipole model) for the pion-baryon interactions with an intention of understanding the various phenomena in a more or less unified way. Of course, we have not any pretension for the generality or approbation of the interpretation of the weak interactions by such simple model. However, one may naively recognize that there should be some common picture behind the diverse current speculations about the nature of the weak interactions. Our main purpose in the present note was to obtain a simple model in the research of the unified picture. In the next paper the non conservation of parity and the universal Fermi interactions will be discussed on the basis of the multipole model.

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Multipole Model of Elementary Particles. II

—Non Conservation of Parity and Baryon-Lepton Interactions—

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The problems of the non conservation of parity and the universal Fermi interaction are discussed on the basis of the multipole model. The non conservation of parity comes from the asymmetric structure in the small region of order 10^{-20}cm . The coupling length $l \sim 10^{-17}\text{cm}$ in the β -decay is derived from the interaction of the zitterbewegung baryons and extended leptons of order 10^{-20}cm . Its coupling becomes, however, of axial vector type instead of vector-axial vector type.

§ 1. Introduction

In the preceding paper,¹⁾ concerning the pion-baryon system, the multipole model was proposed regarding the weak interactions as the second kind interactions. Moreover, not only the above phenomena but also the following two important evidences may support the physical background of the model. That is:

- 1) In the weak interactions parity is not conserved.
- 2) The universal Fermi interactions seem to be of $V-A$ type, and the coupling constant of β -decay is characterized by $l \sim 10^{-17}\text{cm}$.

The interpretation of these two facts will be discussed in the present note. The non conservation of parity is considered as an effect of the asymmetric structure of particles just analogously to the optical activity of asymmetric molecules. The coupling length $l \sim 10^{-17}\text{cm}$ of β -decay is derived from the interaction between the extended lepton and zitterbewegung baryon. The coupling type becomes, however, of axial vector instead of vector-axial vector.

§ 2. Optical activity

Non conservation of parity may be understood by an analogous consideration in the classical optics where the rotation of the polarized plane occurs in the

optically active medium such as crystals or grape sugar. With this intention in mind the mechanism of the optical activity will be briefly reviewed.

As is well known, the rotation of polarization plane results from the difference of the phase velocity of light depending on the reflective index, susceptibility, dipole moment, and so on, of the optically active medium. Such optical activity directly concerns the symmetrical character of molecules or crystal lattices because the induced dipole moment of molecules or crystal lattices essentially depends on the polarization of light, and it is this dipole moment which determines the phase of the scattered waves. However, in the optical activity the two different effects of asymmetric structure appear. That is, in the case of hexaprisms crystals the asymmetry of crystal lattice causes the rotation of light plane. On the other hand, in the case of grape sugar or lactic acid the parity asymmetry of the molecular structure is just the source of their optical activity. Our main purpose is, analogously to the classical optics, to relate the non conservation of parity in the weak interaction with the asymmetrical density distribution in the very small region of order 10^{-20} cm. Therefore, a matter of concern is the optical activity of molecules but not of crystals.

The optical activity in the organic compound appears when its molecular structure is different from its mirror image. For example, the carbon compound composed of different radicals or atoms is not isomorphic with its mirror image. (See Fig. 1). The rotating direction of the polarized light in the molecule is reverse to that of its mirror molecule. That is, if the molecule is laevo rotatory its mirror molecule is dextro rotatory. It must be noted there that the distribution of electric charge which determines the optical activity is not invariant with respect to the mirror image.

Then, how does the electric cloud behave when it enters the incident plane wave? If the wave length λ is sufficiently longer than the molecular length, the most important response will be the oscillation of the total cloud charge $e = \int \rho(z) dz$ regarded as concentrated at the center of mass. ($\rho(z)$: charge density at z). In this case, even if the cloud density is not symmetric with respect to the reflection, the asymmetric effect does not appear at all. Next, in the second order contribution the dipole moment $\mathbf{p} = \int \rho(z) \mathbf{z} dz$ plays a role which depends on the electric distribution of cloud. The effect of the asymmetric character of cloud thus comes into play most markedly in the rotation of molecule forced by the circularly polarized light. The induced rotation of the dipole moment \mathbf{p} of molecule emits

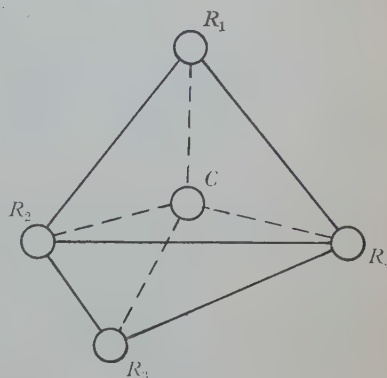


Fig. 1 Optically active carbon compound.

Lactic acid ($R_1=H$, $R_2=OH$,
 $R_3=CH_3$, $R_4=COOH$)
 Amil alcohol ($R_1=H$, $R_2=CH_3$,
 $R_3=CH_2OH$, $R_4=C_2H_5$)

In these compounds the two isomers, laevo and dextro rotatory, are well known.

the secondary circular waves which determine the phase velocity of light in the solution. Therefore, the difference of the dipole moment due to the asymmetric charge distribution of molecule is the very cause of the optical activity.

With regard to the optical activity the following point should be remarked. Even if there exists a right-left asymmetry in the charge distribution its effect does not appear in the lowest order process but appears by retardation in the multipole processes. The reason why we have taken the multipole model lies in such physical insight. Namely, even if the elementary particle has its structure in the small region of order $l \sim 10^{-20}$ cm violating the symmetry with respect to the reflection, its effect does not enter into the monopole interaction (strong interaction) but come to pass only in the multipole interaction (weak interaction).

§ 3. Parity non conservation

Then, how are the two facts, the non conservation of parity and the asymmetric structure of the particles, related to each other? In the preceding paper, the following transition components were required in order to obtain the pion-baryon interactions, both strong and weak, which reproduce the experimental results:

$$A = \frac{1}{4} \int \bar{U}_p(z) U_p(z) d^4z = 0, \quad (1)$$

(conservation of parity in strong interactions)

$$B = \frac{1}{4} \int \bar{U}_p(z) (\gamma_5)_{\rho\sigma} U_\sigma(z) d^4z \neq 0, \quad (2)$$

$$C' = \frac{1}{16l} \int \bar{U}_p(z) (\gamma \cdot z)_{\rho\sigma} U_p(z) d^4z \neq 0, \quad (3)$$

(non conservation of parity in weak interactions)

$$C = \frac{1}{16l} \int \bar{U}_p(z) (\gamma_5 \gamma \cdot z)_{\rho\sigma} U_\sigma(z) d^4z \neq 0. \quad (4)$$

In order to see the meaning of these rules we shall separate the density matrices $D = \bar{U} \gamma_5 U$ in the internal space into two parts, symmetric and asymmetric:

$$\begin{aligned} D_{\rho\sigma}(\mathbf{z}, z_0) &\equiv U_\rho^*(\mathbf{z}, z_0) U_\sigma(\mathbf{z}, z_0), \\ U^*(\mathbf{z}, z_0) &\equiv \bar{U}(\mathbf{z}, z_0) \gamma_5. \end{aligned} \quad (5)$$

$$\begin{aligned} D_{\rho\sigma}(\mathbf{z}, z_0) &= D_{\rho\sigma}^s(\mathbf{z}, z_0) + D_{\rho\sigma}^a(\mathbf{z}, z_0), \\ D_{\rho\sigma}^s(\mathbf{z}, z_0) &\equiv \frac{1}{2} [U_\rho^*(\mathbf{z}, z_0) U_\sigma(\mathbf{z}, z_0) + (U^*(-\mathbf{z}, z_0) \gamma_4)_\rho (\gamma_4 U(-\mathbf{z}, z_0))_\sigma], \\ D_{\rho\sigma}^a(\mathbf{z}, z_0) &\equiv \frac{1}{2} [U_\rho^*(\mathbf{z}, z_0) U_\sigma(\mathbf{z}, z_0) - (U^*(-\mathbf{z}, z_0) \gamma_4)_\rho (\gamma_4 U(-\mathbf{z}, z_0))_\sigma]. \end{aligned} \quad (6)$$

Then, the transition components of baryons are written as

$$A = \frac{1}{4} \text{Sp} \int \gamma_5 D^a(\mathbf{z}, z_0) d^4z = 0, \quad (1')$$

$$B = \frac{1}{4} \text{Sp} \int D^s(\mathbf{z}, z_0) d^4z \neq 0, \quad (2')$$

$$C' = \frac{1}{16l} \text{Sp} \int (\gamma_5 \gamma \cdot \mathbf{z}) D^a(\mathbf{z}, z_0) d^4z \neq 0, \quad (3')$$

$$C = \frac{1}{16l} \text{Sp} \int (\gamma \cdot \mathbf{z}) D^s(\mathbf{z}, z_0) d^4z \neq 0. \quad (4')$$

If the internal structure is right-left symmetric, i. e. the internal function is invariant with respect to the space reflection,

$$\begin{aligned} U(\mathbf{z}, z_0) &= (i\gamma_4) U(-\mathbf{z}, z_0), \\ U^*(\mathbf{z}, z_0) &= U^*(-\mathbf{z}, z_0) (i\gamma_4), \end{aligned} \quad (7)$$

the asymmetric part of density $D^a(\mathbf{z}, z_0)$ becomes identically zero.

$$\begin{aligned} D_{\rho\sigma}^s(\mathbf{z}, z_0) &= D_{\rho\sigma}(\mathbf{z}, z_0), \\ D_{\rho\sigma}^a(\mathbf{z}, z_0) &= 0. \end{aligned} \quad (8)$$

Therefore, as is seen in (3'), the parity non conservation does not appear as long as $D^a(\mathbf{z}, z_0) = 0$. That is, if the internal structure is symmetric with respect to the space reflection the non conservation of parity does not appear in the multipole interactions. This is the same situation as in the optical activity of the molecules. In order to have the non conservation of parity in the weak interactions the baryons must be optically active in the small region $l \sim 10^{-20} \text{cm}$.

On the other hand, the condition of parity conservation in strong interactions in the case of asymmetric structure $U^a(\mathbf{z}, z_0) \neq 0$ is given by

$$A = \frac{1}{4} \text{Sp} \int D^a(\mathbf{z}, z_0) d^4z = 0. \quad (1')$$

This is a weaker condition than condition (1). It is evident that even with this condition the moment of the asymmetric density is in general not zero, i. e., $C' \neq 0$.

(1') is the condition of suppressing the appearance of the asymmetric effect in the total electric charge. In the case of the optical activity of molecules the corresponding condition is derived from the physical situation that the forced displacement of molecules by the external force should be conservative.²⁾ In the same way, the condition (1') might be derived from the mechanical laws in the internal region. Unfortunately, however, we know nothing about details of the internal structure, so we regard condition (1') as a postulate for the moment.

Summarizing the above consideration one may say as follows. In the model where the weak interactions are regarded as the multipole interactions of structured particles of order of 10^{-20}cm the density matrix of structure is in general not symmetric with respect to the space reflection violating the conservation of parity

in the interactions. However, if condition (1') is satisfied the parity is well conserved in strong interactions as expected. The so-called one to one rule in the weak interaction may be expressed that the moments of the symmetric and antisymmetric parts of the internal density matrices are equal in the absolute value.

$$\text{Sp} \int (\gamma \cdot z) D^s(z) d^4z = \pm \text{Sp} \int (\gamma_5) (\gamma \cdot z) D^a(z) d^4z \quad (9)$$

§ 4. Baryon-lepton interactions

In the baryon-lepton interactions the most characteristic feature will be:

- 1) The universal Fermi interactions are of $V-A$ type the coupling of which has the dimension of square of length ($\hbar=c=1$) and the l value in the β -decay is approximately 10^{-17}cm .
- 2) The main selection rules in the reactions are derived from the conservation of lepton number and of neutrino charge. But the fact that the lepton pairs (e^-, e^+), (μ^-, μ^+) do not appear in weak interactions might demand another selection rule for leptons such as the strangeness for baryons.

From our standpoint of the multipole model the leptons will have a structure as do the baryons in the small region $l \sim 10^{-20}\text{cm}$, and the universal Fermi interactions may be interpreted as a van der Waals force between such non local particles. For example, if a field b acts between the baryon and lepton of sizes l_B and l_L respectively, following the same procedure as in the preceding paper, the baryon-lepton Fermi interaction is obtained in the form:

$$V(x) = -\frac{1}{2} f_B f_L l_B l_L \{ \sum \bar{B}^{n+1k}(x) \gamma_\mu (1-\gamma_5) B^{nk}(x) \} \{ \sum \bar{L}^{m+1\lambda}(x) \gamma_\mu (1-\gamma_5) L^{m\lambda}(x) \} \quad (10)$$

where the approximation

$$\partial_\lambda \partial_\nu D_b(x-x') = \frac{2i}{(2\pi)^4} \int \frac{k_\mu k_\nu}{k^2 + m_b^2} \exp(ik(x-x')) \approx \frac{i}{2} \partial_{\mu\nu} \delta(x-x') \quad (11)$$

is used. The suffix m in L is the quantum number for lepton corresponding to the strangeness for baryon. m_b is the particle mass of the b -field, $D_b(x-x')$ its propagator. (See Fig. 2)*

At first sight it seems that such interaction should really be responsible for the Fermi interaction, because the interaction is of the second kind characterized

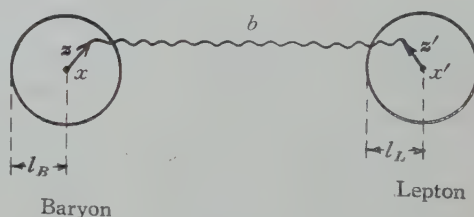


Fig. 2 The b -field acting between the baryon and lepton of sizes l_B and l_L respectively.

* In the monopole interaction between the two extended particles

$$V(x, x') = D_b(x-x') \int \rho_B(x, z) dz \int \rho_L(x, z') dz'$$

the force which gives rise to the shift from the Coulomb scattering in $e-N$ scattering is contained. Then, its force range must be smaller than 10^{-14}cm , that is $\hbar/m_b c < 10^{-14}\text{cm}$.

by the square of length $l_B l_A$ confirming its weakness of coupling due to the smallness of particle sizes l_B and l_L , the parity is not conserved, an explanation of lepton pairs may be possible, and so on. However, the situation is not so simple as first expected. According to Umezawa et al.,³⁾ the 'coupling length' estimated on the hypothesis that all weak interaction is of the second kind is summarized as follows:

$$\begin{aligned}
 \Lambda &\rightarrow p + \pi^- & l &= 3.6 \times 10^{-20} \text{cm} \\
 \Sigma^+ &\rightarrow \pi^+ + n & l &= 4.9 \times 10^{-20} \text{cm} \\
 \Xi^- &\rightarrow \Lambda + \pi^- & l &= 7.0 \times 10^{-20} \text{cm} \\
 K &\rightarrow \mu + \nu & l &= 5.0 \times 10^{-21} \text{cm} \\
 K &\rightarrow \pi + \pi & l &= 8.0 \times 10^{-20} \text{cm} \\
 \pi &\rightarrow \mu + \nu & l &= 3.5 \times 10^{-20} \text{cm},
 \end{aligned} \tag{12}$$

showing nearly the same order for all interactions. On the other hand, in the case of β -interaction the estimated coupling length is much larger than the above cases:

$$n \rightarrow p + e^- + \bar{\nu} \quad \sqrt{l_B l_L} = 6.7 \times 10^{-17} \text{cm}. \tag{13}$$

If the radius of baryon is of order of 10^{-20} cm, the β -decay would indicate the lepton radius $l_L \sim 10^{-14}$ cm. As the classical electron radius e^2/mc^2 is of order of 10^{-13} cm, a possibility of such l_L cannot be excluded. But if l_L were really of 10^{-14} cm, its effect should have appeared already in the strong or electromagnetic interactions.

In the preceding paper we have supposed that the dipole interaction changes the strangeness by one unit. Then in the above dipole-dipole interaction the change of strangeness, if defined, would be expected. However, in the β -decay $n \rightarrow p + e + \bar{\nu}$ the strangeness of baryon is not changed. If the β -interaction is considered through the medium of the b -field the vertex part of baryon must conserve the strangeness, that is, the baryon vertex is the monopole interaction. Probably such difference causes the large value of the coupling length in the β -decay. It will be discussed in the next section.

§ 5. Beta-decay interaction

When the interaction between the extended particles as in Fig. 2 is described by the Lagrangean

$$-L = \int dx dx' dz dz' \rho_B(x, z) A_b(x - x' + z - z') \rho_L(x', z'), \tag{14}$$

if the b -field range $\hbar/m_b c$ is larger than l , we can take the multipole expansion:

$$-L = \int dx dx' V(x, x'), \tag{15}$$

$$V(x, x') = \int \rho_B dz \rho_L dz' \mathcal{A}_b(x - x') \quad (\text{I})$$

$$+ \int z_\mu \rho_B dz \int \rho_L dz' \partial_\mu \mathcal{A}_b(x - x') \quad (\text{II})$$

$$- \int \rho_B dz \int z'_\mu \rho_L dz' \partial_\mu \mathcal{A}_b(x - x') \quad (\text{III}) \quad (16)$$

$$+ \iint (z_u - z'_u)(z_v - z'_v) \rho_B \rho_L dz dz' \partial_\mu \partial_\nu \mathcal{A}_b(x - x'). \quad (\text{IV})$$

(I) is a monopole interaction corresponding to the strong interaction such as $e\text{-}N$ scattering mentioned above. (IV) is the dipole interaction which may be called the Van der Waals interaction. In this interaction the process of strangeness change such as $\Lambda \rightarrow p + e + \bar{\nu}$ is contained but not β -interaction such as $n \rightarrow p + e + \bar{\nu}$. The decay life of $\Lambda \rightarrow p + e + \bar{\nu}$ is very much longer than that of the main decay $\Lambda \rightarrow p + \pi^-$, and the processes described by (IV) will not be so important.

(II) and (III) are the monopole-dipole interactions, and the above β -decay is contained in the process (III) without changing the strangeness of baryon. However, its interaction form,

$$V_{\text{III}}(x, x') = f_B f_L l_L \{ \sum \bar{B}^{nk}(x) \gamma_5 B^{nk}(x) \} \\ \times \{ \sum \bar{L}^{m\pm 1\lambda}(x) \gamma_\mu (1 - \gamma_5) L^{m\nu}(x') \} \partial_\mu \mathcal{A}_b(x - x'), \quad (17)$$

is, as it is, not of Fermi type. Though we know nothing about the field equation of baryons, if the internal structure is neglected the Dirac equation would be the first approximation. Then, the b -field source $\bar{B}(x) \gamma_5 B(x)$ by baryon is expected to be non-locally spread in the low energy processes owing to the zitterbewegung of baryon. If so, the monopole-dipole interaction will effectively be an interaction between the spread of Zitterbewegung of baryon and the structured lepton. This situation is easily seen in the equivalence theorem:

$$\bar{B}(x) \gamma_5 B(x) \mathcal{A}_b(x) \approx \frac{1}{2M_B} \bar{B}(x) \gamma_\mu \gamma_5 B(x) \partial_\mu \mathcal{A}_b(x) + \dots \quad (18)$$

Accordingly, the interaction is reduced to the desired form:

$$V_{\text{III}} = f_B f_L \frac{L}{2M_B} \{ \sum \bar{B}^{nk}(x) \gamma_5 \gamma_\mu B^{nk}(x) \} \\ \times \{ \sum L^{m\pm 1\lambda}(x') \gamma_\nu (1 - \gamma_5) L^{m\lambda}(x') \} \partial_\mu \partial_\nu \mathcal{A}_b(x - x'), \quad (19)$$

$$- L_{\text{III}} = \frac{i}{4} f_B f_L l_L \frac{1}{M_B} \int \{ \sum \bar{B}(x) \gamma_5 \gamma_\mu B(x) \} \{ \sum \bar{L}(x) \gamma_\mu (1 - \gamma_5) L(x') \}. \quad (20)$$

Though the interaction becomes, as is seen, of axial vector type instead of vector-axial vector type, the coupling has the dimension of square length and the order of coupling length is

$$\sqrt{l_L/M_B} \sim \sqrt{(10^{-20} \times 10^{-14}) \text{cm}} = 10^{-17} \text{cm}. \quad (21)$$

just as expected.

§ 6. Discussions

Concerning the non conservation of parity in the weak interaction some diverse models have been proposed. One ascribes it to the nature of neutrino, the other regards it as a local character of the whole universe, and so on. However, it seems rather difficult to imagine that the complicated characters are given on a geometrical point or that the curious behaviours come from the nature of empty space. On the contrary, for a unified understanding of the various phenomena more concrete and acceptable may be that the elementary particles have some small structure whose asymmetry causes the violation of parity in the weak interaction. In nature we see the right- or left-handed spiral shells, tendrils of the climbing plants and the like. In the organic compounds there exist the laevo or dextro rotating isomers. The asymmetric structure of the elementary particles in the micro world may not be so strange.

For the Fermi interaction the situation is not so simple. The coupling length in the β -decay of order of 10^{-17}cm is derived from the interaction between the zitterbewegung baryon and the structured lepton. But the coupling type becomes of axial vector instead of vector-axial vector presumed in the experiments. In addition, in the monopole-dipole interaction the process $\Lambda \rightarrow n + e^+ + e^-$ appears. Such inconveniences prove that the rough representation of the internal structure by the multipole model cannot reproduce the details of the selection rules in the elementary interactions. Our main purpose was, however, first to obtain a unified picture of the various phenomena not analysed hitherto, and for what has been well analysed as the selection rules, only the qualitative description is given for the moment. In any way, it must be remarked that regarding the β -interaction as the interaction between the baryon fluctuation and extended lepton an understanding of its coupling would be given in terms of the concrete particle picture.

§ 7. Concluding remarks

Up to now, the strange particle phenomena have been represented case by case in its own representing frames. But if each description should be united, in the end, the first step towards such direction would be to find the common representing space for the current laws in operation. As an example of such trials we have examined whether the internal space of elementary particles should be the representing space for the curious behaviour of weak interactions. In spite of our rough discussion the internal space of the elementary particles seems to represent the several diverse laws in a unified manner.

The next task is to translate the laws in the charge space into our internal space. A possible elimination of the divergences in our space must be examined

also. For these purposes, apart from the simple model, such as the multipole model, the general character of the internal structure for the representing space of the physical laws must be discussed in detail. A general theory will be given in the next paper.

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Nuclear Collective Motion and the Effective Two-Body Potential

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An effective two-body interaction is uniquely determined, on the condition that the interaction causes a nucleus to deform. This interaction consists of two terms. One is Elliott's Q - Q interaction and the other a restoring term. It is shown that the interaction causes nuclear rotational levels and the moment of inertia is about $I_{rig}/3$ in the case of typical nuclear deformation. Our formalism is the dynamical Hartree method and all results are derived in two-dimensional space.

§ 1. Introduction

The main program in nuclear theory is to derive the properties of the nucleus from the properties of the nuclear force. The first step in this program had been made by Brueckner,¹⁾ Eden²⁾ and Bethe³⁾. Their results indicate that the nucleus may be approximately described in terms of independently moving particles, in spite of the strong correlations between the constituent particles. We suppose that many features of the nucleus should be understood in the light of the above picture.

One of the striking features in low energy nuclear phenomena is the appearance of collective motions. Such nuclear collective motions should be understood on the basis of the independent particle picture, adding another correlations which are lacking in the independent particle picture.

As to the above correlations to be added, however, we know little at present, an example of which being the residual interaction appearing in Brueckner's theory. But it will be difficult to show that this residual interaction is the origin of nuclear collective motions.

On the other hand, more phenomenological treatments have been proposed by many authors, one of which is Inglis' cranking model⁴⁾. In his model, an external force cranks the nuclear body and this cranking causes nuclear collective motion. Thus it might be said that his external force would play a part of nuclear correlation⁵⁾. Therefore if we can replace an external force in the cranking model by an inner effective two-body potential, this modified model will be the next step in the main program.

Now the principal purpose of our paper is to test how far "the independent particles plus the effective two-body potential model" may account for the nuclear deformation and the rotational motion consistently, and to seek for the character of

this effective two-body potential required.

In our model, the collective motion and the effective two-body potential are connected in the following manner. Any particle in a nucleus filling the individual level is excited by the effective two-body potential, and then the nucleus may undergo a density change. A part of the uniform potential at an inner point of the nucleus will be determined by the above two-body potential averaged over the density distribution. Thus the variation of the uniform potential follows from the variation of density. In this case, certain properties of two-body potential will decide the type of the variation of the uniform potential.

The above picture of nuclear collective motion can be represented in the formalism of the time dependent Hartree equations⁶⁾.

We start from giving the solution i.e., the nuclear deformation, and then derive the form and strength of the effective two-body potential, so that the above equations involving the nuclear deformation and this potential are consistent.

In this paper, we consider a two-dimensional oscillator model. A more realistic case will be investigated in a succeeding paper.

In § 2, the time dependent Hartree equations are introduced. In § 3, we express the effective two-body potential in terms of the deformation and the particle configuration, then calculate the rotational energy self-consistently, using thus determined effective two-body potential.

In Appendix I, we briefly comment on a possibility of describing the nuclear collective motion in terms of fluctuating uniform potential, starting from the original nuclear system.

§ 2. Introduction of the self-consistent equations

Following the assumption presented in the preceding section, we introduce here a set of the fundamental equations.

These fundamental equations are*

$$i\hbar \frac{\partial \phi^{(i)}(\mathbf{r}_i, t)}{\partial t} = (T_i + U_0(\mathbf{r}_i) + U_d(\mathbf{r}_i, t)) \phi^{(i)}(\mathbf{r}_i, t) \quad (1)$$

$$U_d(\mathbf{r}_i, t) = \int v(\mathbf{r}_i, \mathbf{r}) \rho(\mathbf{r}, t) d\mathbf{r} \quad (2)$$

$$\rho(\mathbf{r}, t) = \sum_{occ} \phi^{(i)*}(\mathbf{r}, t) \phi^{(i)}(\mathbf{r}, t), \quad (3)$$

where $v(\mathbf{r}, \mathbf{r}')$ is symmetric with respect to \mathbf{r}, \mathbf{r}' and \sum_{occ} means a sum over the occupied states.

The wave function of the system is a normalized Slater determinant of single-particle wave function,

* In these equations, the exchange terms are not contained. We understand these equations as the model equations, which are equivalent to the approximated Fock equations.

$$\Psi(\mathbf{r}_1 \cdots \mathbf{r}_N, t) = |\phi^{(i)}(\mathbf{r}, t)|. \quad (4)$$

Similar systems have already been presented by M. Nogami⁶⁾ and R. A. Ferrell⁶⁾. They have gained some success in their treatments of the plasma oscillations in electron gas and the excited states of O^{16} .

It will be worthwhile to note here that the correlation between two particles in the nucleus, $v(\mathbf{r}, \mathbf{r}')$ in Eq. (3), is not an original inter-particle potential but an effective two-body potential between the particles moving in the uniform potential.

§ 3. The case of 2-dimensional oscillator potential

In this section, the deformation and the rotational motions are treated in the 2-dimensional oscillator model.

For simplicity, we neglect the spin dependence of the potential.

a) Description of the rotational motion

We consider 2-dimensional rotation in y - z plane around a fictitious x -axis. Here we assume the following type of rotational motion exists as the solution of Eqs. (1), (2), and (3),*

$$\phi^{(i)}(\mathbf{r}_i, t) = e^{-(i/\hbar)\Omega L_x t} \phi^{(i)}(\mathbf{r}_i), \quad (5)$$

where L_x is the angular momentum operator and Ω the angular velocity, which we assume to be constant** and $\phi^{(i)}(\mathbf{r}_i)$ describes a stationary state.

Then the transformed functions $\phi^{(i)}(\mathbf{r})$ satisfy

$$i\hbar (\partial \phi^{(i)}(\mathbf{r}) / \partial t) = \{T + e^{i/\hbar \Omega L_x t} (U_0(\mathbf{r}) + U_d(\mathbf{r}, t)) e^{-(i/\hbar)\Omega L_x t} - \Omega L_x\} \phi^{(i)}(\mathbf{r}) \equiv H_t \phi^{(i)}(\mathbf{r}) \quad (6)$$

$$\partial H_t / \partial t = 0, \quad (7)$$

from which we get

$$i\hbar (\partial U_d(\mathbf{r}, t) / \partial t) = [\Omega L_x, U_0(\mathbf{r}) + U_d(\mathbf{r}, t)]. \quad (8)$$

If (7) or (8) is satisfied, we may write

$$(T + U_0(\mathbf{r}) + U_d(\mathbf{r}, 0) - \Omega L_x) \phi^{(i)}(\mathbf{r}) = E_{\Omega}^{(i)} \phi^{(i)}(\mathbf{r}) \quad (9)$$

From Eq. (9), we see that (5) represents a transformation to the body fixed co-ordinate system, in which the system is in stationary state.

b) Consistency of the time variations

As Eq. (2) holds for arbitrary time t , both sides of Eq. (2) must have same time variations. In other word, the time variation of density must be transmitted through the two-body potential to the averaged uniform potential without changes.

* As we shall see later, the expression (5) is the self-consistent solution of Eqs. (1), (2), (3), so this assumption will be justified.

** More generally, Ω may depend on \mathbf{r} and single particle states.

At the same time, the stationary condition (8) is also required.

In the following we seek for the condition of compatibility of the both requirements. Substituting (2) into (8) and using (5), we get*

$$\begin{aligned} & \int d\mathbf{r} [\Omega \mathbf{L}_x, v(\mathbf{r}, \mathbf{r}')] \sum_{occ} \int d\mathbf{r}'' \phi_j^{(i)*}(\mathbf{r}'') e^{(i/\hbar)\Omega \mathbf{L}_{x''} \cdot \mathbf{r}'} \delta(\mathbf{r}' - \mathbf{r}'') e^{-(i/\hbar)\Omega \mathbf{L}_{x''} \cdot \mathbf{r}'} \phi_j^{(i)}(\mathbf{r}'') \\ & + \int d\mathbf{r}' v(\mathbf{r}, \mathbf{r}') \sum_{occ} \int d\mathbf{r}'' \phi_j^{(i)*}(\mathbf{r}') e^{(i/\hbar)\Omega \mathbf{L}_{x''} \cdot \mathbf{r}'} [\Omega \mathbf{L}_{x''}, \delta(\mathbf{r}' - \mathbf{r}'')] \times \\ & \times e^{-(i/\hbar)\Omega \mathbf{L}_{x''} \cdot \mathbf{r}'} \phi_j^{(i)}(\mathbf{r}'') = 0 \end{aligned} \quad (10)$$

where we use $\rho(\mathbf{r}, t) = \sum_{occ} \int \phi_j^{(i)*}(\mathbf{r}_i, t) \delta(\mathbf{r}_i - \mathbf{r}) \phi_j^{(i)}(\mathbf{r}_i, t) d\mathbf{r}_i$.

Using that $\mathbf{L}_x = -i\hbar\partial/\partial\varphi$ in polar coordinate, we can easily perform the partial integrations

$$\int d\mathbf{r}' [\mathbf{L}_x + \mathbf{L}_{x'}, v(\mathbf{r}, \mathbf{r}')] \rho(\mathbf{r}', t) = 0. \quad (11)$$

As this equation should be satisfied for the energy states having the same potential $v(\mathbf{r}, \mathbf{r}')$, we get

$$[\mathbf{L}_x + \mathbf{L}_{x'}, v(\mathbf{r}, \mathbf{r}')] = [\sum_i \mathbf{L}_x^{(i)}, \sum_{k \neq l} v(\mathbf{r}_k, \mathbf{r}_l)] = 0, \quad (12)$$

i.e., the effective two-body potential must be rotational invariant.

c) The uniform potential

In the non-rotating state ($\Omega=0$), the uniform potential is taken to be the deformed oscillator potential

$$-\frac{M}{2}(\omega_y^2 y^2 + \omega_z^2 z^2) \quad (13 \cdot a)$$

which becomes spherical in the absence of the effective two-body potential,

$$U_0(\mathbf{r}) = (M/2)\omega^2(y^2 + z^2), \quad (13 \cdot b)$$

where M is the mass of a particle.

Generally U_a contains the angular velocity Ω . We expand it in the power of Ω/ω , assuming $\Omega/\omega \ll 1$,

$$U_a(\mathbf{r}, 0) = \sum_{n=0}^{\infty} U_a^{(n)}(\mathbf{r}, 0) \quad (14)$$

where $U_a^{(n)}$ means the term of U_a of the order $(\Omega/\omega)^n$.

$U_a^{(0)}$ represents the deformed part of the uniform potential i.e.,

* Later we will consider the case in which U_0 is rotational invariant, so here we use this property of U_0 , i. e., $[\mathbf{L}_x, U_0] = 0$.

$$\begin{aligned} U_d^{(0)}(\mathbf{r}, 0) &= (M/2) (\omega_y^2 y^2 + \omega_z^2 z^2) - (M/2) \omega^2 (y^2 + z^2) \\ &= (M/2) \{ (\omega_y^2 - \omega^2) y^2 + (\omega_z^2 - \omega^2) z^2 \}. \end{aligned} \quad (15)$$

Here we assume the incompressible deformation i.e.,

$$\omega_y \omega_z = \omega^2. \quad (16)$$

From (13) the following relation also results,

$$[\mathbf{L}_z, U_0(\mathbf{r})] = 0. \quad (17)$$

Higher order terms in Ω/ω , $U_d^{(n)} (n \geq 1)$ will be determined later in a self-consistent manner.

d) Non-rotating states

Even in the non-rotating states, a nucleus is forced to deform in the form given by Eq. (15) by the effect of $v(\mathbf{r}, \mathbf{r}')$.

A set of equations in this case ($\Omega=0$) follows from (1), (2), (3),

$$(T_i + U_0(\mathbf{r}_i) + U_d^{(0)}(\mathbf{r}_i, 0)) \phi_0^{(i)}(\mathbf{r}_i) = E_0^{(i)} \phi_0^{(i)}(\mathbf{r}_i) \quad (18)$$

$$U_d^{(0)}(\mathbf{r}_i, 0) = \int v(\mathbf{r}_i, \mathbf{r}) \rho^{(0)}(\mathbf{r}) d\mathbf{r} \quad (19)$$

$$\rho^{(0)}(\mathbf{r}) = \sum_{\text{occ}} \phi_0^{(i)*}(\mathbf{r}) \phi_0^{(i)}(\mathbf{r}). \quad (20)$$

In (18),

$$U_0(\mathbf{r}_i) + U_d^{(0)}(\mathbf{r}_i, 0) = (M/2) (\omega_y^2 y_i^2 + \omega_z^2 z_i^2).$$

The solutions of Eq. (18) are

$$\phi_0^{(i)}(\mathbf{r}) = u_{mn}(\mathbf{r}) = N_{mn} H_m(\alpha y) H_n(\beta z) e^{-1/2(\alpha^2 y^2 + \beta^2 z^2)} \quad (21)$$

$$\alpha = \sqrt{\frac{M\omega_y}{\hbar}}, \quad \beta = \sqrt{\frac{M\omega_z}{\hbar}}, \quad \rho^{(0)}(\mathbf{r}) = \sum_{mn \text{ occ}} u_{mn}^2(\mathbf{r}). \quad (22)$$

Next let us look for the form of $v(\mathbf{r}, \mathbf{r}')$ satisfying the equality (19).

At first, we rewrite $v(\mathbf{r}, \mathbf{r}')$ in the form of power series of y, z, y' and z' ,

$$v(\mathbf{r}, \mathbf{r}') = \sum_{mnlk} a_{nmkl} y^n z^m y'^l z'^k, \quad (23)$$

so the integral of the right-hand side of (19) vanishes if l or k is odd.

Considering the above result together with the symmetricity of $v(\mathbf{r}, \mathbf{r}')$ with respect to \mathbf{r}, \mathbf{r}' , $v(\mathbf{r}, \mathbf{r}')$ may be written as follows,

$$v(\mathbf{r}, \mathbf{r}') = \left(\sum_{m,n,l,k=\text{even}} + \sum_{\substack{n,l=\text{even} \\ m,k=\text{odd}}} + \sum_{\substack{n,l=\text{odd} \\ m,k=\text{even}}} + \sum_{m,n,l,k=\text{odd}} \right) a_{nmkl} y^n z^m y'^l z'^k \quad (24)$$

For abbreviation, we put

$$\sum_{m,n,l,k=\text{even}} \equiv \sum_{\text{even}}, \quad \sum_{\substack{n,l=\text{even} \\ m,k=\text{odd}}} + \sum_{\substack{n,l=\text{odd} \\ m,k=\text{even}}} + \sum_{m,n,l,k=\text{odd}} \equiv \sum_{\text{odd}}.$$

As $U_a^{(0)}$ is a linear combination of y^2 and z^2 , the equality of the both sides of (19) requires

$$\sum_{\text{even}} = F(y'^2, z'^2)y^2 + G(y'^2, z'^2)z^2. \quad (25)$$

Expanding F, G in powers of y'^2, z'^2 and using the symmetricity condition of $v(\mathbf{r}, \mathbf{r}')$ we get

$$\begin{aligned} \sum_{\text{even}} &= ay^2y'^2 + b(y^2z'^2 + z^2y'^2) + cz^2z'^2 \\ &= \left(\frac{a+c}{2} - b\right)(yy' + zz')^2 + b(y^2 + z^2)(y'^2 + z'^2) + \frac{a-c}{2}(yy' - zz')(yy' + zz') \\ &\quad - (a+c-2b)yy'zz'. \end{aligned}$$

Consequently,

$$\begin{aligned} v(\mathbf{r}, \mathbf{r}') &= A'(yy' + zz')^2 + B'(y^2 + z^2)(y'^2 + z'^2) \\ &\quad + C'(yy' - zz')(yy' + zz') + \sum'_{\text{odd}}, \end{aligned} \quad (26)$$

and in polar coordinate,

$$\begin{aligned} v(\mathbf{r}, \mathbf{r}') &= A'r^2r'^2 \cos^2(\varphi - \varphi') + B'r^2r'^2 \\ &\quad + C'r^2r'^2 \cos(\varphi + \varphi') \cos(\varphi - \varphi') + \sum'_{\text{odd}}. \end{aligned} \quad (27)$$

Here we take account of the rotational invariancy of $v(\mathbf{r}, \mathbf{r}')$, which can be written as

$$\left[\frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \varphi'}, v(\mathbf{r}, \mathbf{r}') \right] = 0$$

in polar coordinate.

Then C' must be zero. Furthermore, \sum'_{odd} is irrelevant in our self-consistency problem. Hence we come to the final expression of the effective part of $v(\mathbf{r}, \mathbf{r}')$,

$$\begin{aligned} v(\mathbf{r}, \mathbf{r}') &= A'r^2r'^2 \cos^2(\varphi - \varphi') + B'r^2r'^2 \\ &= Ar^2r'^2 \cos 2(\varphi - \varphi') + Br^2r'^2. \end{aligned} \quad (28)$$

(28) is similar to Elliott's Q - Q interaction.^{7),8)}

The coefficients A, B in (28) can be determined as follows. Substituting (15), (21), (28) into (19) and equating the coefficients of y^2, z^2 on the both sides respectively, we get

$$\left. \begin{aligned} M(\omega_y^2 - \omega^2) &= \frac{\hbar}{M}(S_m - S_n)A + \frac{\hbar}{M}(S_m + S_n)B \\ M(\omega_z^2 - \omega^2) &= -\frac{\hbar}{M}(S_m - S_n)A + \frac{\hbar}{M}(S_m + S_n)B \end{aligned} \right\} \quad (29)$$

$$\text{where} \quad S_m = \sum_{occ} (2m+1) \quad S_n = \sum_{occ} (2n+1). \quad (30)$$

$$\text{Hence} \quad A = -\frac{M^2}{2\hbar} \frac{\omega^2(\omega_y^2 - \omega_z^2)}{\omega_y S_n - \omega_z S_m} \quad (31)$$

$$B = \frac{M^2}{2\hbar} \frac{\omega^2(\omega_y - \omega_z)^2}{\omega_y S_n + \omega_z S_m} \quad (32)$$

where we used (16).

As is clear from (31) and (32), the strength of effective potential can be fixed uniquely from the deformation and particle configurations.

Further, it will be interesting to express A and B in terms of deformation parameter ϵ defined by $\omega_y/\omega_z = 1 + \epsilon$. For small ϵ

$$A \propto -\epsilon \quad B \propto \epsilon^2.$$

Consequently, it is found that the first term of $v(\mathbf{r}, \mathbf{r}')$ induces the deformation of nucleus and the second restores it.

e) Rotating states

Excited states are the rotating states ($\Omega \neq 0$). Because Ω/ω is assumed to be small, we consider up to the second order terms in Ω/ω .

Equations to be solved are

$$(T_i + U_0(\mathbf{r}_i) + U_d^{(0)}(\mathbf{r}_i, 0) + U_d^{(1)}(\mathbf{r}_i, 0) + U_d^{(2)}(\mathbf{r}_i, 0) - \Omega \mathbf{L}_x) \phi^{(i)}(\mathbf{r}_i) = E_\Omega^i \phi^{(i)}(\mathbf{r}_i) \quad (33)$$

$$U_d^{(n)}(\mathbf{r}_i, 0) = \int v(\mathbf{r}_i, \mathbf{r}) \rho^{(n)}(\mathbf{r}) d\mathbf{r} \quad (34)$$

where the suffix (n) represents the term of order $(\Omega/\omega)^n$. In these equations $U_d^{(1)}$ and $U_d^{(2)}$ are the unknown functions which should be solved in self-consistent manner successively, using the $v(\mathbf{r}, \mathbf{r}')$ given in Eq. (28)

Putting $H_I = U_d^{(1)} + U_d^{(2)} - \Omega \mathbf{L}_x$,

$$\phi^{(i)} = \phi_0^{(i)} + \sum_{j \neq i} a_{ji}^{(1)} \phi_0^{(j)} + \sum_{j \neq i} a_{ji}^{(2)} \phi_0^{(j)} - \sum_{j \neq i} b_{ji}^{(2)} \phi_0^{(j)} - \frac{1}{2} N_i \phi_0^{(i)} \quad (35)$$

$$a_{ji}^{(1)} = \frac{\langle j | H_I | i \rangle}{E_0^{(i)} - E_0^{(j)}} \quad (35.1)$$

$$a_{ji}^{(2)} = \sum_{k \neq i} \frac{\langle j | H_I | k \rangle \langle k | H_I | i \rangle}{(E_0^{(i)} - E_0^{(j)}) (E_0^{(i)} - E_0^{(k)})} \quad (35.2)$$

$$b_{ji}^{(2)} = \frac{\langle i | H_I | i \rangle \langle j | H_I | i \rangle}{(E_0^{(i)} - E_0^{(j)})^2} \quad (35.3)$$

$$N_i = \sum_{j \neq i} \frac{|\langle j | H_I | i \rangle|^2}{(E_0^{(i)} - E_0^{(j)})^2} \quad (35.4)$$

where $E_0^{(i)}$, $\phi_0^{(i)}$ etc., are the energies and the eigenfunctions of the unperturbed system.

$$\begin{aligned}
\rho = & \sum_{i, l, c c'} [\phi_0^{(i)*} \phi_0^{(l)} + \sum_{j \neq i} \{ a_{j i}^{(1)*} \phi_0^{(j)*} \phi_0^{(i)} + a_{j i}^{(1)} \phi_0^{(i)*} \phi_0^{(j)} \} \\
& - N_i \phi_0^{(i)*} \phi_0^{(i)} + \sum_{\substack{j \neq i \\ k \neq i}} a_{j i}^{(1)*} a_{k i}^{(1)} \phi_0^{(j)*} \phi_0^{(k)} \\
& + \sum_{j \neq i} \{ (a_{j i}^{(2)*} - b_{j i}^{(2)*}) \phi_0^{(i)*} \phi_0^{(j)} + (a_{j i}^{(2)} - b_{j i}^{(2)}) \phi_0^{(i)} \phi_0^{(j)*} \}].
\end{aligned} \quad (36)$$

We begin with investigating the first order terms. As $v(\mathbf{r}, \mathbf{r}')$ is a linear combination of y^2 , z^2 and yz , $U_a^{(1)}$ can be written generally as

$$U_a^{(1)}(\mathbf{r}, 0) = \left(\frac{\Omega}{\omega} \right) a y z + \left(\frac{\Omega}{\omega} \right) \left(\frac{M\omega}{\hbar} \right) (\hbar\omega) \{ b y^2 + c z^2 \}. \quad (37)$$

In this expression a , b and c are the coefficients to be found.

From (34),

$$U_a^{(1)}(\mathbf{r}, 0) = \int v(\mathbf{r}, \mathbf{r}') \rho^{(1)}(\mathbf{r}') d\mathbf{r}'. \quad (38)$$

Coriolis' force does not give any contributions to $\rho^{(1)}$ because the matrix element $\langle j | \mathbf{L}_x | i \rangle$ is pure imaginary. So Eq. (38) gives a set of the homogeneous equations of a , b and c .

Without loss of generality, a may be set equal to zero, because this term fixes the directions of the principal axis. It will be also shown that b and c must be zero (see Appendix II), so we conclude $U_a^{(1)}(\mathbf{r}, 0) = 0$, and then the excitation energy begins with the second order terms Ω/ω , which guarantees the rotational spectrum.

Next let us examine the second order terms. As in the case of $U_a^{(1)}(\mathbf{r}, 0)$, we put

$$U_a^{(2)}(\mathbf{r}, 0) \equiv (\Omega/\omega)^2 (M\omega/\hbar) (\hbar\omega) (f y^2 + g z^2). \quad (39)$$

Here we omitted a term proportional to yz because it gives contribution to energy only in the fourth order. The second order terms in (36) and (39) are substituted into (34). In this case, the effects of Coriolis' force remain in the right-hand side of (34), which assure the non-zero solutions for f and g .

Detailed calculations will be carried out in Appendix III. Here we quote only the results. The energy of our system is

$$\begin{aligned}
E = & \sum_{o c c'} \frac{\int \phi^{(i)*}(\mathbf{r}, t) i \hbar (\partial/\partial t) \phi^{(i)}(\mathbf{r}, t) d\mathbf{r}}{\int \phi^{(i)*}(\mathbf{r}, t) \phi^{(i)}(\mathbf{r}, t) d\mathbf{r}} \\
= & \sum_{o c c'} \frac{\int \phi^{(i)*}(\mathbf{r}) (T + U_a^{(0)}(\mathbf{r}) + U_a^{(1)}(\mathbf{r}) + U_a^{(2)}(\mathbf{r})) \phi^{(i)}(\mathbf{r}) d\mathbf{r}}{\int \phi^{(i)*}(\mathbf{r}) \phi^{(i)}(\mathbf{r}) d\mathbf{r}},
\end{aligned} \quad (40)$$

of which $(\Omega/\omega)^2$ terms give rotational energy;

$$E^{rot} = - \sum_{i, o c c'} \sum_{j \neq i} \frac{|\langle j | \Omega \mathbf{L}_x | i \rangle|^2}{E_0^{(i)} - E_0^{(j)}} + \frac{1}{2} \sum_{i, o c c'} \langle i | U_a^{(2)} | i \rangle. \quad (41)$$

Since in evaluating the expectation value of $U_a^{(2)}$ the effect of $v(\mathbf{r}, \mathbf{r}')$ is counted

twice, $1/2$ factor is added in the second term.

In (41) the first term denoted by E_c has been given by Inglis^(4),9) and others.⁽¹⁰⁾ which represents the pure effect of Coriolis' force. On the other hand, by the Coriolis force a nucleus changes its density. This causes the change of uniform potential through the effective two-body potential. As $v(\mathbf{r}, \mathbf{r}')$ gives the equilibrium point of density distribution in the ground state of a deformed nucleus, so the density changes require the corresponding energies. The second term corresponds just to this part of the energy increment.

For the typical particle numbers and the deformation $\omega_y/\omega_z \sim 2.6$ (see Appendix III)

$$\frac{1}{2} \sum_{i \text{ occ}} \langle i | U_d^{(2)} | i \rangle \approx 2E_c,$$

therefore

$$E^{\text{rot}} \approx 3E_c. \quad (42)$$

So far the angular velocity Ω has been left as a free parameter, on the assumption $\Omega/\omega \ll 1$.

The angular velocity can be eliminated, if the angular momentum is known. Unfortunately, our Hamiltonian is not invariant under the rotation. We have no means to estimate the total angular momentum correctly. Several authors have recently discussed on this question,^(8),11),12) but here following the Inglis' version, we identify the expectation value of $\sum_i \mathbf{L}_i^{(i)}$ with the eigenvalues of the angular momentum,

$$\langle \mathbf{L}_x \rangle = -\Omega \sum_{i \text{ occ}} \sum_{j \neq i} \frac{|\langle j | \mathbf{L}_x | i \rangle|^2}{E_0^{(i)} - E_0^{(j)}}. \quad (43)$$

Eliminating Ω in (41) and (43), the excitation energy can be expressed in the form of rotational energy,

$$E^{\text{rot}} \approx 3 \frac{1}{2J_{\text{rig}}} \langle \mathbf{L}_x \rangle^2 = \frac{1}{2J} \langle \mathbf{L}_x \rangle^2, \quad J \approx \frac{1}{3} J_{\text{rig}}. \quad (44)$$

Namely the magnitude of the moment of inertia reduces to about one third of the rigid value.

It may be safely said that the effective two-body potential prevents the density changes and the smaller parts of nuclear matter can participate in the rotational motion.

§ 4. Summary and discussion

In this section the results obtained in the preceding chapters will be summarized and some discussions will be given.

1) The effective two-body potential to be added to the uniform potential can be determined, provided that the deformation and the rotational motions turn out to

be the solutions of the time dependent Hartree equations. The resulting form is similar to Elliott's potential.

2) The assumed uniform potential fixes the shape of this effective two-body potential, the strength of which is expressed uniquely in terms of the magnitude of deformation and particle configurations.

3) The two-body potential thus derived consists of two parts, the inducing and the restoring, and each part is factorizable.

It depends rather upon the positions of individual particles than on their mutual distances. This fact seems to be very remarkable. The random parts of correlations would certainly be averaged out, because the deformation and the rotation of a whole nucleus are of the collective nature.

4) The rotational energy is calculated with the use of the above two-body potential. This differs from Inglis' result in the appearance of an extra term which comes from the fluctuation of the uniform potential. This extra effect reduces the magnitude of the moment of inertia to about one third of the rigid value.

5) The extra effect mentioned in 4) originally comes from the mutual correlations which is lacking in Inglis' model and probably relates to Bohr-Mottelson's interactions introduced to adjust the value of the moment of inertia.

Therefore we presume here that our fundamental picture for the nuclear deformation and rotation might be reasonable.

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Appendix I

Here a possibility mentioned in the last part of the introduction will be shown. The formalism is based on the general wave matrix theory proposed by one of the authors (H. Tanaka¹³⁾).

Let the Hamiltonians and the wave functions of the model and real systems be:

$$H_0 = T + U, \quad H_0 \phi = E_0 \phi \quad (\text{AI} \cdot 1)$$

and

$$H = T + \sum v = H_0 + H_I, \quad H_I = \sum v - U, \quad H \Psi = E \Psi \quad (\text{AI} \cdot 2)$$

respectively, where v expresses the nuclear potential and U the uniform potential.

We introduce the general wave matrix V , defined by

$$\Psi = V \phi \quad (\text{AI} \cdot 3)$$

which satisfy the following equation,

$$H V = V (H_0 + \langle H_I V \rangle), \quad (\text{AI} \cdot 4)$$

where the symbol $\langle O \rangle$ means a diagonal part of the operator O with respect to the eigenstates of the model Hamiltonian.

Then we obtain the following time dependent Schroedinger equation

$$i\hbar \frac{\partial \Phi(t)}{\partial t} = (H_0 + \langle H_I V \rangle) \Phi(t), \quad \Phi(t) = e^{-i/\hbar E_0 t} \Phi. \quad (\text{AI} \cdot 5)$$

$\langle H_I V \rangle$ represents the operator of the energy shift between the model and the real systems, and is supposed in this case, to correspond to a kind of residual interaction in the model system. This term can be eliminated with a time dependent transformation,

$$\Phi(t) = e^{-i/\hbar \langle T + \langle H_I V \rangle \rangle t} e^{i/\hbar T t} \psi(t) = R(t) \psi(t). \quad (\text{AI} \cdot 6)$$

Then $\psi(t)$ satisfy

$$i\hbar \frac{\partial \psi(t)}{\partial t} = (T + R(t)^{-1} U R(t)) \psi(t) \equiv (T + U(t)) \psi(t). \quad (\text{AI} \cdot 7)$$

In this system the uniform potential has certain time variation specified by the residual interaction $\langle H_I V \rangle$.

Instead of deriving $U(t)$ from the nuclear potential, we determined it in the text from the nuclear deformation phenomenologically.

Appendix II

We prove $U_d^{(1)} = 0$. As is written in the text,

$$U_d^{(1)}(\mathbf{r}, 0) = \frac{\Omega}{\omega} \cdot \frac{M\omega}{\hbar} \cdot \hbar \omega (by^2 + cz^2). \quad (\text{AII} \cdot 1)$$

Next by picking up the first order terms of ρ ,

$$\rho^{(1)} = \sum_i \sum_{occ} \sum_{j \neq i} (a_{ji}^{(1)*} \phi_0^{(j)*} \phi_0^{(i)} + a_{ji}^{(1)'} \phi_0^{(i)*} \phi_0^{(j)}) \quad (\text{AII} \cdot 2)$$

where

$$a_{ji}^{(1)'} = \frac{\langle j | U_d^{(1)} | i \rangle}{E_0^{(i)} - E_0^{(j)}}. \quad (\text{AII} \cdot 3)$$

These quantities are substituted into the right-hand side of (38) of the text, then

$$\begin{aligned} U_d^{(1)}(\mathbf{r}, 0) = & V_I \sum_{mn} \frac{\langle m+2, n | U_d^{(1)} | m, n \rangle}{E_{mn} - E_{m+2, n}} \langle m+2, n | y^2 | m, n \rangle \\ & + V_{II} \sum_{mn} \frac{\langle m, n+2 | U_d^{(1)} | m, n \rangle}{E_{mn} - E_{m, n+2}} \langle m, n+2 | z^2 | m, n \rangle, \end{aligned} \quad (\text{AII} \cdot 4)$$

$$V_I \equiv (A+B)y^2 + (B-A)z^2 = Py^2 + Qz^2,$$

$$V_{II} \equiv (B-A)y^2 + (A+B)z^2 = Qy^2 + Pz^2,$$

where \sum' means the sum over the states allowed by the Pauli principle.

Equating the coefficients of y^2, z^2 , on the both sides of Eq. (AII, 4), we get the following homogeneous equation for b and c ,

$$\begin{aligned} \left(1 + \frac{\hbar}{4M^2\omega_y^3} PS_m\right)b + \frac{\hbar}{4M^2\omega_z^3} QS_n \cdot c &= 0 \\ \left(1 + \frac{\hbar}{4M^2\omega_x^3} QS_m\right)b + \frac{\hbar}{4M^2\omega_z^3} PS_n \cdot c &= 0. \end{aligned} \quad (\text{AII} \cdot 5)$$

Then the determinant of the coefficients is easily evaluated and

$$\Delta = \frac{\hbar}{4M^2} (P-Q) \left\{ \frac{S_n}{\omega_z^3} + \frac{4}{4M^2\omega^6} S_n S_m (P+Q) \right\} \neq 0, \quad (\text{AII} \cdot 6)$$

because $P-Q=2A \neq 0$ and $P+Q=2B > 0$.

Hence Eqs. (AII 8) have not non-zero solutions.

Appendix III

We calculate the second order part of $U_d^{(2)}(\mathbf{r}, 0)$ self-consistently. The equation to be solved is

$$U_d^{(2)}(\mathbf{r}, 0) = \int v(\mathbf{r}, \mathbf{r}') \rho^{(2)}(\mathbf{r}') d\mathbf{r}' \quad (\text{AIII} \cdot 1)$$

where

$$U_d^{(2)} \equiv \left(\frac{Q}{\omega}\right)^2 \frac{M\omega}{\hbar} \hbar\omega (fy^2 + gz^2)$$

$$\rho^{(2)} = -Q^2 \sum_{i, occ} \sum_{j \neq i} \frac{1}{(E_0^{(i)} - E_0^{(j)})^2} |\langle j | \mathbf{L}_x | i \rangle|^2 \phi_0^{(i)*} \phi_0^{(i)} \quad (\text{AIII} \cdot 2)$$

$$+ Q^2 \sum_{i, occ} \sum_{j, k \neq i} \frac{\langle j | \mathbf{L}_x | i \rangle^* \langle k | \mathbf{L}_x | i \rangle}{(E_0^{(i)} - E_0^{(j)})(E_0^{(i)} - E_0^{(k)})} \phi_0^{(j)*} \phi_0^{(k)}$$

$$+ Q^2 \sum_{i, occ} \sum_{j, k \neq i} \frac{\langle k | \mathbf{L}_x | j \rangle^* \langle j | \mathbf{L}_x | i \rangle}{(E_0^{(i)} - E_0^{(j)})(E_0^{(i)} - E_0^{(k)})} \phi_0^{(k)*} \phi_0^{(i)}$$

$$+ Q^2 \sum_{i, occ} \sum_{j, k \neq i} \frac{\langle k | \mathbf{L}_x | j \rangle \langle j | \mathbf{L}_x | i \rangle}{(E_0^{(i)} - E_0^{(j)})(E_0^{(i)} - E_0^{(k)})} \phi_0^{(i)*} \phi_0^{(k)} \quad (\text{AIII} \cdot 3)$$

$$+ \sum_{i, occ} \sum_{j \neq i} \frac{\langle j | U_d^{(2)} | i \rangle^*}{E_0^{(i)} - E_0^{(j)}} \phi_0^{(j)*} \phi_0^{(i)} + \sum_{i, occ} \sum_{j \neq i} \frac{\langle j | U_d^{(2)} | i \rangle}{E_0^{(i)} - E_0^{(j)}} \phi_0^{(i)*} \phi_0^{(j)}$$

$$= \rho_n^{(2)} + \rho_{c1}^{(2)} + \rho_{c2}^{(2)} + \rho_{\Delta U}^{(2)}.$$

$\rho_n^{(2)}$, the first term of (AIII, 3), represents the reduction of the initial density, $\rho_{c1}^{(2)}$, $\rho_{c2}^{(2)}$, the second and third terms of (AIII, 3) respectively, are the density changes caused by Coriolis' force and $\rho_{\Delta U}^{(2)}$, the last term, is due to the change of the uniform potential $U_d^{(2)}$ itself.

Each part of $\rho^{(2)}$ produces the corresponding part of $U_d^{(2)}$. For example,

$$U_{\alpha n}^{(2)}(\mathbf{r}, 0) = \int v(\mathbf{r}, \mathbf{r}') \rho_n^{(2)}(\mathbf{r}') d\mathbf{r}'.$$

It will be convenient to divide them into two parts, each proportional to y^2 and z^2 . After simple calculations, we find for each part of $U_{\alpha}^{(2)}$

$$\left. \begin{aligned} U_{ny}^{(2)} = & -\frac{1}{8} \left(\frac{Q}{\omega} \right)^2 \left[P \frac{\hbar}{M\omega_y} \left\{ \left(\frac{\omega_y - \omega_z}{\omega_y + \omega_z} \right)^2 S_n^{(1)} + \left(\frac{\omega_y + \omega_z}{\omega_y - \omega_z} \right)^2 S_n^{(2)} \right\} \right. \\ & \left. + Q \frac{\hbar}{M\omega_z} \left\{ \left(\frac{\omega_y - \omega_z}{\omega_y + \omega_z} \right)^2 S_n^{(3)} + \left(\frac{\omega_y + \omega_z}{\omega_y - \omega_z} \right)^2 S_n^{(4)} \right\} \right] y^2, \\ S_n^{(1)} = & \sum_{m,n}^{(m+1, n+1)} (m+1)(n+1)(2m+1), \quad S_n^{(2)} = \sum_{m,n}^{(m+1, n-1)} (m+1)n(2m+1), \\ S_n^{(3)} = & \sum_{m,n}^{(m+1, n+1)} (m+1)(n+1)(2n+1), \quad S_n^{(4)} = \sum_{m,n}^{(m+1, n-1)} (m+1)n(2n+1), \end{aligned} \right\} \quad (\text{AIII} \cdot 4)$$

$$\left. \begin{aligned} U_{c1y}^{(2)} = & \frac{1}{8} \left(\frac{Q}{\omega} \right)^2 \left[P \frac{\hbar}{M\omega_y} \left\{ \left(\frac{\omega_y - \omega_z}{\omega_y + \omega_z} \right)^2 S_{c1}^{(1)} + \left(\frac{\omega_y + \omega_z}{\omega_y - \omega_z} \right)^2 S_{c1}^{(2)} \right\} \right. \\ & \left. + Q \frac{\hbar}{M\omega_z} \left\{ \left(\frac{\omega_y - \omega_z}{\omega_y + \omega_z} \right)^2 S_{c1}^{(3)} + \left(\frac{\omega_y + \omega_z}{\omega_y - \omega_z} \right)^2 S_{c1}^{(4)} \right\} + 2Q \frac{\hbar}{M\omega_z} S_{c1} \right] y^2, \\ S_{c1}^{(1)} = & \sum_{m,n}^{(m+1, n+1)} (m+1)(n+1)(2m+1), \quad S_{c1}^{(2)} = \sum_{m,n}^{(m+1, n-1)} (m+1)(n+1)(2m+1), \\ S_{c1}^{(3)} = & \sum_{m,n}^{(m+1, n+1)} (m+1)(n+1)(2n+3), \quad S_{c1}^{(4)} = \sum_{m,n}^{(m+1, n-1)} (m+1)n(2n-1), \\ S_{c1} = & \sum_{m,n}^{(m+1, n-1)} (m+1)n(n+1), \end{aligned} \right\} \quad (\text{AIII} \cdot 5)$$

$$\left. \begin{aligned} U_{c2y}^{(2)} = & \frac{1}{8} \left(\frac{Q}{\omega} \right)^2 \left[-P \frac{\hbar}{M\omega_y} \left\{ \frac{\omega_y - \omega_z}{\omega_y} S_{c2}^{(1)} + \frac{\omega_y + \omega_z}{\omega_y} S_{c2}^{(2)} \right\} + Q \frac{\hbar}{M\omega_z} S_{c2} \right] y^2, \\ S_{c2}^{(1)} = & \sum_{m,n}^{(m+1, n+1)} (m+1)(m+2)(n+1), \quad S_{c2}^{(2)} = \sum_{m,n}^{(m+1, n-1)} (m+1)(m+2)n, \\ S_{c2} = & \sum_{m,n}^{(m, n+2)} (m+1)(n+1)(n+2), \end{aligned} \right\} \quad (\text{AIII} \cdot 6)$$

$$\begin{aligned} U_{\Delta L, y}^{(2)} = & -\frac{1}{4} \left(\frac{Q}{\omega} \right)^2 \frac{M\omega}{\hbar} \cdot \hbar\omega \left\{ \left(\frac{\hbar}{M\omega_y} \right)^2 \left(\frac{P}{\hbar\omega_y} \right) f S_y + \left(\frac{\hbar}{M\omega_z} \right)^2 \left(\frac{Q}{\hbar\omega_z} \right) g S_z \right\} y^2, \\ S_y = & 2 \sum_{occ} (2m+1), \quad S_z = 2 \sum_{occ} (2n+1) \end{aligned} \quad (\text{AIII} \cdot 7)$$

where the notations $\sum_{m,n}^{(m+1, n+1)}$, etc., mean the sum over the states m, n so that $m+1, n+1$ states are unoccupied. $U_{n, z}^{(2)}$, etc., can be obtained immediately from the above expressions by replacing y^2 with z^2 , P with Q .

Now we can write

$$U_d^{(2)} = U_n^{(2)} + U_{c1}^{(2)} + U_{c2}^{(2)} + U_{\Delta U}^{(2)}$$

from which the coefficients f and g are obtained. Thus the expectation value of $U_d^{(2)}$ with respect to the ground state is easily evaluated as follows,

$$\begin{aligned} \langle U_d^{(2)} \rangle_{\text{ground}} &= \frac{\hbar}{M\omega^2} \left[(1+K_p)(1+J_p) - K_Q J_Q \right]^{-1} \times \\ &\times \left[\left\{ \omega_z S_m (1+K_p) - \omega_y S_n J_Q \right\} \frac{M\omega_y}{\hbar S_m} \langle \sum_i U_{iy}^{(2)} \rangle_{\text{ground}} \right. \\ &\left. + \left\{ \omega_z S_m (1+J_p) - \omega_z S_m K_Q \right\} \frac{M\omega_z}{\hbar S_n} \langle \sum_i U_{iz}^{(2)} \rangle_{\text{ground}} \right] \end{aligned} \quad (\text{AIII} \cdot 8)$$

where \sum_i means the sum over $U_{ny}^{(2)}$, $U_{c1y}^{(2)}$, etc., and

$$\begin{aligned} S_m &= \sum_{occ} (2m+1), \quad S_n = \sum_{occ} (2n+1) \\ J_p &= \frac{1}{2} \left(\frac{\hbar}{M\omega_y} \right)^2 \frac{P}{\hbar\omega_y} S_m, \quad J_Q = \frac{1}{2} \left(\frac{\hbar}{M\omega_y} \right)^2 \frac{Q}{\hbar\omega_y} S_m \\ K_p &= \frac{1}{2} \left(\frac{\hbar}{M\omega_z} \right)^2 \frac{P}{\hbar\omega_z} S_n, \quad K_Q = \frac{1}{2} \left(\frac{\hbar}{M\omega_z} \right)^2 \frac{Q}{\hbar\omega_z} S_n. \end{aligned}$$

We wish to compare this expectation value with E_c , taking a typical example, say, a fictitious nucleus with 128 particles. In this case,

$$[(1+J_p)(1+K_p) - J_Q K_Q] < 0,$$

and

$$\omega_z S_m (1+K_p) - \omega_y S_n J_Q < 0, \quad \omega_y S_n (1+J_p) - \omega_z S_m K_Q > 0$$

and both are of about the same order of magnitude. Further,

$$Q > 0, \quad P < 0$$

$$\left| \frac{Q}{M\omega_z} \right| > \left| P \frac{\hbar}{M\omega_y} \right|, \quad \left| P \frac{\hbar}{M\omega_z} \right| > \left| Q \frac{\hbar}{M\omega_y} \right|.$$

These results make clear how each term of density $\rho^{(2)}$ contributes to the rotational energy. That is, $\rho_n^{(2)}$, as the matter of course, induces the excitation energy to decrease and the others to increase it. All terms are nearly of the same order of magnitude.

Finally we will give numerical results. For this purpose, we employ the equilibrium condition which gives a good fit with the observed deformation,

$$\frac{\omega_y}{\omega_z} = \frac{S_n}{S_m} = \delta. \quad (\text{AIII} \cdot 9)$$

In fact, starting from a given value of δ , the calculated value of S_m/S_n reproduces the initial value of δ . In the case of our example,

$$\delta \approx 2.6.$$

Using this value of δ , we get after straightforward calculations,

$$\langle U_d^{(2)} \rangle_{\text{ground}} \approx 4 \cdot \frac{1}{4} \left(\frac{\Omega}{\omega} \right)^2 \cdot \hbar \omega_z \cdot S_m (1 + \delta^2). \quad (\text{AIII} \cdot 10)$$

On the other hand, E takes the form^{9),10)}:

$$\begin{aligned} E_C &= -\frac{\hbar}{8} \left(\frac{\Omega}{\omega} \right)^2 \left\{ \frac{(\omega_y - \omega_z)^2}{\omega_y + \omega_z} (S_m + S_n) + \frac{(\omega_y + \omega_z)^2}{\omega_y - \omega_z} (S_n - S_m) \right\} \\ &= \frac{1}{4} \left(\frac{\Omega}{\omega} \right)^2 \hbar \omega_z S_m (1 + \delta^2). \end{aligned} \quad (\text{AIII} \cdot 11)$$

Hence

$$\langle U_d^{(2)} \rangle_{\text{ground}} \approx 4E_C. \quad (\text{AIII} \cdot 12)$$

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Maximal Weak Interactions

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All possible weak processes are systematically enumerated, and the hypothesis of maximal set of interactions, that there is always the weak interaction among any combination of all kinds of particles, is investigated. It is found that this maximal interaction model is equivalent to the minimal interaction model in the sense that the experimental facts are described equally well by these two models, provided that a selection rule which forbids at least the appearance of μ - e pair is added to the former model.

§ 1. Introduction

In constructing models of the weak interaction among leptons, mesons and baryons, it is usual to assume explicitly or implicitly a minimal set of weak interactions between elementary particles, that is to assume a minimum number of interactions which can explain all observed processes. However, as pointed out by Taketani¹⁾, we can imagine another extreme type of model, where we assume that there is always the weak interaction acting among any combination of particles, provided that the transformation among the particles of this combination is allowed by the conservation laws. This model may be called the maximal interaction model and this maximal interaction is analogous to the gravitational interaction which acts between any pair of particles with energy.

As the facts that support this maximal interaction model we take up the following three points: (1) While the strong interaction acts only among the limited kinds of particles, namely baryons, and mesons, the weak interaction acts among all kinds of particles including leptons. (2) The strength of all observed weak interactions is of the same order, and in cases of β -decay of neutron and β -decay of μ -meson, the coincidence of the coupling constant and the interaction type is remarkable²⁾. This suggests not only the universality of the coupling strength and the interaction type but also the universal existence of the weak interactions. (3) In weak interactions at least the parity invariance and the charge conjugation invariance are violated while in strong interactions these invariances hold.²⁾ This suggests that there may be fundamental difference between the strong interaction and the weak interaction and these two will play different roles in the future theory. For example, the weak interaction will be connected closely with the frame of the theory as the universal gravitation is connected with the curved space-time. If this were true, then the weak interaction must, as the gravitation, acts among any com-

bination of all kinds of particles. These three points encourage us to investigate maximal interaction model and at least there is no *a priori* reason to believe that the minimal interaction model is superior to the maximal interaction model.

Before we make this speculation, it is necessary to see whether the experimental materials are really consistent with the maximal interaction model. If the universal weak interactions among any combination of all particles are assumed, the reaction rate of any process can be estimated and according to the results of this estimation all processes are divided into two classes: Processes which have the theoretical possibility of observation and processes the reaction rate of which is too slow to be detected with the present experimental techniques. As to the first class there arise three questions: (1) As to the processes which can be observed experimentally, is the observed reaction rate in accordance with the universality of the coupling strength? (2) Is there any process the experimental detection of which is not yet been carried out? (3) Is there any process the non-existence of which is already established within the accuracy of the present experimental techniques and which demands any selection rule? Concerning the second class there arises a question: (4) Is there any process which is observed contrary to the theoretical prediction? To answer these questions is the main purpose of this paper and the results of our investigation are briefly as follows: as to the question (1) the answer is affirmative as is well known, as to the question (2) there is no such process except for several cases which are in the vicinity of the present limit of experimental techniques, as to the question (3) there are several such unwanted processes which can be forbidden by some additional selection rule, and finally as to the question (4) there is no such process. As the additional selection rule we postulate the following association rule of leptons in the weak interaction which is already pointed out by Sachs and Ogawa¹⁶⁾:

Association rule A: In the weak interactions the association of e and μ is forbidden, or more strongly,

Association rule A': In the weak interaction the permitted associations of leptons are (e, ν) and (μ, ν) , and all other associations (e, μ) , (e, e) , (ν, ν) and (μ, μ) are forbidden.*

Here we do not enter into the deeper meaning of this rule. If this selection rule is admitted, it may be said that the maximal interaction model is consistent with the experimental observations and in this sense is equivalent to the minimal interaction model. Although it seems that the investigations of the above problem were already made and the similar conclusions were already established in the literatures, for example, by Iwata et al. and by Ogawa¹⁷⁾ from the point of view of the universality of weak interactions, all conceivable processes have not been thorough-

* In applying these rules to four lepton process $L_1 + L_2 \rightarrow L_3 + L_4$, there arises an ambiguity as to which combination $(L_1 L_3)(L_2 L_4)$ or $(L_1 L_4)(L_2 L_3)$ must be taken. For definiteness we take the combination of weaker forbiddenness as determining the nature of the process. For example, if $(L_1 L_3)(L_2 L_4)$ is allowed and $(L_1 L_4)(L_2 L_3)$ is forbidden by the rule A, the process is an allowed one.

ly examined due to their points of view different from ours, and it seems necessary to reinvestigate these problems extensively from the maximal interaction point of view.

In this paper the above conclusion will be obtained under the following two limitations. First, the universality of weak interactions is limited to the strength of the coupling and the problem on the universality of the coupling type is disregarded, because except in few cases our experimental knowledge about most of the weak processes is not sufficient to allow discussions on the coupling type. Second, in order to minimize the number of kinds of elementary particles we adopt the composite particle model of elementary particles proposed by Sakata and Markov³⁾. Other models on elementary particles are expected to give essentially the same conclusions as ours.*

In § 2 all theoretically possible weak processes are systematically enumerated and the various experimental situations in detecting these processes are discussed. In §§ 3-5 the possibility of observing each process is considered and the unwanted processes are enumerated. In the final section the conclusion and some future aspects of our theory are presented.

§ 2. Enumeration of all possible processes and generality on experimental detection

In order to investigate whether there always acts the weak interaction among any combination of all particles, it is necessary to enumerate all theoretically possible processes systematically. As stated in § 1, we adopt the composite model of elementary particles proposed by Sakata and Markov.³⁾ In this model among strongly interacting particles only nucleons and Λ are considered to be elementary and other particles such as Σ , Ξ and π - and K -mesons are composite particles in the following way:

$$\begin{aligned}\pi &= (\bar{N}N), \quad K = (\bar{\Lambda}N), \\ \Sigma &= (\Lambda\bar{N}N), \quad \Xi = (\Lambda\Lambda\bar{N}).\end{aligned}$$

All leptons, e , ν and μ are considered elementary. Thus in this model all weak processes are caused by elementary interactions between two baryons N and three leptons. As the spin of all particles is $\frac{1}{2}$, all elementary interactions are 4 fermion interactions. Of course it is possible to consider 6, 8... fermions, but as shown in Appendix 1, except in a few cases these interactions can be derived essentially from the 4 fermion interactions, to which our considerations can be restricted.

Thus in order to enumerate all possible processes, it is sufficient to make all possible combinations of 4 particles among 5 particle: N , Λ , e , ν , μ . But the ob-

* The universality of weak interactions based on the weak boson-fermion interaction and Fermi interaction was considered by Iwata et al. and Ogawa¹⁷⁾.

served processes must obey various conservation laws, among which we take up the laws of conservation of charge and baryon numbers as restricting the type of interaction, not to speak of energy and momentum conservation laws. Then as the number of baryons which take part in the interactions must be even all interactions are classified in three types:

$$(1) L+L \rightarrow L+L, \quad (2) B+B \rightarrow B+B, \quad (3) B+L \rightarrow B+L$$

where B and L mean baryon and lepton respectively.

Before entering into the discussions of each process we make a few remarks on the method of experimental detection. Usually, weak interactions are observed in decay and capture processes, but besides these there are at least two methods of observing the weak processes. The one was pointed out by Heisenberg⁴⁾ and recently emphasized by Taketani¹⁾, Blokhintsev⁵⁾ and Umezawa⁶⁾, and it consists in observing scatterings and reactions caused by weak interactions at extreme high energy. The other was proposed by Goto and Machida⁷⁾ and consists in investigating the role of the weak interaction in the bound state problem. Although these possibilities are very interesting, we do not discuss these methods further in this paper for the simplicity of discussions.

The estimation of decay and capture rates is carried out under the following assumptions which will greatly simplify our qualitative discussions and will be sufficient for our preliminary considerations. (1) As far as possible, we limit our considerations only to the kinematical factor especially to the volume of phase space. The matrix element is assumed to be constant and, in the case of similar processes, to have the same value, so that the rate of different processes can be compared only in respect of the phase space. (2) If it is necessary to consider the composite nature of particles such as of π -meson in $\pi \rightarrow \mu + \nu$ decay, the wave function of composite particle is replaced by its value at the origin. At present this may be permitted because we have no knowledge about this wave function governed by the strong interaction which binds the composite particles. In the processes such as $K \rightarrow \pi + \pi$ which contain more than two composite particles, there arises the overlapping of their wave functions. This overlapping is replaced by a constant parameter of the order ~ 1 . (3) We disregard partial waves other than S -wave and also the spin dependence of the wave functions. Thus we neglect the type of interaction and assume it to be of scalar type. (4) In most cases we neglect the difference due to the various charge states.

In comparing the assumed interactions with experiment we notice that there are two different cases where the experimental observation is difficult. (1) The reaction is too slow for our present experimental techniques. Moreover, in many cases there is the strong interaction which overwhelmingly dominates weak processes. (2) In some cases such as $\mu + \mu \rightarrow \mu + \mu$ the preparation of the initial state is difficult. In the following sections we omit these *dominated* and difficult processes from our considerations.

§ 3. Processes of the type $L+L \rightarrow L+L$

In the first column in Table I all possible interactions of this type are enumerated. In this table the difference of charge states of particles is neglected. For example, e means e^+ and e^- , and the process $e+e \rightarrow e+e$ means $e^++e^+ \rightarrow e^++e^+$, $e^++e^- \rightarrow e^++e^-$ and $e^-+e^- \rightarrow e^-+e^-$. Also the difference between ν and $\bar{\nu}$ is often neglected.

Table I. Possible weak interactions of the type $L+L \rightarrow L+L$

Possible interaction	Experimental situation
L 1 $e+e \rightarrow e+e$	U', Do
L 2 $\mu+\mu \rightarrow \mu+\mu$	U', Di
L 3 $\mu+e \rightarrow e+e$	U
L 4 $\mu+\mu \rightarrow e+\bar{e}$	U, Di
L 5 $\mu+\mu \rightarrow \mu+e$	U, Di
L 6 $e+e \rightarrow \nu+\nu$	Do
L 7 $\mu+e \rightarrow \nu+\nu$	O
L 8 $\mu+\mu \rightarrow \nu+\nu$	Di
L 9 $\nu+\nu \rightarrow \nu+\nu$	U', Di

The meanings of the abbreviations are as follows:

Di: difficult to arrange the initial state experimentally.

Do: dominated by stronger interactions, mainly by the strong interactions and the electromagnetic interaction.

O: observed process.

U: unwanted process which can be forbidden by the association rule A.

U': unwanted process which can be forbidden by the association rule A'.

In the processes of this class only two decay processes are allowed by energy and momentum conservations:

$$\mu^\pm \rightarrow 2e^\pm + e^\mp \quad (3.1)$$

$$\mu^\pm \rightarrow e^\pm + \nu + \bar{\nu} \quad (3.2)$$

If we compare the decay rates of these two processes under the assumptions stated in § 2, we get

$$R(\mu \rightarrow 3e)/R(\mu \rightarrow e + \nu + \bar{\nu}) \sim 1, \quad (3.3)$$

neglecting the mass difference of e and ν . Experimentally this ratio is smaller than 10^{-3} . Then the process L 3 is an unwanted process and is desired to be forbidden by the association rule A or A' stated in § 2.

Next, as capture processes there are three processes to be detected:

$$e^+ + e^- \rightarrow \nu + \bar{\nu} \quad (3.4)$$

$$\mu^+ + e^- \rightarrow e^+ + e^- \quad (3.5)$$

$$\mu^+ + e^- \rightarrow \nu + \bar{\nu} \quad (3.6)$$

The initial state of these processes is the bound state due to the Coulomb attraction. Other processes with the initial state such as $\mu^+ + \mu^-$ or $\nu + \bar{\nu}$ are difficult to realize experimentally, and are not considered here.

The process (3.4) is the 2ν annihilation of positronium and is expected to be dominated by the 2γ annihilation. In fact the ratio of these two processes is given by

$$R(e^+ + e^- \rightarrow 2\nu)/R(e^+ + e^- \rightarrow 2\gamma) = (1/8\pi^2) (f/\alpha)^2 (m_e/m_\pi)^4 \sim 2 \times 10^{-20} \quad (3.7)$$

where $\alpha=1/137$ is the fine structure constant and f is the effective weak coupling constant of the universal weak interactions measured in unit of π -meson mass and f^2 is taken to be 5.4×10^{-13} (See (5.6)). The calculation is given in Appendix 2. The result (3.7) confirms the above expectation.

The processes (3.5) and (3.6) are the annihilation of μ uonium and compete with the decay of a bound μ -meson or the destruction of μ uonium by collisions in matter. If we imagine the very dilute matter, we can neglect the latter process. The annihilations (3.5) and (3.6) have almost the same rate

$$R(\mu^+ + e^- \rightarrow e^+ + e^-) = (1/8\pi^2) \alpha^3 f^2 m_e (m_e/m_\pi)^2 (m_\mu/m_\pi)^2 \sim 2 \times 10^{-5}/\text{sec} \quad (3.8)$$

and are much slower than the decay rate of this bound μ -meson which is almost the same as a free μ -meson. Then reason why the rate (3.8) is very slow is that in μ uonium μ^+ and e^- are too far from each other, and there is an additional factor $\alpha^3/\pi \sim 10^{-7}$ in (3.8). In this connection, it is interesting to note that this factor appears in general when the initial state is a bound state due to the Coulomb attraction and one of the constituent particle can undergo free decay.

In this way we get the experimental situation of each process, which is shown in the second column of Table I.

§ 4. Interaction of the type $B+B \rightarrow B+B$

All possible interactions of this type are enumerated in the first column in Table II. They are divided into two classes: the one which conserves A (hence the strangeness), such as B 1, 3 and 5, and is dominated by the strong interactions as stated in § 2, and the other which does not conserve A and is not observed as elementary process because of negative Q -values. The latter is observed as derived process which contains composite particles. As decay processes we have from B 2 and/or B 4

$$\Lambda \rightarrow N + \pi, \quad \Sigma \rightarrow N + \pi, \quad \Xi \rightarrow \Lambda + \pi \quad (4.1)$$

$$K \rightarrow 2\pi \quad (4.2)$$

$$K \rightarrow 3\pi. \quad (4.3)$$

These are all decay processes that can be derived from B 2 and B 4, and they are all observed.

Table II. Possible weak interactions of the type $B+B \rightarrow B+B$

Elementary interaction	Derived process	Experimental situation
B 1 $N+N \rightarrow N+N$		Do
B 2 $\Lambda+N \rightarrow N+N$	$\Lambda(\Sigma) \rightarrow N+\pi, \Xi \rightarrow \Lambda+\pi$	0
	$K \rightarrow 2\pi, 3\pi$	0
B 3 $\Lambda+N \rightarrow \Lambda+N$		Do
B 4 $\Lambda+\Lambda \rightarrow \Lambda+N$	$\Xi \rightarrow \Lambda+\pi$	0
B 5 $\Lambda+\Lambda \rightarrow \Lambda+\Lambda$		Do

It is well known that their observed rates are compatible with the universal strength of all weak couplings⁸⁾ and the estimation of decay rate based on the composite model was given by Tanaka⁹⁾ who took into account the spin dependence. We give in Appendix 3 similar but more simplified calculation carried out under the assumptions stated in § 2. The obtained results are briefly given here. The decay rate of (4.1) is estimated to be

$$R \sim \pi^{-2} f^2 p E_{\pi} E_N / m_{\pi} m_Y \quad (4.4)$$

where p , E_{π} , E_N and m_Y are the momentum and energy of the final π -meson, the energy of the nucleon and the mass of the hyperon in the rest system of the hyperon. In deriving this formula the radius of the π -meson is taken to be equal to m_{π}^{-1} and the composite nature of Σ and Ξ is neglected. The numerical values are given in Table III and are in rough agreement with observed values.

Table III. Decay rate of $B \rightarrow B+\pi$ (in sec^{-1})

Decay process	Estimated value	Observed value
$\Lambda \rightarrow N+\pi$	$0.88 \times 10^{10}/\text{sec}$	$0.36 \times 10^{10}/\text{sec}$
$\Sigma \rightarrow N+\pi$	3.1×10^{10}	$(0.64 \sim 1.28) \times 10^{10}$
$\Xi \rightarrow \Lambda+\pi$	1.4×10^{10}	10^{10}

The decay rate for the process (4.2) is estimated to be

$$R \sim (2\pi^2)^{-1} f^2 p (m_K / m_{\pi}) \gamma^2 \quad (4.5)$$

where p is the momentum of the π -meson in the rest system of K -meson and γ is a parameter which characterizes the overlapping of wave functions of π - and K -mesons. If we compare (4.5) with the decay rate of $K_1^0, \sim 10^{10}/\text{sec}$, we get $\gamma^2 = 0.3$, which is not an unreasonable value in order of magnitude discussion.

In order to estimate the process (4.3) it is necessary to enter into the details of the strong interaction. So we neglect the composite nature of K - and π -mesons and compare its phase volume with that of β -decay of neutron. We get (Appendix 4)

$$R(K \rightarrow 3\pi) = R(n \rightarrow p + e + \bar{\nu}) (E_{\pi \text{ max}} / \Delta)^5 \times (0.44/0.47) \sim 6.4 \times 10^7 / \text{sec} \quad (4.6)$$

where $\Delta = m_n - m_p$, "0.44" is a factor appearing in the phase space integral of 3π decay of K -meson, and "0.47" is a blocking factor of β -decay of neutron,¹⁰⁾ and $E_{\pi\max}$ is the maximum energy of π -meson in $K \rightarrow 3\pi$ decay. The value of (4.6) is in rough agreement with the observed value.

As to the capture processes which are derived from B 2 and B 4, those by proton or nucleus are dominated by the strong interactions and those by leptons will be discussed in the next section.

Now there is another observed process which can be derived from B 2 and B 4, that is the decay of hyperfragments. In order to obtain the order of their decay rate we consider the decay of the simplest and fictitious hyperfragment ${}_{\Lambda}H^2$ as an example of the hyperfragment decays,

$${}_{\Lambda}H^2 \rightarrow N + N. \quad (4.7)$$

The calculation of decay rate is similar to that of the process (3.4) (See Appendix 2) and gives

$$R = (2\pi^2)^{-1} f^2 p M / m_{\pi} \sim 6 \times 10^{10} / \text{sec} \quad (4.8)$$

where p and M are momentum and mass of the final nucleon. The value (4.8) is in rough agreement with experimental estimation. Thus the interactions of the type $B+B \rightarrow B+B$ cause no obstacles in our maximal interaction model.

§ 5. Interactions of the type $B+L \rightarrow B+L$

All interactions of this type are enumerated in the first column of Table IV. They are clasified in three types according to the number of hyperons involved, zero, one or two.

A. Interactions which involve no hyperon.

As the elementary decay process there is only one case which is compatible with kinematical relations:

$$n \rightarrow p + e + \bar{\nu} \quad (5.1)$$

but there are several processes which are derived from elementary interactions:

$$\Sigma \rightarrow \Lambda(\Sigma') + \text{two leptons} \quad (5.2)$$

$$\pi \rightarrow \text{two leptons} \quad (5.3)$$

$$\pi^{\pm} \rightarrow \pi^0 + \text{two leptons} \quad (5.4)$$

where two leptons mean any combination of identical or different two leptons taken from e , μ and ν . The possible processes must be compatible with the charge conservation and the positive Q -value.

Now the decay rate of neutron can be used as a standard in estimating other decay processes and can be written in the form

$$R(n \rightarrow p + e + \bar{\nu}) = (60\pi^3)^{-1} \times 0.47 \Delta^5 \times \frac{1}{2} f^2 m_{\pi}^{-4} \quad (5.5)$$

Table IV. Possible weak interactions of the type $B+L\rightarrow B+L$

Elementary interaction		Derived process	Experimental situation
BL 1	$N+e\rightarrow N+e$	$\pi^0\rightarrow e^++e^-$, $\Sigma^0\rightarrow\Lambda+e^++e^-$	U', Do
BL 2	$N+\mu\rightarrow N+\mu$	no process	
BL 3	$N+\nu\rightarrow N+\nu$	$\pi^0\rightarrow\nu+\bar{\nu}$, $\Sigma^0\rightarrow\Lambda+\nu+\bar{\nu}$	U', Do
BL 4	$N+\mu\rightarrow N+e$	$\mu^-+p\rightarrow p+e^-$, $\pi^0\rightarrow\mu+e$	U U, Do
BL 5	$N+e\rightarrow N+\nu$	$n\rightarrow p+e^-+\nu$, $\pi^-\rightarrow e+\nu$	0
		$\Sigma\rightarrow\Lambda+e+\nu$?
		$\pi^\pm\rightarrow\pi^0+e+\nu$	Do
BL 6	$N+\mu\rightarrow N+\nu$	$\mu^-+p\rightarrow n+\mu$, $\pi\rightarrow\mu+\nu$	0
BL 7	$\Lambda+e\rightarrow N+e$	$\Lambda(\Sigma)\rightarrow N+e^++e^-$	U', ?
		$\Xi\rightarrow\Lambda(\Sigma)+e^++e^-$	U', ?
		$K^0\rightarrow e^++e^-$	U', ?
BL 8	$\Lambda+\mu\rightarrow N+\mu$	$K^0\rightarrow\mu^++\mu^-$	U', ?
BL 9	$\Lambda+\nu\rightarrow N+\nu$	$\Lambda(\Sigma)\rightarrow N+\nu+\bar{\nu}$, $K^0\rightarrow\nu+\bar{\nu}$	U', ?
BL 10	$\Lambda+\mu\rightarrow N+e$	$\Lambda(\Sigma)\rightarrow N+\mu+e$,	U, ?
		$K\rightarrow\mu+e(+\pi)$	U, ?
BL 11	$\Lambda+e\rightarrow N+\nu$	$\Lambda(\Sigma)\rightarrow N+e+\nu$, $K\rightarrow e+\nu(+\pi)$	0
BL 12	$\Lambda+\mu\rightarrow N+\nu$	$\Lambda(\Sigma)\rightarrow N+\mu+\nu$, $K\rightarrow\mu+\nu(+\pi)$	0
BL 13	$\Lambda+e\rightarrow\Lambda+e$	$\pi^0\rightarrow e^++e^-$, $\Sigma^0\rightarrow\Lambda+e^++e^-$	U', Do
BL 14	$\Lambda+\mu\rightarrow\Lambda+\mu$	no process	
BL 15	$\Lambda+\nu\rightarrow\Lambda+\nu$	$\pi^0\rightarrow\nu+\bar{\nu}$, $\Sigma^0\rightarrow\Lambda+\nu+\bar{\nu}$	U', Do
BL 16	$\Lambda+\mu\rightarrow\Lambda+e$	$\pi^0\rightarrow\mu+e$	U, Do

? means that the experimental situation is not definite.

where $\Delta=m_n-m_p$, the factor 0.47 is the blocking factor of electron, f^2m^{-4} is the square of the effective matrix element, and the factor $\frac{1}{2}$ comes from the "two" component of neutrino. If (5.5) is equated to the observed value $0.96\times10^{-3}/\text{sec}$, we get the effective weak coupling constant

$$f^2=5.4\times10^{-13}. \tag{5.6}$$

In treating various processes expressed by (5.2), we neglect, for simplicity, the

Tablay V. Decay rate of $B\rightarrow B+\text{two leptons}$ (The strangeness is not changed)

Decay process	Estimated value
$\Sigma^+\rightarrow\Lambda+e^++\nu$	$1.4\times10^6/\text{sec}$
$\Sigma^-\rightarrow\Lambda+e^-+\bar{\nu}$	1.9×10^6
$\Sigma^-\rightarrow\Sigma^0+e^-+\bar{\nu}$	4.1
$\Sigma^0\rightarrow\Lambda+e^++e^-$	1.4×10^6
$\Sigma^0\rightarrow\Lambda+\nu+\bar{\nu}$	

composite nature of Σ and estimate their decay rates by comparing their kinematical factor with that of β -decay of neutron. The results are shown in Table V. These are compared with the rate of non-leptonic decay of $\Sigma' \sim 10^{10}/\text{sec}$, which shows that the detection of (5.2) is rather difficult but not impossible.

As to the processes (5.3) there are many calculations and discussions.¹¹⁾ We give here our estimation which is based on the composite model, because this would give us an idea on the correctness of our estimation and will be useful in discussing other processes for which there is no observation or reliable calculation. (See Appendix 5 for the calculation). Assuming the "radius" of the π -meson is m_{π}^{-1} -meson and using the value (5.6) for f^2 , we get the values shown in Table VI. The value for $\pi \rightarrow e + \nu$ is in rough agreement with the observed value, but the calculated value for $\pi^0 \rightarrow e + \nu$ is too much larger than that observed value¹²⁾, which may be attributed to the effect of coupling type. The rate of other processes such as $\pi^0 \rightarrow e^+ + e^-$ and $\pi^0 \rightarrow \nu + \bar{\nu}$ is the same as that of $\pi \rightarrow e + \nu$ in our estimation and these processes are dominated by the observed decay $\pi^0 \rightarrow 2\gamma$ which is caused by strong interactions.

Table VI. Decay rate of $\pi \rightarrow$ two leptons

Decay process	Estimated value	Observed value
$\pi \rightarrow \mu + \nu$	$2.1 \times 10^8/\text{sec}$	$0.4 \times 10^8/\text{sec}$
$\pi \rightarrow e + \nu$	7.6×10^8	$\sim 5 \times 10^8$

As to the process (5.4), the decay rate which is estimated by comparison of the kinematical factor with that of β -decay of neutron is 1.2/sec, which is compared with the value 20/sec obtained by more accurate estimation based on the dispersion-like calculation¹³⁾. This gives also an idea on the correctness of our estimation. This rate is naturally too slow to be observed.

Turning to the capture processes, possible elementary processes are

$$\mu^- + p \rightarrow n + \nu \quad (5.7)$$

$$\mu^- + p \rightarrow p + e^- \quad (5.8)$$

and possible derived processes are

$$\pi^+ + e^- \rightarrow \pi^0 + \nu \quad (5.9)$$

$$\Sigma^+ + e^- \rightarrow \Lambda + \nu. \quad (5.10)$$

It is well known that the process (5.7) is observed as the capture by nucleus and is compatible with the universality of coupling strength of the weak interaction⁸⁾. The process (5.8) has the same decay rate as the process (5.7) in our model and is an unwanted process, which can be forbidden by the association rule A. The calculated rates of both capture processes (5.9) and (5.10) are slower than the quasi-free decay of π^+ and Σ^+ respectively by a factor of the electromagnetic

overlapping probability of the initial state, $\alpha^3/\pi \sim 10^{-7}$, and are difficult to be detected experimentally. (See the discussions at the end of § 3.)

B. Interactions of the type $A+L \rightarrow N+L$.

The elementary and derived decay processes are

$$\Lambda(\Sigma) \rightarrow N + \text{two leptons} \quad (5.11)$$

$$\Xi \rightarrow \Lambda(\Sigma) + \text{two leptons} \quad (5.12)$$

$$K \rightarrow \text{two leptons} \quad (5.13)$$

$$K \rightarrow \pi + \text{two leptons.} \quad (5.14)$$

If the composite nature of Σ and Ξ is neglected, the decay rate of processes (5.11) and (5.12) can be estimated simply by comparing their kinematical factor with that of β -decay of neutron. The results are shown in Table VII where the first two processes were already estimated in reference 8) p. 442. All processes except β - and μ -decays are unwanted processes which can be forbidden by the association rule A or A', and only for some β - and μ -decays there are experimental evidences¹⁴⁾ at the present time.

Table VII. Decay rate of $B \rightarrow B + \text{two leptons}$ (The strangeness is changed)

Decay process	Estimated value
$\Lambda \rightarrow N + 2e(2\nu, e + \nu)$	$1.0 \times 10^8/\text{sec}$
$\Lambda \rightarrow N + \mu + \nu(e)$	1.6×10^7
$\Sigma \rightarrow N + 2e(2\nu, e + \nu)$	5.6×10^8
$\Sigma \rightarrow N + \mu + \nu(e)$	1.1×10^8
$\Xi \rightarrow \Lambda + 2e(2\nu, e + \nu)$	2.2×10^8
$\Xi \rightarrow \Lambda + \mu + \nu(e)$	5.8×10^7
$\Xi \rightarrow \Sigma + 2e(2\nu, e + \nu)$	2.5×10^7
$\Xi \rightarrow \Sigma + \mu + \nu(e)$	5.8×10^8

The decay rate of (5.13) is estimated as in the case of $\pi \rightarrow \text{two leptons}$. Assuming the "radius" of K -meson is m_π^{-1} , we get

$$R(K \rightarrow 2 \text{ leptons}) \sim (1/4\pi^2) m_K^2 m_\pi^{-1} f^2 \sim 4 \times 10^{10}/\text{sec.} \quad (5.15)$$

Among the various processes expressed by (5.13) only $K \rightarrow \mu + \nu$ is observed. As to the other unobserved processes the slowness of $K \rightarrow e + \nu$ may be a result of special effect of the type of interaction as in the case of $\pi \rightarrow e + \nu$ and others such as $K^0 \rightarrow e + \mu$, $2e$, 2μ , and 2ν are unwanted processes which can be forbidden by the association rule A and/or A'.

The decay rate of (5.14) is estimated by comparing their kinematical factor with that of β -decay of neutron and is given by

$$R(K \rightarrow \pi + 2 \text{ leptons}) \sim 10^8/\text{sec.} \quad (5.16)$$

In the observed cases such as $K \rightarrow \pi + e + \nu$ and $K \rightarrow \pi + \mu + \nu$ the agreement of the value (5.16) with experiment is rather good, and in unobserved cases the reason can be attributed to the association rule A and/or A'.

As to capture processes there is no process which has the possibility to be detected, for processes of the type

$$\mu^- + p \rightarrow \text{hyperon} + \text{lepton}$$

are all forbidden by the negative Q -value, and processes of the type

$$e^- + K^+ \rightarrow L + (1, 2 \text{ or } 3\pi)$$

$$e^- + \Sigma^+ \rightarrow N + L + (1 \text{ or } 2\pi)$$

are slower than the quasi-free decay of K^+ -meson and Σ^+ respectively by a factor $\alpha^3/\pi \sim 10^7$ (See the discussions at the end of § 3).

C. Interaction of the type $A + L \rightarrow A + L$

As is seen from Table IV the possible processes derived from this interaction are all covered by the processes which were already considered in A.

The main result of this section are summarized in Table IV.

§ 6. Conclusion

Summarizing the results obtained in § 3~5, we can say that the maximal interaction model does not contradict with the present experimental observations provided that the following reservations are made:

(1) Our discussions are quite limited to qualitative problems and must be extended to quantitative and detailed problems such as the type of interaction and the difference due to charge states. For this purpose it is necessary to have full experimental knowledge on all possible processes, which, however, we have not at present.

(2) It is necessary to postulate, besides the usual selection rules, another selection rule which forbids the appearance of a (e, μ) pair in the weak processes or more strongly the appearance of, besides the (e, μ) pair, (e, e) , (μ, μ) and (ν, ν) pairs. This rule must be confirmed more definitely by further experiments, especially on the processes marked by “?” in Table IV.

The fact that our consideration is based on the special model of composite particles might be thought at first to be a serious restriction to the validity of our conclusion, but this is not the case at least concerning the conclusion of qualitative nature. This is because we used the composite model mainly in the enumeration of all conceivable weak processes and did not use it in the dynamical problems except few cases.

Although these conclusions are nearly the same as those obtained under the usual universal weak interaction hypothesis,¹⁷⁾ it must be noted that our conclusions are obtained after more extensive and more systematic examination of all possible

weak processes from the different point of view from the usual universal weak interaction point of view.

If the above conclusion is correct, then the next problem is firstly to make calculations more precise and to see whether our model holds concerning the details on the type of interaction. Secondly, we must seek deeper meaning of the association rule A or A'. Already there are many works on this subject¹⁶⁾ and especially the trials by Feynman and Gell-Mann, and Sudarshan and Marshak¹⁵⁾ are very interesting. But even if their trials succeed, it is necessary to find their more profound meaning. In this respect, although in § 1 we mentioned the analogy between the weak interaction and the gravitation, the conjecture that there is some direct connection between them does not seem promising, because the effect of the gravitation is too small at least in the perturbation theoretic sense.

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Appendices

1. On 6, 8.....fermion processes

Although it is *a priori* possible that besides 4 fermion interactions there exist 6, 8...fermion interactions, it is not necessary to consider the latter interactions except in a few cases. In other words, almost all processes can be derived from 4 fermion elementary interactions and a few processes are derived from 6 fermion elementary interactions. The proof is as follows. Consider an arbitrary decay process,

$$a \rightarrow b + c \cdots$$

First, among a, b, \cdots , pairs of the same baryons which can annihilate by the strong interaction are excluded. Secondly more than one pair of the same leptons are also excluded, partly because of the simplicity and partly because of the association rule A'. Then, as we have only five elementary particles, there remains only the following 6 fermion interactions which can not be derived from 4 fermion interactions:

$$A \rightarrow N + e + 2\nu + \mu \quad (\text{A1} \cdot 1)$$

and all 8, 10.....fermion interactions can be reduced to 4 and 6 fermion interactions. As there is no principle to give the coupling strength of the above process, we do not consider this further.

2. Calculation of capture rate

The rate of various capture processes is estimated under the assumptions stated in § 2. For example, for the process (3.4) $e^+ + e^- \rightarrow \nu + \bar{\nu}$ the capture rate is given by

$$R=2\pi\rho|M|^2|\varphi(0)|^2 \quad (\text{A2.1})$$

where ρ is the energy density of the final state, M is the matrix element taken between plane wave states and is assumed to be equal to be f/m_π^2 , and $\varphi(0)$ is the value of internal wave functions of the initial state at the origin and is taken $\varphi(0)=(\pi a^3)^{-1/2}$ where a is the radius of positronium, $2/(m_e a)$. Then the decay rate of (3.4) is

$$R(e^+ + e^- \rightarrow 2\nu) = (16\pi^2)^{-1} f^2 \alpha^3 m_e^5 m_\pi^{-4}.$$

If this is divided by the rate of 2γ -annihilation of positronium:

$$R(e^+ + e^- \rightarrow 2\gamma) = \frac{1}{2} \alpha^5 m_e$$

we get (3.7).

3. Decay rate of (4.1) and (4.2)

According to the assumptions stated in § 2 the decay rate of (4.1) can be written in the form

$$R=2\pi\rho|M|^2|\varphi_\pi(0)|^2 \quad (\text{A3.1})$$

where the meanings of ρ and M are similar in (A2.1) and $\varphi_\pi(0)$ is the value of the internal wave function of the π -meson at the origin and is taken as $(\pi a^3)^{-1/2}$ where a is the "radius" of the π -meson and is assumed to be m_π^{-1} .

Similarly, for the process (4.2) we get

$$R=2\pi\rho|M|^2|\varphi_\pi(0)|^2\eta^2 \quad (\text{A3.2})$$

where

$$\eta = \left| \int d\mathbf{x} \varphi_\pi^*(\mathbf{x}) \varphi_\pi(\mathbf{x}) \exp(i\mathbf{p}\mathbf{x}) \right| \quad (\text{A3.3})$$

is assumed to be a constant parameter and ≤ 1 .

4. Phase volume of three-body decay

We define the volume of the phase space of three-body decay by

$$W = (2\pi)^{-6} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \delta(E_1 + E_2 + E_3 - E_0) \quad (\text{A4.1})$$

where $E = (m_n^2 + p_n^2)^{1/2}$, p_n and m_n are momentum and mass of n -th particle. For the process (4.3) we get

$$W = (32\pi^4)^{-1} \times 0.44 E_{\pi \max} \quad (\text{A4.2})$$

where $E_{\pi \max}$ is the maximum energy of π -meson.

For the process (5.1) we get

$$W = (120\pi^4)^{-1} \times 0.47 D^5. \quad (\text{A4.3})$$

From these we get (4.6).

5. Decay of π -meson

The method of estimation of the process $\pi \rightarrow \mu + \nu$ is similar to that of capture rate in Appendix 2. We get

$$R = (2\pi^2)^{-1} f^2 E q^2 m_\pi^{-2}$$

where E and q are the energy and momentum of μ -meson. For the process $\pi \rightarrow e + \nu$, E and q are those of electron.

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Remarks on the Final State Interaction in Fermi's Theory of Multiple Particle Production

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A method of modifying Fermi's statistical formula of multiple particle production is presented. It consists of expressing the "matrix element" of Fermi's theory as an integration of plane wave in the effective interaction volume, and then replacing this free state wave function by that of interacting particles. Its application to the π - N system and deuteron production is given. Belenky's formalism on the same problem is critically discussed.

§ 1. Introduction

Fermi's¹⁾ statistical formula for multiple particle production is based on a tacit assumption that the produced particles are in free states. Actually, however, some of them interact strongly with each other and this fact seems to be one of the main factors that may account for the discrepancy between his theory and experimental results.²⁾

We shall follow Fermi's original idea and express his statistical formula in terms of usual quantum mechanics, as the product of the square of transition matrix element and the final state density, and find that the former is just the probability that all the final particles, which are in free states, are found in the interaction volume Ω . In this way we can extend his formula to the case of final state interaction by modifying the matrix elements, though, as we shall see, the results are not very satisfactory.

Another approach, presented by Belenky³⁾, consists in the change of the final state density. His theory explains experimental results very well,⁴⁾ but it does not seem to us theoretically convincing.

§ 2. Formulation

For simplicity we shall take a system of a nucleon and n pions. According to Fermi, the probability for the formation of these particles is proportional to

$$S_0(W, n, M_0) = \frac{1}{n!} \left[\frac{\Omega}{V} \right]^n \times \left[\frac{V}{(2\pi\hbar)^3} \right]^n \int \prod_{i=1}^n d\mathbf{p}_i \cdot d\mathbf{p} \\ \times \delta(W - \sum_{i=1}^n W_i - W_M) \delta(\sum_{i=1}^n \mathbf{p}_i + \mathbf{p}), \quad (1)$$

where W is the total energy of the system; \mathbf{p}_i , W_i the momentum, energy of i -th meson; \mathbf{p} , W_M the momentum, energy of the nucleon; V , Ω normalization and interaction volume.

This expression can be written in the form:

$$S_0(W, n, M_0) = \int_{\Omega} |\phi_F(r_1 \cdots r_n, r_M)|^2 dV_1 \cdots dV_n \cdot dV_M \times \rho_F = P \times \rho_F. \quad (2)$$

Here ϕ_F represents the wave function of the final state. It is obvious from (2) that S_0 is, apart from the final state density, given by the probability that, in the state under consideration, all the particles happen to be within Ω at the same time. This is in agreement with the idea expressed by Fermi in his original paper. Fermi took as ϕ_F the product of plane wave states of individual particles, without taking account of any final state interactions; that is

$$P = \int_{\Omega} |\phi_F|^2 d\tau = \left(\frac{\Omega}{V} \right)^n,$$

thus we have the expression (1).

In order to introduce the effect of a final state interaction between a nucleon and a pion, it is convenient to slightly rewrite the expression (1).

$$S_0(W, n, M_0) = \left[\frac{\Omega}{(2\pi\hbar)^3} \right]^{n-1} \frac{1}{n!} \frac{\Omega}{(2\pi\hbar)^3} \int_{i=1}^{n-1} \prod d\mathbf{p}_i d\mathbf{p}_c d\mathbf{p}' \\ \times \delta(W - \sum_{i=1}^n W_i - W_M) \delta(\sum_{j=1}^{n-1} \mathbf{p}_j + \mathbf{p}_c), \quad (3)$$

where $\mathbf{p}_c = \mathbf{p}_n + \mathbf{p}$ and $\mathbf{p}' = (M\mathbf{p}_n - \mu\mathbf{p})/(\mu + M)$ are the total momentum and relative momentum of n -th meson and the nucleon system, respectively. We interpret the last factor Ω in (3) as the volume* of relative coordinate space, and transform the expression (3) by putting

$$\Omega = \int_{\Omega} |e^{ik'r/2}|^2 dV = \int_b^a r^2 dr \int d\omega |\exp(ik'r \cos \theta)|^2 \\ = \int_0^a r^2 dr \int d\omega \left| \sum_l (2l+1) i^l j_l(k'r) P_l(\cos \theta) \right|^2 \\ = \int_0^a r^2 dr 4\pi \sum_l (2l+1) j_l^2(k'r) \quad (4)$$

into

$$S_0(W, n, M_0) = \left[\frac{\Omega}{(2\pi\hbar)^3} \right]^{n-1} \frac{1}{n!} \sum_l (2l+1) \int_{i=1}^{n-1} \prod d\mathbf{p}_i \cdot d\mathbf{p}_c \cdot d\mathbf{k}'$$

* Strictly speaking, this prescription deviates slightly from Fermi's original one.

$$\times \int_0^a r^2 dr \frac{1}{2\pi^2} j_i(k'r) \times \delta(W - \sum_{i=1}^n W_i - W_M) \delta(\sum_{i=1}^{n-1} \mathbf{p}_i + \mathbf{p}_e) \quad (5)$$

where $j_i(k'r)$ is the spherical Bessel function, a is the interaction radius ($\Omega = 4\pi a^3/3$ and we do not consider the Lorentz contraction) and $p' = \hbar k'$.

Now it is natural to assume that the modification of (5) due to the final state interaction can be effectuated by replacing $v_i^0 \equiv N_{p'} j_i(k'r) \times r$, which represents the non-interacting states, by another function v_i , which represents interacting particles. ($N_{p'} = \sqrt{2k'^2/R}$ is the normalization constant and R is the normalization radius.) If we define ΔP_i by

$$\Delta P_i(p') \equiv \int_0^a \{ |v_i(k'r)|^2 - |v_i^0(k'r)|^2 \} dr, \quad (6)$$

the probability for the formation of the system will be proportional to

$$\begin{aligned} S^{(1)}(W, n, M_0) &= S_0(W, n, M_0) \\ &+ \sum_i (2l+1) \frac{1}{4\pi^2 \hbar} \int d\mathbf{p}' \left\{ \left[\frac{\Omega}{(2\pi \hbar)^3} \right]^{n-1} \frac{1}{n!} \int_{i=1}^{n-1} \prod d\mathbf{p}_i d\mathbf{p}_e \right. \\ &\quad \left. \times \delta(W - \sum_{i=1}^n W_i - W_M) \delta(\sum_{i=1}^{n-1} \mathbf{p}_i + \mathbf{p}_e) \right\} \frac{R \Delta P_i(p')}{k'^2} \end{aligned} \quad (7)$$

where the second term corresponds to the production through the isobars in the case of Belenky's work.

§ 3. Application

i) π -N interaction

We shall consider only the $(3/2, 3/2)$ state interaction between a pion and a nucleon. According to (6) and (7) it is necessary to work out the integration of the wave function inside the interaction region Ω . The solution of the Schrödinger equation with the interaction taken into account is⁽⁶⁾

$$\phi_a^{(+)} = \phi_a + \frac{1}{E_a + i\epsilon - H_0} H_I \phi_a^{(+)} \quad (8)$$

and our method amounts to replacing ϕ_a by $\phi_a^{(+)}$.

We can prove that if ϕ_a is normalized, then $\phi_a^{(+)}$ is also normalized.⁽⁶⁾ Therefore

$$\int_{\Omega} |\phi_a^{(+)}|^2 dV = 1 - \int_{V-\Omega} |\phi_a^{(+)}|^2 dV. \quad (9)$$

In the region $V-\Omega$, which is outside the interaction volume, the wave function can be expressed only by the phase shift and

$$\int_{r=\Omega}^R |\phi_a^{(+)}|^2 dV = \int_{\alpha}^R |N_{p'} (j_l(k' r) \cos \delta_l - n_l(k' r) \sin \delta_l)|^2 r^2 dr \tag{10}$$

where $N_{p'}^2 = 2k'^2/R$ and $N_{p'}$ is the normalization constant in the case $\delta_l = 0$.

The results of estimation* are as follows: As seen from Fig. 1 $R\Delta P_l(p')$ in (6) is not always positive and depends sensitively on the interaction radius, so the second term in (7) cannot be very significant. It is $\sim 1/10$ of the contribution without interaction.

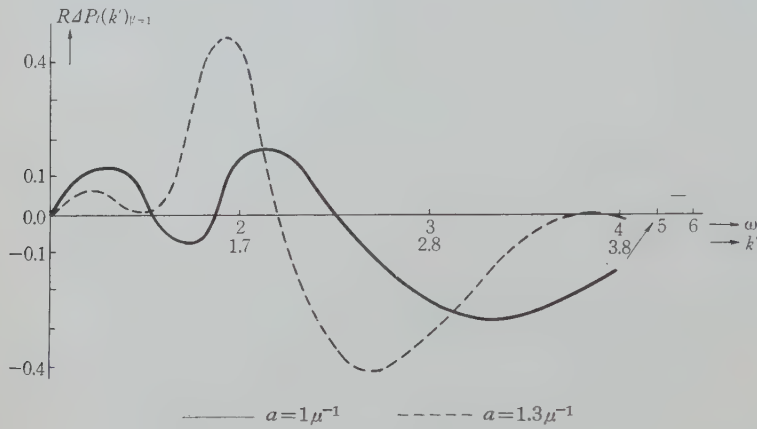


Fig. 1 $R\Delta P_l(k')$

ii) Deuteron production

Let us apply our method to a bound state. We then have

$$S^{(1)}(D) = \int_{\Omega} |\psi_F(\mathbf{r})|^2 dV \times S_0(D) \tag{11}$$

for the probability of deuteron formation. Here $S_0(D)$ is the corresponding expression in Fermi's theory and $\psi_F(r)$ is the wave function of the deuteron. As is well-known the wave function ψ_F is rather spread out and so we get

$$S^{(1)}(D) \approx (\frac{1}{2} \sim \frac{1}{3})^3 S_0(D). \tag{12}$$

Table 1 Production ratio of deuteron

<i>p-p</i> collision ≈ 1 BeV		<i>p-n</i> collision ≈ 1 BeV	
	$d \pi^+ : p n \pi^-$		$d \pi^+ \pi^- : p p \pi^-$
exp. (events)	2 : 92 ⁸⁾	7	97 ⁹⁾
Fermi ($S_0(D)$)	1 : 3	1	2
Eq. (12) ($S^{(1)}(D)$)	1 : 30~90	1	20~60

* The details of the calculation are given in reference 7).

The comparison with experimental data is given in Table 1. and it is seen that our modification gives a fairly good agreement with experiment.

§ 4. Discussion; relation to Belenky's results

Our conclusion concerning the π - N interaction (§ 3, i) differs very much from that of Belenky³⁾, though our standpoint is similar to that of Belenky. Belenky is concerned with the change of the final state density due to interaction. He starts from the formula

$$S_B^{(1)}(W, n, M_0) = \sum_{p'} \left[\frac{\Omega}{(2\pi\hbar)^3} \right]^{n-1} \frac{1}{(n-1)!} \int_{i=1}^{n-1} \prod d\mathbf{p}_i \cdot d\mathbf{p}_c \\ \times \delta(W - \sum_{i=1}^{n-1} W_i - W_c - W') \delta(\sum_{i=1}^{n-1} \mathbf{p}_i + \mathbf{p}_c) \quad (13) \\ \equiv \sum_{p'} S_{p'}^{(1)}$$

and takes a wave function of the nucleon-pion system

$$\psi_r \sim \text{const} \frac{1}{r} \sin \left(\frac{p'}{\hbar} r - \frac{l\pi}{2} + \delta_l \right) \quad \text{for } r \rightarrow \infty \quad (14)$$

with the boundary condition

$$\frac{p'}{\hbar} R - \frac{l\pi}{2} + \delta_l = s\pi \quad s: \text{integer} \quad (15)$$

thus obtaining

$$S_B^{(1)} = \sum_l (2l+1) \int dp' S_{p'}^{(1)} \left(\frac{1}{\pi} \frac{R}{\hbar} + \frac{\partial \delta_l}{\partial p'} \right). \quad (16)$$

He then asserts that the first term corresponds just to Fermi's expression. According to our consideration, however, this term is different from the original Fermi formula by a factor V/Ω . The ratio of the correction term turns out, therefore, to be negligibly small compared with the first term. This could have been seen from (16) because the first term contains factor R while the second does not.

For the above reason we have not considered the change of state density, but restricted ourselves to the change of the matrix element, but unfortunately the results were not satisfactory.

Here we note Amati and Vitale¹⁰⁾ have already used the similar formula as (5) and Kovacs¹¹⁾ have obtained a formula identical to (11) from other view point.

In conclusion the author would like to express his sincere gratitude to Professor Z. Koba for his encouragement and critical comments and to all members of Yukawa laboratory for their continual help.

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Giant Cluster Expansion Theory and Its Application to High Temperature Plasma*

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The conventional virial expansion of thermodynamic functions is converted into a new expansion scheme, similar to the former but more powerful. The new method is particularly suitable to dealing with the interaction of long-range character, such as Coulomb potential, since it suffers from no divergence difficulties contrary to the conventional one. As an application of the method, the equilibrium properties of high temperature plasma is studied and the term of next higher orders than the Debye-Hückel limiting law is obtained exactly. The order estimation indicates that the Debye-Hückel law is accurate within the error of a few per cent in this case. A possible extension of the present method to the theory of non-equilibrium properties of plasma or to quantum statistics is suggested.

§ 1. Introduction

In classical statistical mechanics, thermodynamic functions of an imperfect gas are expanded in a form of power series of particle number density ρ , and coefficients are known to be expressed in terms of irreducible cluster integrals¹⁾. In some cases, however, these expansion formulae are no longer valid, because of the divergence difficulties of cluster integrals. For instance, in an ionic solution or classical plasma, the interaction potentials between particles are of long-range character and all cluster integrals diverge to infinity. Mathematically speaking, this implies that the point $\rho=0$ is a singular point of thermodynamic functions, since these functions cannot be expanded in powers of ρ . Therefore, according to the theory of functions of complex variables, the point $\rho=0$ should be one of the following three cases: the pole, the essentially singular point and the branch point. In any case, the thermodynamic functions are expected to have some singular properties at low densities. Then it is quite clear that the usual virial expansion method cannot afford to describing the low density behaviour of the system.

So far, there have been several attempts to overcome the difficulties mentioned above. Among them, Mayer²⁾ has shown that the long-range interaction can be reduced to the short-range one, if the integral over the intermediate particles of a chain is performed for a given type of prototype graph. Thus, the expansion formula in terms of prototype graphs is free from the divergence difficulties arising from the long-range correlation. In fact, Mayer has proved that the Debye-Hückel limiting law is derived by a summation procedure over the clusters of ring type.

* A short account of this paper was published in Prog. Theor. Phys. 21 (1959), 475.

However, it is important to note that Mayer's procedure brings about new difficulties owing to the short-range correlation. One may encounter with these difficulties if he investigates into the contributions of the complex prototype graphs beyond the ring type. In Mayer's theory, the existence of hard-core potential is assumed and each term corresponding to one prototype graph is calculated without any divergence difficulties. However, if one makes the core radius approach zero, some terms considered by Mayer are shown to diverge to infinity. Thus, the development in terms of prototype graphs is not applicable to the pure Coulomb potential without a hard-core.

The purpose of this paper is to develop a systematic expansion procedure without suffering from difficulties arising from both long- and short-range correlations. In § 2 we shall briefly account for Mayer's expansion method in terms of prototype graphs and discuss how the divergence difficulties arise from the short-range correlation. In § 3 we shall discuss how to avoid these difficulties, taking a simple example of graphs of watermelon type^{(3),(4)}. Generalizing this procedure to the more complex graphs, we shall obtain an expansion formula which has a form similar to the usual one. Each term of new series corresponds to an aggregate of all usual clusters of a given type, and therefore the expansion developed here will be called "giant cluster expansion." The giant cluster integrals corresponding to the usual irreducible cluster integrals are defined and shown to have topological properties similar to the usual one but dependent on ρ . Then § 4 is devoted to an application of the method to an electron gas in a uniform positive ion background and the term corresponding to the usual second virial coefficient is calculated. Calculations are carried out in two different ways: the one based on the use of Bessel functions and the other which will clarify how the lowest order term of giant cluster can be obtained by a summation of the most highly divergent terms of prototype graphs. In § 5 the contributions of higher order giant clusters are briefly discussed and it is verified that the thermodynamic functions are exact up to and including the order considered in this paper. The orders of magnitude are estimated for the high temperature plasma and it is shown that the Debye Hückel law is accurate within the error of a few per cent. It is suggested that the thermodynamic functions of plasma are written as a double power series of λ and $\log \lambda$, with λ the characteristic non-dimensional parameter for plasma. In § 6 is discussed a discrepancy between the results obtained by Ichikawa⁽⁵⁾ and ours. A possible generalization of the present method to the study of non-equilibrium properties of plasma or to quantum statistics is suggested, and these matters will be the subjects of forthcoming papers.

§ 2. Mayer's expansion scheme

In order to deal with the thermodynamic properties of an imperfect gas, it is convenient to introduce a function S defined by

$$S = \sum_{n=1}^{\infty} \beta_n \rho^{n+1} / (n+1) \quad (1)$$

where β_n is the irreducible cluster integral, ρ is the particle number density given by N/V , N being the particle number and V the volume of the system. Then the Helmholtz free energy and the equation of state are given respectively by,¹⁾

$$A/N\kappa T = \log(h^2/2\pi m\kappa T)^{3/2}\rho/e - S/\rho, \quad (2)$$

$$P/\kappa T = \rho + S - \rho \partial S / \partial \rho. \quad (3)$$

Here m stands for the mass of particle, κ Boltzmann's constant, h Planck's constant, T the absolute temperature. All other thermodynamic functions are derived from the S , therefore the investigation of equilibrium properties of an imperfect gas reduces to calculating the function S . In this section we shall give a brief account of Mayer's expansion scheme in terms of prototype graphs.

If we set the potential function between the i -th and j -th particles to be ϕ_{ij} and substitute an explicit expression for β_n in Eq. (1), we have

$$S = \sum_{n=2}^{\infty} \frac{\rho^n}{n!} \frac{1}{V} \int \cdots \int \sum_{n \geq i > j \geq 1} \Pi f_{ij} d\tau_1 \cdots d\tau_n. \quad (4)$$

All products in which all particles
are more than singly connected

Here f function is given, as usual, by

$$f_{ij} = \exp(-\beta\phi_{ij}) - 1 \quad (5)$$

with $\beta = 1/\kappa T$. Usually, each term in Eq. (4) is represented by a bond diagram composed of n points and of bonds representing the f function. Hereafter we shall call this bond the f -bond. If the f is expanded in powers of $-\beta\phi_{ij}$ and if we take the term $(-\beta\phi_{ij})^k/k!$, the f -bond is splitted into k ϕ -bonds as shown in Fig. 1. Then each term in Eq. (4) will be represented by diagrams composed of ϕ -bonds. The difference between the diagrams composed of f and ϕ -bonds is clear: in the former case one and only one bond is permitted to connect the points directly, while in the latter an arbitrary number of bonds are permitted.

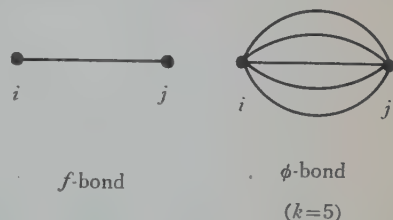


Fig. 1

The diagrams of ϕ -bonds constructed in this way may be classified according to their topological properties. For this purpose, we introduce the terminology "junction," which is defined as a point to which three or more bonds are connected³⁾, and let us consider the diagram of which all points are junctions. Mayer has called such a diagram "prototype graph." Then it is clear that all the diagrams are constructed by adding the points on the bonds between junctions of prototype graphs (except for the diagrams of ring type of which contribution should be calculated separately). Therefore, if we first select one prototype graph and then sum up the terms of all the diagrams derived by this prototype graph, the S is expanded in terms of prototype graphs. In fact, as Mayer²⁾ has proved, S is written as

$$S = S_0 + \sum_{m=2}^{\infty} \frac{\rho^m}{m!} \frac{1}{V} \int \sum'_{m \geq i > j \geq 1} \Pi \frac{(-\beta q_{ij})^{k_{ij}}}{k_{ij}!} d(m). \quad (6)$$

Here S_0 is the contribution of clusters of ring type, and is given by

$$S_0 = -(\rho^2 \beta / 2) \nu(0) + (1/2V) \sum_k \{ -\log[1 + \rho \beta \nu(\mathbf{k})] + \rho \beta \nu(\mathbf{k}) \} \quad (7)$$

with $\nu(\mathbf{k})$ the Fourier component of ϕ defined by

$$\nu(\mathbf{k}) = \int \phi(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x}. \quad (8)$$

Furthermore, \sum' in Eq. (6) implies the summation over prototype graphs, k_{ij} the number of bonds between the i -th and j -th junctions, $d(m)$ the integrations over m junctions, and q_{ij} is given by

$$q_{ij} = (1/V) \sum_k \nu(\mathbf{k}) \exp[i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)] / [1 + \rho \beta \nu(\mathbf{k})]. \quad (9)$$

In the case of electron gas in a uniform positive ion background, $\nu(\mathbf{k})$ is given by

$$\nu(\mathbf{k}) = 4\pi \varepsilon^2 / k^2 \quad (\mathbf{k} \neq 0), \quad \nu(0) = 0, \quad (10)$$

where ε is the absolute value of electronic charge. Then q_{ij} is shown to be

$$q(r) = \varepsilon^2 \exp(-\kappa_0 r) / r \quad (11)$$

with $\kappa_0 = \sqrt{4\pi\beta\rho\varepsilon^2}$. Furthermore from Eqs. (7) and (10) it follows that

$$S_0 = 2\sqrt{\pi} \varepsilon^3 \beta^{3/2} \rho^{3/2} / 3, \quad (12)$$

in this case. From Eqs. (3) and (12), we have

$$P / \rho \kappa T = 1 - \sqrt{\pi} \varepsilon^3 \beta^{3/2} \rho^{1/2} / 3, \quad (13)$$

which is the Debye-Hückel limiting law.

§ 3. Contributions of prototype graphs and giant cluster expansion

We have calculated in § 2 the term S_0 which is the contribution of clusters of ring type, i.e., clusters without junctions. We now proceed in this section to discuss the more complicated clusters with some junctions. As we have mentioned in § 2, the contributions of these clusters can be expressed in terms of prototype graphs, and we shall first consider the graphs with 2 junctions. The simplest one of these is shown in Fig. 2, i.e., the graph with 3 bonds. Its contribution to S is given by

$$\frac{\rho^2}{2!} \frac{1}{V} \int \frac{(-\beta q_{12})^3}{3!} d\tau_1 d\tau_2 = -\frac{2\pi\rho^2\beta^3\varepsilon^6}{3} \int_0^\infty \frac{\exp(-3\kappa_0 r)}{r} dr, \quad (14)$$

from Eqs. (6) and (11). Obviously, this integral is divergent at $r=0$. In a similar manner, it is shown that the contributions of graphs with 4, 5, 6, ... bonds are

all divergent. In Mayer's theory, however, the corresponding results are convergent, since the assumed existence of hard-core potential makes the lower limit of integration the finite value, thus preventing the divergence at $r=0$. However, if we make the core radius tend to zero, the integrals become ∞ as mentioned above. This is the main reason why the development in terms of prototype graphs is not satisfactory for dealing with the pure Coulomb potential without a core.

Let us now discuss how to avoid this divergence difficulty. For this purpose, it may be instructive to consider the procedure in which the f -bond is expanded in powers of ϕ -bonds, as we have done in § 2. If the function ϕ has a dependence $1/r$ near $r=0$, the integration for ϕ -bonds may be divergent at $r=0$, in spite of the convergence of the integral for the f -bond. This in turn suggests that the divergence arising from the short-range character can be avoided by summing up the graphs over the number of bonds. Keeping this in mind, let us sum up all the prototype graphs with 2 junctions as shown in Fig. 3. If we set the contributions of these graphs to S to be S_2 , we obtain

$$\begin{aligned}
 S_2 &= \frac{\rho^2}{2} \frac{1}{V} \int \sum_{k=0}^{\infty} \frac{(-\beta q_{12})^k}{k!} d\tau_1 d\tau_2 \\
 &= 2\pi \rho^2 \int_0^{\infty} \left[e^{-\beta q(r)} - 1 + \beta q(r) - \frac{\beta^2 q^2(r)}{2} \right] r^2 dr. \quad (15)
 \end{aligned}$$

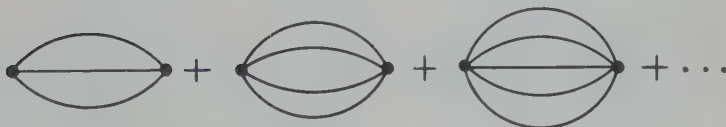


Fig. 3. All the prototype graphs with 2 junctions. The contribution of these graphs to S is S_2 .

From this equation, it is clear that the divergence at $r=0$ no longer appears. The similar equation to this has been obtained by Yukhnovsky⁶⁾, by the use of the method of collective variables. It is also possible to obtain the corresponding equation without expanding the f -bond in terms of ϕ -bonds, as we have shown previously in the theory of watermelon approximation for classical fluids.³⁾

The generalization of the above procedure to the more complicated prototype graphs is straightforward. For example, let us consider the graphs with 3 junctions. The simplest one is that shown in Fig. 4 at the left end. If we sum up all the graphs in Fig. 4, keeping the number of bond between the junctions 1 and 2 to be one, we have from Eq. (6)

$$S_3^{(a)} = \frac{3\rho^3}{3! V} \int w_0(12) w_2(23) w_2(31) d\tau_1 d\tau_2 d\tau_3 \quad (16)$$

as a contribution to S . Here w function is generally defined by

$$w_n = \sum_{k=n}^{\infty} (-\beta q)^k / k!, \quad (n \neq 0) \quad (17)$$

i. e. $w_1 = e^{-\beta q} - 1$, $w_2 = e^{-\beta q} - 1 + \beta q$ and $w_3 = e^{-\beta q} - 1 + \beta q - \beta^2 q^2 / 2$,

and

$$w_0 = -\beta q. \quad (18)$$

Furthermore, the factor 3 in front of the integral corresponds to the three types of graphs as shown in Fig. 5.

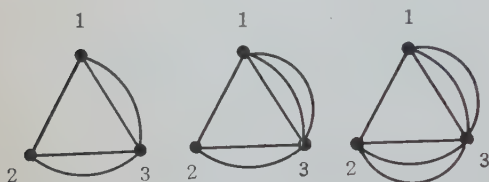


Fig. 4. Prototype graphs with 3 junctions, in which one bond is connected between the junctions 1 and 2. Their contribution to S is $S_3^{(a)}$.

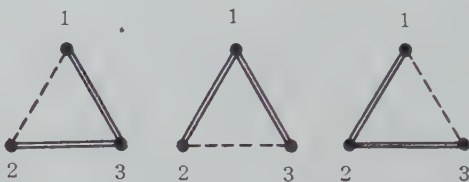


Fig. 5. Three types of graphs in which w_0 bond is connected between junctions. The dotted line is w_0 bond and the double bond is w_2 bond.

If we further consider the prototype graphs in which at least 2 bonds are connected between junctions as shown in Fig. 6, we have

$$S_3^{(b)} = \frac{\rho^3}{3! V} \int w_2(12) w_2(23) w_2(31) d\tau_1 d\tau_2 d\tau_3 \quad (19)$$

as a contribution to S .

It is obvious that the similar procedure can be carried out for prototype graphs with 4, 5, 6... junctions. In general, if we write

$$S = S_0 + \sum_{m=1}^{\infty} \frac{\gamma_m}{m+1} \rho^{m+1} \quad (20)$$



Fig. 6. Prototype graphs in which the number of bonds is larger than 2. Their contribution to S is $S_3^{(b)}$.

γ_m is expressed as an integral of a sum of products of w functions, i.e.,

$$\gamma_m = \frac{1}{m! V} \int \sum_{m+1 \geq i > j \geq 1} \prod w_n(ij) d(m+1). \quad (21)$$

For example, γ_1 and γ_2 are given respectively by

$$\begin{aligned} \gamma_1 &= \frac{1}{V} \int w_3(12) d\tau_1 d\tau_2, \\ \gamma_2 &= \frac{3}{2! V} \int w_0(12) w_2(23) w_2(31) d\tau_1 d\tau_2 d\tau_3 \\ &\quad + \frac{1}{2! V} \int w_2(12) w_2(23) w_2(31) d\tau_1 d\tau_2 d\tau_3. \end{aligned}$$

Comparing Eq. (20) with Eq. (1), one may easily see that γ_m is a counterpart of the usual irreducible cluster integral. However, γ_m corresponds not to a single cluster but to an aggregate of all the usual clusters of a given type. For example, γ_2 represents the contributions of usual clusters of ϕ -bonds shown in Fig. 7. For this reason, we shall call γ_m the giant cluster integral and the expansion given by Eq. (20) the giant cluster expansion. It is quite easy to see that this expansion scheme suffers neither from the divergence difficulty due to the long-range character nor from the one due to the short-range character.

Just as in the usual cluster expansion, the giant cluster integral is represented by the bond diagram composed of w -bonds. For example, the bond diagrams of γ_1 and γ_2 are given in Fig. 8. Furthermore, those of γ_3 are shown in Fig. 9.

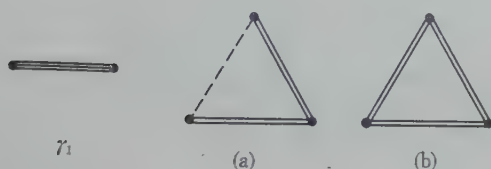


Fig. 8. Bond diagrams of γ_1 and γ_2 . The triple bond is w_3 . (a) and (b) correspond to $S_3^{(a)}$ and $S_3^{(b)}$, respectively.

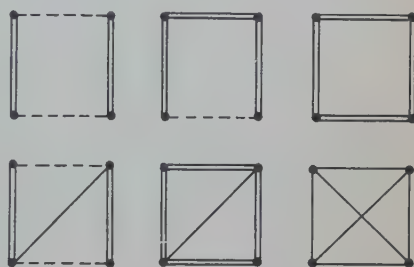


Fig. 9. Some type of bond diagrams contributing to γ_3 . The single bond is w_1 .

From these figures one may see that the bond diagrams representing the giant cluster integrals are quite similar to the usual one. However, the essential difference lies in the fact that they should be drawn so that all the points are junctions in this case. For example, the bond diagrams shown in Fig. 10 do not appear, since these are not consistent with this requirement. As long as this is satisfied, there are several ways in which w -bonds are connected for a given type of bond diagram, as being shown in Fig. 9. Accordingly, there appears, so to speak, the fine structure of usual bond diagram in our case.

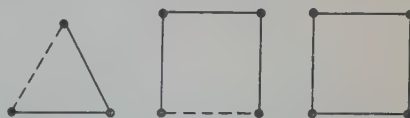


Fig. 10. Bond diagrams not appearing in the giant clusters.

Furthermore, it should be noted that γ_m is dependent on density, contrary to β_n , so that the giant cluster expansion is not a simple power series of density but a more complicated one. But, as will be shown in the next section, it enables one to study the low density behaviour of the system for which β_n diverges.

Before closing this section, we should like to mention that the similar procedure to the present one was discussed by Meeron⁷⁾ in his theory of nodal expansion for the potential of average force and for the distribution function. Furthermore, Morita⁴⁾ has succeeded in a partial summation of giant clusters in his theory of hyper-netted chain approximation. However, it seems that these authors stayed in a rather formal stage. We shall give an actual application of the method to the electron gas in the next section.

§ 4. Application to electron gas

In this section we shall calculate the function S_2 given by Eq. (15) for the electron gas. As can be seen from Fig. 8, this function corresponds to the usual second virial coefficient. In the following, we shall give two different ways of calculating the S_2 : the one based on the use of Bessel functions and the other on the summation of the most highly divergent terms of prototype graphs.

Substituting Eq. (11) in Eq. (15), we have

$$S_2 = 2\pi\rho^2 \int_0^\infty \left[\exp\left(-\frac{\beta\mathcal{E}^2}{r} e^{-\kappa_0 r}\right) - 1 \right] r^2 dr + \frac{\rho}{2} - \frac{\sqrt{\pi} \rho^{3/2} \beta^{3/2} \mathcal{E}^3}{4}.$$

In order to carry out further calculations, it is convenient to introduce a dimensionless parameter characteristic of the classical electron gas:

$$\lambda \equiv \kappa_0 \beta \mathcal{E}^2 = 2\sqrt{\pi} \beta^{3/2} \rho^{1/2} \mathcal{E}^3. \quad (22)$$

Then we have

$$\frac{S_2}{\rho} = \frac{1}{2} - \frac{\lambda}{8} + \frac{\lambda^2}{2} \int_0^\infty \left[\exp\left(-\frac{e^{-\lambda t}}{t}\right) - 1 \right] t^2 dt. \quad (23)$$

If we perform the partial integration, it follows that

$$S_2/\rho = 1/2 - \lambda/8 - (\lambda^2/6) I(\lambda) \quad (24)$$

where $I(\lambda)$ is given by

$$I(\lambda) = \int_0^\infty (t + \lambda t^2) \exp\left(-\lambda t - \frac{e^{-\lambda t}}{t}\right) dt. \quad (25)$$

In calculating this function, we assume that λ is small, that is, we consider the low density and high temperature limit. In such a case, it may be possible to write $-\lambda t - e^{-\lambda t}/t$ as $-\lambda t - 1/t + (1 - e^{-\lambda t})/t$ and expand $\exp[(1 - e^{-\lambda t})/t]$ as a power series of its exponent which is of the order of λ . Then we obtain

$$I(\lambda) = \sum_{n=0}^{\infty} I_n(\lambda)/n! \quad (26)$$

where

$$\begin{aligned}
 I_n(\lambda) &= \int_0^\infty (t + \lambda t^2) e^{-\lambda t - 1/t} \frac{(1 - e^{-\lambda t})^n}{t^n} dt \\
 &= \sum_{r=0}^n \binom{n}{r} (-1)^r \int_0^\infty \frac{(t + \lambda t^2)}{t^n} e^{-\lambda(r+1)t - 1/t} dt.
 \end{aligned} \quad (27)$$

The integral in Eq. (27) can be expressed in terms of Bessel functions, if we notice the formula⁸⁾

$$K_n(z) = \int_0^\infty e^{-z \cosh \theta} \cosh n\theta d\theta.$$

In this way, we find

$$\begin{aligned}
 I_n &= \sum_{r=0}^n \binom{n}{r} (-1)^r \lambda^{n/2} (r+1)^{n/2} \left[\frac{2}{\lambda(r+1)} K_{n-2}(2\sqrt{\lambda(r+1)}) \right. \\
 &\quad \left. + \frac{2}{(r+1)\sqrt{\lambda(r+1)}} K_{n-3}(2\sqrt{\lambda(r+1)}) \right].
 \end{aligned} \quad (28)$$

If we substitute in this equation the explicit expression for $K_n(z)$:

$$\begin{aligned}
 K_n(z) &= \frac{1}{2} \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} \left(\frac{z}{2} \right)^{2m-n} + (-1)^{n+1} \sum_{m=0}^\infty \frac{(z/2)^{n+2m}}{m! (n+m)!} \\
 &\quad \times \{ \log(z/2) - 1/2 \cdot \psi(m) - 1/2 \cdot \psi(m+n) \}
 \end{aligned}$$

$$\psi(m) = -\gamma + 1 + \frac{1}{2} + \dots + \frac{1}{m}, \quad \psi(0) = -\gamma,$$

$$(\gamma: \text{Euler's constant} = 0.57721566\dots),$$

we have

$$I_0 = 3/\lambda^2 - 2/\lambda + 1/2 - 1/2 \cdot \{ \log \lambda - \psi(0) - \psi(2) \} + 0(\lambda),$$

$$I_1 = 5/4 \lambda - 1/2 - \log 2 + 0(\lambda),$$

$$I_2 = 2 \log 2 - \log 3 + 1/3 + 0(\lambda)$$

and

$$I_3 = 0(\lambda).$$

It is easily verified that I_n for $n \geq 3$ can be ignored if we confine ourselves up to the order of λ^0 . Substituting above equations in Eq. (26), we have from Eq. (24)

$$\frac{S_2}{\rho} = \lambda^2 \left(\frac{\log 3}{12} + \frac{\gamma}{6} - \frac{11}{72} \right) + \frac{\lambda^2 \log \lambda}{12} + 0(\lambda^3). \quad (29)$$

On the other hand, the contribution of clusters of ring type, S_0 , is written as

$$S_0/\rho = \lambda/3 \quad (30)$$

from Eqs. (12) and (22). Thus we see that the contribution of giant cluster with 2 junctions is of orders higher than that of clusters without junctions at low density limits. Furthermore, it should be noted that there appears the term $\log \lambda$ in Eq. (29). This implies that the point $\rho=0$ is a branch point of logarithmic character, while in Eq. (30) it is of algebraic one, [see Eq. (22)].

Let us now discuss to derive Eq. (29) in an alternative way. As we have mentioned in § 2, the contribution of each prototype graph diverges to ∞ . However, if the existence of hard-core potential is assumed, it is convergent. Then the contribution of the prototype graph with 2 junctions and 3 bonds is given by

$$\frac{S_{2,3}}{\rho} = -\frac{\lambda^2}{2} \frac{1}{3!} \int_0^\infty \frac{e^{-3\lambda t}}{t} dt$$

where $\delta = a/\beta\epsilon^2$, a being the diameter of hard-core.

Similarly, the contributions of the prototype graphs with 4, 5, ... bonds are given by

$$\frac{S_{2,4}}{\rho} = \frac{\lambda^2}{2} \frac{1}{4!} \int_0^\infty \frac{e^{-4\lambda t}}{t^2} dt, \quad \frac{S_{2,5}}{\rho} = -\frac{\lambda^2}{2} \frac{1}{5!} \int_0^\infty \frac{e^{-5\lambda t}}{t^3} dt, \text{ etc.}$$

Though these terms become ∞ as $\delta \rightarrow 0$, we observe here the most highly divergent terms in this limit. They are given by

$$\begin{aligned} \frac{S_{2,4}}{\rho} &\simeq \frac{\lambda^2}{2} \frac{1}{4!} \frac{1}{\delta} = \frac{\lambda^2}{2} \frac{1}{4!} \int_0^\infty \frac{dt}{t^2}, \\ \frac{S_{2,5}}{\rho} &\simeq -\frac{\lambda^2}{2} \frac{1}{5!} \frac{1}{2\delta^2} = -\frac{\lambda^2}{2} \frac{1}{5!} \int_0^\infty \frac{dt}{t^3}, \text{ etc.} \end{aligned}$$

Then summing up these terms, we obtain

$$\frac{S_2}{\rho} \simeq \frac{\lambda^2}{2} \int_0^\infty \left[e^{-1/t} - 1 + \frac{1}{t} - \frac{1}{2t^2} + \frac{1 - e^{-3\lambda t}}{3! t^3} \right] t^2 dt. \quad (31)$$

Now, this integral is finite as $\delta \rightarrow 0$, and then one may easily verify that the integration leads to the same result as is given by Eq. (29). Thus we see that the lowest order term of contribution of giant cluster can be calculated by summing up the most highly divergent terms of prototype graphs of which the giant cluster is composed. This situation seems somewhat analogous to the case of correlation energy of electron gas as was discussed by Gell-Mann and Brueckner.⁹⁾ We shall use this method in the next section for the order estimation of giant clusters with 3 junctions.

§ 5. Contribution of higher order giant cluster and equation of state of high temperature plasma

We have calculated in § 4 the contribution of giant cluster with 2 junctions. In this section we shall discuss the order estimation of more complicated giant clusters for small λ .

The term expected to be important besides S_0 and S_2 for small λ is $S_3^{(a)}$ given by Eq. (16). The bond diagram corresponding to this term is shown in Fig. 8, and is a counterpart of the usual third virial coefficient. Combining Eq. (16) with Eq. (17), we have

$$\frac{S_3^{(a)}}{\rho} = \frac{\rho^2}{2} \int (e^{-\beta q_{23}} - 1 + \beta q_{23}) (e^{-\beta q_{13}} - 1 + \beta q_{13}) (-\beta q_{12}) d\tau_1 d\tau_2. \quad (32)$$

Carrying out the change of variables and the integration for the angular part, we find

$$\frac{S_3^{(a)}}{\rho} = \frac{\lambda^3}{4} \left\{ \int_0^\infty s e^{-\lambda s} \left[\exp\left(-\frac{e^{-\lambda s}}{s}\right) - 1 + \frac{e^{-\lambda s}}{s} \right] ds \right\}^2 - \frac{\lambda^3}{4} A(\lambda) \quad (33)$$

where

$$A(\lambda) = 2 \int_0^\infty e^{\lambda s} s \left[\exp\left(-\frac{e^{-\lambda s}}{s}\right) - 1 + \frac{e^{-\lambda s}}{s} \right] ds \int_s^\infty e^{-\lambda t} t \left[\exp\left(-\frac{e^{-\lambda t}}{t}\right) - 1 + \frac{e^{-\lambda t}}{t} \right] dt. \quad (34)$$

The calculation of the first term on the right-hand side of Eq. (33) is straightforward, if we use the second method discussed in § 4. Then we obtain for this term

$$\frac{\lambda^3}{4} \left(-\gamma + \frac{3}{4} - \frac{1}{2} \log 3\lambda \right)^2 \quad (35)$$

for small λ . On the other hand, the second term in Eq. (33) is not calculated analytically, yet it is verified that its contribution for small λ is of the same order as is given by Eq. (35). In order to show this, we sum up the most highly divergent terms of prototype graphs, as was discussed in § 4. Then, after some manipulation, we have

$$A(\lambda) \simeq 2 \int_0^\infty s \left(e^{-1/s} - 1 + \frac{1}{s} \right) ds \int_s^\infty \left(e^{-1/t} - 1 + \frac{1}{t} - \frac{1 - e^{-3\lambda t}}{2! t^2} \right) t dt \\ - \frac{1}{2} \int_0^\infty \frac{1 - e^{-\lambda s}}{s} ds \int_s^\infty \frac{e^{-3\lambda t}}{t} dt.$$

Differentiating this equation by λ , we find

$$\frac{dA}{d\lambda} \simeq -\frac{1}{\lambda} \int_0^{\infty} s \left(e^{-1/s} - 1 + \frac{1}{s} \right) e^{-3\lambda s} ds. \quad (36)$$

The right-hand side of Eq. (36) is equal to

$$-1/\lambda \cdot [2(3\lambda)^{-1} K_2(2\sqrt{3\lambda}) - (3\lambda)^{-2} + (3\lambda)^{-1}].$$

Therefore, for small λ Eq. (36) becomes

$$\frac{dA}{d\lambda} \simeq \frac{1}{2\lambda} [\log(3\lambda) - \psi(0) - \psi(2)] + O(\lambda^0).$$

Integrating this equation, we have

$$A(\lambda) \simeq \text{const.} + 1/2 \cdot [\log 3 - \psi(0) - \psi(2)] \log \lambda + 1/4 \cdot (\log \lambda)^2. \quad (37)$$

Comparing the above equation with Eqs. (33) and (35), we see that the contribution of the second term in Eq. (33) is of the order same as the first one. Though the value of constant in Eq. (37) is not determined by the above argument, it is clear that the order of $S_3^{(a)}/\rho$ is λ^3 , if we consider the term $\log \lambda$ as if it were of the order λ^0 . Under the same prescription, S_2/ρ is of the order of λ^2 as can be seen from Eq. (29). Thus, it is verified that the contribution of giant cluster of type (a) shown in Fig. 8 is of orders higher than γ_1 in the same figure.

The calculation of terms of type (b) and of more complex giant clusters is very difficult, and we shall not enter into this problem here. However, as was shown in the above calculation, it is quite probable that their contributions are of orders higher than that of γ_1 . Then, up to the order of λ^2 , we obtain from Eqs. (29) and (30)

$$S/\rho = \lambda/3 + \lambda^2 (\log 3\lambda/12 + \gamma/6 - 11/72) + O(\lambda^3). \quad (38)$$

By the use of Eqs. (3) and (22), the equation of state is written as

$$P/\rho \kappa T = 1 - \lambda/2 \cdot \partial/\partial \lambda \cdot (S/\rho).$$

Therefore, substituting Eq. (38) in the above equation, we have

$$\frac{P}{\rho \kappa T} = 1 - \frac{\lambda}{6} - \lambda^2 \left(\frac{\log 3\lambda}{12} + \frac{\gamma}{6} - \frac{1}{9} \right) + O(\lambda^3). \quad (39)$$

The third term on the right-hand side is graphically shown in Fig. 11. As is seen from this figure, when λ is smaller than about 0.4, the third term makes the $P/\rho \kappa T$ larger than that calculated without this term. This situation is represented in Fig. 12.

In order to estimate orders of magnitude for the high temperature plasma, let us suppose $T \simeq 10^5$ °K and $\rho \simeq 10^{15}$ cm $^{-3}$. Then we have $\lambda \simeq 10^{-2}$. If we set $\lambda = 10^{-2}$, the Debye-Hückel term, $\lambda/6$, in Eq. (39) is 1.6667×10^{-3} . On the other hand, the correction term $\lambda^2 (\log 3\lambda/12 + \gamma/6 - 1/9)$ in Eq. (39) takes the value -0.0308×10^{-3} . Thus we see the latter amounts to only about 2% of the former. The higher the

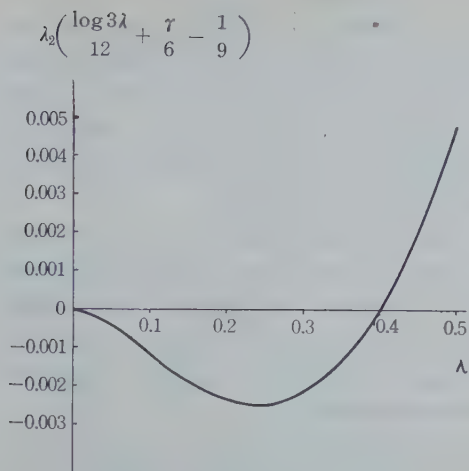


Fig. 11. Graphical representation of $\lambda^2(\log 3\lambda/12 + r/6 - 1/9)$ as a function of λ .

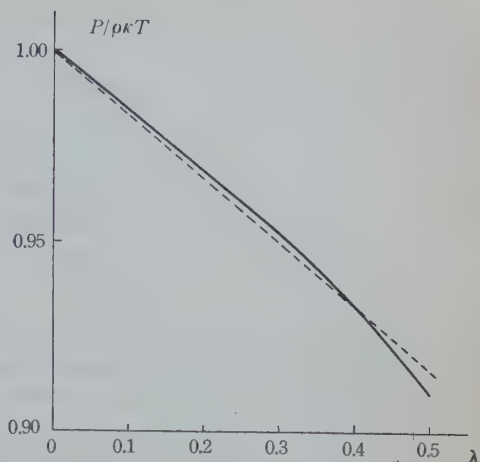


Fig. 12. Equation of state for high temperature plasma. The dotted line is the Debye-Hückel limiting law and the full line is based on our calculation.

temperature is, the smaller the correction becomes. Thus we may conclude that the Debye-Hückel law is accurate within the error of a few per cent for the plasma at sufficiently high temperature.

Here we wish to add some remarks about the functional dependence of S/ρ on λ , which is suggested by the present calculations. We have shown that S_2/ρ includes the terms λ^2 and $\lambda^2 \log \lambda$, $S_3^{(a)}/\rho$ the terms λ^3 , $\lambda^3 \log \lambda$ and $\lambda^3 (\log \lambda)^2$ for small λ . Furthermore, it is easily verified that S_2/ρ is a sum of terms λ^m and $\lambda^m \log \lambda$ ($m=2, 3, \dots$). In all these cases, the S/ρ is expressed as a sum of terms $\lambda^m (\log \lambda)^n$ ($m > n$). If we assume these properties to be more general, S/ρ may be written as

$$S/\rho = \sum_{m>n=0}^{\infty} b_{mn} \lambda^m (\log \lambda)^n$$

where b_{mn} is independent of λ . This implies that the thermodynamic function of plasma is a double power series of λ and $\log \lambda$.

§ 6. Concluding remarks

A systematic expansion scheme quite suitable to dealing with the long-range interaction has been developed. We can safely conclude that the fundamental difficulties associated with the interaction of this sort are completely solved by this method, as far as the behaviour of the system at low densities and high temperatures is concerned. The method has been applied to the study of equilibrium properties of high temperature plasma, and the equation of state beyond the Debye-Hückel limiting law has been obtained exactly up to the order of λ^2 . It in turn indicates that the Debye-Hückel law is exact up to the order of λ .

Now, with respect to this point, we must mention the calculation carried out by Ichikawa.⁵⁾ He has shown that the long-range correlation effects of the Coulomb interaction increase the free energy of the Debye-Hückel law by about 22%. Obviously, his conclusion is not consistent with ours. The origin of this discrepancy is due to the fact that he dealt with the high density behaviour of plasma, though his result is reduced to the Debye-Hückel law at low density limit.

The giant cluster expansion developed here may be generalized to the multi-component system, and more realistic description of plasma in which the positive ion is not smeared may be possible. Moreover, it seems that the similar technique combined with the recent theories¹⁰⁾ on the irreversible process will be usefully applied to the investigation of non-equilibrium quantities such as mobility, relaxation time, of plasma. We shall discuss these problems elsewhere.

Furthermore, it may also be possible to extend the present method to quantum statistics. In this case the expansion scheme corresponding to the classical cluster expansion is "linked cluster expansion" in terms of Feynman graphs¹¹⁾. These graphs have a strong resemblance to the bond diagrams in classical theory, except for the terms representing the exchange effects. In fact, the summation of Feynman graphs of ring type leads to the result derived by Montroll and Ward¹²⁾ in a quite simple way.¹³⁾ It is further possible to develop the expansion scheme in terms of prototype Feynman graphs, in parallel with classical theory. We hope that the quantum statistical treatment of electron gas along the line conjectured here may give a deeper insight into the electron correlation in metals and prove useful in connection with the problems on correlation energy, thermodynamic properties, electromagnetic properties, transport phenomena, superconductivity, etc. In a forthcoming paper we shall undertake a quantum statistical generalization of the present method.

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Note added in proof After I had written this paper I knew that Dr. Harold L. Friedman developed the similar expansion scheme for the multi-component system (*Molecular Phys.* **2** (1959), 23). I would like to express my sincere thanks to Prof. S. Ono at Tokyo University for informing me of this paper.

Effects of s - d Interaction on Transport Phenomena

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The effects of the so-called s - d interaction on transport phenomena of ferromagnetic metals and alloys are discussed. Electrical and thermal conductivities and thermoelectric effect are calculated with the simple molecular field approximation, and an anomalous thermoelectric effect is expected in these substances. In sufficiently low temperatures, the disordering of spin system is described by the collective mode of spin wave. The calculations of transport coefficient by using the spin wave approximation are also carried out.

§ 1. Introduction

In ferromagnetic metals and alloys the most characteristic interaction is the so-called s - d interaction, namely, the spin exchange interaction between the conduction and the unfilled inner shell electrons (hereafter referred to as c - and u -electrons respectively). One of the main effects of this interaction is the effective spin exchange interaction between u -electrons which were treated firstly by Zener¹⁾ and in detail by us²⁾ and Yosida³⁾. The other one of the main effects is the scattering mechanism of c -electrons which has the important contribution to the transport phenomena. The effects on the electrical resistivity are already treated by us⁴⁾ for ferromagnetic metals and by Yosida⁵⁾ for dilute alloys using the simple molecular field approximation. The problems concerning the effect on the superconductivity we treated in another paper⁶⁾. Here we treat the effects of s - d interaction on the thermal resistivity and the thermoelectric force using the molecular field approximation. The anomalous thermoelectric force is expected to occur in these substances. The thermal conductivity by the spin diffusion seems to be also important because of the long range character of the s - d interaction. Furthermore the transport coefficient in sufficiently low temperatures are also calculated with the use of the spin wave approximation.

§ 2. Fundamental standpoints

To solve the Boltzmann Equation rigorously, because for the discussion of the thermoelectric force careful treatment is required, we use here the following simplified model.

(i) We assume that c -electron is represented by one band model with effective mass m and that the effect of s - d interaction is represented by the effective exchange

field H_c . Then the energy spectrum of c -electron is written as

$$E_k^\pm = \frac{\hbar^2 k^2}{2m} \mp H_c, \quad (1)$$

where H_c is approximated by

$$H_c = (N_i/N) (g-1) \bar{j}_z J(0), \quad (2)$$

where \bar{j}_z means the mean value of the z -component of the total angular momentum j_{nz} located in each magnetic ion, $J(0)$ the s - d exchange integral, N_i the number of magnetic ions per unit volume, therefore in metals $N_i = N$, and g the Landé's g -factor. In most cases, H_c is much smaller than the Fermi energy ζ .

(ii) The other scattering mechanisms than the s - d exchange interaction, such as electron-phonon and electron-non magnetic impurity atoms, are neglected in our treatment. These effects are discussed in later sections.

(iii) The Hamiltonian of the s - d interaction is given by

$$\begin{aligned} H_{sd} = & (g-1)/N \cdot \sum_k \sum_{k'} \sum_{R_n} \exp\{i(\mathbf{k}-\mathbf{k}')\mathbf{R}_n\} [\{A(|\mathbf{k}-\mathbf{k}'|)/(g-1) - J(|\mathbf{k}-\mathbf{k}'|)j_n^z\} \\ & a_{k+}^* a_{k'+} + \{A(|\mathbf{k}-\mathbf{k}'|)/(g-1) + J(|\mathbf{k}-\mathbf{k}'|)j_n^z\} a_{k-}^* a_{k'-} \\ & - J(|\mathbf{k}-\mathbf{k}'|)j_n^- a_{k+}^* a_{k'-} - J(|\mathbf{k}-\mathbf{k}'|)j_n^+ a_{k-}^* a_{k'+}], \end{aligned} \quad (3)$$

where $a_{k\nu}$ and $a_{k\nu}^*$ mean respectively the creation and annihilation operators of electron with wave vector \mathbf{k} and spin quantum number $\nu = \pm 1/2$, $J(q)$ the exchange integral and $A(q)$ the interaction except the exchange integral.

(iv) The energy level of $\{j_n\}$ may be very complicated both in metals and dilute alloys. However, we use here the most simplified model that the energy level of j_n is represented by the single molecular field H_0 such as

$$H_{\text{mag}} = - \sum_n H_0 j_n^z, \quad (4)$$

and thus the distribution of j_n^z is given by

$$\omega(j_z) = \exp(H_0 j_z / \kappa T) / \sum_{j_z} \exp(H_0 j_z / \kappa T). \quad (5)$$

(v) The scattering matrix element is given by the Born approximation. The transition probability, where the spin direction of c -electron does not change, namely, the first two processes of Eq. (3), is given by

$$\begin{aligned} W_\pm(q) = & \begin{cases} N^{-1}(g-1)^2 |J(q)|^2 \{\bar{j}_z^2 - j_z^2\}, & \text{in metals} \\ N_i/N^2 \{ |A(q)|^2 + (g-1)^2 |J(q)|^2 \bar{j}_z^2 \mp 2(g-1) \text{Re}(J(q)A(q))\bar{j}_z \}, & \text{in dilute alloys,} \end{cases} \end{aligned} \quad (6)$$

where \bar{j}_z and \bar{j}_z^2 mean the values of j_z and j_z^2 averaged by using Eq. (5). Furthermore, we assumed that $(\bar{j}_z - j_z^2)$ and in dilute alloys the positions of magnetic ions are perfectly at random and have no correlation in each other. On

the other hand, the transition probabilities, where the spin directions change each other, namely, the last two processes of Eq. (3), are given by

$$V_{\pm}(q) = N_i/N^2(g-1)^2 |J(q)|^2 \{j^2 + j - \bar{j}_z^2 \pm \bar{j}_z\}, \quad (7)$$

where \pm mean respectively the processes in which the spin direction of *c*-electron becomes \pm . Also in the above treatment we assumed that there are no correlation between magnetic ions. It is noticeable that between $V_{\pm}(q)$ there is a relation

$$V_{+}(q) = e^x V_{-}(q), \quad (8)$$

where

$$x = H_0/\kappa T. \quad (9)$$

Using these assumptions, both the drift and collision terms of Bloch equations concerning *c*-electrons with \pm spin directions are written as

$$(\partial f_{\pm}(\mathbf{k})/\partial t)_{\text{drift}} = f_1(\varepsilon_{\pm}) v_x/\kappa T \cdot [(e\mathbf{F} - \nabla\zeta) - \varepsilon_{\pm} \kappa \nabla T], \quad (10)$$

$$\begin{aligned} (\partial f_{\pm}(\mathbf{k})/\partial t)_{\text{coll}} = & 2\pi/\hbar \cdot \sum_{\mathbf{k}'} [W_{\pm}(|\mathbf{k}-\mathbf{k}'|) \{f_{\pm}(\mathbf{k}') (1-f_{\pm}(\mathbf{k})) \\ & - f_{\pm}(\mathbf{k}) (1-f_{\pm}(\mathbf{k}'))\} \delta(E_k^{\pm} - E_{k'}^{\pm}) \\ & + \{V_{\pm}(|\mathbf{k}-\mathbf{k}'|) f_{\mp}(\mathbf{k}') (1-f_{\pm}(\mathbf{k})) \\ & - V_{\mp}(|\mathbf{k}-\mathbf{k}'|) f_{\pm}(\mathbf{k}) (1-f_{\mp}(\mathbf{k}'))\} \delta(E_k^{\pm} - E_{k'}^{\mp} \pm H_0)], \end{aligned} \quad (11)$$

where the directions of the electric field \mathbf{F} and the temperature gradient ∇T are chosen in the *x* axis, $f_1(\varepsilon)$ means

$$f_1(\varepsilon) = [(e^{\varepsilon} + 1)(1 + e^{-\varepsilon})]^{-1}, \quad (12)$$

and

$$\varepsilon_{\pm} = (E_k^{\pm} - \zeta)/\kappa T. \quad (13)$$

Further considering that $f_{\pm}(\mathbf{k})$ is written as

$$f_{\pm}(\mathbf{k}) = f_0(\varepsilon_{\pm}) + k_x C_{\pm}(\varepsilon_{\pm}) f_1(\varepsilon_{\pm}), \quad (14)$$

where

$$f_0(\varepsilon) = [e^{\varepsilon} + 1]^{-1}, \quad (15)$$

and restricting to the linear theory, the collision terms become

$$\begin{aligned} (\partial f_{\pm}(\mathbf{k})/\partial t)_{\text{coll}} = & 2\pi/\hbar \cdot \sum_{\mathbf{k}'} [W_{\pm}(|\mathbf{k}-\mathbf{k}'|) f_1(\varepsilon_{\pm}) C(\varepsilon_{\pm}) (k_x' - k_x) \delta(E_k^{\pm} - E_{k'}^{\pm}) \\ & + V_{\mp}(|\mathbf{k}-\mathbf{k}'|) f_0(\varepsilon_{\pm}) f_0(-\varepsilon_{\pm} \mp x) \{k_x' C_{\mp}(\varepsilon_{\pm} \pm x) \\ & - k_x C_{\pm}(\varepsilon_{\pm})\} \delta(E_k^{\pm} - E_{k'}^{\mp} \pm (H_0 - 2H_c))]. \end{aligned} \quad (16)$$

Then, the Bloch equations are solved easily and give the following results

$$C_{\pm}(\varepsilon) = \phi_{\pm}^0(\varepsilon) (e\mathbf{F} - \nabla\zeta) - \phi_{\pm}^1(\varepsilon) \kappa \nabla T, \quad (17)$$

where

$$\phi_{\pm}^{\mu}(\varepsilon) = \frac{\pi \hbar^4}{m^2 \kappa T} \frac{\varepsilon^{\mu} A_{\mp}(\varepsilon \pm x) + (\varepsilon \pm x)^{\mu} B_{\pm}(\varepsilon)}{A_{\pm}(\varepsilon) A_{\mp}(\varepsilon \pm x) - B_{\pm}(\varepsilon) B_{\mp}(\varepsilon \pm x)}, \quad \mu=0, 1, \quad (18)$$

$$A_{\pm}(\varepsilon) = \frac{1}{4k_{\pm}^3} \int_0^{2k_{\pm}} W_{\pm}(q) q^3 dq + \frac{1}{2k_{\pm}} \frac{1+e^{-\varepsilon}}{1+e^{-\varepsilon \mp x}} \int_{\eta_{\pm}}^{\varepsilon_{\pm}} V_{\mp}(q) q dq, \quad (19)$$

$$B_{\pm}(\varepsilon) = \frac{1}{2k_{\pm}} \frac{1+e^{-\varepsilon}}{1+e^{-\varepsilon \mp x}} \int_{\eta_{\pm}}^{\varepsilon_{\pm}} V_{\mp}(q) \left\{ 1 - \frac{q^2}{2k_{\pm}^2} \mp \frac{m(2H_c - H_0)}{\hbar^2 k_{\pm}^2} \right\} q dq, \quad (20)$$

and

$$k_{\pm}^2 = \frac{2m}{\hbar^2} (E \pm H_c) = k_0^2 \left(1 + \frac{\kappa T}{\zeta} \varepsilon \pm \frac{H_c}{\zeta} \right), \quad (21)$$

$$\varepsilon_{\pm} = k_{\pm} \left\{ 1 + \left(1 \mp \frac{2H_c - H_0}{E \pm H_c} \right)^{1/2} \right\}, \quad (22)$$

$$\eta_{\pm} = k_{\pm} \left| 1 - \left(1 \mp \frac{2H_c - H_0}{E \pm H_c} \right)^{1/2} \right|. \quad (23)$$

§ 3. Results

The electric current j_x and the thermal flow Q_x are given by

$$j_x/e = L_{00}(eF - \nu\zeta) - L_{01}\kappa\nu T, \quad (24)$$

$$Q_x/\kappa T = L_{10}(eF - \nu\zeta) - L_{11}\kappa\nu T, \quad (25)$$

where

$$L_{\nu\mu} = \frac{1}{6\pi^2} \frac{\kappa T}{\hbar} \int_{-\infty}^{\infty} \varepsilon^{\nu} (k_{\pm}^3 \phi_{\pm}^{\mu}) f_1(\varepsilon) d\varepsilon, \quad (26)$$

and

$$(k_{\pm}^3 \phi_{\pm}^{\mu}) \equiv k_{+}^3 \phi_{+}^{\mu}(\varepsilon) + k_{-}^3 \phi_{-}^{\mu}(\varepsilon). \quad (27)$$

It is easily seen that Onsager's relations, $L_{\mu\nu} = L_{\nu\mu}$, is satisfied.

Using the conductivity tensor \mathbf{L} , we can calculate the various transport coefficients. But since the formula of \mathbf{L} is still very complicated, we use here the following approximations.

(1) We assume that $W_{\pm}(q)$ and $V_{\pm}(q)$ do not depend on q in the range $0 < q < 2k_0$, where k_0 is the effective wave vector at Fermi surface determined by

$$\zeta = \hbar^2 k_0^2 / 2m. \quad (28)$$

(2) Because J/ζ , H_c/ζ , H_0/H_c and $\kappa T/H_c$, and furthermore J/A in dilute alloys, are sufficiently smaller than unity, we adopt only the leading terms concerning these factors.

Then \mathbf{L} is written as

$$L_{00} = \begin{cases} \frac{2}{3\pi} \frac{\hbar\zeta}{m} \int_{-\infty}^{\infty} \frac{f_1(\varepsilon) d\varepsilon}{W_0 + VF(\varepsilon)}, & \text{in metals} \\ \frac{2}{3\pi} \frac{\hbar\zeta}{m} \frac{1}{W_0} \left\{ 1 + \frac{W_1}{W_0} \frac{H_c}{\zeta} - \frac{V}{W_0} \frac{x}{1-e^{-x}} \right\}, & \text{in dilute alloys} \end{cases} \quad (29)$$

$$L_{01} = L_{10} = \begin{cases} \frac{2}{3\pi} \frac{\hbar\zeta}{m} \int_{-\infty}^{\infty} \frac{\varepsilon f_1(\varepsilon) d\varepsilon}{W_0 + VF(\varepsilon)} \left\{ \frac{\kappa T}{\zeta} \varepsilon + \frac{H_c}{\zeta} \frac{W_0 + 2VF(\varepsilon)}{W_0 + VF(\varepsilon)} \right\}, & \text{in metals} \\ \frac{2\pi}{9} \frac{\hbar\zeta}{m} \frac{1}{W_0} \left\{ \frac{\kappa T}{\zeta} - \frac{3}{\pi^2} \frac{W_1 V}{W_0} \frac{x^2}{1-e^{-x}} \right\}, & \text{in dilute alloys} \end{cases} \quad (30)$$

$$L_{11} = \begin{cases} \frac{2}{3\pi} \frac{\hbar\zeta}{m} \int_{-\infty}^{\infty} \frac{\varepsilon^2 f_1(\varepsilon) d\varepsilon}{W_0 + VF(\varepsilon)}, & \text{in metals} \\ \frac{2\pi}{9} \frac{\hbar\zeta}{m} \frac{1}{W_0} \left\{ 1 + \frac{W_1}{W_0} \frac{H_c}{\zeta} - \frac{V}{W_0} \frac{x + x^3/\pi^2}{1-e^{-x}} \right\}, & \text{in dilute alloys} \end{cases} \quad (31)$$

where

$$W_0 = \begin{cases} N^{-1}(g-1)^2 J^2(0) \{ \bar{j}_z^2 - (\bar{j})_z^2 \}, & \text{in metals} \\ N_s/N^2 \cdot \{ A^2(0) + (g-1)^2 J^2(0) \bar{j}_z^2 \}, & \text{in dilute alloys} \end{cases} \quad (32)$$

$$W_1 = N_s/N^2 \cdot 2(g-1) A(0) J(0) \bar{j}_z, \quad \text{in dilute alloys} \quad (33)$$

$$V = N_s/N^2 \cdot (g-1)^2 J^2(0) \{ j^2 + j - \bar{j}_z^2 - \bar{j}_z \}, \quad (34)$$

$$F(\varepsilon) = (1 + e^{-\varepsilon}) / (1 + e^{-\varepsilon - \varepsilon_c}), \quad (35)$$

and \bar{j}_z , \bar{j}_z^2 , etc., are given by

$$\bar{j}_z = j + \frac{2j+1}{e^{(2j+1)x} - 1} - \frac{1}{e^x - 1}, \quad (36)$$

$$\bar{j}_z^2 = j^2 - \frac{2j-1}{e^x - 1} + \frac{2}{(e^x - 1)^2} - \frac{2j+1}{e^{(2j+1)x} - 1} - \frac{2(2j+1)}{(e^x - 1)(e^{(2j+1)x} - 1)}, \quad (37)$$

$$\bar{j}_z^2 - (\bar{j}_z)^2 = \frac{1}{(e^x - 1)(1 - e^{-x})} - \frac{(2j+1)^2}{(e^{(2j+1)x} - 1)(1 - e^{-(2j+1)x})}, \quad (38)$$

$$j^2 + j - \bar{j}_z^2 - \bar{j}_z = \frac{2}{e^x - 1} \left\{ j + \frac{2j+1}{e^{(2j+1)x} - 1} - \frac{1}{e^x - 1} \right\}. \quad (39)$$

(1) Electrical conductivity σ .

σ is given by

$$\sigma = e^2 L_{00}, \quad (40)$$

and L_{00} is given by Eq. (29). The formula of L_{00} differs from that of the earlier papers⁴⁾⁵⁾ because the degrees of the approximation to solve the Bloch equations are not equal. Nevertheless, the main character that σ increases with the increasing

magnetic order, that is the increase of x , is of course equal. The electrical resistivity at absolute zero temperature and in the state of no magnetic order are given by

$$(\sigma^{-1})_{x=0} = \begin{cases} 0 & \text{in metals} \\ \frac{3\pi}{2} \frac{m}{\hbar\zeta} \frac{N_f}{N^2 e^2} \left\{ A^2(0) + (g-1)^2 j^2 J^2(0) - 2(g-1)jJ(0)A(0) \frac{H_c}{\zeta} \right\}, & \text{in dilute alloys,} \end{cases} \quad (41)$$

and

$$(\sigma^{-1})_{x=0} = \begin{cases} \frac{3\pi}{2} \frac{m}{\hbar\zeta} \frac{1}{Ne^2} (g-1)^2 j(j+1) J^2(0), & \text{in metals,} \\ \frac{3\pi}{2} \frac{m}{\hbar\zeta} \frac{N_i}{N^2 e^2} \{ A^2(0) + (g-1)^2 j(j+1) J^2(0) \}, & \text{in dilute alloys.} \end{cases} \quad (42)$$

It is also easily seen from the above mentioned character that the electrical resistivity decreases when the external magnetic field is applied because x increases by the application of the external magnetic field.

In metals, if $F(\varepsilon)$ is replaced approximately by unity, the result of [I] is obtained, namely

$$\sigma^{-1} = \frac{3\pi}{2} \frac{m}{Ne^2} \frac{(g-1)^2 J^2(0)}{\hbar\zeta} (j^2 + j - \bar{j}_z^2 - \bar{j}_z), \quad \text{in metals.} \quad (43)$$

(2) Thermal conductivity κ_0 and Lorentz number L .

κ_0 and L are given by

$$\kappa_0 = \kappa^2 T \frac{L_{11}L_{00} - L_{10}L_{01}}{L_{00}}, \quad (44)$$

$$L = \frac{\kappa_0}{\sigma T} = \frac{\kappa^2}{e^2} \frac{L_{11}L_{00} - L_{10}L_{01}}{L_{00}^2}. \quad (45)$$

As is seen in Eqs. (29), (30) and (31), L_{10} and L_{01} are much smaller than L_{00} and L_{11} and thus we can write

$$L = \begin{cases} \frac{\kappa^2}{e^2} \int \frac{\varepsilon^2 f_1(\varepsilon) d\varepsilon}{W_0 + VF(\varepsilon)} \bigg/ \int \frac{f_1(\varepsilon) d\varepsilon}{W_0 + VF(\varepsilon)}, & \text{in metals,} \\ \frac{\pi^2}{3} \frac{\kappa^2}{e^2} \left\{ 1 - \frac{(g-1)^2}{\pi^2} \frac{J^2(0)}{A^2(0)} \frac{x^3}{1-e^{-x}} (j^2 + j - \bar{j}_z^2 - \bar{j}_z) \right\}, & \text{in dilute alloys.} \end{cases} \quad (46)$$

In metals L has a nearly constant value $(\pi^2/3)(\kappa/e)^2$ over the whole range of x . This result is caused by the assumption of the isotropic scattering. As is seen in later section, if this assumption is not adopted L changes markedly from the value $(\pi^2/3)(\kappa/e)^2$.

(3) Absolute thermoelectric power \mathcal{S} .

\mathcal{S} is given by

$$\mathfrak{S} = \frac{\kappa}{e} \frac{L_{01}}{L_{00}}. \quad (47)$$

In metals, neglecting W_0 compared to $VF(\varepsilon)$, we obtain (see Appendix I) the following simplified approximated expression,

$$\mathfrak{S} = \frac{\kappa}{e} \left\{ \frac{\pi^2}{3} \frac{\kappa T}{\zeta} + 2 \frac{H_c}{\zeta} \frac{1 - e^{-x}}{1 + e^{-x}} \right\}. \quad (48)$$

The first term corresponds to the ordinary mechanism, namely this is characteristic of the isotropic scattering. The second term is characteristic of the *s-d* interaction and it is noticeable that this term is dominant than the first in sufficiently low temperature and remains to be of the finite value in absolute zero temperature. However, in general, there exists a some kind of lattice imperfection and thus W_0 remains the finite value B_0 . Then in sufficiently low temperature B_0 becomes larger than $VF(\varepsilon)$ and \mathfrak{S} is written as

$$\mathfrak{S} = \frac{\kappa}{e} \left\{ \frac{\pi^2}{3} \frac{\kappa T}{\zeta} + \frac{V^2}{B_0^2} \frac{H_c}{\zeta} x e^x \right\}. \quad (49)$$

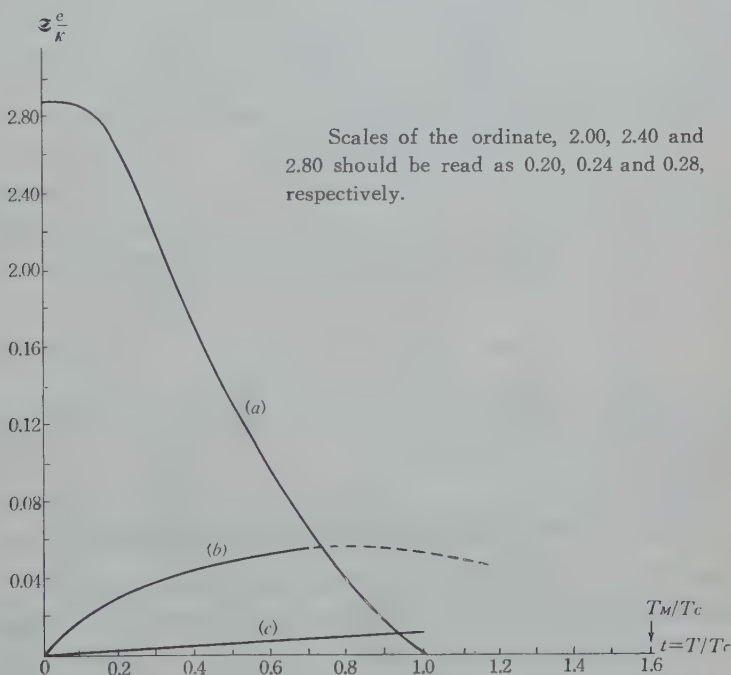


Fig. 1 The graph of $\mathfrak{S}(e/\kappa)$ versus T/T_c (Gd)

- (a) means $\mathfrak{S}_{\text{ano}}^{\text{mol}}$, namely the anomalous part of \mathfrak{S} calculated by simple molecular field approximation.
- (b) means $\mathfrak{S}_{\text{ano}}^{\text{spin}}$, namely the anomalous part of \mathfrak{S} calculated by spin wave approximation.
- (c) means $\mathfrak{S}_{\text{nor}}$, namely the normal part of \mathfrak{S} and is given by $0.011 t$.

The temperature range where the above equation is applicable corresponds to that where the electrical resistance is nearly determined by the residual resistance. In Eq. (45) it is noticeable that the second term is proportional to $(V/B_0)^2$ and the term proportional to V/B_0 vanishes. This result is owing to the assumption that $J(q)$ does not depend on q . In general, if $J(q)$ depends on q the term proportional to (V/B_0) exists.

In dilute alloys, \mathfrak{S} is written as

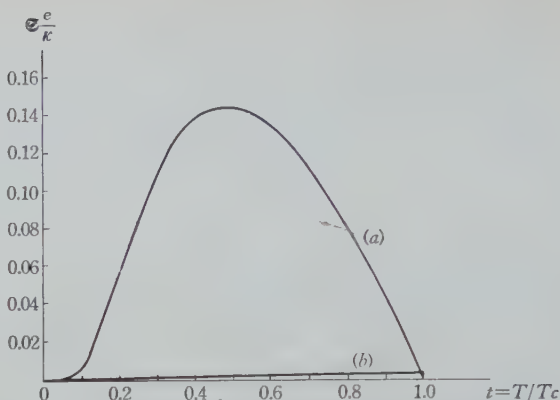


Fig. 2. The graph of $\mathfrak{S}^e(\kappa)$ versus T/T_c (the dilute alloy of 1.8% Cn-Mn)

(a) means the anomalous part of \mathfrak{S} .

(b) means the normal part of \mathfrak{S} and is given by $0.0008 t$.

$$\mathfrak{S} = \frac{\kappa}{e} \left\{ \frac{\pi^2}{3} \frac{\kappa T}{\zeta} - \frac{2(g-1)^3 J^3(0)}{A^3(0)} \bar{j}_z (j^2 + j - \bar{j}_z^2 - \bar{j}_z) \frac{x^2}{1 - e^{-x}} \right\}. \quad (50)$$

As mentioned above, if $J(q)$ depends on q the term proportional to V/W_0 should appear. For example, if we assume that $J(q)$ is written by

$$J^2(q) = J^2(0) - J_1^2 q^2/k_0^2, \quad (51)$$

the following term appears.

$$-\frac{\kappa}{e} \frac{1}{3} \frac{H_c}{\zeta} \frac{(g-1)^2 J_1^2}{A^2(0)} (j^2 + j - \bar{j}_z^2 - \bar{j}_z) \frac{x^2}{1 - e^{-x}}. \quad (52)$$

This term is nearly of the same order of magnitude as the second term of Eq. (50).

The more detailed comparison between the normal term and the anomalous terms are given in Figs. 1 and 2. (see § 5)

§ 4. Calculations by spin wave approximation

In the preceding sections we calculated transport coefficients using the simplest molecular field model. But in sufficiently low temperatures, the magnetic ordered state may be represented by some kind of collective mode. In dilute alloys, however, it is a very complicated problem to treat the ordered state on the collective description. On the other hand, in metals the collective description of the ordered state is possible as was done in the previous paper.²⁾ Therefore we treat in this section only the metallic ferromagnetics such as rare earth metals.

To perform the calculation we use the following model. (i) The magnetic spin system is always in the local thermal equilibrium. The justification of this

assumption may be fairly complicated because in a rigorous treatment the phonon system should be also considered, and then the so-called Umklapp process becomes important. (2) We assume here that the collective mode of the spin system is represented by the so-called spin wave mode and that the energy spectrum of the spin wave with wave vector \mathbf{k} is written by $\varepsilon(k)$.

(iii) Because we consider only sufficiently low temperature range, we adopt only the process in which only the one spin wave quantum is emitted or absorbed.

(iv) The square of the transition matrix element is given by

$$V_{\pm}(q) = \frac{2j(g-1)^2}{N} |J(q)|^2 \begin{bmatrix} N(q)+1 \\ N(q) \end{bmatrix}, \quad (53)$$

where \pm correspond respectively to the emission and the absorption of the spin wave, and $N(q)$ means the distribution function of the spin wave with wave vector \mathbf{q} in the thermal equilibrium, namely

$$N(q) = \left[\exp \frac{\varepsilon(q)}{\kappa T} - 1 \right]^{-1}. \quad (54)$$

Then we can write the Bloch equations concerning the electrons of \pm spin directions. The drift terms are same as Eq. (10). In collision terms of Eq. (11), the first term proportional to $W_{\pm}(q)$ vanishes because it represents two spin wave processes, and only the last two terms are available substituting Eq. (53) for $V_{\pm}(q)$ and $\varepsilon(q)$ for H_0 . Now, however, because it is impossible to solve these Bloch equations rigorously, we use here the technique of the total balance method.⁷⁾ The assumptions adopted for the following calculation are as follows. (v) We adopt only the lowest terms concerning H_c/ζ and $\kappa T/\zeta$. (vi) In Eq. (14) we assume that $C_{\pm}(\varepsilon)$ is approximately written as

$$C_{\pm}(\varepsilon) = C_{\pm}^0 + C_{\pm}^1 \varepsilon. \quad (55)$$

Then the total balance method means that the Bloch equations are multiplied respectively by k_x and $\varepsilon_{\pm} k_x$ and then integrated by $d\mathbf{k}$. These equations mean respectively that the change of the total value of k_x and $\varepsilon_{\pm} k_x$ (or the total current and thermal flow) by drift effect and collision mechanisms are balanced in steady state, and from these four equations we can determine C_{\pm}^0 and C_{\pm}^1 . It is easily shown that in the case of electron-phonon interaction this method gives the same result as that of the second order variational method obtained by Sondheimer and Wilson.^{7,8)} (see Appendix II). Here we omit the detailed process of the calculation (see Appendix II) and give only the final results. Rewriting C_{\pm}^{ν} such as

$$C_{\pm}^{\nu} = \phi_{\pm}^{\nu 0} (eF - \tau \zeta) - \phi_{\pm}^{\nu 1} \kappa T, \quad \nu = 0, 1, \quad (56)$$

conductivity tensor \mathbf{L} in Eq. (26) is obtained as follows,

$$L_{00} = \frac{1}{6\pi^2} \frac{\kappa T}{\hbar} (\phi_{\pm}^{00} k_{\pm 0}^3), \quad (57)$$

$$L_{11} = \frac{1}{18} \frac{\kappa T}{\hbar} (\phi_{\pm}^{11} k_{\pm 0}^3), \quad (58)$$

$$L_{01} = \frac{1}{6\pi^2} \frac{\kappa T}{\hbar} \left\{ (\phi_{\pm}^{01} k_{\pm 0}^3) + \frac{\pi^2}{2} \frac{\kappa T}{\zeta} (\phi_{\pm}^{11} k_{\pm 0}^3) \right\}, \quad (59)$$

$$L_{10} = \frac{1}{18} \frac{\kappa T}{\hbar} \left\{ \frac{3}{2} \frac{\kappa T}{\zeta} (\phi_{\pm}^{00} k_{\pm 0}^3) + (\phi_{\pm}^{10} k_{\pm 0}^3) \right\}, \quad (60)$$

where

$$(\phi_{\pm}^{\nu\mu} k_{\pm 0}^3) = k_{\pm 0}^3 \phi_{\pm}^{\nu\mu} + k_{\pm 0}^3 \phi_{\pm 0}^{\nu\mu} \quad (61)$$

$$k_{\pm 0}^2 = k_0^2 \left(1 \pm \frac{H_c}{\zeta} \right), \quad (62)$$

and adopting only the leading terms with respect to $\kappa T / \epsilon(q_0)$, where q_0 means the maximum wave vector of spin wave mode, we obtain

$$(\phi_{\pm}^{00} k_{\pm 0}^3) = \frac{2k_0^6}{[q^2 x]}, \quad (63)$$

$$(\phi_{\pm}^{11} k_{\pm 0}^3) = \frac{2\pi^2}{3} k_0^4 \frac{[x]}{[x][x^3] - [x^2]^2}, \quad (64)$$

$$\begin{aligned} (\phi_{\pm}^{01} k_{\pm 0}^3) &= \frac{2\pi^2}{3} k_0^6 \frac{\kappa T}{\zeta} \frac{1}{[q^2 x]} \\ &+ \frac{\pi^2}{3} k_0^4 \frac{H_c}{\zeta} \left(\frac{[x^2]}{[x][x^3] - [x^2]^2} + 3 \frac{[q^2 x^2]}{[q^2 x]} \frac{1}{4\pi[x] + [x^3]} \right), \end{aligned} \quad (65)$$

where $[f(q, x)]$ such as $[q^2 x]$ means the following value,

$$[f(q, x)] = \frac{1}{4\pi} \frac{m^2 \kappa T}{\hbar^4} \frac{2(g-1)^2 j}{N} \int_{k_0 H_c / \zeta}^{q_0} \frac{f(q, x) q |J(q)|^2}{(e^x - 1)(1 - e^{-x})} dq. \quad (66)$$

For example, we calculate transport coefficients in some simple cases.

Case A. $\epsilon(q) = \epsilon_0 q / q_0$, $J(q) = J_0$.

In this case $[f(q, x)]$ is written as

$$[f(q, x)] = \frac{1}{2\pi} \frac{(g-1)^2 j J_0^2}{N} \frac{m^2 \kappa T}{\hbar^4} q_0^2 t^2 \int_{x_0}^{t^{-1}} \frac{x f(q_0 t x, x)}{(e^x - 1)(1 - e^{-x})} dx, \quad (67)$$

where

$$t = \kappa T / \epsilon_0, \quad x_0 = \frac{1}{t} \frac{H_c}{\zeta} \frac{k_0}{q_0}, \quad (68)$$

and from Eqs. (40), (57) and (63) we obtain

$$\sigma = \frac{4}{3\pi} \frac{Ne^2}{m} \frac{\hbar \zeta}{j(g-1)^2 J_0^2} \left(\frac{k_0}{q_0} \right)^4 \frac{1}{t^4} \frac{1}{\phi_4(t^{-1}, x_0)}, \quad (69)$$

$$L = \frac{\kappa^2}{e^2} \frac{\pi^4}{9} \left(\frac{q_0}{k_0} \right)^2 t^2 \left\{ 1 - \frac{\phi_3^2(t^{-1}, x_0)}{\phi_2(t^{-1}, x_0) \phi_4(t^{-1}, x_0)} \right\}, \quad (70)$$

$$\begin{aligned} \mathfrak{S} = & \frac{\kappa}{e} \frac{\pi^2}{3} \frac{\kappa T}{\zeta} \left\{ 1 + \frac{3}{2} \left(\frac{q_0}{k_0} \right)^2 \frac{H_c}{\epsilon_0} t \left(\frac{1}{3} \frac{\phi_3(t^{-1}, x_0) \phi_4(t^{-1}, x_0)}{\phi_2(t^{-1}, x_0) \phi_4(t^{-1}, x_0) - \phi_3(t^{-1}, x_0)^2} \right. \right. \\ & \left. \left. + \frac{\phi_5(t^{-1}, x_0)}{4\pi^2 \phi_2(t^{-1}, x_0) + \phi_4(t^{-1}, x_0)} \right) \right\}, \quad (71) \end{aligned}$$

and where

$$\phi_n(t^{-1}, x_0) = \int_{x_0}^{t^{-1}} \frac{x^n dx}{(e^x - 1)(1 - e^{-x})}. \quad (72)$$

When x_0 is sufficiently smaller and t^{-1} is sufficiently larger than unity, we can replace $\phi_n(t^{-1}, x_0)$ by $\phi_n(\infty, 0)$. The values of $\phi_n(\infty, 0)$ are given as follows,

$$\begin{aligned} \phi_2(\infty, 0) &= 3.29, \quad \phi_3(\infty, 0) = 7.22 \\ \phi_4(\infty, 0) &= 25.9, \quad \phi_5(\infty, 0) = 124.4. \end{aligned} \quad (73)$$

Then σ is proportional to T^{-4} , L to T^2 and \mathfrak{S} is written as

$$\mathfrak{S} = \frac{\kappa}{e} \frac{\pi^2}{3} \frac{\kappa T}{\zeta} \left\{ 1 + 4 \left(\frac{q_0}{k_0} \right)^2 \frac{H_c \kappa T}{\epsilon_0^2} \right\}. \quad (74)$$

When x_0 is sufficiently larger than unity, that is, in very low temperature range, we can replace $\phi_n(t^{-1}, x_0)$ by

$$\begin{aligned} \phi_n(t^{-1}, x_0) &\Rightarrow \int_{x_0}^{\infty} x^n e^{-x} dx \\ &= (x_0^n + n x_0^{n-1} + \dots + n!) e^{-x_0} \end{aligned} \quad (75)$$

and thus σ is proportional to e^{-x_0} . On the otherhand, the denominators of L and \mathfrak{S} vanish if we take only the leading term of Eq. (75). Therefore we must calculate the values of L and \mathfrak{S} in such a very low temperature range using the full expressions as given in Appendix II. However, in such a very low temperature range, we must consider the following facts that (i) The approximation to replace $C_{\pm}(\epsilon)$ by $C_{\pm}^0 + C_{\pm}^1 \epsilon$ becomes very poor. (ii) The Raman process in which one spin wave is absorbed and the other one emitted becomes important because in this process there are no lower limit with respect to q . (iii) Furthermore, in actual case, the transport phenomena of such a low temperature range are determined mostly by residual resistance. The more detailed calculation by considering such a circumstance may be carried out in future.

Case B $\epsilon = \epsilon_0 q^2 / q_0^2, J(q) = J_0.$

In this case $[f(q^2, x)]$ is written as

$$[f(q^2, x)] = \frac{1}{4\pi} \frac{j(g-1)^2 J^2(0)}{N} \frac{m^2 \kappa T}{\hbar^4} q_0^2 t \int_{x_0}^{t^{-1}} \frac{f(q_0^2 t x, x)}{(e^x - 1)(1 - e^{-x})} dx, \quad (76)$$

$$\text{where } t = \frac{\kappa T}{\epsilon_0}, \quad x_0 = \frac{1}{t} \left(\frac{H_c}{\zeta} - \frac{k_0}{q_0} \right)^2, \quad (77)$$

and thus we obtain

$$\sigma = \frac{8}{3\pi} \frac{Ne^2}{m} \frac{\hbar \zeta}{j(g-1)^2 J^2(0)} \left(\frac{k_0}{q_0} \right)^4 \frac{1}{t^2} \frac{1}{\phi_2(t^{-1}, x_0)}, \quad (78)$$

$$L = \frac{\kappa^2}{e^2} \frac{\pi^4}{9} \left(\frac{q_0}{k_0} \right)^2 t / \left\{ \frac{\phi_3(t^{-1}, x_0)}{\phi_2(t^{-1}, x_0)} - \frac{\phi_2(t^{-1}, x_0)}{\phi_1(t^{-1}, x_0)} \right\}, \quad (79)$$

$$\begin{aligned} \mathfrak{S} = \frac{\kappa}{e} \frac{\pi^2}{3} \left[1 + \frac{1}{2} \left(\frac{q_0}{k_0} \right)^2 \frac{H_c}{\epsilon_0} \left\{ \left(\frac{\phi_1(t^{-1}, x_0) \phi_3(t^{-1}, x_0)}{\phi_2(t^{-1}, x_0)^2} - 1 \right)^{-1} \right. \right. \\ \left. \left. + 3 \left(4\pi^2 \frac{\phi_1(t^{-1}, x_0)}{\phi_3(t^{-1}, x_0)} + 1 \right)^{-1} \right\} \right] \frac{\kappa T}{\zeta}. \end{aligned} \quad (80)$$

When x_0 is sufficiently smaller and t^{-1} larger than unity we can replace $\phi_n(t^{-1}, x_0)$ by $\phi_n(\infty, 0)$ as was shown in Eq. (73). However, this replacement is impossible for the function $\phi_1(t^{-1}, x_0)$ because $\phi_1(t^{-1}, x_0)$ diverges logarithmically in the limit of $x_0 \rightarrow 0$. Considering the above fact, σ is proportional to T^{-2} and L proportional to T . While \mathfrak{S} is written as

$$\mathfrak{S} = \frac{\kappa}{e} \frac{\pi^2}{3} \frac{\kappa T}{\zeta} \left\{ 1 + 1.5 \left(\frac{q_0}{k_0} \right)^2 \frac{H_c}{\epsilon_0} \frac{1}{\phi_1(t^{-1}, x_0)} \right\}, \quad (81)$$

and in this case, too, the anomalous term becomes large because the value of H_c/ϵ_0 is in general of the order of 10 and $\ln x_0$ is not so large (see § 5).

In the case of antiferromagnetic ordering the same treatment is possible (see Appeneix II). For example, the explicit results of the case A are given as follows.

$$\sigma = \frac{2}{3\pi} \frac{Ne^2}{m} \frac{\hbar \zeta}{j(g-1)^2 J_0^2} \left(\frac{k_0}{q_0} \right)^4 \frac{1}{t^4} \frac{1}{\phi_4(t^{-1})}, \quad (82)$$

$$L = \frac{\kappa^2}{e^2} \frac{\pi^4}{9} \left(\frac{q_0}{k_0} \right)^2 t^2 / \left[1 + \frac{\pi^2}{3} \left(\frac{q_0}{k_0} \right)^2 t^2 \left\{ 1 - \frac{1}{2\pi^2} \phi_6(t^{-1}) / \phi_4(t^{-1}) \right\} \right], \quad (83)$$

$$\mathfrak{S} = \frac{\kappa}{e} \frac{\pi^2}{3} \frac{\kappa T}{\zeta} \frac{1 + \pi^2 (q_0/k_0)^2 t^2 \{ 1 - (1/4\pi^2) \phi_6(t^{-1}) / \phi_4(t^{-1}) \}}{1 + (\pi^2/3) (q_0/k_0)^2 t^2 \{ 1 - (1/2\pi^2) \phi_6(t^{-1}) / \phi_4(t^{-1}) \}}, \quad (84)$$

where

$$\phi_n(t^{-1}) = \int_0^{t^{-1}} \frac{x^n dx}{(e^x - 1)(1 - e^{-x})}, \quad (85)$$

and $t = \kappa T / \epsilon_0$.

In antiferromagnetic ordering, H_c becomes zero and thus the anomalous term of \mathfrak{S} proportional to H_c/ζ disappears.

§ 5. Conclusions and discussions

(i) By *s-d* interaction, anomalous thermoelectric force as well as electrical and thermal resistivities appears both in ferromagnetic metals and alloys.

(ii) For the more qualitative discussions we take Gd as an example of ferromagnetic metals. Then using a free electron model we can obtain the following values.

$$\begin{aligned} n/N=3, \quad \zeta &= 1.2 \times 10^{-11} \text{ erg.}, \quad J_0 = 2.5 \times 10^{-13} \text{ erg.}, \\ g=2, \quad j &= 7/2, \\ H_c &= 1.75 \times 10^{-12} \bar{J}_z / j \text{ erg.} \end{aligned} \quad (86)$$

If we assume that there is only the *s-d* exchange interaction, H_0 is written as

$$H_0 = H_m \bar{J}_z, \quad (87)$$

and H_m is determined experimentally by the paramagnetic Curie temperature T_p such as

$$\kappa T_p = \frac{j(j+1)}{3} H_m \quad (88)$$

and thus in Gd we obtain

$$T_p = 300^\circ \text{ K}, \quad H_m = 8 \times 10^{-15} \text{ erg} = 57\kappa. \quad (89)$$

ϵ_0 is also determined experimentally by T_M as follows,

$$\epsilon_0 = \left\{ \frac{3}{2j} \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right) \right\}^{2/3} \kappa T_M \dot{=} \kappa T_M, \quad (90)$$

where T_M is obtained as an effective Curie temperature by extrapolating the magnetization curve of the range where $T^{3/2}$ law is applicable in a good approximation. In Gd we obtain the following results.

$$\begin{aligned} T_M &= 470^\circ \text{ K}, \quad \epsilon_0 = 5.8 \times 10^{-14} \text{ erg.}, \\ x_0 &= \frac{\epsilon_0}{\kappa T} \left(\frac{H_c}{\zeta} \frac{k_0}{q_0} \right)^2 = \frac{13}{T}, \end{aligned} \quad (91)$$

and the $T^{3/2}$ law of the magnetization curve is applicable still up to 200° K . In Fig. 1 the curve of \mathfrak{S} is plotted as a function of T . Experimental value of \mathfrak{S} is, however, not available.

(iii) As is well-known, the electrical resistivity by the ordinary electron phonon interaction, say ρ_{ep} , is proportional to T^5 in temperature sufficiently lower than the Debye temperature. While the resistivity by *s-d* exchange interaction, say ρ_{sd} , is proportional to T^2 from Eq. (78). Therefore the following behaviour of ρ is expected that in sufficiently low temperature ρ_{sd} overcomes ρ_{ep} , and ρ is proportional to T^2 . In Gd, however, the temperature dependence of ρ differs from

the expected behaviour. This result seems to be owing to the fact that in rare earth metals the temperature dependence of ρ_{ep} may be fairly different from that of the ordinary theory because of the complicated band structure and thus it is very difficult to separate ρ into ρ_{ep} and ρ_{sd} in good accuracy. For example, the electrical resistivity of La at sufficiently low temperature is rather proportional to T . Recently Mannari pointed out that in Fe ρ changes with T^2 in low temperature.⁹⁾ In the transition metals, too, ρ_{ep} seems to be fairly different from that of the ordinary theory as is seen in Pd and W and thus it is still ambiguous whether this temperature dependence is due to the s - d interaction or not.

(iv) In rare earth metals the thermal conductivity has not so far been measured. In the above calculations we neglected the thermal flow due to spin diffusion or spin wave flow. But it seems for me that the effect of spin diffusion may be important because of the long range character of the s - d interaction. The more detailed discussion about this point may be made in future.

(v) For an example of dilute alloys we take up here the Cu-Mn alloys as one of the simplest substances because Mn^{++} is in the state of 6S and has no orbital momentum. Notwithstanding of this circumstance the actual magnetic state of Cu-Mn alloys seems to be fairly complicated and the interactions other than the s - d interaction seems to be important for the magnetic ordering. However, with respect to the electrical and thermal resistivities the results of Eqs. (40) and (44) do not depend so sensitively on the character of the magnetic ordering and the agreement with the experimental results is fairly good as was shown by Yosida.⁵⁾ While the anomalous part of \mathfrak{S} depends so sensitively on the character of the magnetic ordering. For example, if the magnetic spins are in a state of some kind of the antiferromagnetic ordering, the anomalous part of \mathfrak{S} disappears. However, on the other hand, for the applicability of Eq. (50) it is not necessary that there appears the resultant magnitude of the magnetization because Eq. (50) does not depend on the direction of \bar{j}_z . For example, Eq. (50) is applicable even in the case that the ferromagnetic ordering exists only in the limited range of the mean free path of the conduction electron's spin. Therefore the measurement of \mathfrak{S} may give a certain knowledge about the magnetic ordering of dilute alloys. For a more quantitative discussions we take up the sample of 1.8 atomic per cent $Mn^{10)}$ which was chosen by Yosida for the discussion of the anomalous electrical resistivity. Then using the free electron model we obtain the following values.

$$\begin{aligned}
 (\sigma^{-1})_{T=0} &= 4.6\mu \text{ ohm} = 5.1 \times 10^{-18} \text{ e.s.u.}, \\
 (\sigma^{-1})_{r=0} &= 5\mu \text{ ohm} = 5.55 \times 10^{-18} \text{ e.s.u.}, \\
 g &= 2, \quad j = 5/2, \quad N_i/N = 0.018, \\
 \zeta &= 1.1 \times 10^{-11} \text{ erg.}, \quad N = 8.5 \times 10^{22} \text{ c.c.}^{-1}, \\
 A(0) &= 3.4 \times 10^{-12} \text{ erg.}, \quad J(0) = 0.7 \times 10^{-12} \text{ erg.},
 \end{aligned}
 \tag{92}$$

$$H_c = 3.15 \times 10^{-14} \text{ erg.}, \quad T_p \sim 20^\circ \text{ K.}$$

From the above value it is easily seen that the term proportional to H_c/ζ in Eq. (52) is smaller than that proportional to $J(0)/A(0)$ and thus we can use Eq. (50) in a good approximation. The graph of \mathfrak{S} given by Eq. (50) is plotted in Fig. 2. The anomalous thermoelectric effect is very much larger than the normal part. For example, the maximum value of the anomalous part is 350 times of the normal part at 10°K .

Even in the case of non-ferromagnetic ordering, the anomalous part of \mathfrak{S} is observable by applying strong magnetic field. For example, using a simplest model, we can replace H_0 by

$$H_0 = g\beta H, \quad (93)$$

and taking only the linear terms with respect to the external field H , we obtain

$$\mathfrak{S} = \frac{\kappa}{e} \left\{ \frac{\pi^2}{3} \frac{\kappa T}{\zeta} - \frac{4}{9} g^2 (g-1)^2 j^2 (j+1)^2 \left(\frac{J(0)}{A(0)} \right)^3 \left(\frac{\beta H}{\kappa T} \right)^2 \right\}. \quad (94)$$

For example, taking the above data of 1.8% Mn-Cu alloy, we obtain

$$\mathfrak{S} = \frac{\kappa}{e} \frac{\pi^2}{3} \frac{\kappa T}{\zeta} \left\{ 1 - 13 \frac{(10^{-4} H)^2}{(10^{-1} T)^3} \right\}, \quad (95)$$

and we can see that when the external field of 10^4 gauss is applied the anomalous part dominates the normal part in the temperature range lower than 22°K .

Measurement of \mathfrak{S} in various Cu alloys has been carried out so far¹¹⁾ and the anomalously large values of \mathfrak{S} were observed in samples which exhibit resistance anomalies. It seems to me that one of the mechanisms of such anomalous thermoelectric power may be what is considered in this paper. However, because \mathfrak{S} depends sensitively on the ordered states of impurity atoms or the band structure of Cu and these are considered as fairly complicated in Cu alloys (the sign of \mathfrak{S} in pure Cu is inverse from that predicted by the free electron model), the detailed qualitative comparison between our theoretical and experimental values is not tried here. Measurement in a strong magnetic field or in alkaline metals is desired.

(vi) The mechanism of anomalous thermoelectric power may be recognized as follows. When temperature gradient ∇T exists, the electron distribution function $f(\mathbf{k})$ changes from the thermal equilibrium values as shown in Fig. 3a. On the other hand, the inelastic scattering causes a mixing as shown in Fig. 3a and thus the stationary distribution function becomes such as shown in Fig. 3b. If there are no other different circumstances between the systems of \pm spin electron, the situation of Fig. 3b exhibits no anomalous thermoelectric current. However, if there are some differences between \pm spin electrons, for example, the difference of the transition probability of elastic scattering in dilute alloys or the existence of the effective field in metallic ferromagnetics, the anomalous thermoelectric current flows as shown in Fig. 3c and thus \mathfrak{S}_{ano} appears.

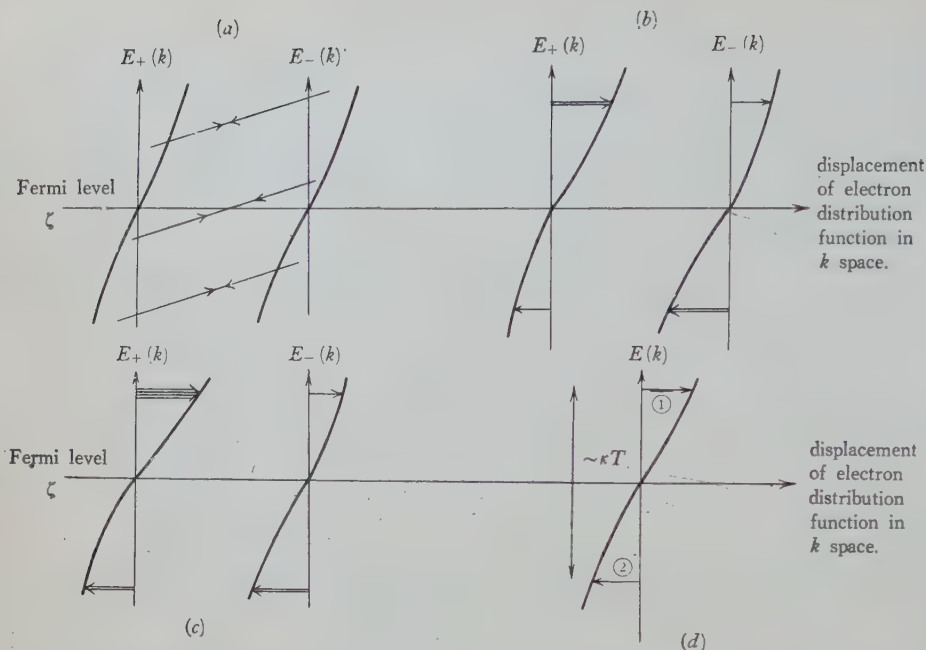


Fig. 3. Illustration of the mechanism of the anomalous thermoelectric power. (a), (b) and (c) show the mechanism of anomalous thermoelectric power. (d) shows the mechanism of normal thermoelectric power \mathcal{S}_{nor} . \mathcal{S}_{nor} is proportional to the difference of the displacements ① and ②, and thus usually of the order of $\kappa T/\zeta$.

Appendix I

The integrals to be solved are such as

$$I = \int_{-\infty}^{\infty} \frac{1 + e^{-\varepsilon - x}}{(e^{\varepsilon} + 1)(1 + e^{-\varepsilon})^2} \varepsilon^n d\varepsilon, \quad n=0, 1, 2. \quad (A1)$$

When n means an even number, we can write the integrand as follows,

$$\frac{\varepsilon^n}{(e^{\varepsilon} + 1)(1 + e^{-\varepsilon})} - (1 - e^{-x}) \frac{\varepsilon^n}{(e^{\varepsilon} + 1)^2(1 + e^{-\varepsilon})}. \quad (A2)$$

The first integral is easily performable. The second integral is also performable using the following relation,

$$\frac{1}{(e^{\varepsilon} + 1)^2(1 + e^{-\varepsilon})} + \frac{1}{(e^{\varepsilon} + 1)(1 + e^{-\varepsilon})^2} = -\frac{1}{(e^{\varepsilon} + 1)(1 + e^{-\varepsilon})}. \quad (A3)$$

Thus I_n is obtained as follows,

$$\begin{aligned} I_0 &= 1/2 \cdot (1 + e^{-x}), \\ I_2 &= \pi^2/6 \cdot (1 + e^{-x}), \text{ etc.} \end{aligned} \quad (A4)$$

When n means an odd number, we can calculate I_n as follows,

$$\begin{aligned}
 I_n &= -1/2 \cdot (1 - e^{-x}) \int_{-\infty}^{\infty} \varepsilon^n \frac{d}{d\varepsilon} \left\{ \frac{1}{(e^\varepsilon + 1)(1 + e^{-\varepsilon})} \right\} d\varepsilon \\
 &= \frac{n}{2} (1 - e^{-x}) \int_{-\infty}^{\infty} \frac{\varepsilon^{n-1}}{(e^\varepsilon + 1)(1 + e^{-\varepsilon})} d\varepsilon.
 \end{aligned} \tag{A5}$$

Thus

$$I_1 = \frac{1}{2} (1 - e^{-x}). \tag{A6}$$

Appendix II

The equations of the total balance with respect to k_x and $\varepsilon_\pm k_x$ are written as

$$d_\pm^\nu = \mathcal{D}_\pm^\nu, \quad \nu = 0, 1 \tag{A7}$$

where

$$d_\pm^0 = \frac{1}{6\pi^2 \hbar} \left\{ k_{\pm 0}^3 (eF - \nu \zeta) - \frac{\pi^2}{2} \frac{\kappa T}{\zeta} k_0^3 \kappa \nu T \right\}, \tag{A8}$$

$$d_\pm^1 = \frac{1}{6\pi^2 \hbar} \left\{ \frac{\pi^2}{2} \frac{\kappa T}{\zeta} k_0^3 (eF - \nu \zeta) - \frac{\pi^2}{3} k_{\pm 0}^3 \kappa \nu T \right\}, \tag{A9}$$

$$\begin{aligned}
 \mathcal{D}_\pm^\nu &= \frac{1}{12\pi^3} \frac{m^2 \kappa T}{\hbar^5} \frac{2j(g-1)^2}{N} \int_{-\infty}^{\infty} \varepsilon^\nu d\varepsilon \int_{k_0 H_c / \zeta}^{q_0} dq \frac{\pm q |J(q)|^2}{(e^{\varepsilon \pm x} + 1)(1 - e^{-\varepsilon})(1 - e^{\mp x})} \\
 &\times \left[\left(\frac{q^2}{2} - k_0^2 \mp k_0^2 \frac{x}{2} \frac{\kappa T}{\zeta} - k_0^2 \varepsilon \frac{\kappa T}{\zeta} \right) (C_{\mp}^0 + (\varepsilon \pm x) C_{\mp}^1) \right. \\
 &\quad \left. + \left(k_{\pm 0}^3 + k_0^2 \varepsilon \frac{\kappa T}{\zeta} \right) (C_{\pm}^0 + \varepsilon C_{\pm}^1) \right].
 \end{aligned} \tag{A10}$$

Because, as is seen from Eq. (61), the quantities we require are $(k_{\pm 0}^3 \phi_{\pm}^{\nu\mu})$, we transform $\{C_{\pm}^\nu\}$ into $\{\alpha_\nu, \beta_\nu\}$ as follows,

$$k_{+0}^3 C_{+}^\nu + k_{-0}^3 C_{-}^\nu \equiv (k_{\pm 0}^3 C_{\pm}^\nu) = \alpha_\nu, \tag{A11}$$

$$k_{+0}^3 C_{+}^\nu - k_{-0}^3 C_{-}^\nu = \beta_\nu. \tag{A12}$$

Then the matrix to determine $\{\alpha_\nu, \beta_\nu\}$ is written by using the approximation to adopt only the leading terms with respect to $\kappa T/\zeta$ and H_c/ζ as follows.

$\alpha_0,$	$\beta_0,$	$\alpha_1,$	β_1	$\frac{eF - \nu \zeta}{\kappa \nu T}$
$\frac{q^2}{2} x,$			$-\frac{q^2}{4} x^2,$	k_0^6
	$-2x\mathcal{A} - \frac{x^2}{2} \delta,$	$x^2\mathcal{A} + \frac{x^3}{4} \delta,$		$-\frac{\pi^2}{2} \frac{\kappa T}{\zeta} k_0^6$
	$-\left(2k_0 - \frac{q^2}{2}\right) x,$	$\left(2k_0 - \frac{q^2}{2}\right) \frac{x^2}{2},$		0

$$\begin{array}{ccc}
2x\Delta + \frac{x^2}{2}\delta, & -x^2\Delta - \frac{5x^3+8\pi^2x}{12}\delta, & 0 \\
\hline
-\left(2k_0 - \frac{q^2}{2}\right)\frac{x^2}{2}, & \frac{1}{6}\left\{3k_0^3x^3 + q^2(\pi^2x - \frac{x^3}{2})\right\}, & \frac{\pi^2}{2}\frac{\kappa T}{\zeta}k_0^6 \\
\hline
x^2\Delta + \frac{x^3}{4}\delta, & -\frac{2}{3}(x^3 + \pi^2x)\Delta - \frac{x^4 + 2\pi^2x}{4}\delta, & -\frac{\pi^2}{3}k_0^6 \\
\hline
\frac{q^2x^2}{4}, & -\frac{1}{6}\left\{k_0^2(x^3 + 4\pi^2x) - q^2\left(\pi^2x - \frac{x^3}{2}\right)\right\}, & 0 \\
\hline
-x^2\Delta - \frac{5x^3+8\pi^2x}{12}\delta, & \frac{2}{3}(x^3 + \pi^2x)\Delta + \frac{x^4 + 2\pi^2x^2}{4}\delta, & 0 \\
\hline
\end{array} \tag{A13}$$

where

$$\Delta = k_0^2 \frac{H_c}{\zeta}, \quad \delta = k_0^2 \frac{\kappa T}{\zeta}, \tag{A14}$$

and $f(q, x)$ means (in Eq. (66) it is written as $[f(q, x)]$)

$$\frac{1}{2\pi} \frac{m^2 \kappa T}{\hbar^4} \frac{j(g-1)^2}{N} \int_{k_0 H_c / \zeta}^{q_0} \frac{f(q, x) q |J^2(q)|^2}{(e^x - 1)(1 - e^{-x})} dq. \tag{A15}$$

From the above matrix, we can easily obtain the required quantities as follows.

$$(i) \quad (\phi_{\pm}^{00} k_{\pm 0}^3) = k_0^6 \frac{-k_0^2(4\pi^2x + x^3) + q^2(\pi^2x - x^3/2)}{D_a}, \tag{A16}$$

where

$$D_a = \begin{vmatrix} \frac{q^2x}{2}, & -\frac{q^2x^2}{4} \\ \frac{q^2x^2}{4}, & -\frac{k_0^2(4\pi^2x + x^3) - q^2(\pi^2x - x^3/2)}{6} \end{vmatrix}. \tag{A17}$$

Because the available wave vector q of the spin wave is much smaller than k_0 in low temperature, we pick up only the leading term with respect to q/k_0 . Then we obtain

$$(\phi_{\pm}^{00} k_{\pm 0}^3) = \frac{2k_0^6}{q^2x}. \tag{A18}$$

$$(ii) \quad (\phi_{\pm}^{11} k_{\pm 0}^3) = \frac{\pi^2}{3} k_0^6 \frac{-(2k_0^2 - q^2/2)x}{D_b}, \tag{A19}$$

where

$$D_b = \begin{vmatrix} -\left(2k_0^2 - \frac{q^2}{2}\right)x, & \left(2k_0^2 - \frac{q^2}{2}\right)\frac{x^2}{2} \\ -\left(2k_0^2 - \frac{q^2}{2}\right)\frac{x^2}{2}, & \frac{1}{2}k_0^2x^3 - \frac{1}{6}q^2(\pi^2x - x^3/2) \end{vmatrix}, \quad (\text{A20})$$

and taking only the leading terms respecting q/k_0 we obtain

$$(\phi_{\pm}^{11} k_{\pm 0}^3) = \frac{2\pi^2}{3} k_0^4 \frac{[x]}{[x][x^3] - [x^2]^2}. \quad (\text{A21})$$

$$\begin{aligned} (\text{iii}) \quad (\phi_{\pm}^{01} k_{\pm 0}^3) &= \frac{\pi^2}{2} - \frac{\kappa T}{\zeta} (\phi_{\pm}^{00} k_{\pm 0}^3) \\ &+ \frac{\pi^2}{3} \frac{k_0^6}{D_a D_b} \begin{vmatrix} -2x\Delta - \frac{x^2}{2}\delta, & x^2\Delta + \frac{x^3}{4}\delta, & -\frac{q^2x^2}{4} \\ -\left(2k_0^2 - \frac{q^2}{2}\right)x, & \left(2k_0^2 - \frac{q^2}{2}\right)\frac{x^2}{2}, & 0 \\ -x^2\Delta - \frac{5x^3 + 8\pi^2x}{12}\delta, & \frac{2}{3}(x^3 + \pi^2x)\Delta + \frac{x^4 + 2\pi^2x^2}{4}\delta, & \\ & & -\frac{k_0^2(4\pi^2x + x^3) + q^2(\pi^2x - x^3/2)}{6} \end{vmatrix}, \end{aligned} \quad (\text{A22})$$

and taking only the leading terms with respect to q/k_0 , we obtain the result of Eq. (65).

If one wishes to obtain the ordinary results of electron-phonon interaction, the following simplification should be made. (i) The behaviours of the electron systems with \pm spin directions are equivalent and thus in the matrix of Eq. (A13) β_0 and β_1 should be put zero and the second and fourth lines of the matrix be omitted. (ii) The effect of the scattering mechanisms becomes twice that in the above case because in the present case the two mechanisms corresponding to the emission and absorption of phonons are available. (iii) The transition probability given by Eq. (53) should be replaced by the ordinary value of electron-phonon interaction. Then the matrix to determine α_0 and α_1 becomes such as

$$\begin{array}{cccc} \alpha_0 & \alpha_1 & eF - \nabla \zeta & \kappa \nabla T \\ \hline q^2x, & k_0^2 \frac{x^3}{2} \frac{\kappa T}{\zeta}, & k_0^6 & -\frac{\pi^2}{2} \frac{\kappa T}{\zeta} k_0^6 \\ k_0^2 \frac{x^3}{2} \frac{\kappa T}{\zeta}, & \frac{1}{3} \left\{ 3k_0^2x^3 + q^2(\pi^2x - \frac{x^3}{2}) \right\}, & \frac{\pi^2}{2} \frac{\kappa T}{\zeta} k_0^6, & -\frac{\pi^2}{3} k_0^6 \end{array} \quad (\text{A23})$$

and this gives just the same result as that calculated by Sondheimer and Wilson using the second order variational method.

In the case of the antiferromagnetic ordering, the situation is just the same as that of the electron phonon interaction and the matrix of (A23) is also applicable. The explicit forms of $(\phi_{\pm}^{\nu\mu} k_{\pm 0}^3)$ are given as follows.

$$(\phi_{\pm}^{00} k_{\pm 0}^3) = \frac{k_0^6}{q^2 x}, \quad (\text{A24})$$

$$(\phi_{\pm}^{11} k_{\pm 0}^3) = \frac{\pi^2 k_0^6}{3k_0^2 x^3 + q^2 (\pi^2 x - x^3/2)}, \quad (\text{A25})$$

$$(\phi_{\pm}^{01} k_{\pm 0}^3) = \frac{\pi^2}{2} k_0^6 \frac{2k_0^2 x^3 + q^2 (\pi^2 x - x^3/2)}{[q^2 x][3k_0^2 x^3 + q^2 (\pi^2 x - x^3/2)]}. \quad (\text{A26})$$

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Electromagnetic Structure of the Nucleon. I[†]

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The electromagnetic structure of the nucleon is investigated in detail. As for its isovector part, the distribution functions of charge and anomalous magnetic moment are obtained with use of two different methods, the second order relativistic perturbation and the static approximation, and the S -wave and recoil effects and higher order corrections are examined by comparing these results. The meson structure correction in the fourth order perturbation is also calculated, and it is shown that its contribution is rather small. As for the isoscalar part, the 3π -state contribution is calculated by the lowest order perturbation (order g^6), and its rough numerical evaluation is carried out. The result obtained is favorable for explaining the recent experimental results (but not conclusive). Hyperon closed loop effects are also discussed.

§ 1. Introduction

Since Yukawa's original proposal, meson theory has been very successful in the P -wave scattering and the theory of nuclear force, and it has been established that meson theory is valid at least about the phenomena to which the outside region of the nucleon mainly contributes. However, we can hardly expect that this theory would be valid in any small region, because it has the well-known divergence difficulty characteristic of the present field theory. Therefore it will be an important problem to clarify the limitation of the applicability of meson theory.

As a crucial test for the validity of meson theory it will be useful to investigate the electromagnetic structure of the nucleon¹⁾, which has recently been experimentally clarified to a considerable extent by high-energy electron-proton scattering by Hofstadter *et al.*,²⁾ and low-energy neutron-atom scattering. The experimental results are as follows.*

[†] The preliminary results were reported in Prog. Theor. Phys. **21** (1959), 726.

* The notation adopted here is explained in the next section.

$$\begin{aligned}
 \text{(i)} \quad & \langle r^2 \rangle_1^S = \langle r^2 \rangle_1^V = [(0.80 \pm 0.04) \times 10^{-13} \text{ cm}]^2 \\
 & \simeq \langle r^2 \rangle_2^V, \\
 & \mu^S \langle r^2 \rangle_2^S \ll \mu^V \langle r^2 \rangle_2^V.
 \end{aligned} \tag{1.1}$$

- (ii) The charge distribution of the proton can be explained by the exponential model or Clementel-Villi model³⁾ (with $\eta=1.2$).

Here we must notice that in the analysis of these experiments some very special models, which could hardly be expected from the current meson theory, were assumed. Hence it should be understood that the successful models stated in (ii) inform to us only a crude approximate form in the outer region $r \gtrsim 0.4 \times 10^{-13} \text{ cm}$. Indeed if Clementel-Villi model with $\eta=1.2$ were completely valid, that would mean the breakdown of the concept of probability in the small distance (though such a possibility naturally can not be excluded *a priori*). Because if probability is well-defined, the bare probability and the clothed one of the proton can take value only between 0 and 1. The former corresponds to the δ -singularity at the origin (i.e. the centre-of-mass of the proton), while the latter corresponds to the distributed charge. Therefore the coefficient of the δ -function must be between 0 and e . (This intuitive argument will be verified rigorously under some weak postulates in § 2). Thus it is inconsistent with the concept of probability that this coefficient would take the negative value $(1-\eta)e$.

Our main purpose is to clarify the limitation of the applicability of meson theory, by investigating the electromagnetic structure of the nucleon which is, as is seen from (1.1), the phenomena in a little more inside region than $r \approx 1/\mu$. So, *our aim is not the fit with (1.1) itself but to check whether or not the theoretical values which are calculated as faithfully to meson theory as possible are consistent with (1.1)*. But in this analysis it is important to notice that we should explicitly distinguish between the contributions from the outside and the inside region. As for the very inside region, since we have no reliable approximation method in the present stage of our ability, we should reserve the analysis even if meson theory were completely valid. For convenience, we roughly divide the distribution of the nucleon into three regions: $r \gtrsim 1/\mu$, $1/\mu \gtrsim r \gtrsim 1/M$, and $1/M \gtrsim r \gtrsim 0$.

i) In the first region $r \gtrsim 1/\mu$, we may regard that meson theory is valid and also its perturbational approach is quantitatively reliable as was suggested by the success of the theory of nuclear force.⁴⁾ Thus we can obtain the distribution functions in this region by perturbation theory. We will then be able to diminish the arbitrariness of the models for $r \gtrsim 1/\mu$ in the experimental analysis.

ii) In the second region $1/\mu \gtrsim r \gtrsim 1/M$ the validity of meson theory is not well established, and so the analysis in this region is our main purpose. Since the r.m.s. radius receives the main contribution from this region thanks to a weight factor r^2 , we can check the validity by comparing theoretical values with (1.1). We take the standpoint that perturbation theory is semi-quantitatively reliable even in this region. Of course, it is necessary that this point is critically checked.

The most crucial point is whether or not meson theory can predict the large $\langle r^2 \rangle_1^S$ as (1.1), but no significant calculation has so far been carried out on this problem. A detailed calculation for $\langle r^2 \rangle_1^S$ is our most important result.

iii) In the third region $1/M \gtrsim r \geq 0$, we are almost completely ignorant. Therefore we do not expect to predict the magnitude of the anomalous magnetic moment* which may receive a large contribution from this region. In order not to reflect our ignorance on the results obtained from the second region, we always use the experimental values for $\mu^{S,V}$ as the normalizations of $\langle r^2 \rangle_2^{S,V}$. This means that we phenomenologically modify the distribution functions in this region. $\langle r^2 \rangle_1^{S,V}$ are likewise normalized by $1/2$.

Though many authors have so far calculated the theoretical values of the r.m.s. radii, it seems that they do not sufficiently recognize the limitation of the applicability of meson theory. In this paper we investigate the electromagnetic structure of the nucleon from the above-mentioned standpoint in detail. In § 2, we state the general formulae on the nucleon structure in order to explain the notation used, and prove a theorem on a limitation to the total amount of the distributed charge. In § 3, the results of the second order relativistic perturbation are presented in more detail than so far given. In § 4, the distribution functions are obtained by the static approximation, and we examine the recoil and S -wave effects by comparing them with the relativistic results. In § 5, we present the lowest order terms of perturbation (order g^6) of 3π -state which is probably the main contributor to $\langle r^2 \rangle_{1,2}^S$, and the lowest order of the isovector part corrected by meson structure. Since these calculations are very bulky, the technical details will be reported in the succeeding paper. In the final section, we discuss the validity of the approximations employed here, and some conclusions are presented.

§ 2. General remarks

As is well known, the electromagnetic vertex function of the nucleon can be written as

$$\Gamma_\mu(q) = F_1(q^2) \cdot e \bar{u}'(p') \gamma_\mu u(p) + F_2(q^2) \cdot (e/2M) \bar{u}'(p') \sigma_{\mu\nu} q_\nu u(p) \quad (2.1)$$

on account of Lorentz covariance and gauge invariance. Here $u(p)$ and $u'(p')$ respectively stand for the initial and final wave functions of the free nucleon, e the observed charge, M the nucleon mass, $q = p' - p$ the transferred momentum and $\sigma_{\mu\nu} = (i/2)(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$. The form factors $F_1(q^2)$ and $F_2(q^2)$ are usually divided into the isoscalar and isovector parts:

$$F_{1,2}(q^2) = F_{1,2}^S(q^2) + \tau_3 F_{1,2}^V(q^2). \quad (2.2)$$

Their values at the origin are

* Hereafter we briefly write it as a. m. m..

$$\begin{aligned}
F_1^S(0) &= F_1^V(0) = 1/2, \\
F_2^S(0) &= \mu^S \equiv (\mu_p + \mu_n)/2 = -0.06, \\
F_2^V(0) &= \mu^V \equiv (\mu_p - \mu_n)/2 = 1.85,
\end{aligned} \tag{2.3}$$

where μ_p and μ_n denote the magnitudes of a.m.m. (in units of nuclear magneton) of the proton and neutron, respectively.

In order to understand the nucleon structure more intuitively, we introduce the fourier transforms of the form factors:

$$\begin{aligned}
\rho_{1,2}^{S,V}(r) &= \frac{1}{(2\pi)^3} \int F_{1,2}^{S,V}(q^2) e^{iq \cdot r} d\mathbf{q}, \\
&\text{(with } q_0=0)
\end{aligned} \tag{2.4}$$

which are called distribution functions. Then r^2 -moments are defined by

$$\langle r^2 \rangle_{1,2}^{S,V} = \frac{4\pi \int_0^\infty r^4 \rho_{1,2}^{S,V}(r) dr}{4\pi \int_0^\infty r^2 \rho_{1,2}^{S,V}(r) dr} = -6 \frac{F_{1,2}^{S,V}(0)}{F_{1,2}^{S,V}(0)}. \tag{2.5}$$

It is proved in every order of perturbation expansion⁵⁾ (but not yet by the axiomatic formulation⁶⁾) that the spectral representations with expected thresholds are possible for the form factors $F_{1,2}^{S,V}(q^2)$. If we assume the boundedness conditions as is expected from perturbation theory, we have

$$\begin{aligned}
F_1^S(q^2) &= \frac{1}{2} - q^2 \int_{(3\mu)^2}^\infty \frac{\alpha_1^S(m^2)}{m^2(q^2 + m^2)} dm^2, \\
F_1^V(q^2) &= \frac{1}{2} + q^2 \int_{(2\mu)^2}^\infty \frac{\alpha_1^V(m^2)}{m^2(q^2 + m^2)} dm^2, \\
F_2^S(q^2) &= \int_{(3\mu)^2}^\infty \frac{\alpha_2^S(m^2)}{q^2 + m^2} dm^2, \\
F_2^V(q^2) &= \int_{(2\mu)^2}^\infty \frac{\alpha_2^V(m^2)}{q^2 + m^2} dm^2,
\end{aligned} \tag{2.6}$$

where μ is the pion mass, and the bar stands for $-i\epsilon$. The distribution functions $\rho_{1,2}^{S,V}(r)$ can be easily obtained by using the spectral functions $\alpha_{1,2}^{S,V}(m^2)$:

$$\begin{aligned}
4\pi r^2 \rho_1^{S,V}(r) &= \left(\frac{1}{2} - \int \frac{\alpha_1^{S,V}(m^2)}{m^2} dm^2\right) \delta(r) + r \int \alpha_1^{S,V}(m^2) e^{-mr} dm^2, \\
4\pi r^2 \rho_2^{S,V}(r) &= r \int \alpha_2^{S,V}(m^2) e^{-mr} dm^2.
\end{aligned} \tag{2.7}$$

From these expressions we see that the contributions from high massive states rapidly damp for larger r thanks to the exponential factor. We therefore may

neglect the contributions from such states, if we are interested only in the tails of the distribution functions. As was pointed out by Chew *et al.*,⁷⁾ the spectral functions are expressed in the order of the thresholds of mass spectra as follows:

$$\begin{aligned}\alpha^S &= \alpha_{(3\pi)} + \alpha_{(5\pi)} + \cdots + \alpha_{(K\bar{K})}^S + \cdots + \alpha_{(N\bar{N})}^S + \cdots, \\ \alpha^V &= \alpha_{(2\pi)} + \alpha_{(4\pi)} + \cdots + \alpha_{(K\bar{K})}^V + \cdots + \alpha_{(N\bar{N})}^V + \cdots.\end{aligned}\quad (2.8)$$

It was shown by Goldberger *et al.*,⁸⁾ that $\alpha_{(K\bar{K})}$ and $\alpha_{(N\bar{N})}$ are indeed not important as is expected. We may expect that the main contributor to $\langle r^2 \rangle_{1,2}^S$ is 3π -state ($\alpha_{(3\pi)}$) and that to $\langle r^2 \rangle_{1,2}^V$ is 2π -state ($\alpha_{(2\pi)}$). Hence we shall mainly investigate them in the following sections.

Now, as is seen from (2.7), the proton's charge distribution consists of δ -singularity at the origin with a charge

$$e \left[1 - \int \frac{\alpha_1^S(m^2) + \alpha_1^V(m^2)}{m^2} dm^2 \right] \quad (2.9)$$

and of continuous distribution whose total amount is

$$e \int \frac{\alpha_1^S(m^2) + \alpha_1^V(m^2)}{m^2} dm^2. \quad (2.10)$$

Since the δ -singularity at the origin corresponds to the bare proton, it is expected that (2.9) is equal to $Z_{2p}e$, where Z_{2p} is the probability of finding the bare proton. Indeed, such an equality holds as is proved in the following.

We, of course, adopt $ps(ps)$ meson theory which is renormalizable. If the theory is consistent, we may assume that all renormalized quantities are finite and the bare charge e_0 is also finite. The unrenormalized vertex function is

$$\begin{aligned}\Gamma_\mu^0(q) &= e_0 \bar{u} \gamma_\mu u \cdot \left[\frac{1 + \tau_3}{2} + \int - \frac{A_1^S + \tau_3 A_1^V}{m^2} dm^2 - q^2 \int \frac{A_1^S + \tau_3 A_1^V}{m^2(q^2 + m^2)} dm^2 \right] \\ &+ \frac{e_0}{2M} \bar{u} \sigma_{\mu\nu} q_\nu u \cdot \int \frac{A_2^S + \tau_3 A_2^V}{q^2 + m^2} dm^2 + \varepsilon_\mu(\Lambda),\end{aligned}\quad (2.11)$$

which is cut off by Λ in order to avoid possible divergences. The unrenormalized quantities $A_{1,2}^{S,V}$ are functions m^2 , Λ and e_0 . $\varepsilon_\mu(\Lambda)$ is a possible non-gauge-invariant term owing to the artificial cut-off. For simplicity, we define Z_1 by

$$Z_1 = Z_{1p} \frac{1 + \tau_3}{2} + Z_{1n} \frac{1 - \tau_3}{2}. \quad (2.12)$$

Carrying out the renormalization after Dyson's prescription, we have

$$\begin{aligned}\Gamma_\mu^0(q) &= Z_1^{-1}(\Lambda) \left\{ e_0 \bar{u} \gamma_\mu u \cdot \left[\frac{1 + \tau_3}{2} - q^2 \int \frac{\alpha_1^S + \tau_3 \alpha_1^V}{m^2(q^2 + m^2)} dm^2 \right] \right. \\ &\left. + \frac{e_0}{2M} \bar{u} \sigma_{\mu\nu} q_\nu u \cdot \int \frac{\alpha_2^S + \tau_3 \alpha_2^V}{q^2 + m^2} dm^2 + \varepsilon'_\mu(\Lambda) \right\}\end{aligned}\quad (2.13)$$

with

$$\varepsilon_\mu'(A) \rightarrow 0 \text{ for } A \rightarrow \infty.$$

Here $\alpha_{1,2}^{S,V}$ depend on m^2 , A and e . Since e_0 is finite, (2.13) shows that $Z_1(A)\Gamma_\mu^0$ remains finite for $A \rightarrow \infty$. Hence

$$\varepsilon_\mu(A) = o(Z_1^{-1}(A)).$$

Comparing (2.11) with (2.13), we obtain

$$\begin{aligned} Z_1^{-1}(A) \frac{1+\tau_3}{2} &= \frac{1+\tau_3}{2} + \int \frac{A_1^S + \tau_3 A_1^V}{m^2} dm^2 + o(Z_1^{-1}(A)), \\ Z_1^{-1}(A) (\alpha_1^S + \tau_3 \alpha_1^V) &= A_1^S + \tau_3 A_1^V + o(Z_1^{-1}(A)), \\ Z_1^{-1}(A) (\alpha_2^S + \tau_3 \alpha_2^V) &= A_2^S + \tau_3 A_2^V + o(Z_1^{-1}(A)). \end{aligned} \quad (2.14)$$

Inserting the second equation into the first, and taking the limit $A \rightarrow \infty$, we find

$$\frac{1+\tau_3}{2} = Z_1 \frac{1+\tau_3}{2} + \int \frac{\alpha_1^S + \tau_3 \alpha_1^V}{m^2} dm^2, \quad (2.15)$$

that is,

$$\begin{aligned} 1 &= Z_{1p} + \int \frac{\alpha_1^S + \alpha_1^V}{m^2} dm^2, \\ 0 &= \int \frac{\alpha_1^S - \alpha_1^V}{m^2} dm^2, \end{aligned} \quad (2.16)$$

which can be rewritten as

$$\int \frac{\alpha_1^S}{m^2} dm^2 = \int \frac{\alpha_1^V}{m^2} dm^2 = \frac{1-Z_{2p}}{2} \quad (2.17)$$

because of Ward's identity. Therefore (2.9) is equal to $Z_{2p}e$ as was expected. From Lehmann's proof⁹⁾ we know

$$0 \leq Z_{2p} < 1, \quad (2.18)$$

provided that negative-norm state is absent. Thus we may conclude from (2.17) that the total amount of the distributed charge of the isoscalar part is equal to that of the isovector part, and it must take a value between 0 and $e/2$. The latter requirement is usually not satisfied in perturbation calculation. So, it should be remedied by a cut-off.

§ 3. Second order relativistic perturbation

The Feynman graphs of the second order perturbation are shown in Fig. 1 (apart from strange particles). The graph (a) has a mass spectrum for $m^2 \geq (2\mu)^2$ and contributes to the isovector part alone, while the graph (b) has that for $m^2 \geq (2M)^2$ and contributes to both of the isoscalar and isovector parts. Of course,

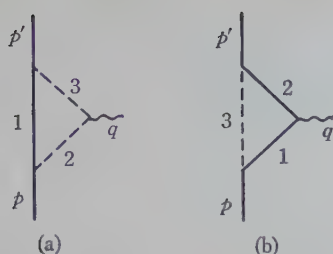


Fig. 1

the former is more important because its threshold is lowest. We calculate the distribution functions due to these graphs, following our standpoint stated in the introduction. Though we rely only on the distribution functions in the outer region from the physical point of view, we will present here also the expressions concerning the inner part for the sake of completeness.*

The spectral functions of the graph (a) are easily calculated by the usual Feynman method, and results are**

$$\alpha_1^V(m^2) = \frac{1}{\pi} \cdot \frac{g^2}{4\pi} \left[-M^2 \int x_1^2 \delta(-V(-m^2, x)) dx + \frac{1}{2} \int \theta(-V(-m^2, x)) dx \right],$$

$$\alpha_2^V(m^2) = \frac{1}{\pi} \cdot \frac{g^2}{4\pi} M^2 \int x_1^2 \delta(-V(-m^2, x)) dx,$$

where g stands for the meson-nucleon coupling constant in the rational unit, and

$$V(q^2, x) = x_1^2 M^2 + (x_2 + x_3) \mu^2 + x_2 x_3 q^2, \\ dx = \delta(1 - \sum_{i=1}^3 x_i) \prod_{i=1}^3 dx_i, \quad x_i \geq 0. \quad (3.2)$$

The integrations in (3.1) are analytically carried out***:

$$\alpha_1^V(m^2) = \frac{1}{4\pi} \frac{g^2}{4\pi} \theta(m^2 - 4\mu^2) \frac{1}{m(4M^2 - m^2)^2} \cdot \\ \cdot [(8M^2 + m^2)(m^2 - 2\mu^2) \sqrt{(m^2 - 4\mu^2)} - \{(8M^2 + m^2)(m^2 - 2\mu^2)^2 + \\ + m^2(4M^2 - m^2)(m^2 - 4\mu^2)\} \varphi(m^2)], \\ \alpha_2^V(m^2) = \frac{1}{\pi} \frac{g^2}{4\pi} \theta(m^2 - 4\mu^2) - \frac{M^2}{m(4M^2 - m^2)^2} \cdot \\ \cdot [-3(m^2 - 2\mu^2) \sqrt{(m^2 - 4\mu^2)} + \{3(m^2 - 2\mu^2)^2 + \\ + (4M^2 - m^2)(m^2 - 4\mu^2)\} \varphi(m^2)], \quad (3.3)$$

where

$$\varphi(m^2) = \frac{1}{\sqrt{(4M^2 - m^2)}} \tan^{-1} \sqrt{\frac{(4M^2 - m^2)(m^2 - 4\mu^2)}{m^2 - 2\mu^2}} \quad \text{for } m^2 < 4M^2 \\ = \frac{1}{\sqrt{(m^2 - 4M^2)}} \log \frac{m^2 - 2\mu^2 + \sqrt{(m^2 - 4M^2)(m^2 - 4\mu^2)}}{m^2 - 2\mu^2 - \sqrt{(m^2 - 4M^2)(m^2 - 4\mu^2)}} \quad \text{for } m^2 > 4M^2. \quad (3.4)$$

* As for old works on the second order perturbation, see Fried's paper.¹⁰⁾

** $\theta(t) = 1$ for $t > 0$
 $= 0$ for $t < 0$.

*** These expressions were given also by Chew *et al.*⁷⁾

$\alpha_{1,2}^V(m^2)$ are analytic also at $m^2=4M^2$:

$$\begin{aligned}\alpha_1^V(4M^2) &= \frac{1}{\pi} \frac{g^2}{4\pi} \left[\frac{(M^2 - \mu^2)^{3/2}}{3M(2M^2 - \mu^2)} - \frac{8M(M^2 - \mu^2)^{5/2}}{15(2M^2 - \mu^2)^3} \right], \\ \alpha_2^V(4M^2) &= \frac{1}{\pi} \frac{g^2}{4\pi} \frac{8M(M^2 - \mu^2)^{5/2}}{15(2M^2 - \mu^2)^3}\end{aligned}\quad (3.5)$$

The behaviors near the threshold are

$$\begin{aligned}\alpha_1^V(m^2) &= \frac{1}{3\pi} \frac{g^2}{4\pi} \varepsilon^{3/2}, \\ \alpha_2^V(m^2) &= \frac{8}{15\pi} \frac{g^2}{4\pi} \frac{M^2}{\mu^2} \varepsilon^{5/2}\end{aligned}\quad (3.6)$$

for $m^2=4\mu^2(1+\varepsilon)$ ($\varepsilon>0$: infinitesimal). For $m^2\rightarrow\infty$ we have

$$\begin{aligned}\alpha_1^V(m^2) &\rightarrow \frac{1}{4\pi} \frac{g^2}{4\pi}, \\ \alpha_2^V(m^2) &\sim \frac{1}{\pi} \frac{g^2}{4\pi} \frac{M^2}{m^2} (-3 + 2 \log m^2/M^2).\end{aligned}\quad (3.7)$$

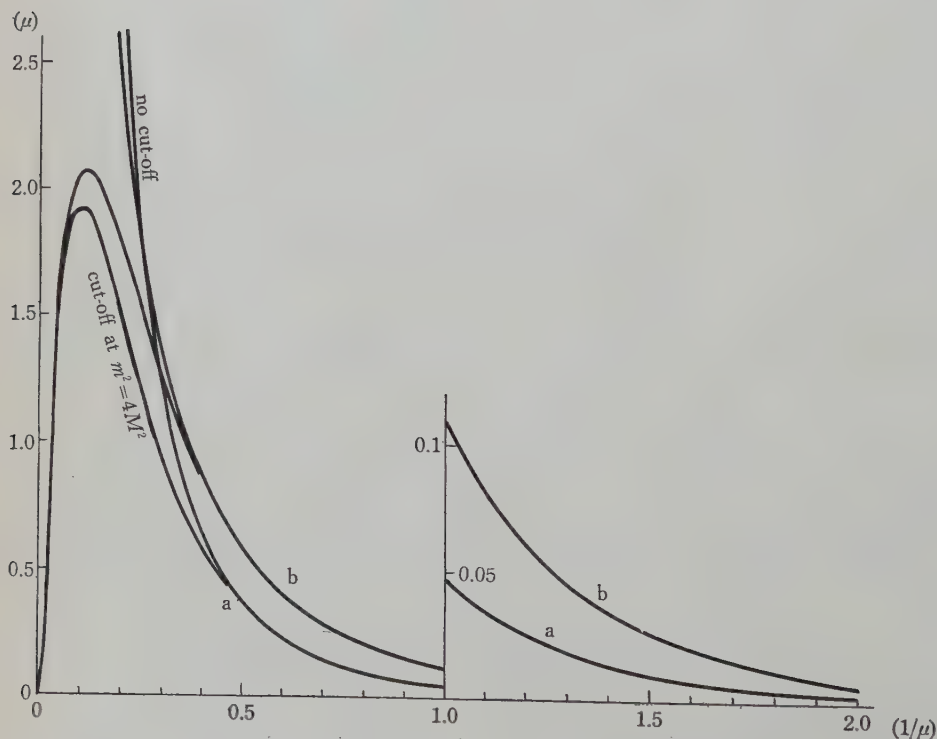


Fig. 2 Distribution functions of the 2π -state contribution by the lowest order relativistic perturbation.

(a) charge distribution $4\pi r^2 \rho_1 V(r)$

(b) a. m. m. distribution $4\pi r^2 \rho_2 V(r)$

From (2.7) and (3.3) we can calculate the distribution functions not analytically but numerically. The result obtained without cut-off and that with cut-off by $m^2=4M^2$ are shown in Fig. 2. It is seen that the distribution functions are quite insensitive to cut-off for $r \gtrsim 0.5/\mu$. The asymptotic forms for $r \rightarrow \infty$ are calculated from (3.6)

$$\begin{aligned} 4\pi r^2 \rho_1^V(r) &\sim \frac{1}{\sqrt{\pi}} \frac{g^2}{4\pi} \mu \frac{e^{-2\mu r}}{(\mu r)^{3/2}} \\ 4\pi r^2 \rho_2^V(r) &\sim \frac{4}{\sqrt{\pi}} \frac{g^2}{4\pi} \cdot \frac{M^2}{\mu} \cdot \frac{e^{-2\mu r}}{(\mu r)^{5/2}} \end{aligned} \quad (3.8)$$

The charge distribution near the origin is divergent as

$$4\pi r^2 \rho_1^V(r) \sim \frac{1}{2\pi} \frac{g^2}{4\pi} \frac{1}{r}. \quad (3.9)$$

The numerical values Q^V , $\langle r^2 \rangle_1^V$, μ^V and $\langle r^2 \rangle_2^V$ for no cut-off and for various (square) cut-offs are summarized in Table I, where Q^V denotes the total amount of the distributed charge. In the numerical evaluations we use the value $g^2/4\pi=14.6$ and 10^{-26}cm^2 unit for $\langle r^2 \rangle$ throughout this paper. The values in the parentheses are obtained by discarding contributions from $m^2 > 4M^2$. As is expected Q^V and μ^V are sensitive to the cut-off while $\langle r^2 \rangle_{1,2}^V$ are not.

Table I

	unit	no cut-off	cut-off					
			$r_c=1/2M$		$r_c=1/M$		$r_c=2/M$	
Q^V	e	∞ (0.64)	1.31	(0.55)	0.65	(0.41)	0.26	(0.21)
$\langle r^2 \rangle_1^V$	10^{-26}cm^2	0.46 (0.37)	0.44	(0.37)	0.41	(0.36)	0.33	(0.32)
μ^V	$e/2M$	1.61 (0.85)	1.05	(0.77)	0.73	(0.62)	0.38	(0.35)
$\langle r^2 \rangle_2^V$	10^{-26}cm^2	0.21 (0.20)	0.21	(0.20)	0.20	(0.20)	0.19	(0.18)

Now, the contributions from the nucleon current of the graph (b) are likewise calculated as follows.

$$\begin{aligned} \alpha_1^S(m^2) &= -3\alpha_1^V(m^2) \\ &= \frac{3}{8\pi} \frac{g^2}{4\pi} \left[\int \left\{ (x_1+x_2)^2 M^2 + x_1 x_2 m^2 \right\} \delta(-V(-m^2, x)) dx + \right. \\ &\quad \left. + \int \theta(-V(-m^2, x)) dx \right], \\ \alpha_2^S(m^2) &= -3\alpha_2^V(m^2) \\ &= -\frac{3}{4\pi} \frac{g^2}{4\pi} M^2 \int (x_1+x_2)^2 \delta(-V(-m^2, x)) dx, \end{aligned} \quad (3.10)$$

where

$$V(q^2, x) = (x_1 + x_2)^2 M^2 + x_3 \mu^2 + x_1 x_2 q^2, \quad (3.11)$$

namely

$$\begin{aligned} \alpha_1^S(m^2) &= -3\alpha_1^V(m^2) \\ &= \frac{3}{8\pi} \frac{g^2}{4\pi} \theta(m^2 - 4M^2) \frac{1}{m\sqrt{m^2 - 4M^2}} \left[\frac{m^2}{2} - \mu^2 - \frac{12\mu^2 M^2}{m^2 - 4M^2} \right. \\ &\quad \left. + \left\{ 4M^2 + \mu^2 + \frac{12\mu^2 M^2}{m^2 - 4M^2} \right\} \frac{\mu^2}{m^2 - 4M^2} \log \frac{m^2 - 4M^2 + \mu^2}{\mu^2} \right], \\ \alpha_2^S(m^2) &= -3\alpha_2^V(m^2) \quad (3.12) \\ &= -\frac{3}{4\pi} \frac{g^2}{4\pi} \theta(m^2 - 4M^2) \frac{M^2}{m\sqrt{m^2 - 4M^2}} \left[1 - \frac{6\mu^2}{m^2 - 4M^2} + \right. \\ &\quad \left. + \left(1 + \frac{3\mu^2}{m^2 - 4M^2} \right) \frac{2\mu^2}{m^2 - 4M^2} \log \frac{m^2 - 4M^2 + \mu^2}{\mu^2} \right], \end{aligned}$$

and for $m^2 \rightarrow \infty$

$$\begin{aligned} \alpha_1^S(m^2) &= -3\alpha_1^V(m^2) \rightarrow \frac{3}{16\pi} \frac{g^2}{4\pi} \\ \alpha_2^S(m^2) &= -3\alpha_2^V(m^2) \sim -\frac{3}{4\pi} \frac{g^2}{4\pi} \frac{M^2}{m^2}. \end{aligned} \quad (3.13)$$

From (3.7) and (3.13) we can confirm the first equality of (2.17) relating to the leading terms of the second order perturbation.

§ 4. Static approximation

The static approximation has been applied by a number of authors¹¹⁾ to calculate a.m.m. and $\langle r^2 \rangle_{1,2}^*$. But to our knowledge the distribution functions themselves have not fully been examined so far. As was stressed in the introduction, we are interested rather in the distributions in outer region than in these averaged quantities. In this section we shall examine the distributions of charge and a.m.m. due to the pion cloud using the static approximation.

Though the static approximation has succeeded in explaining quantitatively the low energy P -wave pion-nucleon reactions and will probably be useful also for the qualitative understanding of the nucleon structure, it is not clear to what extent this approximation is quantitatively reliable for this problem. Recoil effects, which have been proved to be rather small in the low-energy pion-nucleon reactions, may be important in the nucleon structure.¹²⁾ This point will be examined by comparing the result of the static approximation with that of the relativistic calculation.

* Throughout this section, as only the isovector part is concerned, the superscript V is omitted.

4-1 Distribution functions

By the static limit we mean to neglect the terms other than those of the lowest power with respect to $1/M$. Then the small components of the nucleon spinors vanish, and the vertex function $\Gamma_\mu(q^2)$ becomes as follows.

$$\begin{aligned}\lim \Gamma_0(q^2) &= e\chi^*\chi \cdot \tau_3 \lim F_1(q^2), \\ \lim \Gamma(q^2) &= (k/2M)\chi^*i[\boldsymbol{\sigma} \times \mathbf{q}]\chi \cdot \tau_3 \lim F_2(q^2),\end{aligned}\quad (4.1)$$

where "lim" means to take the static limit, and χ denotes the two-component Pauli spinor. Following Salzman,¹¹⁾ we identify the $\lim \Gamma_\mu(q^2)$ with the charge and current densities, $e\rho_{st}(r)$ and $(e/2M)\mathbf{j}_{st}(r)$, of the nucleon as

$$\begin{aligned}\lim F_1(q^2) &= \int \rho_{st}(r) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}, \\ i[\boldsymbol{\sigma} \times \mathbf{q}] \lim F_2(q^2) &= \int \mathbf{j}_{st}(r) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}.\end{aligned}\quad (4.2)$$

Relations with $\rho_{1,2}(r)$ defined in (2.4) are

$$\begin{aligned}\rho_{st}(r) &= \lim \rho_1(r), \\ \mathbf{j}_{st}(r) &= -[\boldsymbol{\sigma} \times \mathbf{r}](1/r)(\partial/\partial r) \lim \rho_2(r).\end{aligned}\quad (4.3)$$

The a.m.m. density $\mathbf{M}(r) = (e/2M)[\mathbf{r} \times \mathbf{j}_{st}(r)]/2$ is given by

$$\begin{aligned}\mathbf{M}(r) &= (e/2M)\mu_{st}(r)\boldsymbol{\sigma} \\ \text{with} \quad \mu_{st}(r) &= -(r/3)(\partial/\partial r) \lim \rho_2(r).\end{aligned}\quad (4.4)$$

Therefore the following relations hold.

$$\begin{aligned}4\pi \int_0^\infty r^4 \rho_{st}(r) dr &= 4\pi \int_0^\infty r^4 \lim \rho_1(r) dr = \lim \langle r^2_1 \rangle \\ 4\pi \int_0^\infty r^2 \mu_{st}(r) dr &= 4\pi \int_0^\infty r^2 \lim \rho_2(r) dr = \lim \mu \\ 4\pi \int_0^\infty r^4 \mu(r) dr &= (5/3) 4\pi \int_0^\infty r^4 \lim \rho_2(r) dr = (5/3) \lim \mu \langle r^2 \rangle_2\end{aligned}\quad (4.5)$$

The charge and current operators for the pion field are $-e(\pi_1\phi_2 - \pi_2\phi_1)$ and $-e(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1)$. The distribution functions, $\rho_{st}(r)$ and $\mu_{st}(r)$, of the P -wave pion cloud are given by the expectation values of these operators in the physical nucleon state. They are calculated by the familiar Chew-Low method. We quote here only the results¹¹⁾:

$$\begin{aligned}\rho_{st}(r) &= \rho_{st}^0(r) + \rho_{st}^1(r), \\ \mu_{st}(r) &= \mu_{st}^0(r) + \mu_{st}^1(r)\end{aligned}\quad (4.6)$$

with

$$\begin{aligned}\rho_{st}^0(r) &= \frac{3}{\pi^3} \left(\frac{f}{\mu} \right)^2 \int dk dk' \frac{F(kr) F(k'r)}{\omega_k \omega_{k'} (\omega_k + \omega_{k'})}, \\ \rho_{st}^1(r) &= \frac{2}{\pi^3} \int dk dk' \frac{F(kr) F(k'r)}{\omega_k + \omega_{k'}} I_c(\omega_k, \omega_{k'}),\end{aligned}\quad (4.7)$$

and

$$\begin{aligned}\mu_{st}^0(r) &= \frac{4M}{3\pi^3} \left(\frac{f}{\mu} \right)^2 \int dk dk' \frac{F(kr) F(k'r)}{\omega_k^2 \cdot \omega_{k'}^2}, \\ \mu_{st}^1(r) &= \frac{4M}{3\pi^3} \int dk dk' \frac{F(kr) F(k'r)}{\omega_k \omega_{k'} (\omega_k + \omega_{k'})} I_m(\omega_k, \omega_{k'}).\end{aligned}\quad (4.8)$$

The notations used here are as follows. The superscripts 0 and 1 respectively mean the lowest order term and the rescattering correction; $f^2 = (\mu/2M)^2 (g^2/4\pi) = 0.08$ is the renormalized coupling constant; $\omega_k = \sqrt{\mu^2 + k^2}$ and

$$F(kr) = k^3 j_1(kr) v_k \quad (4.9)$$

where $j_1(kr)$ is a spherical Bessel function, v_k being a cut-off factor; and

$$\begin{aligned}I_c(\omega_k, \omega_{k'}) &= -\frac{1}{36\pi^2} \int_{\mu}^{\infty} \frac{d\omega_p}{p} \cdot \frac{\sigma_{33}(p)}{(\omega_k + \omega_p)(\omega_{k'} + \omega_p)}, \\ I_m(\omega_k, \omega_{k'}) &= \frac{1}{36\pi^2} \int_{\mu}^{\infty} \frac{d\omega_p}{p} \cdot \frac{(\omega_k + \omega_{k'} + \omega_p) \sigma_{33}(p)}{(\omega_k + \omega_p)(\omega_{k'} + \omega_p)},\end{aligned}\quad (4.10)$$

where σ_{33} stands for the cross section of the pion-nucleon scattering in the $(3/2, 3/2)$ state. Cross sections in other states are neglected.

There is great arbitrariness as for the form of the cut-off factor. If we use the square cut-off, it gives rise to an oscillation of the distribution functions for large values of r and its behavior in the outer region depends appreciably on the cut-off momentum. Physically considering, such a situation seems unreasonable. Hence a smoothly decreasing function should be chosen. We adopt here the Gaussian type:

$$v_k = \exp(-k^2/2K^2), \quad (4.11)$$

K being referred to as the cut-off momentum.

Results obtained for $K=6\mu$ and ∞ (no cut-off) are shown in Figs. 3 and 4. We see that for $r \gtrsim 0.5/\mu$ the results are quite insensitive to the cut-off factor. The effects of rescattering, $\rho_{st}^1(r) (< 0)$ and $\mu_{st}^1(r) (> 0)$, are of relatively small magnitude everywhere, and the ratios of these effects to the lowest order ones gradually decrease with increasing r . For $K \rightarrow \infty^*$, we have

* The limit $K \rightarrow \infty$ should be taken after integration.

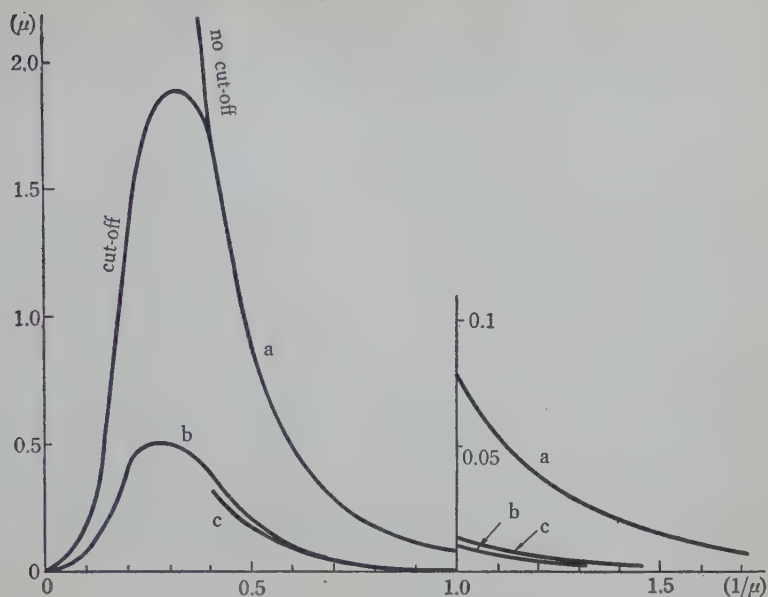


Fig. 3. Charge distribution functions of the 2π -state contribution by the static approximation.

- (a) lowest order (P -wave) $4\pi r^2 \rho_{st}^0(r)$
- (b) rescattering correction (P -wave) $-4\pi r^2 \rho_{st}^1(r)$
- (c) S -wave contribution $4\pi r^2 \rho_{S-wave}(r)$

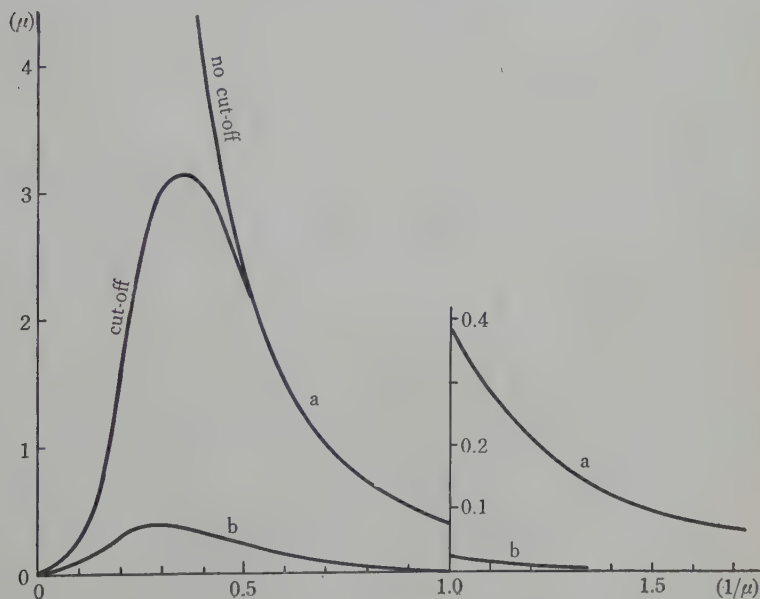


Fig. 4. A.m.m. distribution functions of the 2π -state contribution by the static approximation.

- (a) lowest order $4\pi r^2 \mu_{st}^0(r)$
- (b) rescattering correction $4\pi r^2 \mu_{st}^1(r)$

$$\begin{aligned}\rho_{sl, K \rightarrow \infty}^0(r) &= \frac{1}{2\pi^2} \frac{f^2}{\mu} [(5 + 2\mu^2 r^2) \cdot K_1(2\mu r) + 5\mu r K_0(2\mu r)] \frac{1}{r^4} \\ \mu_{sl, K \rightarrow \infty}^0(r) &= \frac{M}{3\pi} \left(\frac{f}{\mu}\right)^2 (1 + \mu r)^2 \frac{e^{-2\mu r}}{r^4},\end{aligned}\quad (4.12)$$

where $K_{0,1}(x)$ are modified Bessel functions, whose asymptotic forms are $K_0(x) \sim K_1(x) \sim (\pi/2x)^{1/2} e^{-x}$.

The calculated values of Q , $\langle r^2 \rangle_1$, μ and $\langle r^2 \rangle_2$ are shown in Table II. Q and μ are rather sensitive to the cut-off energy, while $\langle r^2 \rangle_{1,2}$ are not. The rescattering corrections for $\langle r^2 \rangle_{1,2}$ are small.

Table II

	unit	lowest order			including rescattering	
		no cut-off	$K=6\mu$	$K=5\mu$	$K=6\mu$	$K=5\mu$
Q	e	∞	0.80	0.50	0.58	0.39
$\langle r^2 \rangle_1$	10^{-26}cm^2	∞	0.68	0.62	0.58	0.52
μ	$e/2M$	∞	1.70	1.26	1.85	1.36
$\langle r^2 \rangle_2$	10^{-26}cm^2	0.58	0.42	0.40	0.44	0.42

In the above calculation only the P -wave pion contributions are taken into account. But the S -wave pion also contributes to the charge distribution. This effect is included in $\lim \rho_1(r)$ which follows from the relativistic calculation. Namely, the leading term of $\alpha_1(m^2)$ given by (3.3) in the $(1/M)$ -expansion gives

$$\lim \rho_1(r) = \rho_{sl(P\text{-wave})}(r) + (f^2/2\pi^2) K_2(2\mu r)/r^3$$

in the cut-off independent region. The additional term of the right-hand side is the contribution from the S -wave pion in the lowest order perturbation, which is plotted in Fig. 3. This is interpreted, in terms of time ordered graphs shown in Fig. 5, as follows. The graphs (P) are reduced to the P -wave contributions in the



Fig. 5.

static limit, while the graphs (S) to the S -wave ones. If the usual static approximation is used from the outset, the graphs corresponding to (S) do not appear. The lowest order perturbation for the S -wave effect will not be very misleading

because the ϕ^2 -term of the non-relativistic Hamiltonian of the S -wave pion-nucleon interaction plays no role in the present case. It is worthwhile to note that an inequality

$$\frac{\lim \rho_1(r) - \rho_{st(P)}^0(r)}{\rho_{st(P)}^0(r)} \lesssim \left(5 + \frac{2\mu^2 r^2}{1 + \mu r}\right)^{-1} \quad (4.14)$$

holds in the cut-off independent region $r \gtrsim 0.5/\mu$. Therefore the S -wave contribution is less than 20% of the P -wave effect in this region, and decreases with increasing r .

The S -wave effect on Q and $\langle r^2 \rangle_1$ was calculated by Ando and Miyazawa¹³⁾, who used the dispersion relations in the static approximation. According to their result, the S -wave contribution to Q is 0.48, and that to $\langle r^2 \rangle_1$ is $0.055 \times 10^{-26} \text{cm}^2$ (square cut-off with $K=6\mu$).

As for a.m.m., since an S -wave pion does not produce magnetic moment, such a difference as mentioned above does not appear as is easily confirmed by (4.4) and (3.3).

4-2 Ambiguity concerning gauge invariance

The interaction Hamiltonian density of the pion-nucleon system is proportional to the gradient of the pion field operator. When the electromagnetic field is included, the well-known prescription of replacing $\nabla \phi \rightarrow (\nabla - ie\mathbf{A})\phi$ and $\nabla \phi^* \rightarrow (\nabla + ie\mathbf{A})\phi^*$ guarantees the gauge invariance, provided that the source nucleon is a point. However, in the static approximation the source should be considered to have a finite extension, and hence the above prescription becomes insufficient. In this case, gauge invariance is guaranteed by introducing filaments of current in the region of the source, as has been discussed by Capps and Holladay.¹⁴⁾ Then we have the following additional term to the interaction Hamiltonian.

$$H' = -(4\pi)^{1/2} (ef^0/\mu) \int d\mathbf{r} \, u(r) \cdot \left[\left(\int_0^r \mathbf{A}_s ds \right) (\tau_1 \boldsymbol{\sigma} \nabla \phi_2 - \tau_2 \boldsymbol{\sigma} \nabla \phi_1) + (\tau_1 \phi_2 - \tau_2 \phi_1) \boldsymbol{\sigma} \mathbf{A} \right] \quad (4.15)$$

Here f^0 is the unrenormalized coupling constant, and $u(r)$ is the source function. $\int_0^r \mathbf{A}_s ds$ is a line integral of the vector potential \mathbf{A} from the origin (the center of the source) to the point \mathbf{r} in the source where pions are created or absorbed. The path of the integration may be chosen arbitrarily; if a straight line is taken as this path then this line current does not produce magnetic moment, while if a circulating current is assumed it will contribute to the magnetic moment.

The last term of (4.15), proportional to $(\tau_1 \phi_2 - \tau_2 \phi_1) \boldsymbol{\sigma} \mathbf{A}$, is familiar as the leading actor in the photo-production of S -wave pion near the threshold, and is necessary to secure the Kroll-Ruderman theorem in the static approximation. If

the source is a point, this term does not contribute to the magnetic moment. But when the source has a finite extension, it produces a considerably large magnetic moment, $\mu_{\text{int}}\tau_3$, as was noted by Hara and Kawarabayashi.¹⁵⁾ They obtained, using the square cut-off, $\mu_{\text{int}} = -1.27$ for $K=5\mu$ and $\mu_{\text{int}} = -1.54$ for $K=6\mu$.

It should be noted, however, that all the above mentioned ambiguities come out from the region of the source extension. Therefore the distribution of a.m.m. in the outer region is free from these ambiguities. For instance, the distribution $\mu_{\text{int}}(r)$ produced by the last term of (4.15) is

$$\mu_{\text{int}}(r) = -\frac{8M}{3\pi} u(r) r \int dk \frac{F(kr)}{\omega_k} \left[\frac{f^2}{\omega_k} + \frac{1}{24\pi^2} \int dk' \frac{\sigma_{33}(k')}{\omega_{k'}(\omega_k + \omega_{k'})} \right], \quad (4.16)$$

which evidently vanishes in the outer region where $u(r)=0$. $\langle r^2 \rangle_2$ also hardly suffers the trouble of gauge invariance. Indeed the contribution from $\mu_{\text{int}}(r)$ to $\langle r^2 \rangle_2$ is $-0.12 \times 10^{-26} \text{cm}^2$ for $K=5\mu$ and $-0.14 \times 10^{-26} \text{cm}^2$ for $K=6\mu$ (Gaussian cut-off).

4-3 Remarks on the recoil effect

Confining ourselves to the lowest order perturbation, we examine the recoil effect and the reliability of the static approximation. In Fig. 6 $4\pi r^2 \rho_1(r)$ and $4\pi r^2 \lim \rho_1(r)$ are compared. It is seen that the recoil effect is very important

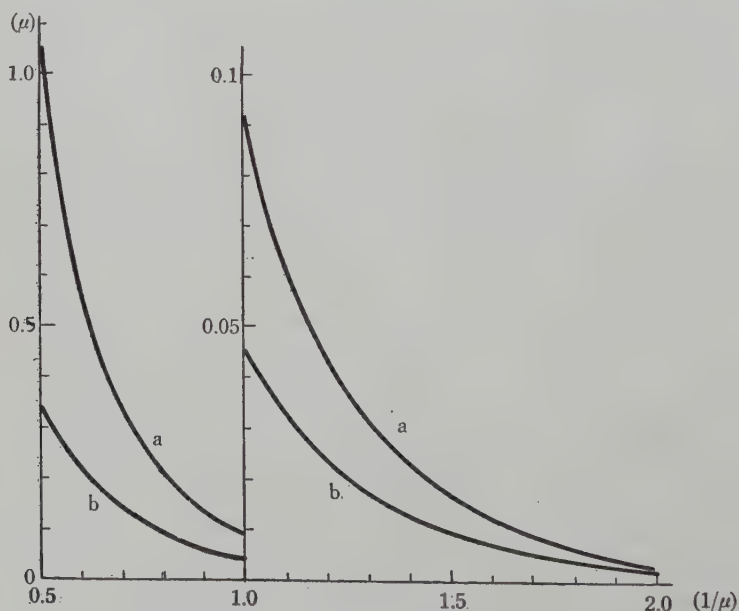


Fig. 6. Charge distribution functions by the lowest order perturbation.

(a) static limit $4\pi r^2 \lim \rho_1(r)$

(b) relativistic $4\pi r^2 \rho_1(r)$

and in fact surprisingly large. The discrepancy between $\rho_1(r)$ and $\lim \rho_1(r)$ or between $\alpha_1(m^2)$ and $\lim \alpha_1(m^2)$ is a factor two or more. Therefore the static approximation for the charge distribution should not quantitatively be relied upon, though it is useful for the qualitative argument. A similar situation is found for the a.m.m., too. We feel there is a contrast: in the low energy reactions such as the scattering, rescattering effects are essential but recoil effects are not, while in the problem of the nucleon structure, recoil effects are very important but rescattering effects are not.

A more unpleasant situation arises at m^2 near the threshold or at the asymptotic behavior for very large r . For $m^2 = 4\mu^2(1 + \varepsilon)$ as (3.6), we have

$$\begin{aligned}\lim \alpha_1(m^2) &= (f^2/\pi) \cdot \varepsilon^{1/2}, \\ \lim \alpha_2(m^2) &= (f^2 M/2\mu) \cdot \varepsilon.\end{aligned}\quad (4.17)$$

The asymptotic forms of $\lim \rho_{1,2}(r)$ are

$$\begin{aligned}4\pi r^3 \lim \rho_1(r) &\sim (2f^2 \mu/\sqrt{\pi}) \cdot e^{-2\mu r}/(\mu r)^{1/2} \\ 4\pi r^2 \lim \rho_2(r) &\sim 4f^2 M \cdot e^{-2\mu r}/(\mu r),\end{aligned}\quad (4.18)$$

which differ very much from (3.8). This discrepancy may seem somewhat puzzling, since for m^2 near the threshold the recoil effect would be considered to be quite unappreciable. The trouble comes out from the inadequate limiting process "lim". Namely, the "lim" of

$$\tan^{-1} \sqrt{\frac{(4M^2 - m^2)(m^2 - 4\mu^2)}{m^2 - 2\mu^2}} = \tan^{-1} \left[\frac{2}{\mu} \sqrt{(M^2 - \mu^2)\varepsilon} + \dots \right]$$

in (3.3) is put to be $\pi/2$, while (3.6) was obtained by expanding this arctangent like $(2/\mu)\sqrt{(M^2 - \mu^2)\varepsilon} + \dots$. Such a discrepancy cannot be remedied by taking account of $(1/M)$ -order correction for the static approximation. Indeed, for very large r , we have

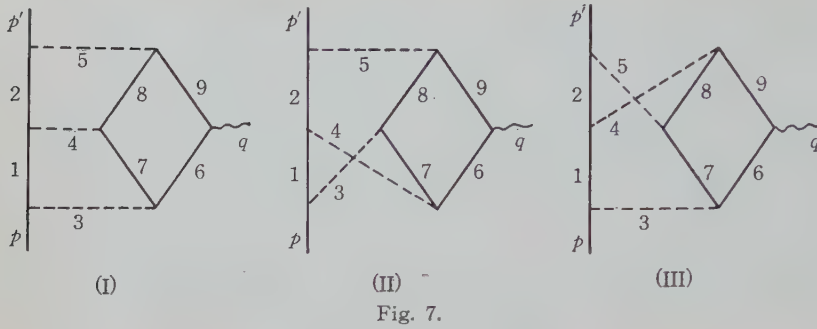
$$\begin{aligned}(1/M)\text{-order correction of } 4\pi r^2 \rho_1(r) &\sim -(f^2 \mu^2/M) \cdot e^{-2\mu r}, \\ \text{and that of } 4\pi r^2 \rho(r) &\sim -(4f^2 \mu/\sqrt{\pi}) \cdot e^{-2\mu r}/(\mu r).\end{aligned}\quad (4.19)$$

Therefore these corrections exceed the leading terms for very large r . Thus $(1/M)$ -expansion does not uniformly converge, and therefore it is inadequate not only for very small r but also for very large r .

§ 5. Isoscalar part and meson structure

5-1 3π -state contributions

We have so far been concerned with the 2π -state contributions, but in order to investigate the isoscalar part we must calculate the 3π -state which is probably the main contributor.^{7) 8)} We do not know any approximation method except for the relativistic perturbation theory, whose lowest order terms are already of order g^6 .



The Feynman graphs of the lowest order are shown in Fig. 7. These calculations are elegantly carried out without extreme complexity by means of the general integral formula presented by one of the authors (N.N.).¹⁶⁾ Detailed account of calculations will be stated in the succeeding paper. We write here only the final results :

$$\begin{aligned}
 F_1^S(q^2) &= \sum_{I \text{ II III}} \frac{3}{16\pi^3} \left(\frac{g^2}{4\pi} \right)^3 M^2 \cdot (-q^2) \cdot \int \frac{\xi(x) \sum_6^9 x_i}{U^3(x) \bar{V}^2(q^2, x)} dx, \\
 F_2^S(q^2) &= \sum_{I \text{ II III}} \frac{3}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 M^2 \left[-M^2 \int \frac{\xi(x) \sum_6^9 x_i}{U^3(x) \bar{V}^2(q^2, x)} dx + \right. \\
 &\quad \left. + \int \frac{\sum_6^9 x_i}{U^3(x) \bar{V}(q^2, x)} dx \right], \quad (5.1)
 \end{aligned}$$

where

$$\begin{aligned}
 V(q^2, x) &= \xi(x) M^2 + \sum_3^5 x_i \cdot \mu^2 + \sum_6^9 x_i \cdot M^2 + \zeta(x) q^2, \\
 dx &= \delta \left(1 - \sum_1^9 x_i \right) \prod_1^9 dx_i, \quad x_i \geq 0. \quad (5.2)
 \end{aligned}$$

The explicit expressions for $U(x)$, $\xi(x)$ and $\zeta(x)$ for the respective graphs are rather lengthy, and so they will be presented in the succeeding paper. We give here only the following exact inequalities, which are proved by the general theory.^{16) 17)}

$$\begin{aligned}
 0.12 &> U(x) \geq 0, \\
 x_1 + x_2 &\geq \xi(x) \geq 0, \\
 1/4 &\geq \zeta(x) \geq 0, \\
 V(q^2, x) &\geq 0 \text{ for } q^2 \geq -(3\mu)^2.
 \end{aligned} \quad (5.3)$$

We enumerate the main characteristics of the expressions (5.1).

- i) They contain no divergence. Hence no renormalization is necessary. This fact is very favorable to numerical estimations.
- ii) The contributions from graph II and graph III are identical. This is easily seen by noticing that they are interchanged by the exchange $p \leftrightarrow p'$, and that Γ_μ depends only on $p + p'$ and $(p - p')^2$ on account of gauge invariance.
- iii) The three graphs contribute with the same sign. This fact favors a large isoscalar part.
- iv) The total charge vanishes. This is obvious from (5.1), but we can prove it more directly. Consider the self-energy graphs which are obtained by eliminating the photon line from Fig. 7. Since the closed loop has three γ_5 's, these contribution identically vanish. Of course, this property is not changed by arbitrarily adding a constant momentum k to the propagator of the line 6. Differentiating with respect to k and putting $k=0$, we obtain just $\Gamma_\mu^S(0)$, and therefore $\Gamma_\mu^S(0)=0$. Thus we find $F_1^S(0)=0$. This remains valid also for the higher order graphs of the 3π -state contribution.
- v) $F_1^S(q^2)$ consists of a single term, while $F_2^S(q^2)$ is a difference of two positive terms. The contrast between this and the isovector part (3.1) is noteworthy.
- vi) We can define without ambiguity the approximation in which we regard the mass in the closed loop as very large.* This corresponds graphically to shrinking the closed loop to one point. We can see the distribution functions for large r from this approximation, in which the contributions from the three graphs completely coincide.

Now, in order to understand the results (5.1) more intuitively, we transform them into distribution functions by means of (2.7):

$$\begin{aligned}
 4\pi r^2 \rho_1^S(r) &= \sum_{\text{I II III}} \frac{3}{16\pi^3} \left(\frac{g^2}{4\pi} \right)^3 M^2 \int \frac{\xi(x) \sum_6^9 x_i}{U^3(x) V^2(0, x)} \kappa^4(x) \cdot \\
 &\quad \cdot \left(\frac{\kappa(x) r^2}{2} - r \right) e^{-\kappa(x)r} dx, \\
 4\pi r^2 \rho_2^S(r) &= \sum_{\text{I II III}} \frac{3}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 M^2 \int \left[\frac{\sum_6^9 x_i}{U^3(x) V(0, x)} \kappa^3(x) r \right. \\
 &\quad \left. - \frac{M^2 \xi(x) \cdot \sum_6^9 x_i}{U^3(x) V^2(0, x)} \cdot \frac{\kappa^3(x) r^2}{2} \right] e^{-\kappa(x)r} dx, \quad (5.4)
 \end{aligned}$$

where

* This limiting procedure is uniformly convergent in contrast with the static approximation discussed in § 4.3.

$$\kappa(x) = [V(0, x)/\zeta(x)]^{1/2} \geq 3\mu. \quad (5.5)$$

The form of the charge distribution is displayed *qualitatively* in Fig. 8, which should not be taken too seriously.* The amount of the positive charge is equal to that of negative one as was stated in the remark iv), and the sign change occurs near $r \approx 1/M$ (or a little more inward).

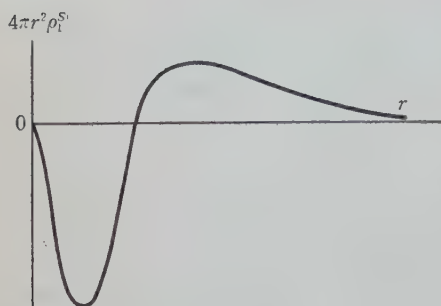


Fig. 8. Qualitative form of the charge distribution function of the 3π -state contribution by the lowest order relativistic perturbation.

We can in principle evaluate $Q^S, \langle r^2 \rangle_1^S, \mu^S$ and $\langle r^2 \rangle_2^S$ (without and with cut-off) from the distribution functions (5.4). But a straightforward numerical evaluation is prohibitively tedious because these are eightfold integrals. We therefore have evaluated these integrals by a statistical method. We first transformed the integration domain into the unit 8-dimensional cube.

And taking ten points 0.05, 0.15, ..., 0.95 for each variable, we regarded the set of the values of the integrand at these 10^8 lattice points as a population universe, and estimated its average value by sampling 100 points at random. Since the population universe is not in the normal distribution, the estimation of statistical error is rather difficult. Hence we estimated it by the deviations of the four average values over 25 samples. The results obtained are summarized in Table III.

Table III.

	unit	no cut-off	cut-off	
			$r_c=1/M$	$r_c=2/M$
Q^S	e	0	6.6 ± 4	1.1 ± 0.5
$\langle r^2 \rangle_1^S$	10^{-26}cm^2	1.8 ± 0.9	1.6 ± 0.6	0.65 ± 0.3
μ^S	$e/2M$	60 ± 30	12 ± 8	1.5 ± 1
$\mu^S \langle r^2 \rangle_2^S$	$e/2M \cdot 10^{-26} \text{cm}^2$	2.0 ± 1.2	1.3 ± 0.7	0.5 ± 0.4

These numerical values are surprisingly large. Especially μ^S for no cut-off is extremely huge. The value of $\langle r^2 \rangle_1^S$ for no cut-off is confirmed by another approximate integration method, which gives a lower bound to it. Unfortunately, since a large number of the samples give large values to $\kappa(x)$, the cut-off values, in which we are more interested, are less reliable. We are planning a more reliable evaluation.**

5-2 Isovector part with meson structure

In the last subsection we have obtained unexpectedly large values for the

* The form of the a.m.m. distribution drawn in our preliminary report is not correct at large distances.

** Unfortunately such a calculation takes a very long time.

isoscalar part. The counterpart to Fig. 7 in the isovector part will be the graph shown in Fig. 9 rather than the lowest order one, Fig. 1 (a). Hence we are interested in the contributions from Fig. 9, which we may call the lowest order graph corrected by meson structure.

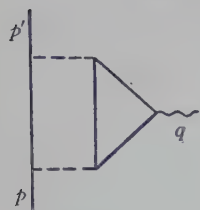


Fig. 9.

Though this graph is of order g^4 , unfortunately it contains an internal divergence, and so needs a renormalization of the internal part, which makes its order-of-magnitude estimation rather troublesome. Its calculation is carried out by the same technique as in the last subsection,¹⁶⁾ and the result obtained will be stated in the succeeding paper. We here notice only that this result exhibits a remarkable similarity to that of order g^2 more than expected.

The parameter integrals are fivefold, but we can easily carry out two integrations only as for $\langle r^2 \rangle_1^V$, μ^V and $\langle r^2 \rangle_2^V$ for no cut-off. We have evaluated $\langle r^2 \rangle_1^V$ and $\langle r^2 \rangle_2^V$, which are shown in the left column of Table IV, by Simpson's method.*

Table IV

	unit	no cut-off	approximation	
			$4\mu^2 \rightarrow M^2$	$4\mu^2 \rightarrow 4M^2$
$\langle r^2 \rangle_1^V$	10^{-26}cm^2	0.57	0.05	0.26
$\langle r^2 \rangle_2^V$	10^{-26}cm^2	0.13	0.03	0.09

The cut-off values, in which we are more interested, are not calculated, because the corresponding integrals are fivefold. Instead, we can calculate the spectral functions $\alpha_{1,2}^V(m^2)$ by expanding them in powers of the inverse square of the closed-loop mass. Retaining only the first non-vanishing terms**, we obtain

$$\alpha_{1,2}^V(m^2) = \frac{1}{\pi} \frac{g^2}{4\pi} \frac{m^2}{6M^2} \alpha_{1,2}^{V(\text{lowest})}(m^2), \quad (5.6)$$

where $\alpha_{1,2}^{V(\text{lowest})}$ are given by (3.3). This approximation is, of course, not reliable for $m^2 \gtrsim M^2$. The numerical values of $\langle r^2 \rangle_1^V$ and $\langle r^2 \rangle_2^V$ in this approximation are tabulated in the right two columns of Table IV (which should be compared with Table V in § 6-1).

Thus we may conclude that the contributions from Fig. 9 are all positive and rather small at least in the outer region.

5-3 Hyperon effects

In the last two subsections we have considered only the nucleons as the particles in the closed loop, but we must, in general, take into account hyperon

* In the numerical evaluation care must be taken because the integrand takes extremely large values near some end points. Since the results are differences of large terms, error may amount to 50%.

** The lowest order term of this expansion cancels with the renormalization term as is expected.

closed loops. The corresponding form factors are easily obtained by a slight modification, but there their relative signs are important. We investigate in this subsection the contributions from the graphs which contain hyperon closed loops.

First, we consider $N = \begin{pmatrix} p \\ n \end{pmatrix}$ and $\Xi = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}$. The interaction Hamiltonian consists of

$$H_s = ig_N \bar{N} \gamma_5 \tau_j N \cdot \pi_j + ig_\Xi \bar{\Xi} \gamma_5 \tau_j \Xi \cdot \pi_j \quad (j=1, 2, 3)$$

$$\text{and} \quad H_e = -ie \left(\bar{N} \gamma_\mu \frac{1+\tau_3}{2} N - \bar{\Xi} \gamma_\mu \frac{1-\tau_3}{2} \Xi \right) A_\mu, \quad (5.7)$$

where $g_N \equiv g$. Here N and Ξ respectively couple with the electromagnetic field A_μ through $(1+\tau_3)/2$ and $-(1-\tau_3)/2$. Thus the relative signs of the form factors for N -loop and Ξ -loop are the same for the isovector part and opposite for the isoscalar part* (apart from the relative signs of g_N and g_Ξ).

Next, we consider Λ and Σ . We assume that Λ and Σ have the same parity. Then the interaction Hamiltonian consists of

$$H_s = ig_\Sigma \bar{\Sigma} \gamma_5 \rho_j \Sigma \cdot \pi_j + ig_\Lambda (\Sigma_j \gamma_5 \Lambda + \bar{\Lambda} \gamma_5 \Sigma_j) \pi_j$$

$$\text{and} \quad H_e = -ie (\bar{\Sigma}^+ \gamma_\mu \Sigma^+ - \bar{\Sigma}^- \gamma_\mu \Sigma^-) A_\mu \\ = -e \bar{\Sigma} \gamma_\mu \rho_3 \Sigma \cdot A_\mu, \quad (5.8)$$

$$\text{where} \quad \Sigma = \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \end{pmatrix}, \quad \Sigma^\pm = (\Sigma_1 \mp i\Sigma_2)/\sqrt{2}, \quad \Sigma^0 = \Sigma_3 \quad \text{and}$$

$$\rho_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \rho_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{with} \quad \rho_i \rho_j - \rho_j \rho_i = i\rho_k \quad (i, j, k: \text{cyclic}).$$

For simplicity we neglect the Λ - Σ mass difference.

i) The case of global symmetry ($g_\Lambda = g_\Sigma$).¹⁸⁾

$$\text{Putting } Y = \begin{pmatrix} \Sigma^+ \\ Y^0 \end{pmatrix}, Z = \begin{pmatrix} Z^0 \\ \Sigma^- \end{pmatrix} \text{ where } Y^0 = (\Lambda^0 - \Sigma^0)/\sqrt{2} \text{ and } Z^0 = (\Lambda^0 + \Sigma^0)/\sqrt{2},$$

we can rewrite (5.8) as

$$H_s = ig_\Lambda (\bar{Y} \gamma_5 \tau_j Y \cdot \pi_j + \bar{Z} \gamma_5 \tau_j Z \cdot \pi_j) \\ H_e = -ie \left(\bar{Y} \gamma_\mu \frac{1+\tau_3}{2} Y - \bar{Z} \gamma_\mu \frac{1-\tau_3}{2} Z \right) A_\mu, \quad (5.9)$$

which is completely analogous to (5.7). Thus Λ and Σ contribute to the isovector part with the same sign as N , while they mutually cancel in the isoscalar part.

For simplicity, we hereafter consider only the lowest order graphs Figs. 7

* This N - Ξ cancellation in the case of global symmetry was first noticed by Miyazawa.¹⁹⁾

and 9.

ii) The case of no $A(g_A=0)$.

As for the isovector part, it suffices to replace $g_N^2 \cdot \text{Sp}(\tau_i \tau_j \tau_3)$ in the N -closed loop by

$$g_\Sigma^2 \text{Sp}(\rho_i \rho_j \rho_3) = g_\Sigma^2 \cdot i \varepsilon_{ijk} = g_\Sigma^2 \cdot \text{Sp}(\tau_i \tau_j \tau_3),$$

i.e. g_N^4 by $g_N^2 g_\Sigma^2$.

As for the isoscalar part, the contribution from Σ vanishes.

iii) The general case ($g_A/g_\Sigma=a$).

The form factors must be linear with respect to a^2 on account of isotopic-spin conservation. Determining the coefficients from the above results for $a=1$ and $a=0$, we find that

$$g_N^4 \rightarrow g_N^2(g_A^2 + g_\Sigma^2) \quad \text{for the isovector part,}$$

and

$$0 \quad \text{for the isoscalar part.}$$

Summarizing the results obtained, we have the following replacements for the inclusion of hyperon-loop graphs.*

$$g_N^4 L^V(M_N) \rightarrow g_N^2 \{g_N^2 L^V(M_N) + (g_A^2 + g_\Sigma^2) L^V(M_{\Lambda, \Sigma}) + g_\Xi^2 L^V(M_\Xi)\} \quad \text{for Fig. 9, (5.10)}$$

$$g_N^6 L^S(M_N) \rightarrow g_N^3 \{g_N^3 L^S(M_N) - g_\Xi^3 L^S(M_\Xi)\} \quad \text{for Fig. 7,}$$

where $L^{S,V}(M)$ is the contribution from the closed loop of the baryon having mass M (i.e. Spur part).

It is worthwhile to remark that the contributions from the baryon loop are all additive for the isovector part, but whether those from N and Ξ are additive or subtractive for the isoscalar part depends on the relative signs of g_N and g_Ξ .

We have numerically estimated these effects in the case of global symmetry. The contributions from the Ξ -loop graphs for the isoscalar part (Fig. 7) amount to

$$\begin{aligned} \langle r^2 \rangle_1^S: & -(60 \pm 15) \%, \\ \mu^S \text{ and } \langle r^2 \rangle_2^S: & -(80 \pm 20) \%, \end{aligned} \quad (\text{for no cut-off})$$

and those for very large r are about -35% . The contributions for the isovector part (Fig. 9) are inversely proportional to the squares of the baryon masses for large r .

§ 6. Discussions and conclusions

We have so far mainly carried out relativistic perturbation calculations. Our standpoint is that perturbation theory is semi-quantitatively reliable in the second region $1/\mu \gtrsim r \gtrsim 1/M$, but it must certainly be critically examined. We will discuss

* For Fig. 7, (5.10) holds as it is even when A and Σ have different parities.

higher order corrections, which we classify into the rescattering part and the closed loop part.

6-1 Rescattering correction

The rescattering effect can be calculated for the isovector part alone. The calculations with use of the static approximation have so far been performed by many authors¹¹⁾, and the rescattering correction was shown to be relatively small ($\lesssim 20\%$). We confirmed this result by a more detailed calculation, and showed that the distribution functions of the rescattering correction, as a whole, is quite smaller (and slightly more inward) than that of the lowest order.

The relativistic calculation of the rescattering correction was carried out by Chew *et al.*⁷⁾ and Goldberger *et al.*,⁸⁾ who made use of the dispersion theoretical approach. But the former neglected the higher order terms with respect to not only μ/M , but also m/M , hence obtained essentially the same result as the static approximation. On the other hand, Goldberger *et al.* carried out the spectral integration for $(2\mu)^2 \leq m^2 \leq (2M)^2$ without using the $(1/M)$ -expansion. As a result, they found an unexpectedly large rescattering effect. For large m^2 , however, the Legendre polynomial method in the analytical continuation, which they made use of, will converge poorly and further, as is seen from (2.8), the 4μ -state contribution may become rather significant. Hence we have recalculated their result, dividing the integration domain into the two parts, $4\mu^2 \rightarrow M^2$ and $M^2 \rightarrow 4M^2$.

Table V

	unit	lowest order			including rescattering		
		$4\mu^2 \rightarrow M^2$	$M^2 \rightarrow 4M^2$	$4\mu^2 \rightarrow 4M^2$	$4\mu^2 \rightarrow M^2$	$M^2 \rightarrow 4M^2$	$4\mu^2 \rightarrow 4M^2$
Q^V	e	0.13	0.51	0.64	0.06	-0.27	-0.21
$\langle r^2 \rangle_1^V$	10^{-26}cm^2	0.20	0.17	0.37	0.11	-0.05	0.06
μ^V	$e/2M$	0.31	0.54	0.85	0.40	1.39	1.80
$\langle r^2 \rangle_2^V$	10^{-26}cm^2	0.14	0.05	0.20	0.20	0.12	0.32

We see from Table V that the rescattering effect estimated by this approach becomes dominant for large m^2 . The charge spectral function $\alpha_1^V(m^2)$ changes its sign near $m^2=M^2$, and the total distributed charge Q^V becomes negative. This is quite unnatural. Therefore we can hardly believe that their result is reliable even for $m^2>M^2$. Then we may conclude that the rescattering correction is not so significant ($\lesssim 30\%$).

6-2 Closed loop correction

The meson's self-energy part is not important, because the renormalized meson propagator including higher order corrections has the mass spectrum which consists of $Z_3\delta(m^2-\mu^2)$ and a continuous spectrum $m^2 \geq (3\mu)^2$, the former being already included in the lowest order.

The closed loop correction for photon-pion vertex is usually referred to as meson structure. The corrections by meson form factor were investigated in detail by means of dispersion relations by Goldberger *et al*⁽⁸⁾. They showed that the part through 2π -state may contribute as

$$\langle r^2 \rangle_1^V \approx 0.16 \times 10^{-26} \text{cm}^2, \quad (6.1)$$

while the part through N -pair state is limited by

$$|\langle r^2 \rangle| \leq 0.09 \times 10^{-26} \text{cm}^2 \quad (6.2)$$

from the unitarity requirement.

We have calculated the lowest order correction by meson structure, Fig. 9 in § 5-2, and found that this contribution is rather small. Thus the perturbational result is not inconsistent with (6.2) except for the contribution from inner region. But this meson-structure correction is considerably enhanced by hyperon effects as stated in § 5-3.

As for the isoscalar part, the lowest order graphs for 3π -state, Fig. 7, already contain the closed loop part. Though the estimation of higher order corrections is technically impossible in the present stage, we conjecture that they will not drastically change the outer forms of the distribution functions, from the fact that the three topologically different graphs in Fig. 7 contribute nearly equally (by the numerical estimation) and coincide asymptotically.

6-3 Takeda-Kato's criticism

Takeda and Kato gave the following comments to our preliminary results on the 3π -state contributions.

The amount of the positive charge in Fig. 8 may be very large as is seen from Table III. The localization of such a large amount (in a spherical shell) will violate the conservation of probability. The graphs calculated by us, Fig. 7, can contain at most two pairs in the closed loop part at an arbitrary time t . Therefore denoting by ψ the wave function at t , we can write

$$\psi = c_0 \psi_0 + c_1 \psi_1 + c_2 \psi_2 \quad (6.3)$$

with

$$|c_0|^2 + |c_1|^2 + |c_2|^2 = 1,$$

where $\psi_j (j=0, 1, 2)$ is the j -pair state. Hence the maximum localizable charge is

$$e(|c_1|^2 + 2|c_2|^2) \leq 2e. \quad (6.4)$$

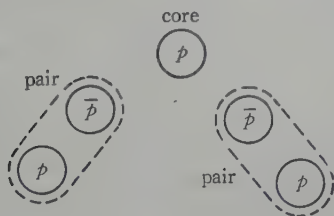


Fig. 10.

The most favorable case is shown in Fig. 10. Thus if an amount larger than $2e$ localizes, our perturbational calculation will be inadequate.

It should be noticed that the above criticism naturally depends essentially on the order of approximation, and so the upper bound (6.4) is by no

means a strict one. What can be asserted is that the result is rather doubtful if it exceeds $2e$. Therefore probably we should not believe semi-quantitatively the perturbational result for the region $r \lesssim (1/M \sim 2/M)$, though we cannot yet definitely say so because our numerical evaluations are very crude and the experimental error of the coupling constant is rather large ($\sim 40\%$ for g^0).

6-4 Conclusions

[A] *Isovector part.* We may conclude as follows.

- i) perturbation theory is not misleading in the S -wave effect ($\lesssim 20\%$), contrary to the scattering problems.
- ii) The recoil effect is unexpectedly large, and it changes even the asymptotic forms. The static approximation predicts larger values for $\langle r^2 \rangle_{1,2}^V$ by about factor two.
- iii) Higher order corrections contribute mainly to the inner region.

Therefore they are not important for $\langle r^2 \rangle_{1,2}^V$. Thus the relativistic perturbation theory is fairly good except for in the inner region.* That is further confirmed by the fact that the fourth order correction by meson structure is indeed not inconsistent with the dispersion theoretical result. The numerical value of $\langle r^2 \rangle_1^V$ is reduced by the rescattering correction, but is recovered to a considerable extent by meson structure correction. The resultant will not be too small to explain the experimental data (1.1). As for $\langle r^2 \rangle_2^V$, the theoretical value is rather small, though it is enhanced by both the rescattering and meson structure corrections. But the disagreement with (1.1) is not conclusive, because the experimental analysis of $\langle r^2 \rangle_2^V$ is quite model-dependent.

[B] *Isoscalar part.* Our results for the 3π -state contribution are qualitatively favorable to explain the large $\langle r^2 \rangle_1^S$ (and probably the small $\mu^2 \langle r^2 \rangle_2^S$) as in (1.1). But the numerical results are much too large, and hardly reliable in the region $r \lesssim 1/M$ from Takeda-Kato's criticism. Unfortunately, our numerical evaluation is a very crude one, hence no quantitative conclusion can be deduced.

At any rate, if the charge distribution function of the isoscalar part becomes negative in the inner region $r \lesssim 1/M$, as was indicated in Fig. 8(a), also for the exact theoretical value, then the situation will be very favorable to the experimental result $\langle r^2 \rangle_1^S = \langle r^2 \rangle_1^V$. This is because we know the following facts:

- i) We may safely expect (perturbationally and intuitively) that the charge distribution function of the isovector part is positive definite.
- ii) The total distributed charge of the isoscalar part is equal to that of the iso-

* Contrary to our standpoint, there is the opinion that the static theory would be better than the relativistic theory even in the problem of nucleon structure. But we can hardly agree with it, because such a non-Lorentz invariant and non-gauge invariant theory can not be expected to be valid in the rather inner region $1/M \lesssim r \lesssim 1/\mu$, and indeed the static theory (without an unnaturally large core¹¹⁾) predicts $\langle r^2 \rangle_1^S = 0$ which is completely inconsistent with the experimental evidence (1.1).

vector part (cf. (2.17)).

iii) The threshold of the isoscalar part ($m=3\mu$) is higher than that of the isovector part ($m=2\mu$).

Therefore the charge distribution function of the isoscalar part will be small for large r and large by a considerable amount for middle r in comparison with that of the isovector part. Then the neutron's charge distribution will take the form suggested by Schiff.²⁰⁾

We may thus conclude that meson theory need not be modified drastically in the second region $1/\mu \gtrsim r \gtrsim 1/M$, and probably the introduction of a new field (such as Nambu's neutral vector meson²¹⁾) is unnecessary.

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Tensor Force of the Pion-Theoretical Potential and the Doublet Splitting in n -He⁴ Scattering

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In order to investigate whether the spin-orbit force in the theory of nuclear shell structure is due to the tensor force of the pion-theoretical potential, the doublet splitting of the p -phase shifts in low energy n -He⁴ scattering is analysed. We get the conclusion that the major part of the experimental doublet splitting can be reproduced by the strong tensor force of the pion-theoretical potential. Also it is shown qualitatively what features of the pion-theoretical potential are important to the binding energy of He⁴ and the discontinuity of the binding energies between He⁴ and the system of He⁴ plus one nucleon.

§ 1. Introduction

Recent developments in researches on nuclear forces, made it clear that all the nucleon-nucleon phenomena up to about 150 Mev are well explained by the pion theory of nuclear forces¹⁾. The pion-theoretical potential has been established quantitatively in the outer region ($r \gtrsim 2 \times 10^{-13}$ cm) and at least qualitatively in the intermediate region ($r \simeq 1 \sim 2 \times 10^{-13}$ cm), where r is the inter-nucleon distance. The inner part ($r \lesssim 1 \times 10^{-13}$ cm) of nucleon-nucleon interaction, to which the present day pion theory can not give any reliable prediction, has been determined by the comparison with experimental data. Thus we know of the phenomenological effective potential corresponding to main features of nucleon-nucleon interaction in this inner region. At present, our knowledge about nuclear forces is sufficient to attack problems of nuclei on the basis of two-body interaction.

The characteristic features of the pion theoretical potential differ essentially from those of phenomenological potentials conventionally adopted so far as will be shown in § 2. The most remarkable one of them is the strong tensor force due to one-pion-exchange process. It is thus very interesting to investigate the relations between the strong tensor force and characteristic properties of nuclei. However, there has yet been no attempt taking into account this feature of two-body interaction. One of important problems in connection with the strong tensor force is whether the spin-orbit coupling force in the shell model can be accounted for by this strong tensor force*.

* As will be discussed in § 2, the two-body spin-orbit potential predicted by the pion theory is too weak to produce the spin-orbit potential in the shell model.

At the present stage, there exist many difficulties in general treatment of this strong tensor force, because one has to take account of the following situations: The mixing of states plays an important role in this case, hence the perturbational approach becomes questionable. Furthermore, significant contributions from the tensor force may appear at the nuclear surface, and it is desirable to treat the nucleus as a finite system not as an infinite medium.

In the case of lightest nuclei, the above mentioned difficulties do not appear and we can treat the problems directly by adopting the pion-theoretical potential.

The investigation of the doublet splitting of He^5 and Li^5 is the crucial test of the problem whether the strong tensor force is the origin of the spin-orbit coupling force in the shell model. The effects of the spin-orbit force in He^5 and Li^5 appear in the most direct fashion as the wide splitting of the doublet p -phase shifts in the low energy nucleon scattering by He^4 . The main purpose of this work is to investigate qualitatively the relation between this wide doublet splitting and the strong tensor force of the pion-theoretical potential. In the course of this investigation we also consider the binding energy of He^4 and the reason why there is no bound state in the system of He^4 plus one nucleon, while He^4 is a tightly bound system. This feature is closely related to the binding energy discontinuity at the closed shell.

Many authors investigated the effect of a tensor force on the doublet splitting. Dancoff²⁾ estimated the doublet splitting of He^5 in the second order perturbation. Feingold³⁾ also calculated it in the variation-perturbational way. In these works, besides the defect of the perturbational approach, there exists the unsatisfactory point that the values of the splitting and even its sign depend seriously on the parameters of the wave function, for they treat He^5 as the bound system. In another type of approach⁴⁾, the doublet splitting is calculated on the basis of the Fermi gas model in the second order Born approximation with respect to a tensor potential or a modified tensor potential (t -matrix in Brueckner's theory). In these works, however, there are also unsatisfactory points in treating the nuclear surface effects or the mixing of states due to a tensor force.

Here, we follow the procedure developed by Sugie, Hodgson and Robertson⁵⁾. This approach seems to be most reasonable for investigating the spin-orbit coupling force resulting from the tensor force in He^5 and Li^5 . However, they got only the small splitting of the p -phase shifts (about 30% of the experimental value), because they used the phenomenological potential with a weaker tensor force than a central one. As discussed by Sugie et al., the main part of the interaction term responsible for the splitting of the phase shifts is proportional to the strength of a tensor force and the mixing ratio of the D -state of He^4 due to a tensor force. Therefore, the strong tensor force characteristic of the pion-theoretical potential is expected to account for the wide experimental splitting. Indeed, it is shown that the major part ($\sim 60\%$) of the experimental value of the splitting can be reproduced by the pion-theoretical potential.

It is to be noted that, although such a strong tensor force to reproduce the experimental splitting is believed to reduce the binding energy of He^4 utterly, the pion-theoretical potential gives its reasonable value mainly due to the strong attractive force of the two-pion-exchange potential in the singlet even state.

The essential difference between the work of Sugie et al. and ours lies in the properties of the two-body potential. In § 2, we shall show the characteristic features of the pion-theoretical potential. In § 3, we shall recapitulate the procedure deriving the spin-orbit coupling term and discuss the approximations used. The determination of the parameters of the He^4 wave function and the calculation of the binding energy of He^4 will be made in § 4. The qualitative features of the spin-orbit coupling term and the numerical results derived from the pion-theoretical potential will be presented in § 5. § 6 will be devoted to discuss various corrections affecting numerical results. In § 7, we shall summarize the main results obtained.

§ 2. Characteristic features of nucleon-nucleon interaction

In this section we summarize the characteristic features of nucleon-nucleon interaction clarified by the analyses on two-nucleon problems¹⁾. It should be noted that the features of the outer and intermediate parts ($r \gtrsim 1 \times 10^{-13}$ cm) have been established pion-theoretically, while those of the inner part ($r \lesssim 1 \times 10^{-13}$ cm) have been determined by the comparison with experimental data. These features are shown in Fig. 1.

(i) In the outer region ($r \gtrsim 2 \times 10^{-13}$ cm), the tensor potential is very strong compared with the central one. This feature, the most characteristic one of the pion-theoretical potential, results from the one-pion-exchange potential.

$$V^{(1\pi)} = \left(\frac{g_e^2}{4\pi} \right) \mu c^2 \frac{(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)}{3} \left\{ (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) + S_{12} \left(1 + \frac{3}{\kappa r} + \frac{3}{(\kappa r)^2} \right) \right\} \frac{e^{-\kappa r}}{\kappa r}, \quad (2.1)$$

where

$$S_{12} = 3r^{-2} (\boldsymbol{\sigma}_1 \cdot \mathbf{r}) (\boldsymbol{\sigma}_2 \cdot \mathbf{r}) - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2),$$

and $\kappa^{-1} = \hbar/\mu c = 1.415 \times 10^{-13}$ cm (μ is the pion mass)*

$$g_e^2/4\pi \simeq 0.08.$$

The potential in this region is completely described by $V^{(1\pi)}$.

(ii) In the intermediate region ($r \simeq 1 \sim 2 \times 10^{-13}$ cm), the contributions from the two-pion-exchange potential $V^{(2\pi)}$ become important in addition to $V^{(1\pi)}$. The tensor part of $V^{(1\pi)}$ is important also in this region. The qualitative features of $V^{(2\pi)}$ have been verified, although there remain some quantitative ambiguities due to different choices of methods in the derivation of $V^{(2\pi)}$. The most essential feature of $V^{(2\pi)}$ is noticed in the strong attractive potentials in the central part of

* We adopt the value $\mu c^2 = 139.4$ Mev.

the charge triplet states (1E and 3O)*. The central potentials in 3E and 1O and the tensor potential of $V^{(2\pi)}$ are not very effective and their effects can be expressed by the suppression of magnitude of $V^{(1\pi)}$ in this region. The comparison with experimental data also supports this feature¹⁾.

(iii) The two-body spin-orbit coupling potential predicted by the pion theory is very small compared with the static potential in the intermediate region⁶⁾⁷⁾⁸⁾. This potential is not strong enough to produce the spin-orbit coupling force in the shell model. It is of the wrong sign in the recent calculation using the dispersion relation⁸⁾. Also, the nucleon-nucleon scatterings up to about 150 Mev can be reproduced by the pion-theoretical potentials without spin-orbit potentials as predicted by Signell and Marshak⁹⁾ and by Gammel and Thaler¹⁰⁾. Therefore, the two-body spin-orbit potential cannot play any essential role, and we can neglect its effect in qualitative discussions on problems of nuclei.

(iv) The main features of the exchange character of the pion-theoretical potential can be represented, from the properties (i) and (ii), as follows:

$$V = (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) V_c^{(1\pi)} + (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) S_{12} V_t^{(1\pi)} - \frac{3 + (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)}{4} \left\{ \frac{3 + (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)}{4} V_c^{(2\pi)}({}^3O) + \frac{1 - (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)}{4} V_c^{(2\pi)}({}^1E) \right\}. \quad (2.2)$$

$V_c^{(1\pi)}$ are the radial parts of $V^{(1\pi)}$. $V_c^{(2\pi)}({}^3O)$ and $V_c^{(2\pi)}({}^1E)$ are those of the two-pion-exchange-central potentials in 3O and 1E respectively. The minus sign in the third term is added to make $V_c^{(2\pi)}({}^3O)$ and $V_c^{(2\pi)}({}^1E)$ positive for convenience's sake.

(v) In the inmost part ($r \lesssim 0.5 \times 10^{-13}$ cm), there exists the hard-core-like repulsive interaction in all the states. Through comparison with experiments, it has been shown that the effective potential just outside this hard-core can be roughly given by the straight cut-off potential of $V^{(1\pi)} + V^{(2\pi)}$ at the region, $r \cong 0.5 \sim 1 \times 10^{-13}$ cm¹⁾.

In choosing the detailed forms of $V_c^{(1\pi)}$, etc., in the intermediate and inner regions, the following should be taken into account. In 3E , $V_t^{(2\pi)}$ is very small and $V^{(1\pi)}$ with the hard-core cut-off is the most reasonable potential for reproducing the deuteron data¹⁰⁾¹¹⁾. As will be discussed in § 3, $V_t({}^3E)$ plays an essential role in the phenomena of He^4 and the system of He^4 plus one nucleon, while $V_t({}^3O)$ does not. So, in the case of¹⁾ the present paper, it is allowable to take $V_t = V_t^{(1\pi)}$ in the region, $r \geq 1 \times 10^{-13}$ cm*. $V_c^{(2\pi)}({}^1E)$ is stronger than $V_c^{(2\pi)}({}^3O)$, i. e. $V_c^{(2\pi)}({}^1E)/V_c^{(2\pi)}({}^3O) \cong 3/2^{14)1f)}$.

Thus on the basis of the features shown in (i)~(v) and the situations discussed above, we adopt the potential given by (2.2) as the two-body potential outside the core region. For convenience of analyses, we take the following ap-

* We use the following simple notation for the classification of the two-nucleon states; 3E (the triplet even state), 1E (the singlet even state), 3O (the triplet odd state) and 1O (the singlet odd state).

** Strictly speaking, $V_t^{(1\pi)}({}^3O)$ should be modified due to $V_t^{(2\pi)}({}^3O)$ so as to be damped in the region, $r \lesssim 1.4 \times 10^{-13}$ cm^{1f)}.

proximate form* to the pion-theoretical potential.

$$V_c^{(1\pi)} = v_c^{(1)} r^2 \exp[-\mu^{(1)} r^2],$$

$$V_t^{(1\pi)} = v_t^{(L)} r^2 \exp[-\nu^{(L)} r^2] + v_t^{(S)} r^2 \exp[-\nu^{(S)} r^2], \quad (2.3)$$

$$V_c^{(2\pi)}(^1E) = v_c^{(2)} r^2 \exp[-\mu^{(2)} r^2] \quad \text{and} \quad V_c^{(2\pi)}(^3O) = (2/3) V_c^{(2\pi)}(^1E)$$

$$v_c^{(1)} = 6.86, \quad v_c^{(2)} = 700, \quad v_t^{(L)} = 7.45 \quad \text{and} \quad v_t^{(S)} = 456 \quad (\text{in unit of } \text{Mev} \times 10^{26} \text{ cm}^{-2}),$$

$$\mu^{(1)} = 0.600, \quad \mu^{(2)} = 1.94, \quad \nu^{(L)} = 0.388 \quad \text{and} \quad \nu^{(S)} = 1.76 \quad (\text{in unit of } 10^{26} \text{ cm}^{-2}).$$

In order to obtain a good approximate form to $V_t^{(1\pi)}$, we use the sum of potentials with different ranges. The long range part corresponds to $(1/3)(g_e^2/4\pi)\mu c^2(e^{-\kappa r}/\kappa r)$ and the short range part to the remainings. The errors caused by these approximations are very small (several per cent) except at the tail ($r \gtrsim 3 \times 10^{-13}$ cm), where the potentials themselves are vanishingly small, as shown in Fig. 1.**

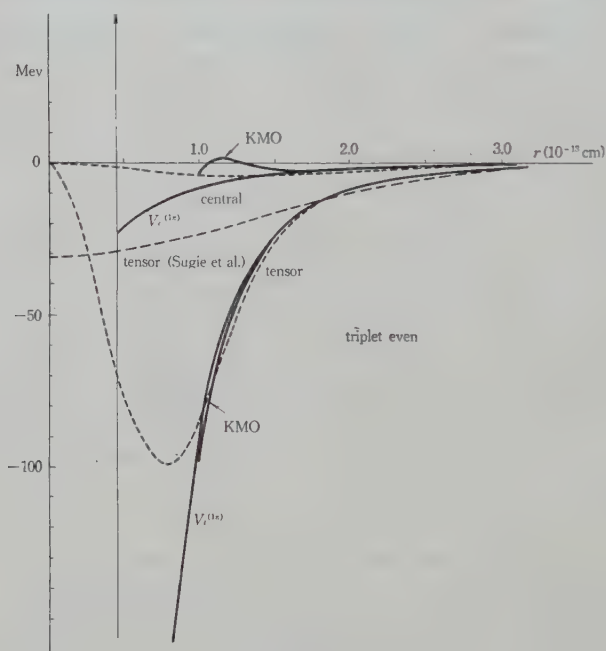


Fig. 1 (a)

* The reason why we use such a particular form comes from the following situation: If we choose this form, we can analytically perform the calculation of the binding energy of He^4 and the derivation of the interaction kernels in $n\text{-He}^4$ scattering. Otherwise cumbersome numerical calculations are needed after eliminating exactly the motion of center of mass. Including the r^2 -factor, we can avoid the procedure of Eq. (26) in Sugie et al. The r^2 -factor plays the role as damping factor in the inner region, which corresponds to the situation discussed in (v). Also, we can avoid overestimation of the contributions of the potential in the core region ($r \lesssim 0.5 \times 10^{-13}$ cm) to the binding energy of He^4 and the scattering potential in $n\text{-He}^4$, even in the case where no short-range correlation function is introduced.

** Because the results depend on the overlap integral of the forces and the wave function, the errors in the results that arise from the small incorrectness of the tail may be negligible.

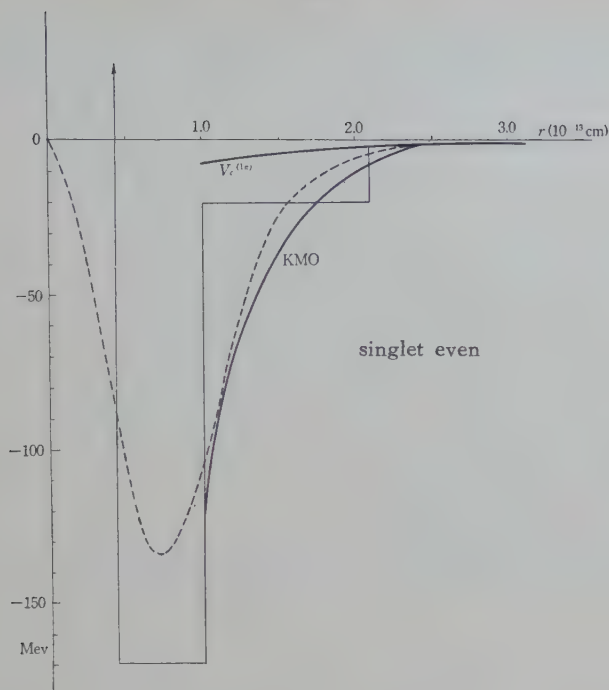


Fig. 1 (b)

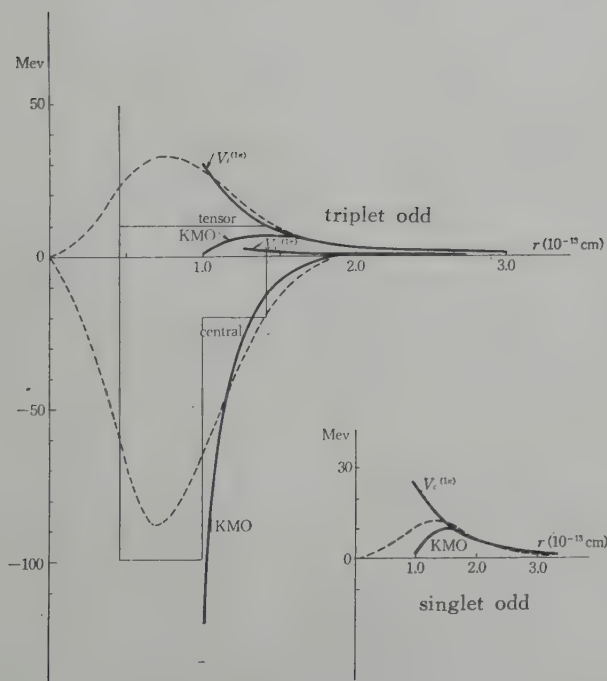


Fig. 1 (c)

Fig. 1 (d)

Fig. 1. Two-body potential. $V_c(1\pi)$ and $V_t(1\pi)$ are the central and tensor parts of the one-pion-exchange potential, respectively. The curves denoted by KMO are the one- plus two-pion-exchange potentials of ref. 1b). For other pion-theoretical potentials, see ref. 1a). Also, the effective potentials in the inner region determined by comparison with the experimental data are shown. The dotted curves are the approximate forms given by (2.3) to the pion-theoretical potential.

1(a) Potential in the triplet even state.

1(b) Potential in the singlet even state.*

1(c) Potential in the triplet odd state.*

1(d) Potential in the singlet odd state.

* The square well potentials with the tails of the one-pion-exchange potential are the effective potentials which explain the experimental data.

In the present paper, we perform the calculation using the exchange operators instead of $(\sigma_1 \cdot \sigma_2)$ and $(\tau_1 \cdot \tau_2)$. If we use the conventional notation for the exchange operators,

$$V = \sum_{i=1,2} (w^{(i)} + m^{(i)} P_M + b^{(i)} P_B + h^{(i)} P_H) V_c^{(i\pi)}(r) + S_{12} (w^{(t)} + m^{(t)} P_M) V_t^{(1\pi)}(r) \quad (2.4)$$

and put

$$V_c^{(2\pi)}(r) = V_c^{(2\pi)}(^1E),$$

the exchange character given by (2.2) is rewritten as follows:

$$\begin{aligned} V_c^{(1\pi)}: w^{(1)} &= 1/3, \quad m^{(1)} = -4/3, \quad b^{(1)} = -2/3 \quad \text{and} \quad h^{(1)} = 2/3, \\ V_c^{(2\pi)}: w^{(2)} &= -5/12, \quad m^{(2)} = -1/12, \quad b^{(2)} = 1/12 \quad \text{and} \quad h^{(2)} = 5/12, \\ V_t^{(1\pi)}: w^{(t)} &= -1/3 \quad \text{and} \quad m^{(t)} = -2/3 \end{aligned} \quad (2.5)$$

§ 3. Spin-orbit term

In this section, we outline the method to derive the spin-orbit term*. We discuss what is important to cause the wide splitting of the p -phase shifts in n -He⁴ scattering.

First of all, we assume that He⁴ remains in the ground state during the scattering process. This will be justified in n -He⁴ scattering at low energy ($\lesssim 5$ Mev), because the first excited state of He⁴ is believed to be very high**. Hereafter, particles 1, 2 and 3 are neutrons, and 4 and 5 are protons. The totally antisymmetric wave function Ψ of the system is then written in the form

$$\Psi = \phi(-1)\phi(1) + \phi(-2)\phi(2) + \phi(-3)\phi(3), \quad (3.1)$$

where $\phi(-i)$ is the antisymmetric wave function of He⁴ which does not contain the i -th neutron and $\phi(i)$ describes the i -th neutron in the scattering state. The ground state of He⁴ is considered to be principally the ¹S₀-state. Due to the tensor force it has a small admixture of ⁵D₀-states. As the first approximation we neglect the ³P₀-state probability. And, of all the possible spin-angular wave functions, only the principal ¹S₀- and ⁵D₀-state wave functions are considered¹²⁾. We take this approximation, because our main aim is to see the qualitative feature which the pion-theoretical potential shows in the lightest nuclei. Then the wave function of He⁴ is of the form

* The method of derivation is the same as the one given by Sugie et al.⁵⁾, on the whole. Hence our notations follow what they used in most of the cases. Sometimes, we may omit the description of meanings of notations when they seem obvious. The reader who is not familiar with the notations is advised to see the paper by Sugie et al.

** This assumption is justified by the following experimental data. The behavior of the $p_{3/2}$ phase shift is well accounted for by the one level formula. The proton reduced width is above 75% of the sum rule limit. (R. K. Adair, Phys. Rev. **86** (1952), 155; D. C. Dodder and J. L. Gammel, Phys. Rev. **88** (1952), 520.) The $p_{1/2}$ level is much broader. This shows that for $E < 7$ Mev, the present assumption is not so bad.

$$\phi(-1) = \{g_s \chi(\widetilde{23}, \widetilde{45}) + C g_D w_D \chi(\widetilde{23}, \widetilde{45})\} / \sqrt{1+C^2}, \quad (3.2)$$

where g_s and g_D represent the normalized spatial parts of the wave functions for the principal 1S_0 - and 5D_0 -state, respectively. χ is $\frac{1}{2} \{\alpha(2)\beta(3) - \beta(2)\alpha(3)\} \{\alpha(4)\beta(5) - \beta(4)\alpha(5)\}$ and, if the operand is χ ,

$$w_D = \sum_{i>j=2}^5 r_{ij}^2 S_{ij} \\ = 3(\sigma_2 \cdot r_{23})(\sigma_4 \cdot r_{45}) + 3(\sigma_2 \cdot r_{45})(\sigma_4 \cdot r_{23}) - 2(\sigma_2 \cdot \sigma_4)(r_{23} \cdot r_{45}). \quad (3.3)$$

The wave function ϕ is obtained from the Schrödinger equation of the n -He⁴ system

$$(\sum_i T_i + \sum_{i>j} V_{ij})\Psi = (E_\alpha + E)\Psi. \quad (3.4)$$

Decomposing the wave function ϕ into partial waves

$$\phi(1) = \sum_J (f_{lJ}(r)/r) \chi_{lJ}^m(\theta\varphi, s), \quad (3.5)$$

we get a set of uncoupled differential equations, each of which is specified by two good quantum numbers, J and l , since we neglect the virtual excitation of He⁴. The equation* for $f_{lJ}(r)$ is

$$\frac{\hbar^2}{2M'} \left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 \right] f_{lJ}(r) \\ = \int d\tau_{-1} d\Omega_1 \bar{\psi}(-1) \bar{\chi}_{lJ}^m(1) \sum_{l'=2}^5 V_{lJ} \phi(-1) \chi_{l'J}^m(1) f_{l'J}(r) \\ + 2 \int d\tau_{-1} d\Omega_1 \bar{\psi}(-1) \bar{\chi}_{lJ}^m(1) \sum_{j=2}^5 V_{lJ} \phi(-2) \chi_{l'J}(2) \frac{f_{l'J}(r')}{r'} r \\ + 2 \frac{\hbar^2}{2M'} \int d\tau_{-1} d\Omega_1 \bar{\psi}(-1) \bar{\chi}_{lJ}^m(1) [-\nabla_{1,2345}^2 - k^2] \phi(-2) \chi_{l'J}^m(2) \frac{f_{l'J}(r')}{r'} r, \quad (3.6)$$

where

$$k^2 = (2M'/\hbar^2)E, \quad M' = (4/5)M, \\ \mathbf{r} = \mathbf{r}_1 - (1/4)(\mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_5) \\ \mathbf{r}' = \mathbf{r}_2 - (1/4)(\mathbf{r}_1 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_5). \quad (3.7)$$

and

The explicit form of $\phi(-1)$ is given in the next section. We calculate the S - S terms and the S - D cross terms, neglecting the effect of the D - D terms.

* In deriving the equation, we assume $H_\alpha \psi(-1) = E_\alpha \psi(-1) \dots (a)$. Sugie et al. assumed only $(\psi H_\alpha \psi) = E_\alpha \dots (b)$. By doing so, they included the contribution from S_{24} - and S_{34} -terms in their Eq. (34). However, the splitting should be given only by S_{14} - and S_{15} -terms if we could find the exact wave function of He⁴, and so these terms are of physical significance. Then we assume the relation (a), though in actual calculations we use the approximate form for $\psi(-1)$.

Equation (3.6) can be expressed in the following abbreviated form*

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} \right\} f_{lj}(r) = W(r) f_{lj}(r) + \int k^l(r, r') f_{lj}(r') dr' + \int k_{sp}^{lj}(r, r') f_{lj}(r') dr'. \quad (3.8)$$

The first term in the right-hand side is the so-called potential term. This term is derived from the central potentials without the space exchange operator in the direct terms and those with the space exchange operators in the antisymmetrized terms. The kernel of the second term arises from (1) the remaining parts of the central potentials in the direct and antisymmetrized terms, (2) the J -independent parts from the tensor force in the antisymmetrized terms and (3) the term related to the kinetic energy in the antisymmetrized terms. The J -dependence of the second term in the right-hand side of Eq. (3.8) is caused only by that of the wave function $f_{lj}(r)$. Hence this term does not give direct contribution to the spin-orbit splitting. The kernel $k_{sp}^{lj}(r, r')$ itself in the last term depends on J , hence it contributes directly to the splitting. In the following, we discuss this term in detail.

First of all we must mention that the direct tensor terms do not give any contribution. Then the antisymmetrization is essential in the spin-orbit splitting. After integrating the exchange tensor term over all coordinates except \mathbf{r} and \mathbf{r}' , there remain the scalar quantities of the following types:

$$(I) \quad (\text{scalar function of } \mathbf{r} \text{ and } \mathbf{r}')$$

and

$$(II) \quad (\boldsymbol{\sigma} \cdot (\mathbf{r} \times \mathbf{r}')) \cdot (\text{scalar function of } \mathbf{r} \text{ and } \mathbf{r}')^{**}.$$

It is evident that the spin-orbit splitting results only from the terms of the type (II). Expanding the scalar functions of \mathbf{r} and \mathbf{r}' in terms of the Legendre polynomials $P_l(\cos(\widehat{\mathbf{r}, \mathbf{r}'}))$, we obtain the linear combination of the following expressions*** by integrating the terms belonging to the type (II) over $d\Omega$ and $d\Omega'$;

$$\begin{aligned} \int \bar{\chi}(-1) \bar{\chi}_{lj}^m(1) i(\boldsymbol{\sigma}_3 \cdot (\mathbf{r} \times \mathbf{r}')) \cdot \sum_M Y_L^M(\Omega) \bar{Y}_L^M(\Omega') \chi_{lj}^m(2) \chi(-2) d\Omega d\Omega' \\ = -\frac{1}{2} r r' C_{ljL}, \end{aligned} \quad (3.9)$$

where

$$C_{l, l+1/2, l \mp 1} = \pm \frac{l}{2l+1}, \quad C_{l, l-1/2, l \mp 1} = \mp \frac{l+1}{2l+1}, \quad \text{for } l \neq 0.$$

* The explicit expressions of each term are given in §5 and the Appendix.

** This type does not result from the exchange S - D term but from the exchange D - S term in our treatment.

*** This relation is derived by Sugie et al.⁵⁾

The exchange D - S tensor term with Majorana exchange character $\langle D(-1) | S_{14} P_{M,14} | S(-2) \rangle$ is equal to that of Wigner type, since the wave function $g_s(-2) \chi(-2) f_{ij}(r')/r'$ is invariant under the exchange of the coordinates $\mathbf{r}_1 \leftrightarrow \mathbf{r}_4$. Thus, the splitting term has the factor $(w^{(t)} + m^{(t)})$, which means that only the tensor force in 3E contributes.

We get the following summary as to the splitting kernel $k_{sp}^{lJ}(r, r')$.

- (i) Antisymmetrization is essential to the splitting kernel.
- (ii) The splitting kernel is expressed as

$$k_{sp}^{lJ}(r, r') = (w^{(t)} + m^{(t)}) \left(\frac{-C}{1+C^2} \right) v_t({}^3E) \|\boldsymbol{\sigma} \cdot \mathbf{l}\| A(\alpha\beta\nu l; r, r'). \quad (3.10)$$

Here $\|\boldsymbol{\sigma} \cdot \mathbf{l}\|$ is the eigen-value of the operator $\boldsymbol{\sigma} \cdot \mathbf{l}$, resulting from the numerator of C_{lJ+1} ;

$$\|\boldsymbol{\sigma} \cdot \mathbf{l}\| = \begin{cases} l & \text{for } J=l+1/2 \\ -(l+1) & \text{for } J=l-1/2. \end{cases}$$

α and β mean the spreads of the 1S_0 - and 5D_0 - wave functions of He^4 , respectively, and ν is the range of the tensor force. From the expression (3.10), we see that the tensor force contributes to the splitting in two ways: (1) proportionally to the strength of the tensor force and (2) through the D -state mixing ratio C of He^4 . The strong tensor force characteristic of the pion-theoretical potential is then expected to be favorable to the wide splitting. The remaining part of this paper will show this is the case. As shown in § 4 and § 5, the sign of k_{sp}^{lJ} is the same as what the shell model assumes,, because

$$(w^{(t)} + m^{(t)}) v_t({}^3E) < 0, \quad C < 0 \quad \text{and} \quad A > 0.$$

It will be worth mentioning that also in the case of more general nuclei we expect to get the spin-orbit coupling of the type (3.9) from a tensor force, when we take into account the antisymmetrization and the mixing of the core states due to tensor forces between core nucleons.

§ 4. Wave function and binding energy of He^4

The available data on He^4 are as follows: Its spin is zero, parity even and the experimental binding energy ~ 28 Mev. Because its spin is zero, we cannot get any information from the E2 and M1 moments. The high energy electron scattering experiments showed that the charge distribution of He^4 can be best fitted by the Gaussian radial distribution with r.m.s. radius $1.61 \times 10^{-13} \text{ cm}^{13}$.

As in Eq. (3.2), we chose the spin-angular wave functions. Taking account of the results of the high energy electron scattering experiments, Gaussian radial wave functions are chosen, namely

$$g_s = N_s \exp\left\{-\frac{1}{2}\alpha \sum_{i>j=2}^5 r_{ij}^2\right\}; \quad g_p = N_p \exp\left\{-\frac{1}{2}\beta \sum_{i>j=2}^5 r_{ij}^2\right\}. \quad (4.1)^*$$

Usually, the parameters of the wave function are determined by the variational calculation. But we do not follow this procedure, because it may be meaningless to determine the values of the parameters, particularly of α , by this method when the wave function has no short-range correlation. Instead of this, we use the result of the high energy e -He⁴ scattering experiments in determining α . As the r.m.s. radius $\sqrt{\langle r^2 \rangle}$ is mainly determined by the ¹S₀-state wave function and the contribution from the ⁵D₀-state is estimated to be smaller than 5%, we determine parameter α to fit the experimental value of the r.m.s. radius $\sim 1.61 \times 10^{-13}$ cm.

Regardless of the finite charge distribution of proton, we obtain $\alpha \sim 0.11 \times 10^{26}$ cm⁻² from the relation $\langle r^2 \rangle^{p'} = 9/32\alpha$. But through the electron scattering experiments, it has also been shown that the r.m.s. radius of the charge distribution of proton is about $0.7 \sim 0.8 \times 10^{-13}$ cm¹³⁾. Taking this fact into account, we obtain the next formula:

$$\langle r^2 \rangle^{fs} = \langle r^2 \rangle^{p'} + \langle r^2 \rangle_p,$$

where $\langle r^2 \rangle^{fs}$ is the mean square radius of the charge distribution of He⁴ including the effect of the proton finite size, and $\langle r^2 \rangle_p$ is that of proton. Substituting the values 1.61×10^{-13} cm and 0.7×10^{-13} cm for $\sqrt{\langle r^2 \rangle^{fs}}$ and $\sqrt{\langle r^2 \rangle_p}$ respectively, we obtain $\sqrt{\langle r^2 \rangle^{p'}} = 1.4 \times 10^{-13}$ cm, from which we determine $\alpha = 0.14 \times 10^{26}$ cm⁻². This value is considerably smaller than those obtained so far by other authors using the variational calculation without the short-range correlation. Later we shall discuss this point.

Fixing the parameter α to this value, we determine the other parameters β and C by the variational calculation. With nuclear forces of ($r^2 \times$ Gauss) type radial dependence, the variational expression of the total energy of He⁴ becomes

$$\begin{aligned} E_a = & \frac{1}{1+C^2} \left[\frac{\hbar^2}{2M} (18\alpha + 26\beta C^2) + \sum_{1\pi, 2\pi} v_c \left\{ 9(w+m) A^{3/2} \frac{1}{2\alpha+\mu} \right. \right. \\ & + C^2 \left(\frac{15}{4} (w+m+b+h) A^{13/2} + \frac{15}{2} (w-m+b-h) A^{15/2} \right. \\ & + \frac{7}{4} (w+m+b+h) A^{17/2} \left. \right) \frac{1}{2\beta+\mu} \left. \right\} - \sum_{L,S} v_L \left\{ 6\sqrt{5} C (w^{(L)} + m^{(L)}) B' \frac{1}{\alpha+\beta+\nu} \right. \\ & \left. \left. + C^2 \left(\frac{21}{2} (w^{(L)} - m^{(L)}) B^{5/2} + \frac{7}{2} (w^{(L)} + m^{(L)}) B^{7/2} \right) \frac{1}{2\beta+\nu} \right\} \right] \end{aligned}$$

where

* Such functions have no two-body correlation: for example, $\exp\{-\frac{1}{2}\alpha \sum r_{ij}^2\} = \exp\{-2\alpha \sum r_i^2\}$ where r_i is the coordinate of the i -th nucleon relative to the center of mass of He⁴.

$$A = \frac{2\alpha}{2\alpha + \mu}; \quad B = \frac{2\beta}{2\beta + \nu};$$

$$A' = \frac{2\beta}{2\beta + \mu}; \quad B' = \frac{2\alpha}{\alpha + \beta} \left(\frac{2\beta}{\alpha + \beta} \right)^2 \left(\frac{2\sqrt{\alpha\beta}}{\alpha + \beta + \nu} \right)^{5/2}. \quad (4.2)$$

This is minimized with respect to β and C . The results are shown in Table 1.

Table 1 Binding energy of He^4 calculated using the pion-theoretical potential without the short-range correlation from Eq. (4.2).

$\beta = 0.35 \times 10^{26} \text{ cm}^{-2}; \quad C = -0.30 \quad (\alpha = 0.14 \times 10^{26} \text{ cm}^{-2})$	
$SS \langle \text{K. E.} \rangle$	46 (Mev)
$\langle \text{Central } 1\pi \rangle$	-11
$\langle \text{Central } 2\pi \rangle$	-55
$SD \langle \text{Tensor } 1\pi \rangle$	-21
$DD \langle \text{K. E.} \rangle$	16
$\langle \text{Central } 1\pi \rangle$	-0.3
$\langle \text{Central } 2\pi \rangle$	-5.9
$\langle \text{Tensor } 1\pi \rangle$	-1.9
Total	-33 (Mev)

As seen in Table 1, $V_c^{(2\pi)}(^1E)$ plays an important role, and $V_l^{(1\pi)}(^3E)$ also yields large contribution. Without them He^4 will hardly be bound. It is noted that $V_c^{(2\pi)}(^3O)$ and $V_l^{(1\pi)}(^3O)$ do not contribute to the SS and SD terms, respectively. Obviously, the total binding energy is too large*. This is due to the fact that we take no account of the short-range correlation, the main effect of which is to increase the $SS \langle \text{K.E.} \rangle$. Although it is difficult to say anything about this effect quantitatively, we estimate this roughly in § 6 using a trial correlation function $\prod_{i>j=2}^5 (1 - \exp(-\gamma r_{ij}^2))$. From this result we may say qualitatively that: 1) we get the reasonable minimum total energy, 2) the value of α minimizing the total energy tends to be much smaller than that determined variationally without correlation ($\alpha = 0.50 \times 10^{26} \text{ cm}^{-2}$; B.E. = 120 Mev. See Fig. 5(d)), 3) the values of C and β are insensitive to the correlation function in our case.

So far, many authors^{(12), (14)} have calculated the binding energy of He^4 to determine the "consistent" phenomenological potentials. From their results it is seen that we could not obtain sufficient binding energy if the tensor force was predominant in 3E . In our case, the pion-theoretical potential has two central parts with different ranges, i.e. $V_c^{(1\pi)}$ and $V_c^{(2\pi)}$; so, although its tensor force is strong and the contribution of $V_c^{(1\pi)}$ is small, it can reproduce the binding energy of He^4 reasonably as shown above. Moreover, the strong tensor force results in a large value of

* Since, in the intermediate region, $V_o^{(2\pi)}$ is known only qualitatively, so our choice of $V_o^{(2\pi)}$ and the value of the total energy should not be taken seriously. The latter is very sensitive to the choice of the detailed form of the former.

$C^2 \sim 9\%$, which is very advantageous for reproducing the wide splitting of the p -phase shifts in n -He⁴ scattering, while the value of C^2 is about 4% according to other authors. Also, by our method of determining the value of α , the bremsstrahlung-weighted cross section¹⁵⁾ in the γ -He⁴ reaction is naturally reproduced.

About the corrections due to the correlation and additional 5D_0 states, we shall discuss briefly in § 6.

§ 5. Effective potentials and phase shifts

In this section we discuss the p - and s -phase shifts in n -He⁴ scattering and show the numerical results.

5.a) Effective potentials

The explicit expression of the integro-differential equation is derived from Eq. (3.6) using Eqs. (2.3), (3.2) and (4.1). The result is given in the Appendix.

In order to find out the characteristic features of the interaction terms we rewrite the terms which contain kernels in the form of the effective potential. The abbreviated form for $l=1$ is

$$\frac{\hbar^2}{2M'} \left(\frac{d^2}{dr^2} + k^2 - \frac{2}{r^2} \right) f_J(r) = (W(r) + W_J'(r) + W_{sp}^J(r)) f_J(r) \quad (5.1)$$

where

$$W_J'(r) = \frac{\int k(r, r') f_J^0(r') dr'}{f_J^0(r)} \quad (5.2)$$

$$W_{sp}^J(r) = \frac{\int k_{sp}^J(r, r') f_J^0(r') dr'}{f_J^0(r)}. \quad (5.3)$$

$f_J^0(r)$ is a solution in the square well potential reproducing nearly the experimental splitting. This procedure is allowable, if f_J^0 does not differ much from f_J inside the force range and then the convergence of the iteration in solving the integro-differential equation is good. In our case, we can find such f_J^0 as seen in the following. These potentials and the phase shifts are plotted in Fig. 2.

In $W(r)$, $V_c^{(1\pi)}$ vanishes exactly because of its exchange character. Consequently, $V_c^{(2\pi)}$ is essential in this scattering problem because it composes the whole $W(r)$ which is the main part of the effective potentials. Besides $V_c^{(1\pi)}$, the tensor force, which is important in binding four nucleons, has no effect on extra neutron in the direct term. These circumstances seem to be the reason why five nucleons do not bind, although the $p_{3/2}$ -level of this system is just above the zero energy. $W(r)$ is shown in Fig. 3 (a) with the use of the parameters decided in § 4.

In $W_J'(r)$ the main part arises from the $V_c^{(2\pi)}$, V_t and the kinetic energy in the antisymmetrized effect (the last term in Eq. (3.6)). They cancel each other to some extent in $r \lesssim 3.0$. Out of this region, only the kinetic energy term is

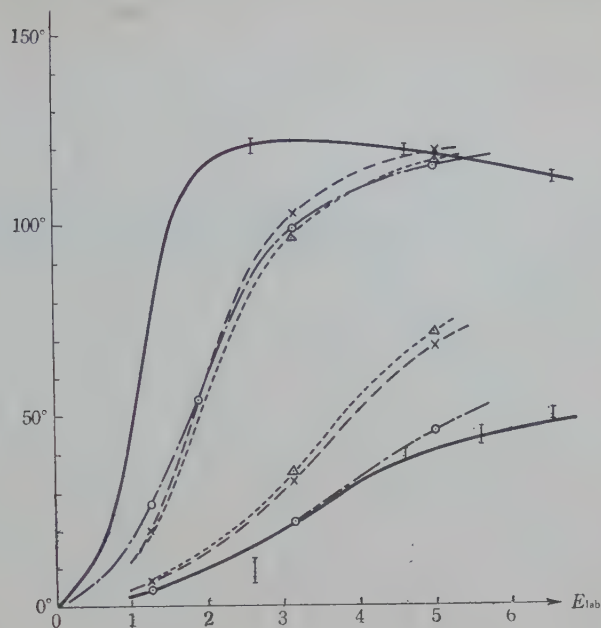


Fig. 2(a). p -phase shift.

- I Experiment of n -He⁴ (J. D. Seagrave, Phys. Rev. **92** (1953) 1222; Levintov et al., Soviet Phys. JETP **5** (1957) N 2258)
- Calculated from experiments of p -He⁴. (K. W. Brockman, Jr. Phys. Rev. **102** (1956), 391. We wish to thank Dr. Brockman for sending his data prior to the publication.
- Calculated by the zeroth order wave function.
- x--- Theoretical values at the first step.
- △--- Theoretical values at the second step. At the third step they scarcely change.

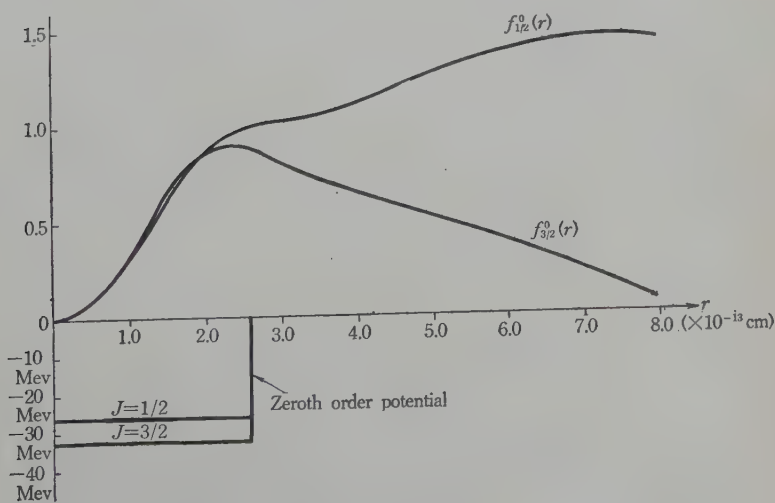


Fig. 2(b). Zeroth order potentials and their wave functions for $l=1$.

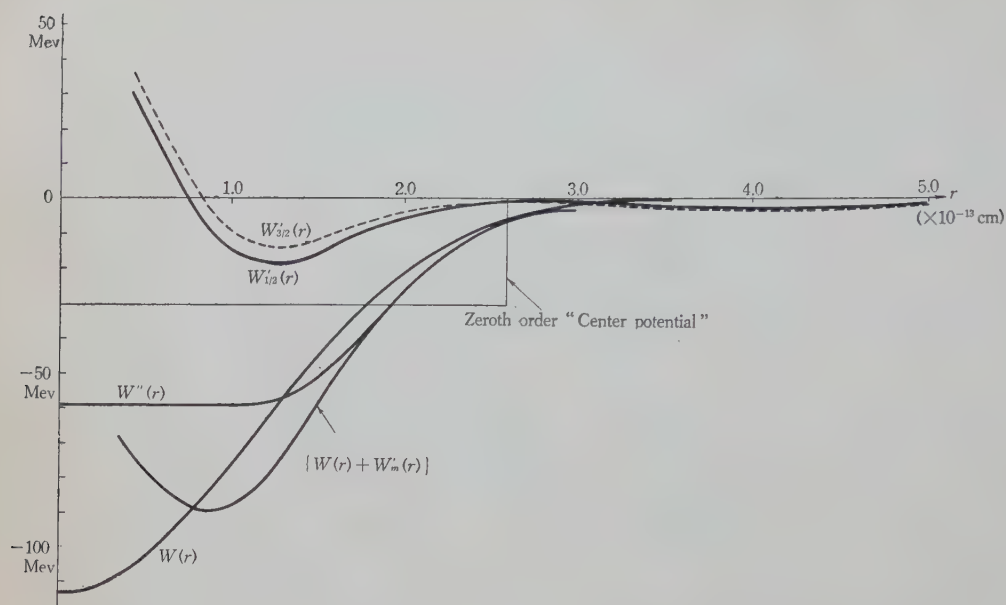


Fig. 3(a). Effective potentials $W(r)$, $W_J'(r)$ and "center potential" $W''(r)$ for $l=1$.

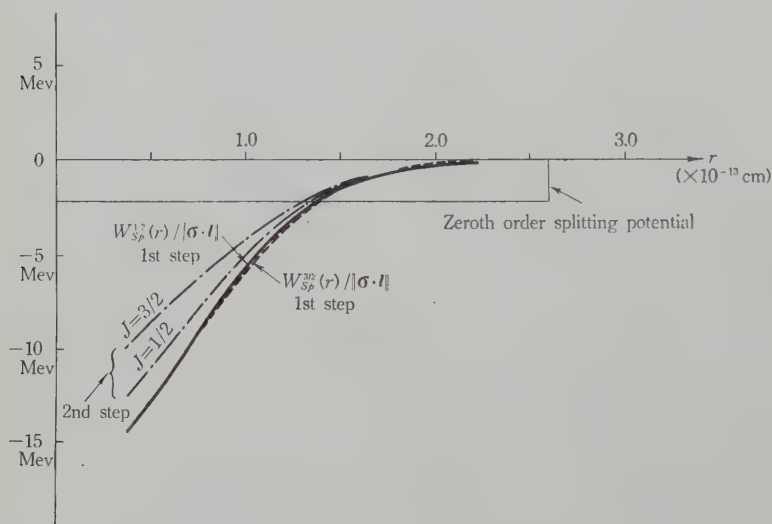


Fig. 3(b). Effective spin-orbit potential $W_{Sp}^J(r)$ for $l=1$.

effective but small and attractive. As $k(r, r')$ itself has no J -dependence, the J -dependence of $W_J'(r)$ is small. Its effect to the splitting is reductive in $r \lesssim 3.0$ and constructive in $r \gtrsim 3.0$. This is shown in Fig. 3 (a).

In the last term of Eq. (5.1) the kernels $k_{Sp}^J(r, r')$ themselves split depending

on the total angular momentum J , proportionally to the factor $\|\boldsymbol{\sigma} \cdot \mathbf{L}\|$. Now $W'_{sp}(r)/\|\boldsymbol{\sigma} \cdot \mathbf{L}\|$ is shown in Fig. 3 (b). There we find that the effective spin-orbit potential $W'_{sp}(r)$ has only small J -dependence except the kinematical factor $\|\boldsymbol{\sigma} \cdot \mathbf{L}\|$.

The J -dependence of the effective potentials through that of $f_J(r)$ is small. This feature comes from the situation that in spite of the wide splitting of the p -phase shifts the wave functions $f_{3/2}(r)$ and $f_{1/2}(r)$ are not very different inside the potential range ($r \lesssim 2.5$) below 5 Mev of the incident energy, (e. g. see Fig. 2. (a)), and outside the range the kernels are small.

5.b) Numerical calculation

In the next place, we solve the integro-differential equation (3.8) and calculate the p -phase shifts. As the first step we consider Eq. (5.1) given in terms of effective potentials. Further steps will be discussed in § 6.

The energy dependence of $f_J(r)$ inside the potential range is very small below 5 Mev of the incident energy. Consequently, in this energy range, we use Eq. (5.1) with the effective potentials defined by Eqs. (5.2) and (5.3) at a definite energy.

The wide splitting of the p -phase shifts is related to the "critical" situation that the $p_{3/2}$ -level is just above the zero energy, while the $p_{1/2}$ -level is not so. Therefore, the p -phase shifts are very sensitive to the details of the potentials, particularly in the $p_{3/2}$ -state. Then, we can hardly obtain the reasonable "center potential" to reproduce approximately the weighted mean value of the experimental p -phase shifts, unless we have the very detailed knowledge of two-body interactions and of the treatment of the system. It is readily seen in practice that our effective potentials fail to reproduce the experimental mean phase shift, mainly because $W(r)$ is somewhat too strong. It is, however, noted that we have not taken into consideration the hard-core of the nuclear force and the short-range correlation between two nucleons. Consequently, if we took account of these points, the potential $W(r)$ should be reduced by some amount, particularly for small r . However, we can hardly estimate this effect definitely. So we are obliged to decide this "center potential" phenomenologically by the following procedure.

At first we neglect the outside part ($r \geq 3.5 \times 10^{-13}$ cm) of $W'_J(r)$ because of its smallness, and for $r \leq 3.5 \times 10^{-13}$ cm adopt the mean value $W'_m(r)$ of $W'_{3/2}(r)$ and $W'_{1/2}(r)$, neglecting the small J -dependence of $W'_J(r)$. We then solve the following differential equation

$$\frac{\hbar^2}{2M'} \left(\frac{d^2}{dr^2} + k^2 - \frac{2}{r^2} \right) f(r) = W''(r) f(r), \quad (5.4)$$

where $W''(r)$ has the same form as $W(r) + W'_m(r)$ at $r \gtrsim 2 \times 10^{-13}$ cm but the inner part of $W''(r)$ is to be decided to give the "center potential". Second, we solve the following differential equation

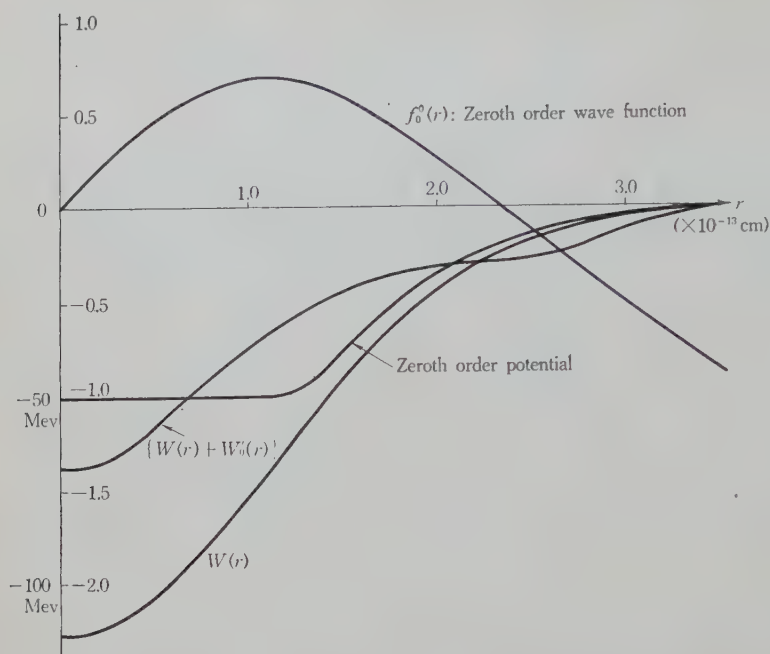


Fig. 4(a). Zeroth order potential, its wave function and effective potentials for $l=0$.

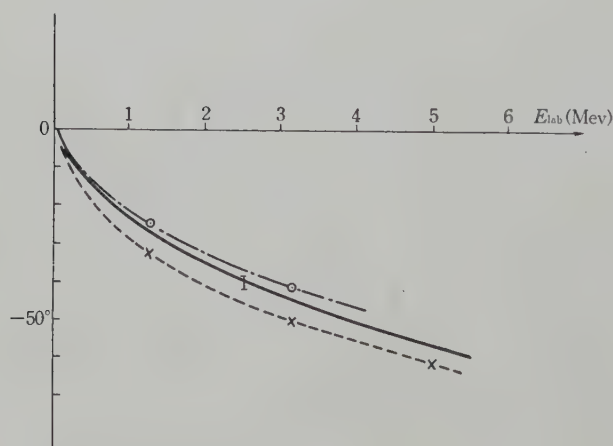


Fig. 4(b). s -phase shift.

- I Experiment of n -He⁴. See the caption of Fig. 2 (a)
 ———— Calculated from experiments of p -He⁴. See the caption of Fig. 2 (a)
 - - - ⊙ - - - Calculated by 0-th order wave function.
 - - - x - - - Theoretical value.

$$\frac{\hbar^2}{2M'} \left(\frac{d^2}{dr^2} + k^2 - \frac{2}{r^2} \right) f_J(r) = (W''(r) + W_{sp}''(r)) f_J(r). \quad (5.5)$$

In spite of our approximations we can get the essential feature so far as the spin-orbit coupling is concerned. In order to obtain better solutions, we continue the iteration in the original equation, (3.8).

The results are shown in Fig. 3 (a) and Fig. 2 (b). In Fig. 3 (a) the "center potential" $W''(r)$ is shown. It is seen that the tensor force of the pion-theoretical potential gives about 60% of the experimental splitting at the second step iteration. This result is reasonable in comparison with that of Sugie et al. obtaining 30% by the weak tensor force in Fig. 1 (a). The values of the $W_{sp}''(r)/\|\sigma \cdot l\|$ at the third step (obtained by using the second step solution) are almost equal to those at the second step. This shows that the convergence of the iteration is good.

Finally, we calculate the s -phase shift. Also in this case the discussion is pushed in the same manner as the p -phase shifts, and the differential equation becomes

$$\frac{\hbar^2}{2M'} \left(\frac{d^2}{dr^2} + k^2 \right) f_0(r) = (W(r) + W_0'(r)) f_0(r), \quad (5.6)$$

where

$$W_0'(r) = \frac{\int k(r, r') f_0^0(r') dr'}{f_0^0(r)}. \quad (5.7)$$

$W(r)$ is the same as that in (5.1) and $k(r, r')$ is abbreviated from the kernels for $l=0$ in (A.1). $f_0^0(r)$ is solved by the similar potential to the "center potential" for $l=1$, reproducing the experimental s -phase shift. In this case, $W_0'(r)$ is mainly repulsive contrary to the p -waves and so the effective potential $W(r) + W_0'(r)$ has a reasonable strength. Although the p -waves are very sensitive to the details of the interaction terms, the s -phase shift is almost determined by the main feature of the effective potential. This is because the s -state is not at the "critical" situation. The potentials and the phase shifts are shown in Fig. 4. It can be concluded that the pion-theoretical potential explains the s -phase shift.

§ 6. Supplementary discussions

6.a) Short-range correlations

In the previous calculation the nuclear size parameter α was fixed from the information of the high energy e -He⁴ scattering. In the following we show that our value of α is consistent with that determined by the variational method, if the short-range correlation between two nucleons is taken into account.

Let us use the following correlation function so that the calculation can be performed analytically:

$$\prod_{i>j} (1 - \exp(-\gamma r_{ij}^2)). \quad (6.1)$$

Generally, introducing the correlation function, we obtain an additional kinetic energy as the most important effect. Since this additional energy increases with α , the value of α which minimizes the binding energy is to be reduced. However, this type of the correlation function may not be realistic in the core region of nuclear force, because the actual wave function vanishes there. The additional kinetic energy derived from this correlation function has a (negative) contribution in the core region, so we should eliminate this unrealistic contribution by cutting off.

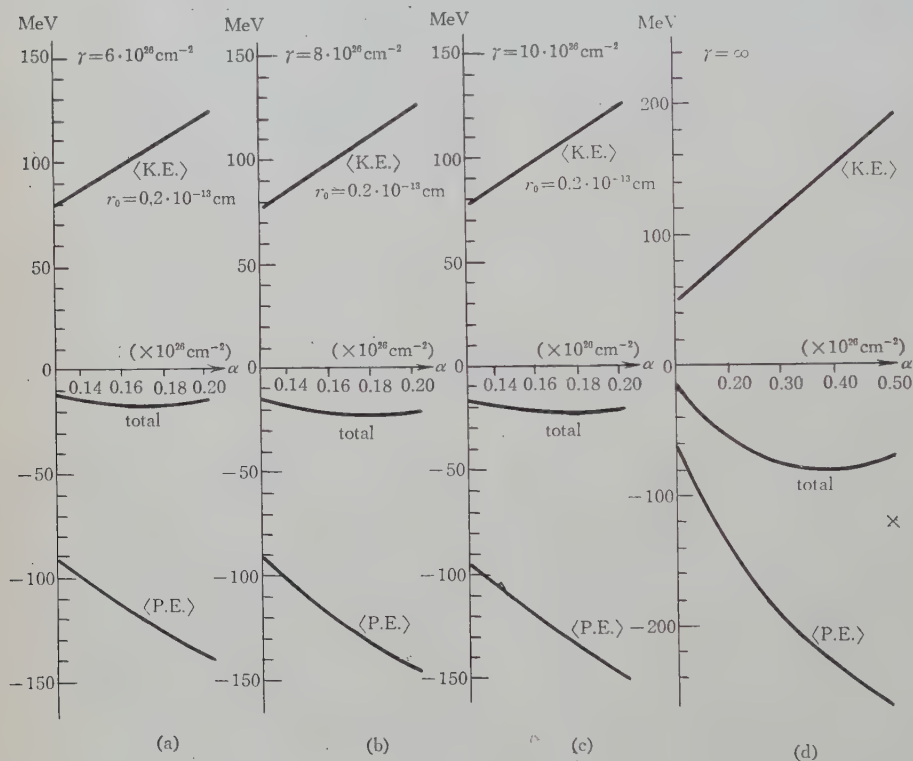


Fig. 5. The α 's minimizing the total energy of He^4 are shown for several cases. Solid lines are obtained by fixing β and C to $0.35 \times 10^{26} \text{ cm}^{-2}$ and -0.30 , respectively (Table 1).

Figs. (a), (b) and (c) show the results with short-range correlation function $\prod_{i>j=2} (1 - \exp(-\gamma r_{ij}^2))$. (a), (b) and (c) correspond to $\gamma = 6, 7$ and $8 \times 10^{26} \text{ cm}^{-2}$, respectively. r_0 is the range of cutting off. In these calculations, we include only two-body clusters, since the effects from the higher clusters are negligible in our cases. The values of C and β are insensitive to the correlation function when $\alpha \leq 0.2 \times 10^{26} \text{ cm}^{-2}$ and $\gamma \geq 6 \times 10^{26} \text{ cm}^{-2}$.

5 (d) shows the results without short-range correlation ($\gamma = \infty$). The variational calculation with respect to all parameters α , β and C gives the result shown by the cross point (B. E. = 120 Mev, $\alpha = 0.5 \times 10^{26} \text{ cm}^{-2}$, $\beta = 0.8 \times 10^{26} \text{ cm}^{-2}$ and $C = -0.5$).

Using (6.1), we obtain the following results. Fig. 5 shows the values of α that give the minimum binding energy for fixed β , C , the range of cutting off and several values of γ^* . It is seen from Fig. 5 that owing to the introduction of the correlation function the value of α minimizing the binding energy is much reduced to $\alpha < 0.2 \times 10^{26} \text{ cm}^{-2}$ from $\alpha = 0.5 \times 10^{26} \text{ cm}^{-2}$ without correlation. In spite of above rough estimation, we can expect to get $\alpha \sim 0.14 \times 10^{26} \text{ cm}^{-2}$ if we adopt a more realistic correlation function. The situation is same also in the three-body problem, as already studied by Kikuta et al.¹⁶, that is, adopting a correlation function which is consistent with the hard core, the parameter (corresponding to our α) becomes smaller than the value without the correlation.

Moreover, it is to be noted that the short-range correlation does not affect our determination of the value of α through the r.m.s. radius, because in the expectation value of r^2 the contribution from the region, where the correlation is important, is negligibly small.

From the above discussions we conclude that the value $0.14 \times 10^{26} \text{ cm}^{-2}$ of α is consistent with the whole variational treatment with the hard core and it is rather appropriate to $n\text{-He}^4$ scattering problem.

Speaking of $n\text{-He}^4$ scattering in relation to α , the magnitude of the splitting of the scattering potential and so the p -phase shifts are rather insensitive to the change of α near $\alpha \sim 0.14 \times 10^{26} \text{ cm}^{-2}$ as seen from Eq. (A.6)**.

6.b) Corrections to the numerical value of the splitting

(1) *The additional 5D_0 states in He^4 .* According to Abraham et al.^{14b)}, the additional 5D_0 states have some effect on the binding energy of He^4 . So, introducing these states in our problem, they would make the binding energy and the splitting of the p -phase shifts larger, since their effect seems to be at least additive to the principal 5D_0 state.

(2) *So-called S_{24} and S_{34} terms discussed in § 3.* There remain two independent terms, S_{24} and S_{34} , with respect to the tensor terms, if the wave function of He^4 is not exact. While S_{34} term is independent of the splitting, S_{24} becomes the same as S_{14} after the interchange $1 \leftrightarrow 2$, hence in the kernel from S_{24} the coordinates \mathbf{r} and \mathbf{r}' exchange each other as compared with that from S_{14} . Then, speaking about the splitting, we now get $k_{sp}^J(r r') + k_{sp}^J(r' r)$ instead of $k_{sp}^J(r r')$ in (3.8). This additional kernel may increase the splitting of the p -phase shifts.

* In the next step, we estimated the effect of short-range correlation to β and C fixing α to $\alpha = 0.14 \times 10^{26} \text{ cm}^{-2}$. We found that C and β are insensitive to the correlation function in our case.

** It is because, firstly, the integrand of the splitting potential, in which α appears in the form of $\alpha + \beta$, is not so affected by the change of α on account of $\alpha < \beta$ in our case, and secondly, the fractional variation with respect to α of the other factor, which depends on r , is nearly equal to $\Delta\alpha/\alpha$ over the region where the contribution to the splitting is most important.

As for the effect to the result by the change of β , the effect may be estimated to be small as seen in Sugie et al., provided that our β should be replaced for their 2α because $\alpha < \beta$ in our case.

However, the contribution of all kernels resulting from inaccuracy of the wave function of He^4 , including this additional splitting kernel, tends to vanish as the wave function of He^4 becomes exact. Therefore, we cannot say anything definite about this effect.

(3) *J-dependence of W_J' and iteration.* Finally we speak about the approximations in the calculation of phase shifts. In § 5, we neglected the small J -dependence of $W_J'(r)$. It is reductive in $r \lesssim 3.0 \times 10^{-13}$ cm and constructive in $r \gtrsim 3.0 \times 10^{-13}$ cm to the splitting. Its net effect on the splitting of the p -phase shifts is estimated to be reductive, 15% at most. Our result in the previous section was obtained at the second step of the iteration method. At the third step it is found that the effective potentials $W_{sp}'(r)/\|\sigma \cdot l\|$ reconstructed by using the second step solution scarcely change. There is a reason* to consider that the successive effective splitting potentials fall inside the bounded region between the second step ones for $J=1/2$ and $J=3/2$. Thus we estimate the error of our result which arises by successive iteration to be negligible.

After all, as for the splitting, we conclude that these various corrections will not change essentially the results that the tensor force of the pion-theoretical potential explains about 60% of the splitting of the p -phase shifts.

§ 7. Concluding remarks

Applying the pion-theoretical potential to He^4 and $n\text{-He}^4$, we obtained valuable information on the relations between the characteristic features of nucleon-nucleon interaction and important properties of nuclei. We summarize the main results in the following.

(i) The wide splitting of the p -phase shifts in the low energy $n\text{-He}^4$ scattering can be explained by the strong tensor force of the one-pion-exchange potential in the triplet even state, if we take into account the Pauli principle and the mixing of of 5D_0 -states of He^4 due to this tensor force. Basing upon this results, therefore, we can expect that the spin-orbit coupling in the shell model is originated from the

* At the second step, the reduction of $W_{sp}^{1/2}(r)$ is mainly due to the reduction in the overlap integral $\int k_{sp}^{1/2} f_{1/2}^{(1)} dr'$ arising from the change of the form of the effective potential. Although this reduction of $W_{sp}^{1/2}(r)$ makes the total scattering potential a few Mev deeper for $r < 1.2 \times 10^{-13}$ cm and so the wave function $f_{1/2}^{(2)}$ somewhat larger at this region, the change of $W_{sp}^{1/2}(r)$ constructed by this $f_{1/2}^{(2)}$ is much smaller than that of the first step to the second step. The trend is to make $W_{sp}^{1/2}(r)$ smaller because of the division of the larger value of $f_{1/2}^{(2)}$, but, of course, $W_{sp}^{1/2}$ does not become smaller than $W_{sp}^{3/2}(r)$ at the second step. On the other hand, the reduction of $W_{sp}^{3/2}(r)$ is partly due to that in the overlap integral and partly, in $r < 1.2 \times 10^{-13}$ cm, due to the division by $f_{3/2}^{(1)}$, which is larger than $f_{3/2}^{(0)}$ because of the large discrepancy between the effective scattering potentials there. The smaller $W_{sp}^{3/2}(r)$ becomes, the smaller $f_{3/2}$ becomes and then $W_{sp}^{3/2}(r)$ at the next step becomes larger, but the absolute value of the change is very small. Thus the effective potential $W_{sp}^{3/2}(r)$ will converge at a slightly larger absolute value than that of the second step.

strong tensor forces between an extra nucleon and core particles, through the exchange effects among these particles and the mixing of the core states due to the tensor forces acting between core particles.*.**

(ii) The binding energy of He^4 can be explained by the following main features of nucleon-nucleon interaction: the strong short-range attractive force of the two-pion-exchange potential in the singlet even state, the strong tensor force of the one-pion-exchange potential in the triplet even state and the short-range repulsion. The parameter representing the spread of the wave function is consistent with the experimental data of $e\text{-He}^4$ scattering, if we properly take into account the short-range correlation effects due to the hard-core-like repulsive interaction. In three-body system, essential contributions to the energy seem to come from the above mentioned three parts of nucleon-nucleon interaction. Therefore, we expect that the pion-theoretical potential may explain all data of the nuclei with $A \leq 4$.

(iii) The strong tensor force gives an important contribution to the binding energy of He^4 . On the other hand, in the system of He^4 plus one nucleon, the direct (non-exchange) contribution from the tensor force to the interaction acting on the extra nucleon vanishes exactly, even though we include the 5D_0 -state. The main contribution from the one-pion-exchange central potential also vanishes. These situations help us to explain the discontinuity in the binding energy at the closed shell. Also in this case we see the importance of the attractive force of the two-pion-exchange potential in the two-nucleon charge triplet states.

The authors are indebted to professor M. Kobayasi and the members of his laboratory for their helpful discussions.

Appendix

Eq. (3.6) is rewritten with the use of Eqs. (2.3), (3.2) and (4.1) as follows:

$$\begin{aligned} \frac{\hbar^2}{2M'} \left\{ \frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} \right\} f_{lj}(r) = W(r) f_{lj}(r) + \int k'_{KE}(r, r') f_{lj}(r') dr' \\ + \int k'_c(r, r') f_{lj}(r') dr' + \int k'_t(r, r') f_{lj}(r') dr' + \int k'_{sp}(r, r') f_{lj}(r') dr', \end{aligned} \quad (\text{A-1})$$

where $k'_{KE}(r, r')$ is derived from the kinetic energy in the antisymmetrized term, $k'_c(r, r')$ is from the central potentials and $k'_t(r, r')$ is from the tensor potential. These have no J -dependence.

* This program is now being pushed in the case of general nuclei by Takagi, Watari and Yasuno.

** T. Terasawa and T. Terasawa and A. Arima obtained about half the doublet splittings of the energy levels of He^6 , N^{15} and O^{17} in the second order perturbation, by taking into account the same effect.

$k_{sp}^{l,r}(r, r')$ is the splitting kernel derived from the tensor potential. The expressions of $W(r)$ and these kernels are given in the following.

$$W(r) = \sum_{i=1,2} (4w^{(i)} + 2b^{(i)} - m^{(i)} - 2h^{(i)}) \frac{v_c^{(i)}}{1+C^2} p_{(i)}^{5/2} \left(p_{(i)} r^2 + \frac{9}{32\alpha} \right) \exp(-p_{(i)} r^2),$$

$$p_{(i)} \equiv \frac{16\alpha}{16\alpha + 3\mu^{(i)}}. \quad (\text{A.2})$$

$$k_{K.E.}^l(r, r') = \frac{1}{1+C^2} \frac{32}{15} \left(\frac{3\alpha}{\pi} \right)^{1/2} \zeta^2 \left\{ \left(16\alpha^2 + r'^2 - \frac{8(l+3)}{\zeta} + \frac{4}{\zeta^2} k^2 \right) \mathcal{J}_{l+1/2}(\zeta r r') \right. \\ \left. + 8rr' \mathcal{J}_{l+3/2}(\zeta r r') \right\} \exp \left\{ -\frac{17}{16} \zeta (r^2 + r'^2) \right\},$$

$$\zeta \equiv 6 \left(\frac{8}{15} \right)^2 \alpha. \quad (\text{A.3})$$

$\mathcal{J}_{l+1/2}(x) \equiv x i^l j_l(ix)$, where j_l is the spherical Bessel function of order l .

$$k_e^l(r, r') = k_e^{l\kappa}(r, r') + k_e^{l\eta}(r, r'),$$

$$k_e^{l\kappa}(r, r') = \sum_{i=1,2} (4m^{(i)} + 2h^{(i)} - w^{(i)} - 2b^{(i)}) \frac{v_c^{(i)}}{1+C^2} \left(\frac{16}{15} \right)^3 \left(\frac{3\alpha}{\pi} \right)^{1/2} \frac{18\alpha}{4\alpha - 3\mu^{(i)}} \\ \times \left\{ \left(r^2 + r'^2 + \frac{2l}{\xi^{(i)}} \right) \mathcal{J}_{l+1/2}(\xi^{(i)} r r') - 2rr' \mathcal{J}_{l+3/2}(\xi^{(i)} r r') \right\} \\ \times \exp \left\{ -\frac{8}{75} (17\alpha + 6\mu^{(i)}) (r^2 + r'^2) \right\},$$

$$k_e^{l\eta}(r, r') = - \sum_{i=1,2} (w^{(i)} + m^{(i)}) \frac{v_c^{(i)}}{1+C^2} \frac{128}{5} \left(\frac{3\alpha}{\pi} \right)^{1/2} \frac{2\alpha}{2\alpha + \mu^{(i)}} q_{(i)}^{3/2} \left[\left\{ \left(\frac{4}{15} \right)^2 q_{(i)} \right. \right. \\ \times \left. \left(16r^2 + r'^2 - \frac{8l}{\eta^{(i)}} \right) + \frac{1}{4\alpha} \right\} \mathcal{J}_{l+1/2}(\eta^{(i)} r r') + 2 \left(\frac{8}{15} \right)^2 q_{(i)} r r' \mathcal{J}_{l-3/2}(\eta^{(i)} r r') \right] \\ \times \exp \left[-\frac{4}{75} q_{(i)} \{ (34\alpha + 27\mu^{(i)}) r^2 + (34\alpha + 7\mu^{(i)}) r'^2 \} \right],$$

$$\xi^{(i)} \equiv \frac{32}{75} (4\alpha - 3\mu^{(i)}), \quad \eta^{(i)} \equiv \frac{128}{25} \frac{2\alpha + \mu^{(i)}}{6\alpha + \mu^{(i)}} \alpha,$$

$$q_{(i)} \equiv \frac{6\alpha}{6\alpha + \mu^{(i)}}. \quad (\text{A.4})$$

$$k_t^l(r, r') = \sum_{j=1,2} k_t^{l(j)}(r, r'),$$

$$k_t^{l(j)}(r, r') = - \sum_{l,s} (w^{(i)} + m^{(i)}) \frac{C}{1+C^2} v_t \frac{2^6 \alpha^{9/4} \beta^{13/4}}{(\alpha + \beta)^6} b^{1/2} d \left[\left\{ S_l^{(j)} - 3 \left(\frac{16}{15} \right)^2 \right. \right. \\ \times (\alpha + \beta) (T_l^{(j)} r^2 + T_l'^{(j)} r'^2) + 9 \left(\frac{16}{15} \right)^4 (\alpha + \beta)^2 (U^{(j)} r^4 + 2U'^{(j)} r^2 r'^2 + U''^{(j)} r'^4) \left. \right\} \\ \times \mathcal{J}_{l+1/2}(\lambda r r') + 3 \left(\frac{16}{15} \right)^2 (\alpha + \beta) r r' \left\{ -V^{(j)} + 3 \left(\frac{16}{15} \right)^2 (\alpha + \beta) (W^{(j)} r^2 \right.$$

$$+ W^{(\beta)} r'^2 \} \left[\mathcal{J}_{l+3/2}(\lambda r r') \right] \exp \{ -\gamma^{(\beta)} r^2 - \delta^{(\beta)} r'^2 \},$$

$$\lambda \equiv 3 \left(\frac{8}{15} \right)^2 (1+2a)(\alpha+\beta)$$

$$a \equiv \frac{\nu}{3(\alpha+\beta)+\nu} \quad b \equiv \frac{3(\alpha+\beta)}{3(\alpha+\beta)+\nu} \quad d \equiv \frac{\alpha+\beta}{\alpha+\beta+\nu}$$

$$\gamma^{(1)} \equiv \frac{24}{225} (16\alpha+\beta) + 3 \left(\frac{16}{15} \right)^2 (\alpha+\beta) a$$

$$\gamma^{(2)} \equiv \frac{24}{225} (\alpha+16\beta) + 3 \left(\frac{16}{15} \right)^2 (\alpha+\beta) a$$

$$\delta^{(1)} \equiv \frac{24}{225} (\alpha+16\beta) + \frac{3}{16} \left(\frac{16}{15} \right)^2 (\alpha+\beta) a$$

$$\delta^{(2)} \equiv \frac{24}{225} (16\alpha+\beta) + \frac{3}{16} \left(\frac{16}{15} \right)^2 (\alpha+\beta) a$$

$$S_l^{(1)} \equiv \frac{5}{2} b^2 + \frac{5}{6} l b d (25a+17b) + 2l(l-1) a d^2 \left(7a + \frac{17}{3} b \right)$$

$$S_l^{(2)} \equiv \frac{5}{2} b^2 + \frac{20}{3} l b d (2a+b) + 8l(l-1) a d^2 \left(a + \frac{2}{3} b \right)$$

$$T_l^{(1)} \equiv \frac{5}{6} b^2 (5a+b) + l a b d \left(9a + \frac{19}{3} b \right)$$

$$T_l^{(2)} \equiv \frac{10}{3} b^2 (2a+b) + 8 l a b d \left(a + \frac{2}{3} b \right)$$

$$T_l'^{(1)} \equiv \frac{5}{24} b^2 (5a+4b) + \frac{1}{16} l a b d \left(19a + \frac{49}{3} b \right),$$

$$T_l'^{(2)} \equiv \frac{5}{24} b^2 (2a+b) + \frac{1}{2} l a b d \left(a + \frac{2}{3} b \right),$$

$$U^{(1)} \equiv a b^2 \left(a + \frac{1}{3} b \right), \quad U^{(2)} \equiv 2 a b^2 \left(a + \frac{2}{3} b \right),$$

$$U'^{(1)} \equiv \frac{1}{32} a b^2 (21a+17b), \quad U'^{(2)} \equiv \frac{3}{16} U^{(2)},$$

$$U''^{(1)} \equiv \frac{1}{128} a b^2 \left(3a + \frac{8}{3} b \right), \quad U''^{(2)} \equiv \frac{1}{256} U^{(2)},$$

$$V^{(1)} \equiv \frac{5}{24} b^2 (25a+17b) - a b d \left(7a + \frac{17}{3} b \right),$$

$$V^{(2)} \equiv \frac{5}{3} b^2 (2a+b) - 4 a b d \left(a + \frac{2}{3} b \right),$$

$$W^{(1)} \equiv \frac{1}{4} a b^2 \left(9a + \frac{19}{3} b \right), \quad W^{(2)} \equiv U^{(2)},$$

$$W^{(1)} = \frac{1}{64} ab^2 \left(19a + \frac{49}{3} b \right), \quad W^{(2)} = \frac{1}{16} U^{(2)}. \quad (\text{A} \cdot 5)$$

In $k_i^l(a, r')$, (1) and (2) do not denote the number of exchanged pion in potentials.

$$\begin{aligned} k_{sp}^{lj}(r, r') = & - \sum_{L, S} (w^{(l)} + m^{(l)}) \frac{C}{1+C^2} v_l \frac{64}{15} \left(\frac{3}{5\pi} \right)^{1/2} \frac{2^8 \alpha^{9/4} \beta^{13/4}}{(\alpha + \beta)^5} 3b^{3/2} dr r' \\ & \times \exp \left\{ -\gamma^{(1)} r^2 - \delta^{(1)} r'^2 \right\} \sum_{L=I \pm 1} C_{LJK} \left[\left\{ X_L - 3 \left(\frac{16}{15} \right)^2 (\alpha + \beta) (Y r^2 + Y' r'^2) \right\} \right. \\ & \times \mathcal{J}_{L+1/2}(\lambda r r') - 3 \left(\frac{16}{15} \right)^2 (\alpha + \beta) r r' Z \mathcal{J}_{L+3/2}(\lambda r r') \left. \right], \\ X_L = & \frac{8}{3} b + L \frac{32}{15} ad, \quad Y = \frac{16}{15} ab, \quad Y' = \frac{1}{16} Y, \quad Z = \frac{1}{2} Y. \end{aligned} \quad (\text{A} \cdot 6)$$

C_{LJL} is defined in Eq. (3.9)

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

High Energy n -He⁴ Scattering

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Received May 8, 1959

In the previous reports¹⁾²⁾³⁾⁴⁾, hereafter referred to as I, we calculated the n - d scattering at high energy (near 100 Mev neutron energy). In those papers we found that the impulse approximation can well reproduce the experimental result of the differential cross section at least $\theta \lesssim 90^\circ$. It will be of interest to examine the validity of the impulse approximation in applying to the n -He⁴ scattering at the same energy. In the previous papers, I, we found that the both effects, (1) the hard core and (2) the D -state in the target nucleus (deuteron), are small in the scattering. (It is effective only at large angles.) In another paper⁵⁾ due to one of the present authors (T.S.), it was found that the Gaussian type wave function, the S -state of which is expressed by

$$\psi_a = N_0 \exp \left\{ - (1/2) \alpha \sum_{i>j} (\mathbf{r}_i - \mathbf{r}_j)^2 \right\}$$

$$\alpha = 0.14 \times 10^{20} \text{ cm}^{-2}, \quad (1)$$

can well give the binding energy of He⁴ besides that it can well reproduce the wide

splitting of the p -He⁴ system when we take meson theoretical two-body interaction.

In the present report, we calculate the differential cross section and the polarization for n -He⁴ elastic scattering at 90 Mev, adopting exactly the same approximation as the previous papers I. As the wave function of He⁴, we take one given by the Equation (1).

The effect of the D -state is neglected. Following the same notations as in the previous papers I, the differential cross section is given by

$$I(\theta) = (8/5)^2 (1/2)$$

$$\times \sum_{i=1}^2 \left[(2(M_{12} + M_{14}) |\chi_i\rangle^\dagger \right.$$

$$\times (2(M_{12} + M_{14}) |\chi_i\rangle) \left. S(\Delta k) \right]$$

$$= (8/5)^2 (1/2)$$

$$\times \left[(1/16) |N + G + B|^2 + |C|^2 \right]$$

$$\times S(\Delta k). \quad (2)$$

Here 1 is the incident neutron, 2 the neutron and 4 the proton in He⁴, respectively. $S(\Delta k)$ is the sticking factor.

In the present case, it is given by

$$S^{1/2}(\Delta k) = \exp \left[- (3/16\alpha) \right.$$

$$\left. \left\{ (\mathbf{k}_f - \mathbf{k}_i)/2 \right\}^2 \right]. \quad (3)$$

N , etc., are the sum of the coefficients

of the spin matrix of the two-body scattering,

$$N = 2(N_{nn} + N_{np}), \text{ etc.} \quad (4)$$

The result of calculation is given in Fig. 1, together with the experimental result⁶⁾. (The experiment is the scattering of 93 Mev protons by He^4 .) We get the following results.

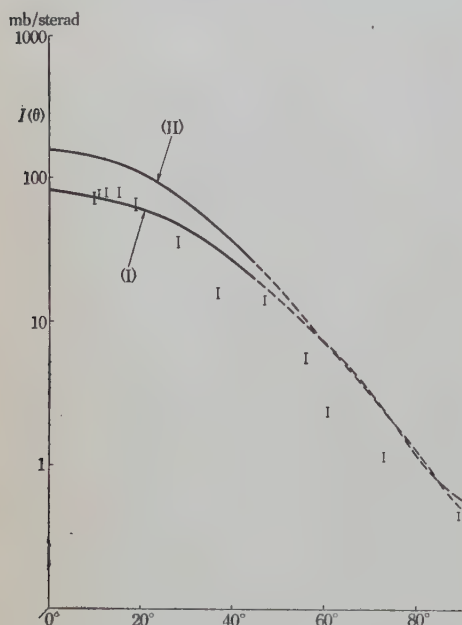


Fig. 1 Calculated value of the differential cross-section of $n\text{-He}^4$ scattering at 90 Mev. Curve I is obtained by using the meson-theoretical phase shifts and Curve II by using the phase shifts analysed by Gammel and Thaler. Experimental value shows the differential cross section of $p\text{-He}^4$ scattering at 93 Mev.

(a) The impulse approximation can reproduce well the angular distribution of $p\text{-He}^4$ scattering, especially at the forward direction. This fact seems to indicate the superiority of the impulse approximation to the Born approximation. Heidmann⁷⁾ calculated the $n\text{-He}^4$ scattering using the Born approximation.

However, he obtained the unsatisfactory result. His result is about five times larger at 20° and about three times larger at 30° than the experimental values. This large discrepancy seems to come from the use of the Born approximation. The small discrepancy between the present calculation and the experimental results may be reconciled by taking into account the $D\text{-state}$ ¹⁾⁵⁾ of He^4 .

(b) The result obtained by making use of the phase shifts derived meson theoretically is better than those obtained by using the phase shifts derived phenomenologically in scattering problems⁹⁾. This is consistent with our previous conclusion¹⁾ about $n\text{-d}$ scattering.

(c) If we adopt some smaller value as the mean square radius of He^4 , we shall get unsatisfactory result in repro-

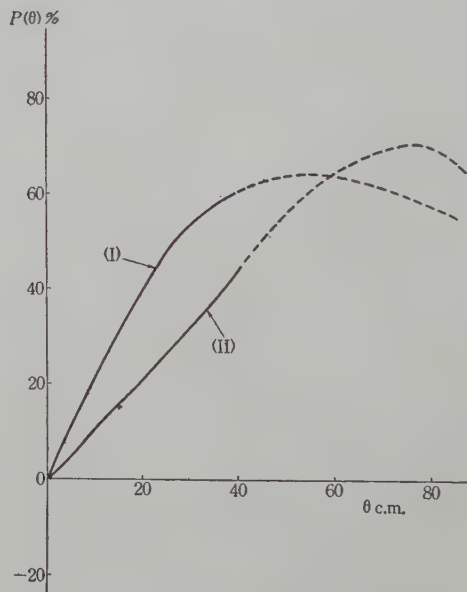


Fig. 2. Polarization of the neutron scattered from He^4 . The difference of the chosen phase shifts of the two-body scattering appears clearly. Curve I and Curve II correspond to the calculated results in Fig. 1.

ducing the angular distribution.

The polarization is given by

$$P(\theta) = \frac{\text{Re } (1/2)C^*(N+G+B)}{(1/16)|N+G+B|^2 + |C|^2} \quad (5)$$

The calculated results is shown in Fig. 2.

The polarization of the $n\text{-He}^4$ scattering may manifest the difference of the nature in the various sets of the phase shifts of the two-body scattering more clearly than that of the $n\text{-d}$ scattering¹⁾. Therefore the measurment of the polarization of $n\text{-He}^4$ scattering may be used as the check of the two-body scattering phase shifts.

The authors wish to thank Prof. M. Kobayashi for his continuous interest in this problem. Also one of the authors (Y. S.) is indebted to the Yomiuri-Yukawa Fellowship for financial aid.

The Triplet Odd State Potential of the Two-Nucleon System*

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The present authors have verified in a series of works the validity of the pion theory of nuclear forces in all low energy two-nucleon phenomena below about 20 MeV¹⁾, taking properly into account the reliability of the theory as proposed by Taketani et al.²⁾ It is, of course, not legitimate to extend the method straightforwardly to the high energy region, since there is no *a priori* justification of sticking to the static pion-theoretical potential at higher energies. But we may expect that the characteristics of this static potential will not change drastically in the outer region at an energy around *one* hundred MeV. It is therefore worth while to ascertain the point in the actual analysis of the nucleon-nucleon scattering.

At first, some of the present authors

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* Main contents of the work were reported at the Tokyo meeting of the Physical Society of Japan, April 1959.

Table 1. Shape of the potentials.

$$V = V_c + V_T S_{12},$$

The internucleon distance is measured in the unit of the pion Compton wave length $\hbar/\mu c$.
 $V^{(1)}$ means the one-pion-exchange potential. (OPEP).

Case I.	Case II
$V_c = V_c^{(1)}(x)$	$V_c = V_c^{(1)}(x)$
$V_T = V_T^{(1)}(x)$	$V_T = V_T^{(1)}(x)$
$\left. \begin{array}{l} V_c = -20 \text{ MeV} \\ V_T = V_T^{(1)}(1.0) \end{array} \right\} x < 1.0,$	$\left. \begin{array}{l} V_c = -20 \text{ MeV} \\ V_T = V_T^{(1)}(x) \end{array} \right\} x > 1.0,$
$\left. \begin{array}{l} V_c = -100 \text{ MeV} \\ V_T = 0 \end{array} \right\} 1.0 > x > 0.7,$	$\left. \begin{array}{l} V_c = -100 \text{ MeV} \\ V_T = V_T^{(1)}(1.0) \end{array} \right\} 1.0 > x > 0.7,$
$\left. \begin{array}{l} V_c = -100 \text{ MeV} \\ V_T = 0 \end{array} \right\} 0.7 > x > x_c,$	$\left. \begin{array}{l} V_c = -100 \text{ MeV} \\ V_T = V_T^{(1)}(1.0) \end{array} \right\} 0.7 > x > x_c,$
$V = +\infty, \quad x_c > x,$	$V = +\infty, \quad x_c > x$
$x_c = 0.34 \quad (90 \text{ MeV})$	$x_c = 0.36 \quad (90 \text{ MeV})$
$= 0.28 \quad (150 \text{ MeV}).$	$= 0.31 \quad (150 \text{ MeV}).$

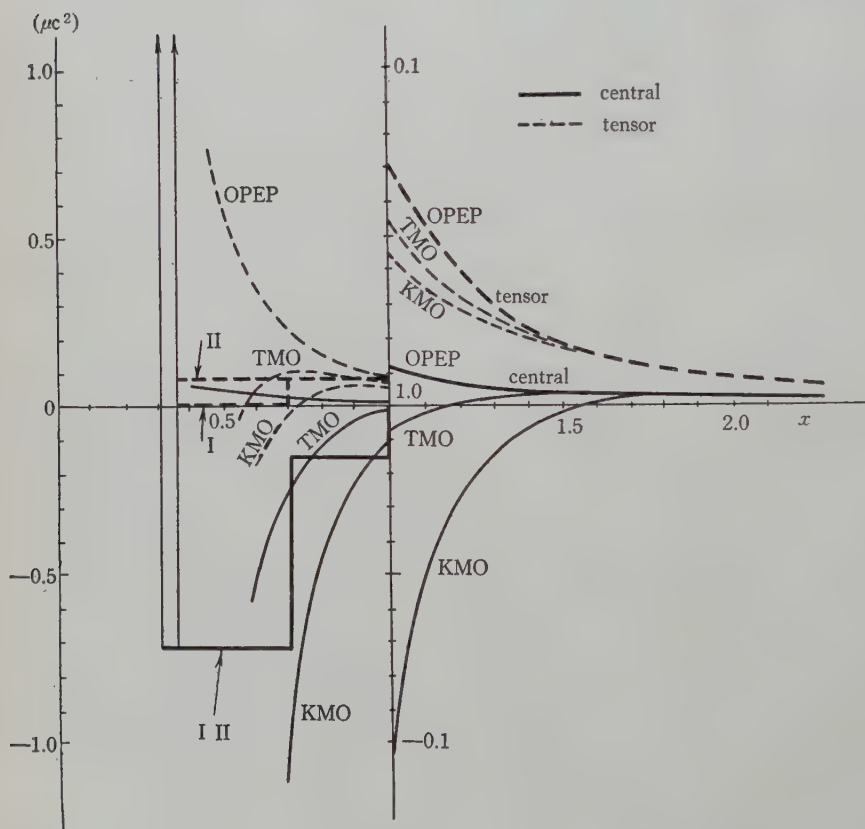


Fig. 1. Potentials in the triplet odd state. For the two-pion exchange potentials, see ref. 4).

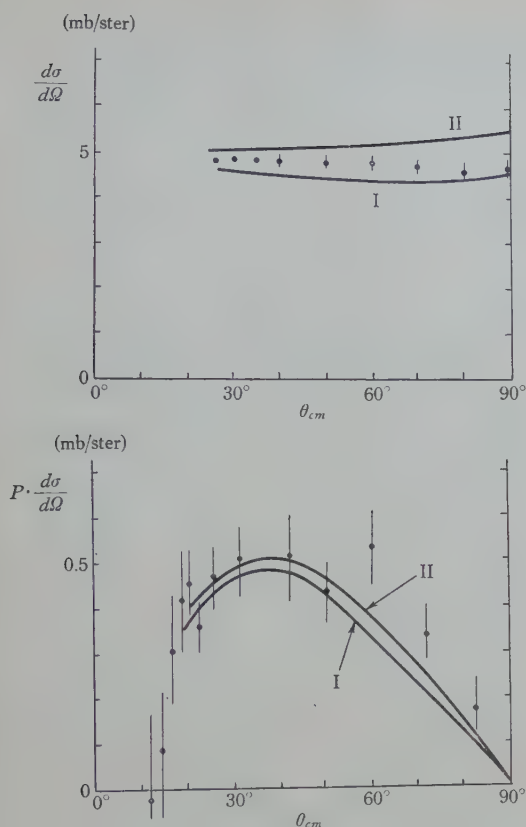


Fig. 2. Angular distribution and polarization cross section at 90 MeV. Theoretical curves are due to potentials of Fig. 1.

attempted a kind of phase shift analysis for the p - p and n - p scattering at 90 MeV and 150 MeV to see whether there exists any possibility to reproduce the data by the phase shifts based on the pion-theoretical potential, and obtained a set of phase shifts which has the characteristic features of the strong tensor force due to the one-pion-exchange potential. Then, we have tried to find some sets of potentials with the one-pion-exchange tail to reproduce the p - p single and double scattering data at 90 and 150 MeV. Examples of the potentials thus selected out are given in Fig. 1

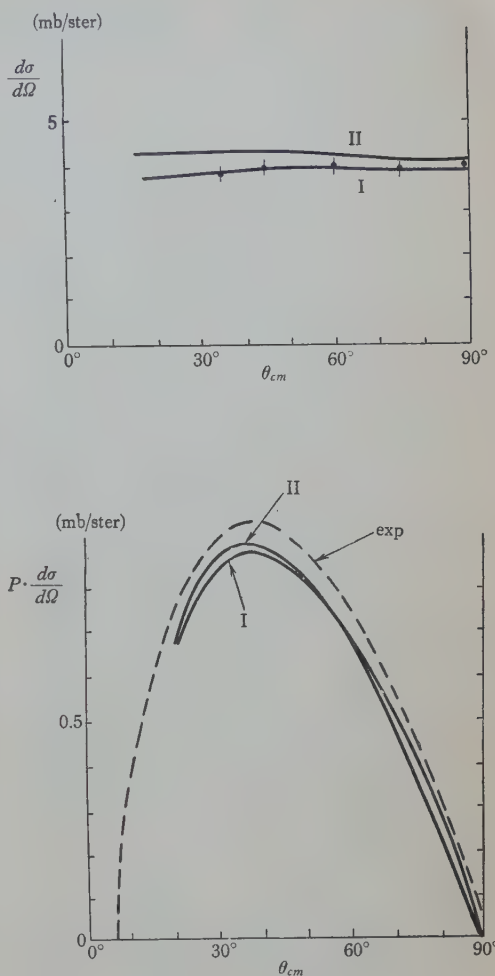


Fig. 3. Angular distribution and polarization cross section at 150 MeV.

and Table 1. The first are shown in Figs. 2-3, together with the predicted phase shifts in Table 2. Care was taken not to destroy the fit obtained at low energies. As seen in Table 1, there is a very weak energy dependence of a parameter of the inner-most potential, that is, the hard core radius. The possibility is not excluded that the energy dependence would disappear if we adopt-

Table 2. Predicted phase shifts at 90 MeV and 150 MeV.
(Blatt-Biedenharn phase shift in radian.)

	${}^3\delta_0^\tau$	${}^3\delta_1^\beta$	${}^3\delta_2^\alpha$	${}^3\delta_2^\tau$	ε_2	${}^1\delta_0$	${}^1\delta_2$	${}^3\delta_3^\beta$	${}^3\delta_4^\alpha$	${}^3\delta_4^\tau$	ε_4
90MeV											
I	0.53	-0.18	0.11	-0.02	-0.50	0.20	0.05	-0.028	0.011	-0.004	-0.60
II	0.56	-0.19	0.11	-0.02	-0.52	0.10					
150MeV											
I	0.64	-0.13	0.22	-0.004	-0.34	-0.10	0.08	-0.034	0.028	-0.005	-0.58
II	0.68	-0.17	0.20	-0.01	-0.40	0.05					

ed the more realistic shape for the potential around the poin range. In the course of our analysis, we did not find indications of the necessity of strong LS potential in the outer region. The point was already discussed in other places.³⁾ We also notice that the shapes of the proposed potentials are quite similar to that of the two-poin-exchange potential⁴⁾, which suggests the qualitative validity of the static poin-theoretical potential even at these high energies.

The detailed accounts will be published in the Progress of Theoretical Physics in the near future.

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- 4) See Ref. 1. Part III, and IV.

Note on Electrodisintegration of He^4

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June 11, 1959

After we published the previous paper,¹⁾ Prof R. Hofstadter has kindly sent us his unpublished experimental data on electrodisintegration of He^4 nucleus in which the scattering angles cover a wider range of 45° to 105° at 400MeV electrons, compared with the previous observations (45° and 60°). In order to get additional confirmation of our previous theory¹⁾ in comparision with the experimental result mentioned above we have extended the numerical work of the cross section to the mentioned range of angles and obtained the following result.

Since the observed cross sections are not given in absolute magnitude but in arbitrary scale, the calculated variations of the cross section with the scattered electron energies have been fitted to those of the observed ones through adjusting the scale factors and taking the appropriate width-parameters of He^4 and residual nucleus (μ_a and μ_t). The scale factors are estimated by comparing the integrated cross sections with the area bounded by the experimetal curves. The results are described in Fig. 1~5 in which the critical energies for meson production are indicated by arrow symbols.

Table 1. The values of μ_α and μ_t adopted in the numerical calculation.

No. of the curves	$\mu_\alpha^{-1/2}(10^{-13}\text{cm})$	$\mu_t^{-1/2}(10^{-13}\text{cm})$
(1)	3.30	5.25
(1)	3.30	3.30
(2)	3.65	3.65
(2)	3.65	5.25
(3)	3.85	5.25
(3)	3.85	3.85
(4)	4.50	4.50
(5)	5.25	5.25

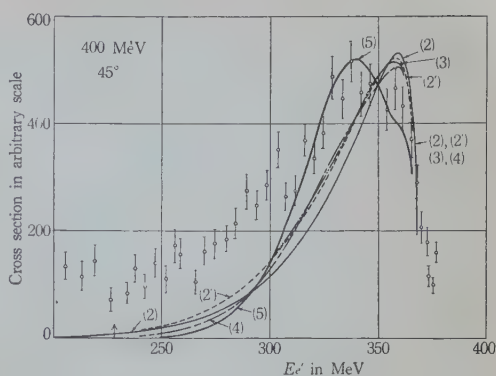


Fig. 1

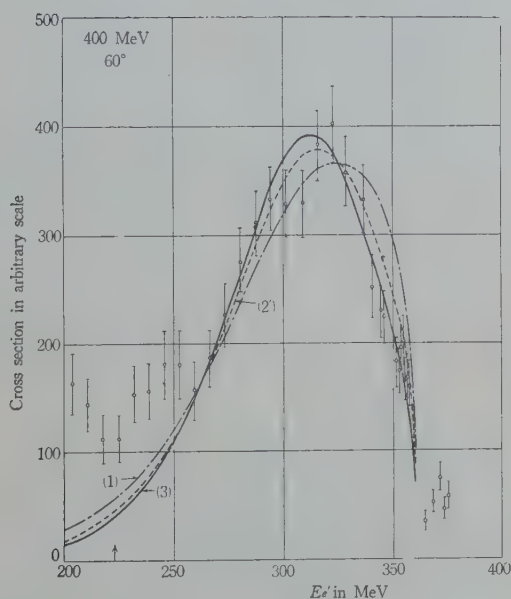


Fig. 2, (i)

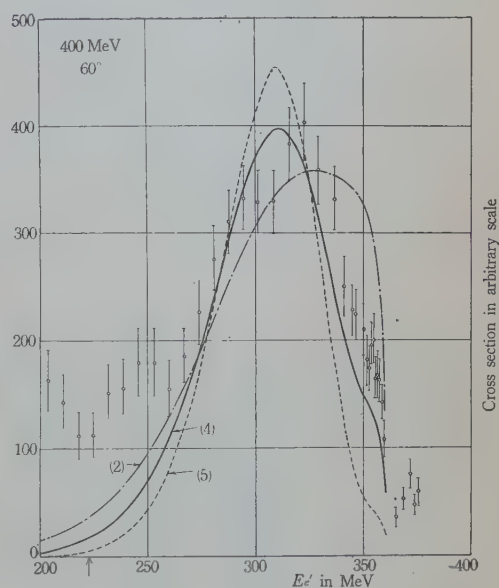


Fig. 2, (ii)

Generally speaking, the agreement of our result with the Stanford experiment 400MeV electrons are rather good except for the 45° data in view of the limit of accuracies involved in the experiment and in our computation, particularly in the assumed form of nuclear wave functions. In the case of 45° data the agreement does not seem to be so good in comparison with the other data, which fact be attributable to the mixing-up of

the other type of nuclear reaction, i.e., deuteron ejection from He^4 . In fact, the deuteron has been seen to become rather dominant in the forward electron scattering according to our unpublished computation.

If the size effect of nucleons determined from the scattering data of a free nucleon is allowed for in the present case the cross sections for all observed scattering angles will increase by small amount in

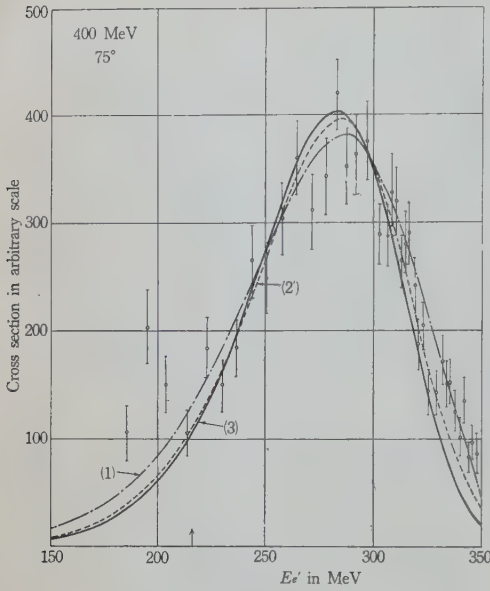


Fig. 3, (i)

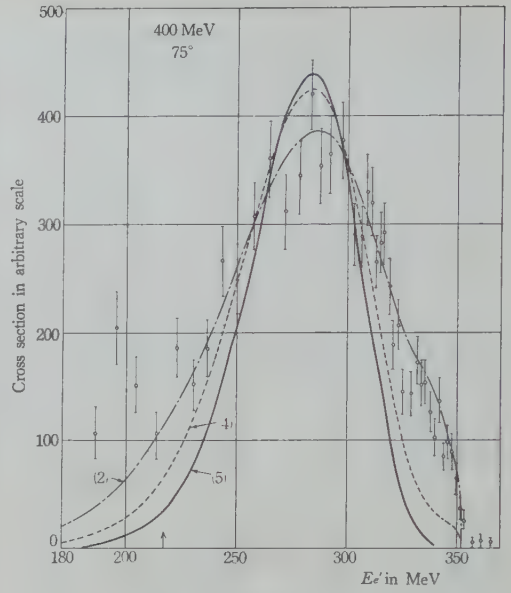


Fig. 3, (ii)

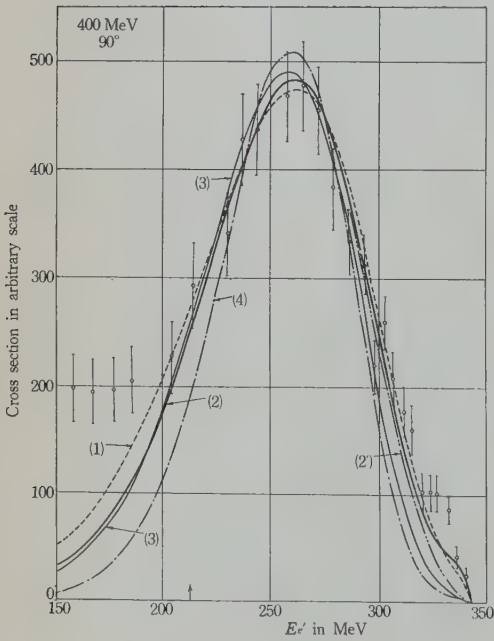


Fig. 4

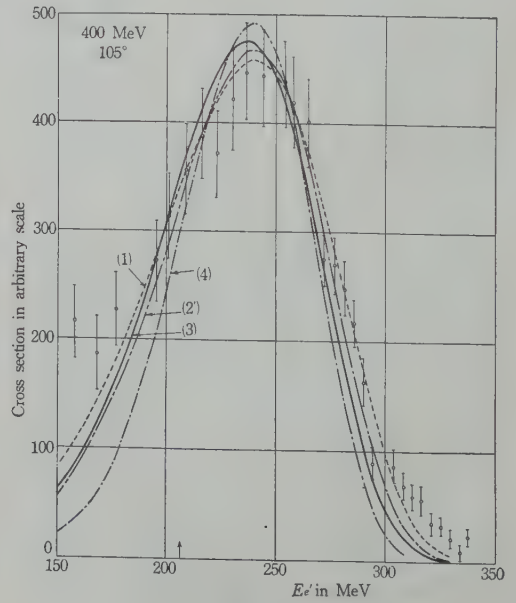


Fig. 5

the lower energy side of the maximum cross section and decrease by roughly equal amount in its high energy side. In the present numerical calculation such nucleon-size effect (Gaussian form with $\langle r \rangle = 0.70$ fermi) has actually been taken into account only in the curves (3), the effect lying within the limit of experimental error.

As for the energy of inelastically scattered electrons corresponding to the maximum cross section, a careful examination of the calculated cross section¹⁾ reveals that the indirect process yields rather minute contribution in the vicinity of the maximum cross section compared with that from the direct process and then the corresponding energy is shown to be approximately determined by the following relation.

$$\frac{3M}{2\hbar^2} \left(\Delta E_e - c^2 \Delta M - \frac{\hbar^2}{8M} \Delta k_e^2 \right) \cong \left(\frac{3}{4} \Delta k_e \right)^2,$$

where $\Delta E_e = E_e - E'_e$ expresses the energy difference between an incident electron and a scattered one, $\Delta \hbar k$ the corresponding momentum change, ΔM the mass difference between He^4 and (a residual nucleus + a nucleon), and M the nucleon mass. The other quantities have the usual meaning. The above relation yields, at $E_e = 400 \text{ MeV}$,

$$\begin{array}{ccc} \theta = 45^\circ, & 60^\circ, & 75^\circ, \\ E_e = 336 \text{ MeV}, & 316 \text{ MeV}, & 284 \text{ MeV}, \\ & 90^\circ, & 105^\circ. \\ & 259 \text{ MeV}, & 238 \text{ MeV}. \end{array}$$

For $\theta \geq 75^\circ$, the above result is seen to be in fair agreement with the observation while, for $\theta = 45^\circ$ and 60° , the observ-

ed energies are found to shift, by small amount, toward the high energy side. This is due to the contribution from the indirect process neglected in deriving the above equation since the mentioned contribution tends to increase in the smaller scattering angles.

The authors would like to express their deep gratitude to Prof. R. Hofstadter for sending us his experimental results prior to publication.

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Magnetic Susceptibility of Ce-Th Alloy

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June 15, 1959

It has been generally accepted that well-localized $4f$ -electrons are promoted into a conduction band in the cubic close packed α -cerium metal. This α -phase succeeds in the lower temperature range to the γ -phase which is also packing structure, and the phase change can also be induced by a hydrostatic pressure. We have previously interpreted the magnetic susceptibility and the anomalous part of specific heat of the α -Ce metal from the viewpoint that half of the $4f$ -electrons are promoted into the conduction band.³⁾

Bates and Newmann observed the susceptibility of Ce-Th alloy and they concluded from the effective Bohr magneton numbers obtained from their experiment that more than one $4f$ -electron might be accommodated in each Ce

atom. However, we rather think that the Ce atoms make nearly complete transition to the α -phase in the alloy. Our consideration is based on the following facts -

(1) Th metal has the structure of cubic close packing.

(2) Th has a slightly smaller atomic radius than Ce ($r_{\text{Th}}=1.80\text{\AA}$, $r_{\text{Ce}}=1.82\text{\AA}$). This difference of the atomic radii can be the cause for giving rise to a large internal stress or pressure.

(3) The inverse magnetic susceptibility curve of each alloy tends to be still flatter as the temperature decreases in comparatively high temperatures and the susceptibility falls rapidly to a small value in low temperatures. This behavior seems to correspond to a gradual transition to the α -phase in the comparatively high temperatures and to a nearly complete transition in the low temperatures.

(4) The hysteresis loops in the susceptibility curves diminish continuously and then vanish when the concentration of Ce is decreased. This phenomenon suggests that it is intimate with the usual α - γ transition of the pure Ce metal.

In order to estimate the degree of the transition we have calculated the susceptibility of the Ce metal in which the fraction α of the $4f$ -electrons are promoted into the conduction band. Several assumptions included in our calculation are as follows:

(1) The Pauli spin paramagnetism due to the conduction electrons is always set to be equal to $\chi_{\text{Laanthanum}}=0.70 \times 10^{-6.5}$ (If there are enough of the $4f$ -electrons the contribution of the Pauli spin paramagnetism to the susceptibility is only

a small fraction so that we have omitted it in our previous calculation on pure Ce metal.)

(2) Expressing the susceptibility due to the $4f$ -electrons in the form of $\chi=C/(T+\theta)$, θ is assumed to change linearly as a function of α and take the values of 12°K for $\alpha=0$ and 10°K for $\alpha=0.5$.

(3) The quantity 4 , which is a measure of the strength of crystalline field³⁾, varies also linearly with α , i.e. $4/k=206 \times (3+\alpha)/3^\circ\text{K}$. This corresponds to the smearing out of valencies.

(4) We neglect a small correction to the Curie constant, which arises from the fact that localized magnetic moment always polarizes the conduction electron through the exchange interaction between them⁶⁾.

(5) We neglect all higher order effects of the crystalline field which mix the higher energy levels in Russell-Saunders's scheme (i.e. the states of different total quantum number J) into the ground state.

The result of our calculation is shown by solid curves in Fig. 1 and also the typical experimental curve (Ce 20%-Th 80%) obtained by Bates and Newmann by dotted curve. That the experimental curve crosses over the solid lines of constant α is interpreted as indicating that the fraction of the promoted $4f$ -electrons increases as the temperature is lowered. The crossing points give the values of α for the respective temperatures. Our curve is, however, sensitive to the contribution from the conduction band, which we actually set to be equal to $\chi_{\text{Laanthanum}}$, in the regions of large α ; hence our calculation should not be

taken too seriously in these regions. A further investigation of the electronic structure of the rare earth metals is now in progress and will be discussed in another paper.

The writer would like to thank Prof. C. Horie and Dr. Y. Osaka of Tohoku University for their profitable discussions.

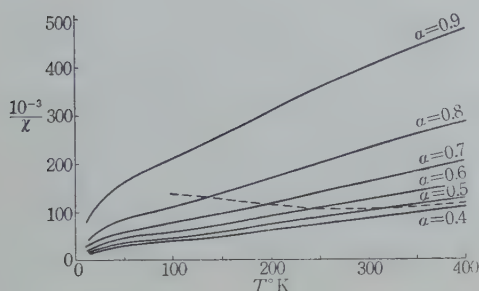


Fig. 1. Calculated susceptibility curves (solid lines) for each value of the fraction α of the promoted 4f-electrons into the conduction band. The dotted curve shows Bates and Newmann's experimental result on Ce 20%-Th 80% sample.

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On the Radiative Muon Decay

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May 18, 1959

Dabis, Roberts, and Zipf have recently reported on an experiment whereby they searched for the event $\mu \rightarrow e + \gamma$ and obtained the value $\sim 10^{-5}$ as the upper limit of the branching ratio $\mu \rightarrow e + \gamma / \mu \rightarrow e + \nu + \bar{\nu}$.¹⁾

The purpose of this note is to suggest that most of the events observed by Davis et al. may be understood as the contaminating process, the internal bremsstrahlung of muon decay ($\mu \rightarrow e + \nu + \bar{\nu} + \gamma$), and the process $\mu \rightarrow e + \gamma$ may be less frequent. We have calculated the decay ratio $\rho_{\gamma} = \mu \rightarrow e + \nu + \bar{\nu} + \gamma / \mu \rightarrow e + \bar{\nu} + \nu$, assuming $A-V$ Fermi interaction²⁾. In the calculation of the numerator, the range of integration over electron energy E is $35 \text{ Mev} \leq E \leq 53 \text{ Mev}$. The lower limit of γ -ray energy k is taken as 20 Mev which is chosen to give nearly the same condition with the experiment. For given E and θ (θ is the angle between electron and γ -ray), the available maximum energy of γ -ray is restricted as

$$k_{\max} = M(W - E) / (M - E + P \cos \theta),$$

Table 1

Range of integration of θ	$\rho_{\gamma}(\theta) \times 10^5$	
	$E_{\min} = 35 \text{ Mev}$	$E_{\min} = 45 \text{ Mev}$
$\theta = \pi \quad 154^{\circ} \leq \theta \leq 180^{\circ}$	0.6	0.2
$\theta = \frac{3}{4}\pi \quad 107.5^{\circ} \leq \theta \leq 162.5^{\circ}$	2.2	0.7
$\theta = \frac{1}{2}\pi \quad 67^{\circ} \leq \theta \leq 113^{\circ}$	2.2	0

where $W = (M^2 + m^2)/2M$. M and m are the masses of muon and electron respectively, and p is the electron momentum. The angular integration is performed over $4\pi \times 0.05$ solid angle around the point $(\theta, \varphi = 0)$. The results are given in Table I. Probably the present estimation should be multiplied by a factor ~ 2 , when we compare it with the experiment of D-R-Z.

It is seen from the first column of Table I that the value of $\rho_\tau(\pi)$ is consistent with the observed value of $\sim 10^{-6}$. This indicates that most of the observed events should be understood as the contamination by the $\mu \rightarrow e + \nu + \bar{\nu} + \gamma$ process and suggests the possibility that careful correction of $\mu \rightarrow e + \nu + \bar{\nu} + \gamma$ process may give much lower limit for the $\mu \rightarrow e + \gamma$ process. To obtain more precise value for the ratio $\mu \rightarrow e + \gamma$, it will be necessary to use the γ -telescope and electron telescope with more high-energy cut and small aperture. For example, if we use the γ -telescope with aperture of a tenth of $4\pi \times 0.05$, then the expected ratio of $\mu \rightarrow e + \nu + \bar{\nu} + \gamma$ will be smaller than a tenth of the $\rho_\tau(\pi)$ of table I, while the process $\mu \rightarrow e + \gamma$ will not decrease so much as the $\mu \rightarrow e + \nu + \bar{\nu} + \gamma$ process. In the second column of Table I we give the ρ_τ when we take 45 Mev as the minimum of E . We have also estimated the angular dependency of the internal bremsstrahlung for the similar experimental situation as that used for the present experiment. These are also given in Table I. The investigation of the angular distribution will supply the information which is useful for supporting the value of ρ_τ at $\theta = \pi$ and can also give the

proof for the type of Fermi interaction³⁾.

Finally, we should like to note that although the fact that the process $\mu \rightarrow e + \gamma$ is rarer than the theoretical value $\sim 10^{-4}$ estimated by Feinberg⁴⁾ may be discouraging for the consideration that the weak Fermi interaction is mediated by the charged vector boson, the rarity of $\mu \rightarrow e + \gamma$ even cannot deny the "charged boson hypothesis" completely, since there is an ambiguity in the theoretical estimate of the branching ratio due to the cut-off parameter and also the selection rule such as Konopinski-Mahmoud's one⁵⁾ might be operating.*

We wish to thank Dr. S. Ogawa for his valuable discussions.

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Note Added :

After we have completed the present article, we saw the paper recently published by Berley, Lee, and Bordon (Phys. Rev. Lett. **2** (1959), 357) in which they obtained more small lower limit for $\mu \rightarrow e + \gamma$ (2×10^{-6}).

* This is not the case when the neutrino is of two-component. However, if we do not take the two-component neutrino (this is possible so long as the present experimental data are concerned), Konopinski-Mahmoud's rule can survive.

Possible Explanation of the Energy Gap in the Excitation Spectra of Nuclei

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June 13, 1959

Bohr, Mottelson and Pines¹⁾ noticed the energy gap in the intrinsic excitation spectrum of nuclei. The energy of first excited intrinsic states is markedly different for the even-even nuclei and the odd mass nuclei. In the even-even nuclei, the first intrinsic excitation appears at about 0.8~1.0 Mev, whereas in the odd mass nuclei it is observed about 0.2 Mev or so. See Fig. 1 of their paper and the table of isotopes.²⁾

The first intrinsic excitation in odd mass nuclei seems to be related closely to the single particle excitation, predicted by the shell model. If the first excited intrinsic state is of the same parity as the ground state, the spin change by this excitation is $\Delta j = \pm 1$ (e.g., Lu¹⁷⁵, Hf¹⁷⁹, Ta¹⁸¹, W¹⁸³, Au¹⁹⁷), with only a few exceptions (e.g., Pu²³⁹). On the other hand, in the even-even nuclei, the spin of the first excited intrinsic state is 0 in most of the cases (e.g., Er¹⁶⁶, Po²¹⁴, U²³⁴, Pu²³⁸), and 2 in other cases (e.g., W¹⁸², W¹⁸⁴, Hg²⁰⁰, Rn²¹⁸, Th²⁸⁸), when we pay attention to those intrinsic states which have the same parity as the ground state. This shows that the mechanism of the intrinsic excitation in the even-even nuclei differs markedly from those in the odd mass nuclei. The purpose of the present note is to give

a possible explanation of the gap in the excitation spectra of nuclei.

First we discuss the odd mass nuclei. Near the top of the Fermi sea, the level distance of the degenerate system (spin up or down) is given by³⁾

$$d_F = (4/3) \cdot (T_F/A). \quad (1)$$

If we take the usual value of 40 Mev³⁾ as T_F , the value of d_F becomes nearly equal to the value predicated by Bohr et al.¹⁾ However, the potential seen by a nucleon is velocity dependent. The mass is reduced perhaps to about 0.6 M at the top of the Fermi sea.³⁾⁴⁾ Then, we must take $(0.6)^{-1}$ times the usual value of the Fermi energy as T_F , when we take the inter-nucleon correlation into account. In the ground state of the odd mass nuclei, the level distance is then $d_F \sim 90A^{-1}$ Mev. There are two sorts of excitations which are degenerated in zeroth order; one-particle-jump and one-hole-jump. But the nuclear force removes this degeneracy in first order. Thus the first intrinsic excitation is reduced, say, to $1/2 \cdot d_F$ in the average. This value is consistent with experiment.

In the even-even nuclei, the rather

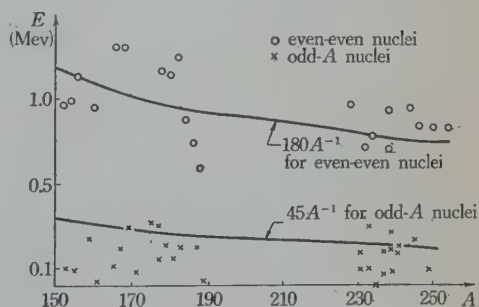


Fig. 1 The experimental value of the first intrinsic excited state collected by Bohr, Mottelson and Pines,¹⁾ and our predicted value as shown by the solid curve

strong pairing energy ($1 \sim 2$ Mev) makes it more probable to make the degenerated two particles jump simultaneously, than to make one particle jump after breaking up the degeneracy. There are two sorts of excitations, which are degenerated in zeroth order; two-neutrons-jump and two-protons-jump. In this case the internuclear two-body forces cannot remove the degeneracy in first order. Then the energy required to this jump, i.e., the first intrinsic energy is $2d_F \sim 180A^{-1}$ Mev. Again this value is consistent with the energy gap in the even-even nuclei.

Such a simple statistical explanation cannot account for the spin change of the type $0^+ \rightarrow 2^+$. However, the mechanism of the intrinsic excitation would be very similar.

nism of the intrinsic excitation would be very similar.

The author wishes to acknowledge many helpful discussions with Professor M. Kobayashi, Professor S. Yoshida, Dr. A. Arima, Dr. A. Iwamoto and Dr. M. Yasuno.

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- 5) The effective mass correction to the level distance is also discussed by F. H. Bakke, Nucl. Phys. **9** (1958), 670. The author wishes to thank Prof. S. Yoshida for his notifying this to the author.

Errata

Molecular Processes induced by μ^- Mesons in Hydrogen Bubble Chamber. II

Yukio Mizuno, Takeo Izuyama and Mikio Shimizu

Prog. Theor. Phys. **21** (1959), 479.

Following corrections should be made.

p. 479, 7th line from the bottom; (44ev) should be replaced by (-44ev)

p. 480, 11th line from the bottom; "at the limit of zero bombarding energy" should be eliminated.

p. 480, 9th line from the bottom; " ϕ_J, ϕ_P and ϕ_S " should be replaced by " ϕ_P and ϕ_S ".

On the Wave Functions of Higher Spin Particles

Shoichi Hori

Prog. Theor. Phys. **21** (1959), 613.

The following corrections should be made.

p. 617, 15th line... $p_0 = (\mathbf{p}^2 + \kappa^2)$ should be read as $p_0 = (\mathbf{p}^2 + \kappa^2)^{1/2}$.

p. 618, 1st line... *however, that* should be read as *however, noted that*.

p. 620, eq. (34)... The second argument of E_{prod} should read m'_Y instead of m'_K .

p. 622, eq. (51)... The second term in the bracket should be replaced by

$$\lambda[(2K+3)/(2K+1)]^{1/2}(KK+10/KK+1L0)(s_Y L m'_Y M | s_Y L s_Y m_Y) W(KK+1 s_Y s_Y; L \frac{1}{2}).$$

p. 622, footnote... The equation should read as follows, $Y_K^{R*} = (-1)^R Y_K^{-R}$.

Charge-Transfer Forces in Molecular Compounds. II*

—Benzene-Iodine Complex—

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(Received January 9, 1959)

It is of much interest to inquire whether the structure of the benzene-iodine complex belongs to the symmetry of C_{2v} or C_{6v} . Here the former is called the *R* (resting) model and the latter the *A* (axial) model. Some recent experiments on the infra-red spectra of this complex and the observation of the crystalline state of the benzene-bromine complex support the *A* model of C_{6v} symmetry. In this article the author concludes from a theoretical consideration that the *A* model is most probable for the structure of the benzene-iodine complex. In the *A* model, the charge-transfer takes place from ϕ_0 , the MO (molecular orbital) of benzene to $\phi(\sigma_u, 5p)$, the MO of iodine. The energy loss of the charge-transfer state which is involved in adopting the *A* model instead of the *R* model, is supplied by the "disscreening effect" and the interaction energy between the no-bond state and the charge-transfer state. This interaction energy is far larger in the *A* model than in the *R* model.

The stabilization energy, the dipole moment, the charge-transfer spectrum and the oscillator strength are calculated, and we have obtained fairly good agreement with the observations.

§ 1. Introduction

Many interesting observations about the benzene-iodine complex have been reported. This complex shows an intense characteristic absorption near $\lambda=2900 \text{ \AA}^{1)}$ which does not appear in case benzene and iodine exist independently. Mulliken²⁾ has explained this absorption theoretically as the charge-transfer spectrum, and estimated the oscillator strength of this spectrum at 0.30 from the experimental value obtained by Green and Ress³⁾. This complex has the dipole moment $0.72D^{2,4)}$, which is an evidence of the existence of the charge-transfer state. The stabilization energy has been measured by Bower⁵⁾ as 1.72 kcal/mol.

To a certain extent Mulliken has succeeded in a semi-empirical explanation of the above mentioned observations in view of the charge-transfer mechanism, by means of the *R* model (Fig. 1). However, Ferguson⁶⁾ has recently concluded from the observation of infra-red spectra of this complex that the *A* model having the C_{6v} symmetry (Fig. 2) is suitable to this complex. Hassel⁷⁾ has observed that the benzene-bromine complex has the axial structure in the crystalline state, in which the distance between the carbon and the next bromine atom is 3.70 Å. We are

* The previous paper published in Prog. Theor. Phys. **20**, 133 (1958) is called "I" of the present series.

now much interested in a theoretical explanation of the benzene-iodine complex having the C_{6v} symmetry.



Fig. 1



Fig. 2

The solid line represents the MO of iodine, $\phi(\sigma_u, 5p)$. The dotted line represents the MO's of benzene, ϕ_{-1} in Fig.1 and ϕ_0 in Fig. 2.

The reasons why Mulliken has given up the A model and adopted the R model are summarized as follows.

The vacant MO of iodine to which an electron of benzene may transfer is

$$\phi(\sigma_u, 5p) = \{2(1 - S_A)\}^{-1/2}(\chi_A - \chi_{A'}), \quad (1)$$

which is a strongly antibonding MO, spreading along the direction of bond. Here χ_A and $\chi_{A'}$ are $5p$ atomic orbitals of iodine, and S_A is the overlap integral

$$S_A = \int \chi_A \chi_{A'} dv. \quad (2)$$

The occupied orbitals of benzene are written as

$$\phi_0 = 6^{-1/2}(\chi_I + \chi_{II} + \chi_{III} + \chi_{IV} + \chi_V + \chi_{VI}) \quad (3)$$

$$\phi_{+1} = 12^{-1/2}(2\chi_I + \chi_{II} - \chi_{III} - 2\chi_{IV} - \chi_V + \chi_{VI}) \quad (4)$$

$$\phi_{-1} = 4^{-1/2}(\chi_{II} + \chi_{III} - \chi_V - \chi_{VI}) \quad (5)$$

where $\chi_I \sim \chi_{VI}$ are $2p\pi$ atomic orbitals of corresponding carbon atoms. In the case of the R model, the charge-transfer takes place from (5) to (1), which is easily verified from symmetry consideration. On the other hand, if we adopt the A model, we cannot but consider the charge-transfer from (3) to (1). This process is energetically more difficult to arise at first glance, because ϕ_0 is lower in energy than ϕ_{-1} .

Next, in the calculation of the energy of the charge-transfer state, the electrostatic attraction between the charged molecules must be taken into account. The mean separation distance between the charged centers of the benzene cation and the iodine anion is greater in the A model than in the R model, and therefore the stabilization in the energy of the charge-transfer state arising from the electrostatic attraction favours the R model. In order to explain theoretically that the A model is suitable to this complex, we must overcome the above mentioned points.

§ 2. Theory with numerical calculation

The ground state wave function of this complex BA (B and A refer to benzene

and iodine respectively) is written as

$$\Psi_N = a\Psi_0 + b\Psi_1, \quad (6)$$

where Ψ_0 is no-bond wave function

$$\Psi_0 = \mathcal{A}(BA) = \mathcal{A} \Psi_B \Psi_A, \quad (7)$$

and \mathcal{A} is the antisymmetrizer. Further

$$\Psi_B \Psi_A = \phi_B \alpha(1) \phi_B \beta(2) [B][A], \quad (8)$$

in which $[B]$ and $[A]$ denote the configuration of electrons in the occupied MO's of B except ϕ_B , and that of electrons in the occupied MO's of A , respectively. In (8)

$$\phi_B = \phi_{-1} \quad \text{for the } R \text{ model,}$$

$$\phi_B = \phi_0 \quad \text{for the } A \text{ model.}$$

Ψ_1 is the wave function describing the charge-transfer state

$$\Psi_1 = \mathcal{A}(B^+ A^-), \quad (9)$$

which is written in detail as

$$\Psi_1 = (2 + 2S_{BA}^2)^{-1/2} (\Psi_I + \Psi_{II}), \quad (10)$$

$$\Psi_I = \mathcal{A} \phi_B \alpha(1) \phi_A \beta(2) [B][A] \quad (11)$$

$$\Psi_{II} = \mathcal{A} \phi_A \alpha(1) \phi_B \beta(2) [B][A], \quad (12)$$

$$S_{BA} = \int \phi_B \phi_A dv. \quad (12)$$

The coefficients of (6) have the relation

$$a^2 + 2abS + b^2 = 1, \quad (13)$$

$$S = \int \Psi_0 \Psi_1 dv = \{2/(1 + S_{BA}^2)\}^{-1/2} S_{BA}. \quad (14)$$

As $[B]$ and $[A]$ are regarded as fixed in the calculation, the effect of these electrons may be treated as a part of the effective Hamiltonian. In (6) the mixing of Ψ_1 into Ψ_0 seems to be very small ($a \gg b$), so the second order perturbation theory is an adequate approximation in this case. The ground state energy of this complex is

$$W_N \cong W_0 - \frac{(H_{01} - SW_0)^2}{W_1 - W_0} \cong W_0 - \frac{H_{01}^2}{W_1 - W_0}, \quad (15)$$

where

$$W_0 = \int \Psi_0 H \Psi_0 dv, \quad W_1 = \int \Psi_1 H \Psi_1 dv, \quad H_{01} = \int \Psi_0 H \Psi_1 dv. \quad (16)$$

The stabilization energy is

$$\Delta W = H_{01}^2 / (W_1 - W_0). \quad (17)$$

It must be determined from the estimation of the stabilization energy whether the *R* model or the *A* model should be suitable to the benzene-iodine complex. In estimating (17) the denominator ($W_1 - W_0$) takes a smaller value in the *R* model than in the *A* model as mentioned in § 1. Here we are led to use the following approximation for H'_{01} , otherwise it must be treated as a semi-empirical parameter.

$$H'_{01} = \beta S / S_\beta, \quad (18)$$

where S_β is the overlap integral between $2p\pi$ orbitals of adjacent carbon atoms in benzene, and β is the resonance integral of the corresponding orbitals, of the magnitude -40 kcal/mol.

$$S_\beta = \int \chi_I \chi_{II} dv = 0.26.$$

The width of the benzene ring 2.41 Å is nearly equal to the interatomic distance of the iodine molecule 2.66 Å. Therefore, in the *R* model, the $2p\pi$ carbon atomic orbital is almost orthogonal to the $5p$ iodine atomic orbital, and the overlap integral between them becomes nearly zero.

In the case of the *A* model, we may easily understand from the orientations of atomic orbitals that the value of S_{BA} becomes largest. Here we ought to use (3) as ϕ_B and the polarized form of (1) for ϕ_A , which is written as

$$\phi_A = N^{-1/2} (\chi_A - \lambda \chi_{A'}). \quad (19)$$

In order to estimate the parameter λ , we may adopt the following consideration as a reasonable one. The positive charge of benzene in the charge-transfer state is assumed to be at the center of the benzene ring, and the negative charge density about *A* or *A'* atom of iodine is thought to be inversely proportional to the distance between the positive charge of benzene and the *A* or the *A'* atom of iodine respectively. Then we obtain $\lambda = 0.82$.

Thus the overlap integral S is calculated at

$$\left. \begin{aligned} S &= 0.0058 && \text{for the } R \text{ model,} \\ S &= 0.11 && \text{for the } A \text{ model,} \end{aligned} \right\} \quad (20)$$

where the Slater orbitals are used for the carbon and the iodine atomic orbitals and the Mulligan approximation⁹⁾ is used for the heteropolar integral. (20) suggest very small interaction energy for the *R* model, therefore we may conclude that the *R* model is not adequate to this complex. In the following our discussion shall be directed to the *A* model and let us examine how to overcome what Mulliken has pointed out.

The interaction energy H'_{01} is estimated by substituting (18) with (20), and we obtain

$$H'_{01} = -0.72 \text{ eV.} \quad (21)$$

The energy of the no-bond state is

$$\begin{aligned}
 W_0 &= \int \Psi_0 H \Psi_0 dv \\
 &= \int \phi_B \alpha(1) \phi_B \beta(2) (H_B + H_A) \phi_B \alpha(1) \phi_B \beta(2) dv \\
 &\cong 2 \int \phi_B H_B \phi_B dv \\
 &\cong -2I_B \\
 &\cong -2(9.2 + 1.7) \text{ eV}.
 \end{aligned} \tag{22}$$

Here the electronic Hamiltonian is the sum of the self-consistent one-electron Hamiltonian and they are considered to be constructed by two parts, part *B* and part *A*. I_B is the second ionization potential of benzene and is estimated to be the sum of the first ionization potential 9.2 eV and $-\beta = 1.7$ eV.

On the other hand, the charge-transfer state, as has been written in (11), has the positive hole in part *B*. In the evaluation of the energy of this state we must take account of the effect of this positive hole in the electronic Hamiltonian, that is to say, the Hamiltonian referred to *B* is to be H_B^+ , then

$$\begin{aligned}
 W_1 &= \int \Psi_1 H \Psi_1 dv \\
 &= \int \phi_B \alpha(1) \phi_A \beta(2) (H_B^+ + H_A) \phi_B \alpha(1) \phi_A \beta(2) dv \\
 &\cong \int \phi_B H_B^+ \phi_B dv + \int \phi_A H_B^+ \phi_A dv + \int \phi_A H_A \phi_A dv,
 \end{aligned} \tag{23}$$

where

$$\int \phi_A H_A \phi_A dv \cong -E_A \cong -1.8 \text{ eV} \tag{24}$$

in which E_A is the electron affinity of the iodine molecule and has been reported at the magnitude of larger than 1.8 eV⁹⁾

$$\int \phi_A H_B^+ \phi_A dv \cong -1/r_e = -3.3 \text{ eV} \tag{25}$$

seems to be a good approximation as the inter-molecular distance is very large, where r_e is the distance between the positive charge of benzene and the mean charge distribution of the iodine molecule. Next,

$$\begin{aligned}
 \int \phi_B H_B^+ \phi_B dv &= \int \phi_B H_B \phi_B dv + \Delta \\
 &= -I_B + \Delta.
 \end{aligned} \tag{26}$$

Here Δ is the effect arisen from that the Hamiltonian has changed from H to H^+ , and called the "discreening effect" hereafter. It is estimated approximately as the difference of the Coulomb interactions between above two states,

$$\Delta \cong (\delta - \delta^+) / r_i. \quad (27)$$

Here r_i is the mean distance between the $2p\pi$ electron and the positive charge at the center of the benzene ring, and δ and δ^+ are the orbital exponents of the neutral $2p\pi$ carbon atom and that of the positive carbon ion respectively. r_i is easily estimated at 1.65 Å (3.12 a.u.), then

$$\begin{aligned} \Delta &\cong (1.625 - 1.80) / 3.12 \\ &= -0.056 \text{ a.u.} = -1.52 \text{ eV.} \end{aligned}$$

In (26), Δ is in appearance the stabilization energy for the electron left in ϕ_B , but the above estimation of Δ in (27) contains all stabilization energies for the electrons not only in ϕ_0 but also in ϕ_{+1} and ϕ_{-1} of benzene. And this treatment is fortunately valid in this problem.*

From the foregoing results we obtain

$$\begin{aligned} W_1 - W_0 &= I_B - E_A - 1/r_e + \Delta \\ &= 4.3 \text{ eV.} \end{aligned} \quad (28)$$

Now we calculate the stabilization energy, using (28) and (21) combined with (17),

$$\Delta W = 0.12 \text{ eV} = 2.8 \text{ kcal/mol.}$$

The coefficients a and b in (6) are evaluated by using

$$b/a = -H_{01}' / (W_1 - W_0) \quad (29)$$

and the normalization condition (13), at

$$a = 0.97, \quad b = 0.16. \quad (30)$$

The dipole moment of the ground state

$$\mu_N = -e \int \Psi_N \sum r_i \Psi_N dv \quad (31)$$

is calculated in quite the same way as Mulliken²⁾ has used, at

$$\mu_N = 0.94 \text{ D.}$$

In order to calculate the charge-transfer spectrum and the oscillator strength associated with it, the excited state shall be considered.

$$\Psi_E = a^* \Psi_1 - b^* \Psi_0, \quad (32)$$

$$a^{*2} - 2a^*b^*S + b^{*2} = 1. \quad (33)$$

The energy of the excited state is given as

* For this discussion the author is very much obliged to Prof. S. Nagakura.

$$W_E \cong W_1 + \frac{(H_{01} - SW_1)^2}{W_1 - W_0} \cong W_1 + \frac{H_{01}^2}{W_1 - W_0} \quad (34)$$

The frequency of the charge-transfer spectrum is given as

$$h\nu = W_E - W_N = W_1 - W_0 + 2H_{01}^2 / (W_1 - W_0) \quad (35)$$

and we obtain

$$\nu = 36,500 \text{ cm}^{-1}.$$

The coefficients a^* and b^* are estimated at

$$a^* = 1.00, \quad b^* = 0.27. \quad (36)$$

The oscillator strength

$$f = (4.704 \times 10^{-7}) \nu \mu_{EN}^2,$$

where μ_{EN} is the transition dipole, is calculated by the similar procedure to that of calculating the ground state dipole moment, and then we obtain

$$f = 0.34.$$

§ 3. Discussion and conclusion

The calculated results are tabulated in Table 1, where we can see fairly good agreement between the calculated results and the observed values. The charge-transfer spectrum $36,500\text{cm}^{-1}$ in the observed values of Table 1 is the value which is obtained by adding $2,000\text{cm}^{-1}$ to the actual observed value as a solvent effect³⁾.

Table 1.

	calc.	obs.
stabilization energy (kcal/mol)	2.8	1.72 ⁶⁾
dipole moment (Debye unit)	0.94	0.72 ²⁾
charge-transfer spectrum (cm^{-1})	36,500	36,500 ¹⁾
oscillator strength	0.34	0.30 ²⁾

From the preceeding calculations we can conclude as follows: The geometrical structure of the benzene-iodine complex is not the R model having the symmetry of C_{2v} which has been postulated by Mulliken, but the A model having the symmetry of C_{6v} which is supported by the recent experiments. We have concluded that the R model is not suitable to this complex by the estimation of the interaction energy between the no-bond state and the charge-transfer state of this complex. Mulliken has not exactly calculated the overlap integral between these two states, but has simply assumed for it the value of 0.1, which is, however, obtained when we adopt the A model. The estimation of the overlap integral is essentially im-

portant, so long as it is assumed that the interaction energy is proportional to the value of the overlap integral, as has been done in (18). Though the Slater orbital of high quantum number, say the $5p$ atomic orbital of iodine, does not seem to be adopted as a fully accurate one, the semi-qualitative treatment may be possible under the approximations assumed in this article. Also we may note that Mulliken has used 1.2 eV for the electron affinity of the iodine molecule (the observed value is larger than 1.8 eV.)

The charge-transfer does not always take place from the highest occupied orbital of the charge-donating molecule to the lowest vacant orbital of the charge-accepting molecule. In other words, we should not expect the simple relation between the first ionization potentials of the charge-donating molecules and the charge-transfer spectra of the complexes. In this problem we cannot but consider the charge-transfer from ϕ_0 of benzene to $\phi(\sigma_u, 5p)$ of iodine, where the former is not the highest occupied orbital of benzene. The energy loss caused by adopting the A model is compensated by the introduction of the "disscreening effect" which is the stabilization energy for the electrons left in the MO's of the charge-donating molecules. This effect is originated from the change of the orbital exponents of the atomic orbitals of the charge-donating molecule, corresponding to the change from the neutral state to the positively charged state of the molecule.

Acknowledgments

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The Pion-Nucleon S-Wave Scattering, the Structure of the Nucleon and the Composite Model of Baryons and Mesons

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It seems impossible that the pion-nucleon s-wave scattering can be explained in the frame of meson theory. It is suggested from the experimental result that the static pion-nucleon potential with the appropriate isospin dependence has the possibility of explaining this phenomenon. The following characteristics of Fermi-Yang's composite model of pions are noticed: (i) Pions are bound states of one nucleon and one antinucleon. (ii) The attractive short range static potential exists between nucleon and antinucleon, and the repulsive short range static potential between nucleon and nucleon. (iii) The numbers of nucleons and antinucleons are independently conserved.

Using this model, we obtain the static potential between a pion and a nucleon. When the interaction type is vector, the static pion-nucleon potential with the charge dependence $\tau \cdot \omega$ is given. In this case, the pion-nucleon force in the state $I=\frac{1}{2}$ becomes attractive, therefore the bound states of two nucleons and one antinucleon may be possible. We consider this bound state of total isospin one half and spin one half as the nucleon core. Then the possibility of understanding the extension and other mysterious properties of the nucleon core exists.

This model is extended to every baryon and meson.

§ 1. Introduction

The experimental results about the pion-nucleon s-wave scattering are expressed by the formulae¹⁾

$$\left. \begin{aligned} \delta_3 &\simeq -0.77(k/M_{ob}), \\ \delta_1 &\simeq +1.1(k/M_{ob}), \end{aligned} \right\} \quad (1.1)$$

where k is the center-of-mass momentum of pions, and M_{ob} is the observed nucleon mass. In spite of the great successes of the meson theory²⁾ for the structure of the meson cloud about the nucleon, we have no satisfactory explanation about the s-wave scattering.³⁾ The similar circumstance exists for the structure of the nucleon core and the electron-neutron interaction.⁴⁾ Several trials⁵⁾ including the effects of the virtual K -particles have been performed. However, the results of these calculations do not seem to be consistent and favorable. At present it is natural to think that these phenomena are essentially different from the P -wave phenomena.

Therefore our starting point is that

(I) The phenomena concerning the pion-nucleon s-wave scattering and the nucleon

core are beyond the limit of the validity of meson theory.

From the behavior of the phase shifts (1.1), we conclude that

(II) No resonance occurs in pion-nucleon s -wave scatterings, and

(III) The pion-nucleon s -wave scattering may be expected to be explained by the static potential with the appropriate isospin dependence.

The phenomenological treatment by the pion-nucleon static potential has too many possibilities and, therefore, is not adequate for our purpose. Thus we must seek the appropriate model satisfying the requirements of (I), (II) and (III).

Fortunately, Fermi and Yang's model⁶⁾ seems to fit these conditions. Their model is characterized by the following hypotheses:

- (i) Pions are the bound states of one nucleon and one antinucleon.
- (ii) Between nucleons and antinucleons the Fermi interactions exist.
- (iii) The interactions between nucleons and antinucleons are replaced by the static potentials with range $1/M$, where M is the "nucleon" mass.

Fermi and Yang replaced the Fermi interaction by the static potential basing upon the consideration that the effects of such processes as $N + \bar{N} \rightarrow N + \bar{N} + N + \bar{N}$ (N represents a "nucleon" and \bar{N} an "antinucleon"), etc., collectively extend the interaction. If the s -wave scattering can be explained by the Fermi-Yang model, (iii) may be considered to have the more meaning than to be only a conventional treatment.

Thus, for the phenomena which are beyond the limit of the validity of meson theory, the composite model may become a clue.

Therefore we postulate as follows:

(IV) Pions are the bound states of a "nucleon" and an "antinucleon", belonging to the 1S_0 state.

The meson-nucleon interactions are charge independent, so it is natural to propose that

(V) The Fermi interactions between nucleons and antinucleons are charge independent.

Following (III), the treatment by the static potential between pions and nucleons may be available for the s -wave scattering; therefore it may be assumed that

(VI) For the phenomena concerning the s -wave scattering and the nucleon core, the Fermi interactions between nucleons and antinucleons are replaced by the static potential. In such a case, the hypothesis that the "nucleon" number and the "antinucleon" number are independently conserved is a good approximation.

In Appendix A the static nucleon-nucleon and nucleon-antinucleon potentials are derived from the scalar and vector interactions.

§ 2. Pion-"nucleon" static potential

The interaction between a "nucleon" and a pion, which is composed of one

“nucleon” and one “antinucleon”, is derived from the static potentials between “nucleons” and “antinucleons”.

We will adopt the following notations. The nucleon and the antinucleon forming a pion are denoted respectively by N_1 and \bar{N}_2 , and the nucleon interacting with the pion is denoted by N_0 . The distance between the center-of-mass of the pion and N_0 is designated by R , the distance of N_0 and N_1 is r_1 , and that of N_0 and \bar{N}_2 is r_2 . The relative coordinate of N_1 and \bar{N}_2 is denoted by \mathbf{r} .

The total Hamiltonian of this system is

$$H = H_0^{(0)} + H_0^{(1)} + H_0^{(2)} + H'^{(0-1)} + H'^{(0-2)} + H'^{(1-2)}, \quad (2.1)$$

where $H_0^{(0)}$, $H_0^{(1)}$, $H_0^{(2)}$, $H'^{(0-1)}$, $H'^{(0-2)}$ and $H'^{(1-2)}$ are, respectively, the free Hamiltonian of N_0 , N_1 and \bar{N}_2 and the interaction Hamiltonian of N_0-N_1 , $N_0-\bar{N}_2$ and $N_1-\bar{N}_2$.

We shall treat the problem non-relativistically.

The wave function of pion $\psi(\mathbf{r})$ satisfies the Schrödinger equation:

$$(H_0^{(1)} + H_0^{(2)} + H'^{(1-2)})\psi(\mathbf{r}) = [2M - (1/M)\Delta + H'^{(1-2)}]\psi(\mathbf{r}) \quad (2.2)$$

$$= \mu\psi(\mathbf{r}),$$

where μ is the pion mass.

Then the total Hamiltonian is reduced to

$$H = H_0^{(0)} + \mu + H'_{\pi-N}(R), \quad (2.3)$$

where $H'_{\pi-N}(R)$ is the interaction potential of the pion and the nucleon N_0 . On the analogy of the classical treatment for the interaction between a charge and an electric dipole, we obtain

$$H'_{\pi-N}(R) = \langle [H'^{(0-1)}(r_1) + H'^{(0-2)}(r_2)] \rangle_{\text{Pion}}$$

$$= \int \psi^*(\mathbf{r}) [H'^{(0-1)}(r_1) + H'^{(0-2)}(r_2)] \psi(\mathbf{r}) d\mathbf{v}. \quad (2.4)$$

On introducing the polar coordinate (see Fig. 1), (2.4) is reduced to

$$H'_{\pi-N}(R) = 2\pi \int_0^\infty \rho(r) \langle [H'_{N-\bar{N}}(\sqrt{(R^2 + rR \cos \theta + (r^2/4))})$$

$$+ H'_{N-N}(\sqrt{(R^2 - rR \cos \theta + (r^2/4))})] \rangle r^2 dr \sin \theta d\theta, \quad (2.5)$$

where $\rho(r) = \psi^*(\mathbf{r})\psi(\mathbf{r})$, and we denote as follows:

$$\left. \begin{aligned} H'_{N-N}(r_1) &= h_{N-N}v(r_1), \\ H'_{N-\bar{N}}(r_2) &= h_{N-\bar{N}}v(r_2), \end{aligned} \right\} \quad (2.6)$$

and the function $v(r)$ satisfies

$$v(0) = 1, \quad v(\infty) = 0. \quad (2.7)$$

For example, in the case of the square well potential,

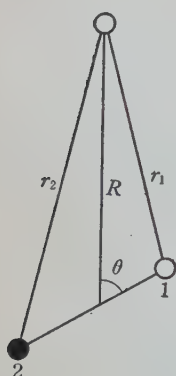


Fig. 1.

$$v(r) = \begin{cases} 1 & (r < b), \\ 0 & (r > b), \end{cases}$$

and in the case of the Gaussian type,

$$v(r) = \exp(-\alpha^2 r^2).$$

Changing $z \equiv \cos \theta \rightarrow -z$ in the second term of the right-hand side of (2.5) and using (2.6), we obtain

$$H'_{\pi-N}(R) = \langle (h_{N-N} + h_{N-\bar{N}}) \rangle_{\text{Pion}} U(R), \quad (2.8)$$

where

$$U(R) \equiv 2\pi \int_0^\infty r^2 dr \rho(r) \int_{-1}^1 dz v(\sqrt{R^2 + (r^2/4) + rRz}) \quad (2.9)$$

has the various forms for the types of the potential $V(r)$. In the case of the square well type,

$$U(R) = 2\pi \int_D \int dz r^2 \rho(r),$$

where D is the integration domain $R^2 + r^2/4 + rRz \leq r^2$. The values of $U(R)$ in this case are given by Fig. 2.

We will calculate the average value of (2.8) for the charge and spin states of a pion. The charge dependence of the "nucleon"- "nucleon" and "nucleon"- "antinucleon" potentials has been derived in Appendix A. In the case of the scalar interaction, we obtain, using (A.10), the following.*

$$\begin{aligned} h_{N-N} + h_{N-\bar{N}} &= [4\eta - \boldsymbol{\tau} \cdot (\boldsymbol{\tau}^{(1)} - \boldsymbol{\tau}^{(2)})] V_0, \\ \langle [h_{N-N} + h_{N-\bar{N}}] \rangle_{\text{Pion}} &= 4\eta V_0. \end{aligned} \quad (2.10)$$

In the case of the vector interaction, we obtain, using (A.11), the following:

$$h_{N-N} + h_{N-\bar{N}} = -2\boldsymbol{\tau} \cdot (\boldsymbol{\tau}^{(1)} + \boldsymbol{\tau}^{(2)}) V_0.$$

Now

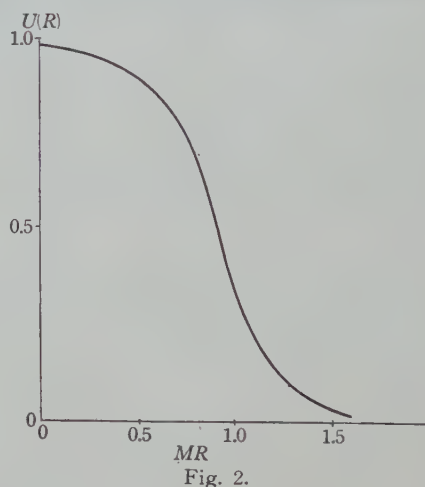
$$\frac{1}{2} \langle (\boldsymbol{\tau}^{(1)} + \boldsymbol{\tau}^{(2)}) \rangle_{\text{Pion}} = \boldsymbol{\omega} \quad (2.11)$$

is the charge operator of a pion, so

$$\langle (h_{N-N} + h_{N-\bar{N}}) \rangle_{\text{Pion}} = -4\boldsymbol{\tau} \cdot \boldsymbol{\omega} V_0. \quad (2.12)$$

Thus in the case of the vector interaction, the $\boldsymbol{\tau}$ -dependent pion- "nucleon" static potential

* V_0 is defined as $V(r) = V_0 v(r)$ in (A.8) and (A.9).



$$H'_{\pi-N}(R) = -4\tau \cdot \omega V_0 U(R) \quad (2.13)$$

is derived. It is needed for this potential to agree with the experimental tendency that $V_0 < 0$. The comparison with the experimental result is discussed in Appendix C.

§ 3. Composite model of the nucleon core

From the result of the preceding section, it is seen that

- (i) for the $I=3/2$ state, the pion-nucleon potential becomes a very strong repulsion with range $1/M$, and
- (ii) for the $I=1/2$ state, a very strong attraction with range $1/M$, where I is the total isospin quantum number of the pion-nucleon system.

This circumstance (i) justifies the $1/M$ cut-off prescription²⁾ for the $I=3/2$ state in meson theory. From the conclusion of (ii), the possibility that two nucleons and one antinucleon make a bound state, exists. The derivation of (2.13) is based on the approximation that the distance of the pion and the nucleon is not so small. When the pion and the nucleon become close to each other, the correlation between two nucleons and one antinucleon acts and the interaction between them becomes stronger. The bound state thus introduced must have the extension of several nucleon Compton wave-length. It is natural to assume that such a bound state is just the nucleon core.

Let us postulate that :

(VII) The nucleon cores are the bound states of two "nucleons" and one "antinucleon" belonging to the $^2S_{1/2}$ state. Here the "nucleon" and "antinucleon" are the bare particles without the meson cloud. We shall denote these "fundamental particles" or "Urmaterie" by N and \bar{N} , or p , n , \bar{n} and \bar{p} . The nucleon core and the antinucleon core are denoted by N and \bar{N} , respectively. The physical nucleon and antinucleon are designated by (N) and (\bar{N}) , respectively.

The composite model of the nucleon thus proposed may explain several phenomena qualitatively.

- (i) Extension of the nucleon core.

The "nucleon"- "nucleon" and "nucleon"- "antinucleon" forces are given in Appendix A. The wave function may, for example, be looked for, using the trial function

$$\Psi = \exp[-\alpha r_{12} - \alpha'(r_{23} + r_{31})], \quad (3.1)$$

where the particles 1 and 2 are nucleons and the particle 3 is an antinucleon. The attractive forces act between 2 and 3, and between 3 and 1. The repulsive force acts between 1 and 2, hence $\alpha' > \alpha$. The spatial configuration is schematically shown in Fig. 3. The extension of the core for the repulsive nucleon-nucleon force is much larger than that for the attractive nucleon-nucleon force. Therefore if the depth and shape of the potential are appropriately given, it may be possible that the extension Δr of the core defined by

$$(\mathcal{J}r)^2 \equiv \frac{1}{3}(\Psi, (r_{12}^2 + r_{23}^2 + r_{31}^2)\Psi) \quad (3.2)$$

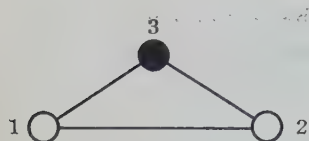


Fig. 3.

becomes about $\frac{1}{3}\mu^{-1}$.

(ii) Nucleon-antinucleon annihilation.

Following Koba and Takeda,⁷⁾ the pion production by the annihilation of the antinucleon is due to the process, that about 2.6 pions produce from the meson clouds of the nucleon and antinucleon and the residual pions from the annihilation of the cores of both particles. In our model of the nucleon, the cores are composite as follows:

$$N: (N, N, \bar{N}) \text{ and } \bar{N}: (\bar{N}, \bar{N}, N). \quad (3.3)$$

Following the hypothesis (VI), the numbers of N and \bar{N} are independently conserved, therefore three pions are produced from the cores:

$$N + \bar{N} \rightarrow (N + \bar{N}) + (N + \bar{N}) + (N + \bar{N}) \rightarrow 3\pi. \quad (3.4)$$

Thus together with the contributions from the meson clouds, $2.6 + 3 = 5.6$ pions are produced. This result is in good agreement with experiment.

(iii) Pion-nucleon s -wave scattering.

Using the static potential (2.13), we obtain the static potential between a nucleon core and a pion as

$$H'_{\pi-N}(R) = \langle [H'_{\pi-N}(r_1) + H'_{\pi-\bar{N}}(r_2) + H_{\pi-\bar{N}}(r_3)] \rangle_{\text{nucleon core}}, \quad (3.5)$$

where r_1 , r_2 and r_3 are, respectively, the distances from the center-of-mass of the pion to the "nucleon" 1 and 2, and the "antinucleon" 3. Roughly estimated, the potential (3.5) reduces to

$$H'_{\pi-N}(R) = -4V_0\tau \cdot \omega U_m(R), \quad (3.6)$$

which becomes the extended interaction, and here

$$\tau = \langle (\tau^{(1)} + \tau^{(2)} + \tau^{(3)}) \rangle_{\text{nucleon core}} \quad (3.7)$$

is the isospin of the nucleon core. $U_m(R)$ is the modified form of $U(R)$. Thus the discussion in the preceding section remains qualitatively unchanged.

(iv) Nucleon-nucleon and nucleon-antinucleon forces.

We obtain the nucleon-nucleon potential using meson theory of nuclear forces in the region of μ^{-1} distance, and in the region of the extension of the nucleon core—about $\frac{1}{3}\mu^{-1}$ distance—by the composite model of the nucleon core. The latter, the N - N force, becomes very strongly repulsive and this effect may be considered as the impenetrable core.²⁾ For the nucleon-antinucleon potential the impenetrable core may be replaced by an absorbing core, owing to the N - \bar{N} strong attractive force. Thus we can lay the foundation for Ball and Chew's phenomenological model.⁸⁾

(v) Multiple production of mesons.

When two nucleons collide with a very great energy, the total energy of this

system is concentrated on the overlapping volume of the two nucleon cores, where the fundamental "nucleons" and "antinucleons" are multiply produced. The interactions between these fundamental particles are non-linear and very strong. The produced energetic ensemble diffuses and many fundamental "nucleons", "antinucleons" and their bound particles, i.e., mesons, are separated out. The separated fundamental particles are extremely unstable and decay into physical nucleons or antinucleons accompanied with the multiple production of mesons.

At comparatively high energies, the correlations between the fundamental particles in the ensemble strongly affect the multiple production, therefore the multiplicity of mesons becomes larger than that of the fundamental particles.

At extremely high energies, the effect of the correlations between the fundamental particles becomes smaller, therefore the multiplicity of mesons is not so larger than that of the fundamental particles, as in the case of comparatively high energy.

(vi) β -decay of a neutron into a proton.

The composite structure of the physical nucleons affects the β -decay of a neutron into a proton. If we assume the V-A weak interaction for the fundamental "neutrons" and "protons" and adopt the wave functions for the neutron and proton cores appropriately, we have the possibility of explaining the observed shift of the β -decay interaction for a physical neutron into a physical proton from the V-A form.

We assume the β -decay interaction for the fundamental "nucleons", for simplicity, non-relativistically, as follows:

$$H_{\beta}' = (g_V + g_A \boldsymbol{\sigma}^{(e)} \cdot \boldsymbol{\sigma}) \frac{\bar{\tau}_1 + i\bar{\tau}_2}{2}, \quad (3.8)$$

where $\boldsymbol{\sigma}^{(e)}$ is the spin vector for an electron and a neutrino, and $\boldsymbol{\sigma}$ and $(\bar{\tau}_1 + i\bar{\tau}_2)/2$ are, respectively, the spin vector and the isospin operator for the fundamental "nucleons." The β -decay ($n \rightarrow p$) occurs for the transitions $n \rightarrow p$ or $\bar{p} \rightarrow \bar{n}$.

It is very difficult to give the spatial wave function, and therefore we replace with two parameters as follows:

(for a neutron)

$$\psi_I = a\varphi_{as}^{-1/2} \chi_s^m + b\varphi_s^{-1/2} \chi_{as}^m, \quad (a^2 + b^2 = 1)$$

(for a proton)

$$\psi_F = a'\varphi_{as}^{1/2} \chi_s^{m'} + b'\varphi_s^{1/2} \chi_{as}^{m'}, \quad (a'^2 + b'^2 = 1),$$

(3.9)

where φ_{as} and φ_s are the isospin eigenfunctions as follows:

$$\begin{aligned} \varphi_{as}^{\pm 1/2} &= (1/\sqrt{2}) \{ \varphi_{1/2}^{1/2}(1) \varphi_{1/2}^{-1/2}(2) - \varphi_{1/2}^{-1/2}(1) \varphi_{1/2}^{1/2}(2) \} \varphi_{1/2}^{\pm 1/2}(3), \\ \varphi_s^{1/2} &= (1/\sqrt{6}) \{ \varphi_{1/2}^{1/2}(1) \varphi_{1/2}^{-1/2}(2) + \varphi_{1/2}^{-1/2}(1) \varphi_{1/2}^{1/2}(2) \} \varphi_{1/2}^{1/2}(3) \\ &\quad - \sqrt{(2/3)} \varphi_{1/2}^{1/2}(1) \varphi_{1/2}^{1/2}(2) \varphi_{1/2}^{-1/2}(3), \\ \varphi_s^{-1/2} &= \sqrt{(2/3)} \varphi_{1/2}^{-1/2}(1) \varphi_{1/2}^{-1/2}(2) \varphi_{1/2}^{1/2}(3) \\ &\quad - (1/\sqrt{6}) \{ \varphi_{1/2}^{1/2}(1) \varphi_{1/2}^{-1/2}(2) + \varphi_{1/2}^{-1/2}(1) \varphi_{1/2}^{1/2}(2) \} \varphi_{1/2}^{-1/2}(3). \end{aligned} \quad (3.10)$$

and the spin eigenfunctions χ_{as} and χ_s are obtained by replacing $\varphi \rightarrow \chi$ in the definitions (3.10).

The β -decay interaction is given by*

$$H'_\beta = \sum_{i=1}^2 \frac{\tau_1^{(i)} + i\tau_2^{(i)}}{2} (g_V + g_A \boldsymbol{\sigma}^{(e)} \cdot \boldsymbol{\sigma}^{(i)}) + \frac{\tau_1^{(3)} + i\tau_2^{(3)}}{2} (g_V - g_A \boldsymbol{\sigma}^{(e)} \cdot \boldsymbol{\sigma}^{(3)}) \quad (3.11)$$

where $\boldsymbol{\sigma}^{(i)}$ ($i=1, 2, 3$) are the spin vector for the fundamental "nucleons" and "antinucleon", which are the constituents of a physical nucleon.

The transition probability for the β -decay of a nucleon is proportional to

$$\begin{aligned} & \frac{1}{2} \sum_{I, F} |\langle I | H'_\beta | F \rangle|^2 \\ &= g_V^2 (aa' + bb')^2 \\ &+ \frac{1}{3} g_A^2 \{ 2(a'b + ab') - (aa' + bb') \}^2 \\ &= \bar{g}_V^2 + 3\bar{g}_A^2. \end{aligned} \quad (3.12)$$

The experimental results show $\bar{g}_A/\bar{g}_V \cong -1.25^9$. We assume V - A symmetry, that is, $g_V = -g_A$. Unfortunately the ratio

$$\frac{\bar{g}_A^2}{\bar{g}_V^2} = \frac{1}{9} \left[\frac{2(a'b + ab') - (aa' + bb')}{aa' + bb'} \right]^2 \quad (3.13)$$

deduced from (3.12) is equal to or smaller than one. The maximum value is given by $a=a'=1/\sqrt{2}$ and $b=b'=-1/\sqrt{2}$.**

§ 4. Composite model of the pions

In the above consideration, we have postulated the criterion (VI) and treated the problem by the particle dynamics. But, of course, the numbers of N and \bar{N} are not independently conserved. This effect is treated only by quantum field theory. However we have not yet a satisfactory and useful formalism of the composite model. Thus we shall set an anticipation which will be solved in the future development of physics, as follows.

(VIII) The presence of the meson cloud surrounding the physical nucleon is due to the effect of the quantum fluctuations in the fundamental nucleon number. The behavior of the meson cloud can be fairly well described by the meson theoretical method in our model.

Then the meson cloud are accompanied not only by nucleons, but also pions and other baryons and mesons. Therefore (IV) must be modified as follows.

(IV') Pions are composed of their own cores, which are constructed from one

* For the fundamental antinucleons, the sign of g_A in (3.8) is inverted,

** Mr. M. Nakagawa kindly informed the author that the $(\bar{g}_A/\bar{g}_V)^2$ ratio may be taken to be larger than one when the P state for the nucleon core is assumed.

“nucleon” and one “antinucleons”, and their meson clouds surrounding the cores.

It is very difficult to investigate the composite structure of pions experimentally, but this may be ascertained in several considerations, for example, the mass difference between charged and neutral pions. In this model, physical pion states are represented as

$$\left. \begin{aligned} |(\pi)^+\rangle &= \alpha|\pi^+\rangle + \beta|\pi^0\rangle, \\ |(\pi)^0\rangle &= \alpha'|\pi^+\rangle + \beta'|\pi^0\rangle + \gamma'|\pi^-\rangle, \\ |(\pi)^-\rangle &= \alpha|\pi^-\rangle + \beta|\pi^0\rangle, \end{aligned} \right\} \quad (4.1)$$

where π^0 and π^\pm are the pion cores composed of

$$\left. \begin{aligned} \pi^+ &: (p, \bar{n}), \\ \pi^0 &: (p, \bar{p}), (n, \bar{n}), \\ \pi^- &: (n, \bar{p}). \end{aligned} \right\} \quad (4.2)$$

It is easily seen that the mass of π^0 is smaller than that of π^\pm (see Appendix B). In order that the mass difference between charged and neutral physical pions is in agreement with the observed value, it is necessary that $\alpha^2 > \beta^2$ and $(\alpha'^2 + \gamma'^2) < \beta'^2$. α^2 and β'^2 are the probabilities that the total charge of the meson cloud surrounding the physical pions is zero. Therefore we expect that

(IX) The probability of the total charge of the meson clouds surrounding the physical pions to be zero is larger than the probability of that to be $\pm e$.

§ 5. Composite model of baryons and mesons

Sakata¹⁰⁾ has extended the composite model to baryons and mesons. The fundamental particles are p, n, \bar{n}, \bar{p} and $\Lambda, \bar{\Lambda}$. Here we extend (IV') to every baryons and mesons as follows:

fundamental nucleon ————— M

(X) Every baryons and mesons are composed of their own cores, which are listed in Table 1, and their π -meson and K -meson clouds.

It is natural to introduce the “global symmetry”¹¹⁾ into our model.

physical nucleon ————— M^*
 ————— M_{ob}

(XI) The masses of Λ and $\bar{\Lambda}$ are equal to the fundamental nucleon mass M , and M is larger than the observed masses of every baryon. The baryon cores have smaller masses than M , and further the meson clouds decrease the core masses to the observed masses. These circumstances are shown in Fig. 4.

pion ————— μ^*
 ————— μ_{ob}

Fig. 4.

It is necessary for explaining the mass difference between pions and K -particles, that the couplings or potential depths of $N-\bar{N}$ and $N-\bar{\Lambda}$ and

Table 1

Particle core	Isospin	Strangeness	Structure	State	Spite
π^+	1	0	(p, \bar{n})	1S_0	0
π^0	1	0	$(n, \bar{n}), (p, \bar{p})$	1S_0	0
π^-	1	0	(n, \bar{p})	1S_0	0
K^+	$\frac{1}{2}$	1	(p, \bar{A})	1S_0	0
K^0	$\frac{1}{2}$	1	(n, \bar{A})	1S_0	0
\bar{K}^0	$\frac{1}{2}$	-1	(A, \bar{n})	1S_0	0
K^-	$\frac{1}{2}$	-1	(A, \bar{p})	1S_0	0
p	$\frac{1}{2}$	0	$(p, n, \bar{n}), (p, p, \bar{p}), (p, A, \bar{A})$	$^2S_{1/2}$	$\frac{1}{2}$
n	$\frac{1}{2}$	0	$(n, n, \bar{n}), (n, p, \bar{p}), (n, A, \bar{A})$	$^2S_{1/2}$	$\frac{1}{2}$
A	0	-1	$(A, n, \bar{n}), (A, p, \bar{p}), (A, A, \bar{A})$	$^2S_{1/2}$	$\frac{1}{2}$
Σ^+	1	-1	(A, p, \bar{n})	$^2S_{1/2}$	$\frac{1}{2}$
Σ^0	1	-1	$(A, n, \bar{n}), (A, p, \bar{p})$	$^2S_{1/2}$	$\frac{1}{2}$
Σ^-	1	-1	(A, n, \bar{p})	$^2S_{1/2}$	$\frac{1}{2}$
Ξ^-	$\frac{1}{2}$	-2	(A, A, \bar{p})	$^2S_{1/2}$	$\frac{1}{2}$
Ξ^0	$\frac{1}{2}$	-2	(A, A, \bar{n})	$^2S_{1/2}$	$\frac{1}{2}$

$A\bar{N}$ are different.

(XII) The coupling or interaction potential depth for $N\bar{A}$ and $A\bar{N}$ is smaller than that for $N\bar{N}$. Similarly, it is clear that

(XIII) The couplings or interaction potential depths for $A\bar{A}$ and $A\bar{A}$ are smaller than that for $\bar{A}N$ and $A\bar{N}$, for explaining the mass difference of Ξ -particles and other baryons.

This corresponds to that the K -meson coupling with nucleons is smaller than the π -meson coupling with nucleons.

However, the "global symmetry" in our model is different from Gell-Mann's model.¹¹⁾ From the hypotheses in this section, the strengths of the interactions between pions and baryons are not equal, because the hyperons contain the fundamental A -particles.

Finally, it is pointed out that there are the difficulties in the particle dynamics of the cores:

(a) The relativistic effects are extremely large.

(b) The results are very sensitive to the shape of the potential.

These circumstances make it difficult to treat the problem quantitatively.

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Appendix A. General formulation

The interaction Hamiltonian is

$$H' = \sum_{\mu, \alpha} \int d\mathbf{x} d\mathbf{x}' \bar{\psi}(\mathbf{x}) \mathcal{O}_{\mu} T_{\alpha} \psi(\mathbf{x}) V(\mathbf{x} - \mathbf{x}') \bar{\psi}'(\mathbf{x}') \mathcal{O}_{\mu} T_{\alpha}' \psi'(\mathbf{x}'). \quad (\text{A} \cdot 1)$$

$\psi'(\mathbf{x})$ and $\bar{\psi}'(\mathbf{x})$ are defined as

$$\left. \begin{aligned} \psi'(\mathbf{x}) &= -i\tau_2 \gamma_4 c \psi^*(\mathbf{x}), \\ \bar{\psi}'(\mathbf{x}) &= i\psi^T(\mathbf{x}) c^{-1} \tau_2, \end{aligned} \right\} \quad (\text{A} \cdot 2)$$

where the operator c represents the charge conjugation, and the suffices $*$ and T mean the conjugate complex and transposed, respectively. The isotopic spins of the "nucleon" and "antinucleon" are assigned as

$$\begin{aligned} p: \quad \tau_3 &= +1, & \bar{n}: \quad \tau_3 &= +1, \\ n: \quad \tau_3 &= -1, & \bar{p}: \quad \tau_3 &= -1. \end{aligned}$$

The isotopic spin operators T_{α} and T'_{α} ($\alpha=1, 2, 3, 4$) are given as

$$\left. \begin{aligned} T_k &= \tau_k, \\ T'_k &= -\tau_k, \\ T_4 &= \gamma_4, \quad T'_4 = 1 \end{aligned} \right\} \quad (k=1, 2, 3). \quad (\text{A} \cdot 3)$$

Thus, using (A.2) and (A.3), H' is rewritten as

$$H' = \sum_{\mu, \alpha} \int d\mathbf{x} d\mathbf{x}' \bar{\psi}(\mathbf{x}) \mathcal{O}_{\mu} T_{\alpha} \psi(\mathbf{x}) V(\mathbf{x} - \mathbf{x}') \bar{\psi}'(\mathbf{x}') \tilde{\mathcal{O}}_{\mu} \tau_2 T_{\alpha}' \tau_2 \psi'(\mathbf{x}'), \quad (\text{A} \cdot 4)$$

where

$$\tilde{\mathcal{O}}_{\mu} \equiv (c^{-1} \mathcal{O}_{\mu} c)^T = c \mathcal{O}_{\mu}^T c^{-1} = \epsilon_{\mu} \mathcal{O}_{\mu}, \quad (\text{A} \cdot 5)$$

with

$$\epsilon_{\mu} = \begin{cases} +1, & \text{when } \mathcal{O}_{\mu} \text{ is scalar, axial vector or pseudoscalar,} \\ -1, & \text{when } \mathcal{O}_{\mu} \text{ is vector or tensor,} \end{cases} \quad (\text{A} \cdot 6)$$

and

$$\left. \begin{aligned} \tau_2 T_k'^T \tau_2 &= \tau_k \quad (k=1, 2, 3), \\ \tau_2 T_4'^T \tau_2 &= 1. \end{aligned} \right\} \quad (\text{A} \cdot 7)$$

Thus the interaction Hamiltonians (A.1) and (A.4) are, in the configuration space, reduced as follows: The "nucleon"- "antinucleon" potential is

$$H'_{N-\bar{N}} = 2(\gamma - \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) (\gamma_4 \mathcal{O}_{\mu})^{(1)} (\gamma_4 \mathcal{O}_{\mu})^{(2)} V(r). \quad (\text{A} \cdot 8)$$

Similarly, the "nucleon"- "nucleon" potential is

$$H'_{N-N} = 2(\gamma + \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \epsilon_{\mu} (\gamma_4 \mathcal{O}_{\mu})^{(1)} (\gamma_4 \mathcal{O}_{\mu})^{(2)} V(r). \quad (\text{A} \cdot 9)$$

In the non-relativistic limit ($\rho_1 \rightarrow 0$, $\rho_2 \rightarrow 0$, $\rho_3 \rightarrow 1$), for the scalar interaction,

$$\left. \begin{aligned} H'_{N-\bar{N}} &= 2(\eta - \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) V(r), \\ H'_{N-N} &= 2(\eta + \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) V(r), \end{aligned} \right\} \quad (\text{A} \cdot 10)$$

and for the vector interaction,

$$\left. \begin{aligned} H'_{N-\bar{N}} &= 2(\eta - \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) V(r), \\ H'_{N-N} &= -2(\eta + \boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) V(r). \end{aligned} \right\} \quad (\text{A} \cdot 11)$$

It is necessary for the composite model of the pion that the "nucleon"- "nucleon" potential is repulsive and the "nucleon"- "antinucleon" potential is attractive.

For simplicity, the potential $V(r)$ has the form of square well. Denoting its range as b ,

$$V(r) = \begin{cases} 0 & (r > b), \\ V_0 & (r < b). \end{cases} \quad (\text{A} \cdot 12)$$

First, we examine the scalar interaction. For the pion state, the attractive force acts between the "nucleon" and "antinucleon", hence $2(\eta - 1)V_0$ must be negative, that is,

$$\eta > 1, \quad V_0 < 0 \quad \text{or} \quad \eta < 1, \quad V_0 > 0. \quad (\text{A} \cdot 13)$$

For the "nucleon"- "nucleon" charge triplet state, the repulsive force acts, hence $2(\eta + 1)V_0$ must be positive, that is,

$$\eta > -1, \quad V_0 > 0 \quad \text{or} \quad \eta < -1, \quad V_0 < 0. \quad (\text{A} \cdot 14)$$

For both conditions (A·13) and (A·14) to be satisfied, it is necessary that

$$1 > \eta > -1 \quad \text{and} \quad V_0 > 0. \quad (\text{A} \cdot 15)$$

Under this condition, $2(\eta - 3)V_0$ becomes negative, therefore the bound state of the "nucleon"- "nucleon" charge singlet state becomes possible. Thus the scalar interaction is not favorable.

Second, we examine the vector interaction. For the pion state, the attractive force acts between the "nucleon" and the "antinucleon", so $2(\eta - 1)V_0$ must be negative, that is, the condition (A·13) must be satisfied. For the "nucleon"- "nucleon" charge triplet state, the repulsive force acts, so $-2(\eta + 1)V_0$ must be positive, that is,

$$\eta > -1, \quad V_0 < 0 \quad \text{or} \quad \eta < -1, \quad V_0 > 0. \quad (\text{A} \cdot 16)$$

For both conditions (A·13) and (A·16) to be satisfied, it is necessary that

$$\eta > 1, \quad V_0 < 0 \quad \text{or} \quad \eta < -1, \quad V_0 > 0. \quad (\text{A} \cdot 17)$$

Then for the "nucleon"- "nucleon" charge singlet state to act the repulsive force between them, it is necessary that

$$\eta > 3, V_0 < 0 \text{ or } \eta < 3, V_0 > 0. \quad (\text{A} \cdot 18)$$

Therefore the final condition is

$$\eta > 3 \text{ and } V_0 < 0, \quad (\text{A} \cdot 19a)$$

or

$$\eta < -1 \text{ and } V_0 > 0. \quad (\text{A} \cdot 19b)$$

For the "nucleon"-"antinuclcon" charge singlet state, the condition (A·19a) brings the attractive force and the condition (A·19b) the repulsive force between them. When the "nucleon"-"antinuclcon" charge singlet force is attractive, the presence of the charge singlet meson may be possible. In Sakata's model, the charge singlet meson has the configuration (A, \bar{A}) in addition to (N, \bar{N}) ,^{(12), (13)} therefore the condition (A·19a) or (A·19b) is not decisive.

Appendix B. Mass difference between charged and neutral pions

Let the "nucleon" and "antinuclcon" be denoted by 1 and 2, respectively. The states of the pion cores are given by

$$\begin{aligned} \left. \begin{aligned} |\pi^+\rangle \\ |\pi^0\rangle \\ |\pi^-\rangle \end{aligned} \right\} &= |\psi(\mathbf{r})\rangle (1/\sqrt{2}) [\chi_{1/2}^{1/2}(1) \chi_{1/2}^{1/2}(2) - \chi_{1/2}^{-1/2}(1) \chi_{1/2}^{1/2}(2)] \\ &\times \begin{cases} \varphi_{11/2}^{1/2}(1) \varphi_{1/2}^{1/2}(2), \\ (1/\sqrt{2}) [\varphi_{1/2}^{3/2}(1) \varphi_{1/2}^{-1/2}(2) + \varphi_{1/2}^{-1/2}(1) \varphi_{1/2}^{1/2}(2)], \\ \varphi_{1/2}^{-1/2}(1) \varphi_{1/2}^{-1/2}(2). \end{cases} \end{aligned}$$

$\psi(\mathbf{r})$ is the 1S_0 wave function. The Coulomb interaction Hamiltonian is

$$H_c = \frac{1}{2} (\tau_3^{(1)} + 1) \frac{1}{2} (\tau_3^{(2)} + 1) \frac{e^2}{4\pi r},$$

hence we easily obtain

$$\begin{aligned} \langle \pi^+ | H_c | \pi^+ \rangle &= \langle \pi^- | H_c | \pi^- \rangle = 0, \\ \langle \pi^0 | H_c | \pi^0 \rangle &= -\frac{e^2}{4\pi} \cdot \frac{1}{2} \int \psi^*(\mathbf{r}) \frac{1}{r} \psi(\mathbf{r}) dv. \end{aligned}$$

Thus the calculated value of the mass difference for the cores of the neutral and charged pions is

$$\Delta\mu^* \equiv \mu_{\text{neutral}} - \mu_{\text{charged}}^* = \langle \pi^0 | H_c | \pi^0 \rangle \simeq -38m_e,$$

when we adopt the square well $N\bar{N}$ potential with range $1/M_{\text{oh}}$.

Appendix C. Semi-phenomenological treatment of the pion-nucleon s-wave scattering

The shape of the pion-nucleon static potential (3·6) is not favorable for calcu-

lating the scattering phase shifts, and further (3.6) is not sufficiently correct quantitatively. Therefore we shall treat the s -wave scattering semi-phenomenologically. Instead of (3.6) we postulate the form of the pion-nucleon interaction as follows:

$$H_{\pi-N}(R) = \tau \cdot \omega \bar{U}(R), \quad (\text{C} \cdot 1)$$

where

$$\bar{U}(R) = \begin{cases} \bar{U}_1 & \text{for } r < r_1, \\ \bar{U}_2 & \text{for } r_1 < r < r_2, \\ 0 & \text{for } r_2 < r, \end{cases} \quad (\text{C} \cdot 2)$$

is a two-step square well potential. We take as $r_1 = 1/2M_{\text{ob}}$ and $r_2 = 1/M_{\text{ob}}$. Both \bar{U}_1 and \bar{U}_2 are positive.

The pion-nucleon interaction (C.1) is repulsive in the total isospin $I=3/2$ state and the phase shift δ_3 is not sensitive to the shape of $\bar{U}(R)$. In order to agree with the experimental value, the depth becomes

$$\bar{U}_1 \simeq 1.1 \times 10^2 M_{\text{ob}}. \quad (\text{C} \cdot 3)$$

In the $I=1/2$ state, the pion-nucleon interaction is attractive and the phase shifts δ_1 is very sensitive to the shape of $\bar{U}(R)$, that is, the ratio \bar{U}_2/\bar{U}_1 . In order to agree with the experimental value, it is needed that

$$\bar{U}_2/\bar{U}_1 \simeq 0.9. \quad (\text{C} \cdot 4)$$

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Electrical Resistance of Ferromagnetic Metals

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We use a model for ferromagnets such that the electrical current is carried by 4s (or 6s) electrons which are assumed to be described in band scheme and the unpaired electrons (3d or 4f) are assumed to be localized on the lattice points. In the temperature region far below the Curie point the spin-disorder in spin orientation, which can be naturally described in terms of spin waves, gives rise to the scattering of conduction electrons through the *s-d* interaction and accordingly contributes to the anomalous part of the resistivity of type T^2 , which agrees fairly well with the experimental results both in order of magnitude and in temperature dependence. The effects of the lattice vibration are also discussed and turn out to give only the small additional contribution of type T^5 .

§ 1. Introduction

It has long been recognized that the electrical resistivity of the transition metals and of rare earth metals consists of two parts^(1),2); the one is the resistivity coming from the usual electron-phonon interaction, and the other part is the anomalous resistivity, which is usually larger than the former part up to the Curie point. This anomalous part has been discussed by several authors^(1),2),3),4),5) whose models are classified generally into two types: The one is spin-disorder resistivity^(1),2),4),6) and the other is the so-called *s-d* transition mechanism^(3),7). Taking the former point of view, we discuss the anomalous part of the resistivity of ferromagnetic metals in the temperature region far below the Curie point.

The *s-d* interaction is written as⁽⁶⁾

$$V = - \sum_{\mathbf{k}\mathbf{k}'} \sum_n J(\mathbf{k}' - \mathbf{k}) \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{R}_n] \{ (a_{\mathbf{k}'\uparrow}^* a_{\mathbf{k}\uparrow} - a_{\mathbf{k}'\downarrow}^* a_{\mathbf{k}\downarrow}) S_n^z + a_{\mathbf{k}'\uparrow}^* a_{\mathbf{k}\downarrow} S_n^- + a_{\mathbf{k}'\downarrow}^* a_{\mathbf{k}\uparrow} S_n^+ \}, \quad (1)$$

where $a_{\mathbf{k}\uparrow}^*$ and $a_{\mathbf{k}\uparrow}$ are usual creation and annihilation operators for the conduction electron with spin upwards moving through the lattice with wave vector \mathbf{k} , and $a_{\mathbf{k}\downarrow}^*$, $a_{\mathbf{k}\downarrow}$ refer to the electron having spin downwards. $J(\mathbf{k}' - \mathbf{k})$ is given by⁽⁶⁾

$$J(\mathbf{k}' - \mathbf{k}) = \iint \varphi_n^*(1) \phi_{\mathbf{k}'}^*(2) \frac{e^2}{r_{12}} \varphi_n(2) \phi_{\mathbf{k}}(1) d\mathbf{r}_1 d\mathbf{r}_2 \exp[i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{R}_n] \quad (2)$$

and is easily shown to be independent of \mathbf{R}_n , where $\varphi_n(\mathbf{r})$ and $\phi_{\mathbf{k}}(\mathbf{r})$ are the wave functions of unpaired electron localized on the *n*-th lattice point and conduction

electron of wave vector \mathbf{k} , respectively. \mathbf{S}_n is the spin operator of the unpaired electron localized at the n -th lattice point, and if the Hund-rule is effective in the unfilled shell on each lattice point the above operator \mathbf{S}_n may be considered to represent the total spin angular momentum at the n -th lattice point and then the above quantity $J(\mathbf{k}' - \mathbf{k})$ must be understood to represent some relevant mean of J over the states occupied in each unfilled shell.

We now classify the types of spin-disorders by which the mean free path of conduction electron is limited. Confining ourselves to the case of regular lattice, we have the two types of spin-disorders, i.e., in spin orientation and in its location. The former type of disorder can be described in terms of spin waves⁸⁾ (magnons) in the low temperature region far below the Curie point and the latter in terms of lattice waves (phonons).

In the low temperature region only the magnons of long wave length are effective in determining the mean free path of conduction electron. Through a single scattering a conduction electron diminishes its wave vector by a small amount of order $k_f \cdot \kappa^2 / k_f^2$, and on the average, k_f^2 / κ^2 collisions are necessary for a conduction electron to lose its wave vector completely, or to travel a distance of order of its mean free path. On the other hand, the probability of occurrence of a single scattering process is proportional to the number of magnons and the magnitude of coupling constant in V , or, explicitly speaking, is proportional to the quantity

$$\int d\kappa n_\kappa |J(\kappa)|^2 / \kappa \propto T,$$

where $J(\kappa)$ is assumed to tend to a constant value with decreasing κ , and n_κ stands for the number of magnons with κ . Further, we have $(\kappa^2 / k_f^2)_{AV} \propto T$ and the mean free path is proportional to T^{-2} . Then we can expect to have the resistivity of type $\text{const.} \times T^2$. In § 3 these and some allied problems are discussed. In the next section as to these scattering processes a more quantitative discussion is given in detail, and the effects of lattice vibration are also discussed.

§ 2. Transition probabilities for the various processes and the electrical resistivity

a) Transition probability in terms of spin correlation and resistivity

The transition probability for the scattering process $|\mathbf{k} \downarrow \rangle \longrightarrow |\mathbf{k}' \uparrow \rangle$

is⁹⁾

$$P(\mathbf{k} \downarrow \rightarrow \mathbf{k}' \uparrow) = \frac{N}{\hbar^2} |J(\mathbf{k}' - \mathbf{k})|^2 f_k^- (1 - f_{k'}^+) \int e^{i(\kappa \cdot \mathbf{r} - \omega t)} \sum_n G_n(\mathbf{r}, t) \gamma_n^+(t) d\mathbf{r} dt \quad (3)$$

with $\kappa = \mathbf{k} - \mathbf{k}'$, $\hbar\omega = E_k^- - E_{k'}^+$ and

$$G_n(\mathbf{r}, t) = \langle \int d\mathbf{r}' \delta(\mathbf{r} + \mathbf{R}_0(0) - \mathbf{r}') \delta(\mathbf{r}' - \mathbf{R}_n(t)) \rangle \quad (4)$$

$$\gamma_n^{+-}(t) = \langle S_0^+(0) S_n^-(t) \rangle, \quad (5)$$

where $\langle B \rangle$ stands for the statistical average of the operator B .

In order to observe the essential features of the resistivity determined by the disorder of spins in their orientations, in the first place we discuss the case of rigid lattice; $G_n(\mathbf{r}, t) = \delta(\mathbf{r} - \mathbf{R}_n)$. In the low temperature region the spin-disorder in orientation can be described in terms of the free spin waves^{(8), (10)}. Then Eq. (3) becomes

$$\begin{aligned} P(\mathbf{k}\downarrow \rightarrow \mathbf{k}'\uparrow) &= \frac{4\pi SN}{\hbar^2} |J(\kappa)|^2 f_k^- (1 - f_{k'}^+) (n_{k-k'} + 1) \delta(\omega_1 - \omega_{k-k'}) \\ P(\mathbf{k}\uparrow \rightarrow \mathbf{k}'\downarrow) &= \frac{4\pi SN}{\hbar^2} |J(\kappa)|^2 f_k^+ (1 - f_{k'}^-) n_{k-k'} \delta(\omega_{-1} + \omega_{k-k'}) \\ P(\mathbf{k}\uparrow \rightarrow \mathbf{k}'\uparrow) &= \frac{4\pi SN}{\hbar^2} |J(\kappa)|^2 f_k^\pm (1 - f_{k'}^\pm) (NS/2 - \sum_\lambda n_\lambda) \delta_{\mathbf{k}\mathbf{k}'} \delta(\omega_0) \end{aligned} \quad (6)$$

in which $\hbar\omega_{\pm 1} = E_k^\mp - E_{k'}^\pm$, $\hbar\omega_0 = E_k^\pm - E_{k'}^\pm$, and $\hbar\omega_\kappa$ is the energy quantum of spin wave κ .

Next we determine the distribution function f_k^\pm of the conduction electron when a uniform external electric field \mathbf{F} is present in the direction of the x -axis by the use of the steady state equation⁽¹¹⁾

$$df_k^\pm/dt|_F + df_k^\pm/dt|_{\text{coll}} = 0, \quad (7)$$

and then we calculate the resistivity. In what follows we make the following assumptions.

(1) $J(\kappa)$ depends only on the magnitude of κ .

(2) In the low temperature region most effective spin waves to the scattering processes are of long wave length and the energy spectrum of these spin waves is $\varepsilon_\kappa = \hbar^2 \kappa^2 / 2\mu$, μ being the effective mass of magnon.

(3) The phonon spectrum is assumed to be $\hbar\omega_q = \hbar sq$, where s is the sound velocity.

The scattering process leaving the spin orientation unchanged (corresponding to the last member of Eqs. (6)) does not bring the change of \mathbf{k} in the low temperature region. Hence this type of process can be neglected in determining the resistivity and we have

$$\begin{aligned} df_k^+ / dt|_{\text{coll}} &= \frac{4\pi SN}{\hbar} \sum_{\substack{\kappa \\ (k' = k + \kappa)}} |J(\kappa)|^2 \{ f_{k'}^- (1 - f_k^+) (n_\kappa + 1) - f_k^+ (1 - f_{k'}^-) n_\kappa \} \\ &\quad \times \delta(E_{k'}^- - E_k^+ - \varepsilon_\kappa). \end{aligned} \quad (8)$$

Because the electrons near the Fermi surface are only effective to these scattering processes, we put as usual

$$f_k^\pm = f_k^0 - F\phi^\pm \, df_k^0/dE_k^\pm \cdot k_x. \quad (9)$$

Leaving the terms up to the first order in ϕ 's, we have that

$$\begin{aligned} & \int_0^{\kappa_0} n(\kappa) |J(\kappa)|^2 \kappa d\kappa \left\{ \phi^+(E_k^+) \frac{df^0(E_k^+)}{dE_k^+} \exp\left(\frac{\varepsilon_\kappa}{k_B T}\right) \frac{f^0(E_k^+ + \varepsilon_\kappa)}{f^0(E_k^+)} \right. \\ & - \phi^-(E_k^+ + \varepsilon_\kappa) \frac{df^0(E_k^+ + \varepsilon_\kappa)}{d(E_k^+ + \varepsilon_\kappa)} \frac{f^0(E_k^+)}{f^0(E_k^+ + \varepsilon_\kappa)} \left[1 - \frac{(1 - m/\mu)\kappa^2}{2k^2} \right] \Big\} \\ & + \frac{\pi \hbar^4 e k}{m^2 N^2 S v_0} \frac{df^0(E_k^+)}{dE_k^+} = 0 \end{aligned} \quad (10)$$

where v_0 stands for the volume of unit cell. Similarly we have that for the electron with spin downwards

$$\begin{aligned} & \int_0^{\kappa_0} n(\kappa) |J(\kappa)|^2 \kappa d\kappa \left\{ \phi^-(E_k^-) \frac{df^0(E_k^-)}{dE_k^-} \frac{f^0(E_k^- - \varepsilon_\kappa)}{f^0(E_k^-)} \right. \\ & - \phi^+(E_k^- - \varepsilon_\kappa) \frac{df^0(E_k^- - \varepsilon_\kappa)}{d(E_k^- - \varepsilon_\kappa)} \exp\left(\frac{\varepsilon_\kappa}{k_B T}\right) \frac{f^0(E_k^-)}{f^0(E_k^- - \varepsilon_\kappa)} \left[1 - \frac{(1 + m/\mu)\kappa^2}{2k^2} \right] \Big\} \\ & + \frac{\pi \hbar^4 e k}{m^2 N^2 S v_0} \frac{df^0(E_k^-)}{dE_k^-} = 0. \end{aligned} \quad (11)$$

As the function $-df^0(E)/dE$ has sharp maximum at E_f and ϕ 's would be very slowly varying function of E , one may regard ϕ 's are constant. Integrating (10), (11) with respect to E_k^\pm and combining the resulting equations, we have

$$\begin{aligned} A + \frac{m}{\mu} B + \frac{4\pi \hbar^4 e k_f^3}{m^2 S v_0 J_2} &= 0 \\ -\frac{m}{\mu} A + \frac{4k_f^3 J_1 - J_2}{J_2} B &= 0, \end{aligned} \quad (12)$$

where

$$\begin{aligned} A &= \phi^+(E_f) + \phi^-(E_f), \quad B = \phi^+(E_f) - \phi^-(E_f) \\ J_s &= N^2 \int_0^{\kappa_0} n(\kappa) |J(\kappa)|^2 \frac{x}{1 - e^{-x}} \kappa^{2s-1} d\kappa \\ x &= \hbar^2 \kappa^2 / 2\mu k_B T. \end{aligned} \quad (13)$$

Upon solving Eq. (12) one finally has (cf. (9)) as resistivity

$$\rho = \frac{3\pi m^2 S v_0}{2e^2 \hbar^3 k_f^6} \cdot J_2 \cdot \left(1 + \left(\frac{m}{\mu} \right)^2 \frac{J_2}{4k_f^2 J_1 - J_2} \right). \quad (14)$$

J_1 reveals a logarithmic divergence at the lower limit of the integration. However, in reality, the actual sample is of finite dimension so the mode $\kappa=0$ has to be excluded from the collective excitations. Moreover, this mode does not have any contribution to limit the mean free path of conduction electron and if one tenta-

tively use the values

$$\kappa_{\min} \cong (Nv_0)^{-1/3} \sim 1\text{cm}^{-1}, \quad \mu/m = 100, \quad T = 40^\circ\text{K},$$

one has that as order of magnitude

$$4k_f^2 J_1/J_2 \simeq 60/\zeta(2) \cdot E_f/k_B T \cdot m/\mu \gg 1.$$

So one can neglect the second term in the parentheses of Eq. (14) and obtains

$$\rho_{\text{spin wave}} = \frac{3\pi^5}{16} \frac{S}{e^2} \left(\frac{\mu}{m} \right)^2 \frac{k_B^2 T^2 N^2 J^2(0)}{E_f^4} \frac{\hbar}{k_f}. \quad (15)$$

In Ni, adopting the following values²⁾,

$$NJ(0) = 0.48\text{ eV}, \quad E_f = 3\text{ eV}, \quad S = 1/2 \quad \text{and} \quad \mu/m = 38,$$

one has that

$$\rho_{\text{spin wave}} = 11 \times 10^{-6} \times T^2 \quad (\text{microhm-cm}). \quad (16)$$

b) *Effects of the lattice vibration*

As has been shown in § 1, $J(\mathbf{k}' - \mathbf{k})$ is independent of \mathbf{R}_n . Therefore if it is assumed that the electronic wave functions relevant to the quantity $J(\mathbf{k}' - \mathbf{k})$ are rigid, J can be regarded merely as a coupling parameter independent of phonon coordinates. As to the inner shell electrons such as $3d$ or $4f$ this may be the case but as to the conduction electrons this assumption is rather serious. However, making this assumption throughout the present calculation, we investigate the effects of lattice vibration whose degrees of freedom are then contained in the quantity G_n only.

In the usual Debye approximation⁹⁾ one has

$$G_n(\mathbf{r}, t) = \left(\frac{1}{2\pi} \right)^3 \int d\mathbf{k} e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{R}_n)} \exp \left\{ - \sum_{\alpha\beta}^{\epsilon, y, z} [M_{\alpha\beta}(0, 0) - M_{\alpha\beta}(\mathbf{R}_n, t)] \kappa_\alpha \kappa_\beta \right\} \quad (17)$$

with

$$M_{\alpha\beta}(\mathbf{R}_n, t) = \frac{\hbar}{2MN} \sum_{qj} \frac{e_{qj}^\alpha e_{qj}^\beta}{\omega_{qj}^p} \left\{ (n_{qj}^p + 1) \exp[-i(\mathbf{q} \cdot \mathbf{R}_n - \omega_{qj}^p t)] + n_{qj}^p \exp[i(\mathbf{q} \cdot \mathbf{R}_n - \omega_{qj}^p t)] \right\} \quad (18)$$

where \mathbf{e}_{qj} is the polarization vector, j specifies three independent modes of lattice waves and the quantities with superscript p are referred to phonon. At low temperatures far below the Debye temperature θ

$$\begin{aligned} \left| \sum_{\alpha\beta} M_{\alpha\beta}(\mathbf{R}_n, t) \kappa_\alpha \kappa_\beta \right| &\lesssim \sum_{\alpha\beta} M_{\alpha\beta}(0, 0) \kappa_\alpha \kappa_\beta \\ &\simeq \frac{3m}{2M} \left(\frac{\kappa}{k_f} \right)^2 \frac{E_f}{k_B \theta} \left(1 + 4\zeta(2) \frac{T^2}{\theta^2} \right) \end{aligned} \quad (19)$$

$$\ll 1,$$

then we can expand $G_n(\mathbf{r}, t)$ in terms of these quantities and obtain

$$\begin{aligned} P(\mathbf{k}', \mathbf{s}; \mathbf{k}) &= \frac{4\pi S N}{\hbar} |J(\boldsymbol{\kappa})|^2 f_{\mathbf{k}}^+ (1 - f_{\mathbf{k}'}^+) \delta(\hbar\omega) \delta_{\mathbf{s},0} e^{-W} \left(\frac{NS}{2} - \sum_{\lambda} n_{\lambda} \right) \\ &+ \frac{2\pi S}{M} |J(\boldsymbol{\kappa})|^2 f_{\mathbf{k}}^+ (1 - f_{\mathbf{k}'}^+) e^{-W} \frac{(\mathbf{e}_{\mathbf{s}} \cdot \boldsymbol{\kappa})^2}{\omega_{\mathbf{s}}^2} \left(\frac{NS}{2} - \sum_{\lambda} n_{\lambda} \right) \\ &\times \{ (n_{\mathbf{k}}^s + 1) \delta(\hbar\omega - \hbar\omega_{\mathbf{s}}^s) + n_{-\mathbf{s}}^s \delta(\hbar\omega + \hbar\omega_{\mathbf{s}}^s) \} \end{aligned} \quad (20)$$

with

$$\hbar\omega = E_{\mathbf{k}} - E_{\mathbf{k}'}, \quad \boldsymbol{\kappa} = \mathbf{k} - \mathbf{k}', \quad W = \sum_{\alpha\beta} M_{\alpha\beta}(0, 0) \kappa_{\alpha} \kappa_{\beta}. \quad (21)$$

Confining ourselves to the lowest order scattering process, hence making use of the second term of Eq. (20), we have the following equation,

$$\begin{aligned} df_{\mathbf{k}}^+ / dt)_{\text{phonon}} &= \frac{2\pi S}{M} \left(\frac{NS}{2} - \sum_{\lambda} n_{\lambda} \right) \sum_{\mathbf{s}} |J(\boldsymbol{\kappa})|^2 e^{-W} \frac{\kappa^2}{\omega_{\mathbf{s}}^2} \\ &\times \{ [f_{\mathbf{k}'}^+ (1 - f_{\mathbf{k}}^+) (n_{-\mathbf{s}} + 1) - f_{\mathbf{k}}^+ (1 - f_{\mathbf{k}'}^+) n_{-\mathbf{s}}] \delta(E_{\mathbf{k}'}^+ - E_{\mathbf{k}}^+ - \hbar\omega_{\mathbf{s}}^s) \\ &+ [f_{\mathbf{k}'}^+ (1 - f_{\mathbf{k}}^+) n_{\mathbf{s}} - f_{\mathbf{k}}^+ (1 - f_{\mathbf{k}'}^+) (n_{\mathbf{s}} + 1)] \delta(E_{\mathbf{k}'}^+ - E_{\mathbf{k}}^+ + \hbar\omega_{\mathbf{s}}^s) \}. \end{aligned} \quad (22)$$

At low temperatures one may put $J(\boldsymbol{\kappa})$ and e^{-W} as constants ($J(\boldsymbol{\kappa}) = J(0)$, $e^{-W} = 1$; cf. (19), (21)). Upon solving the steady state equation we have⁽¹²⁾

$$\begin{aligned} f_{\mathbf{k}}^+ &= f_{\mathbf{k}}^0 - F\phi^+(E_{\mathbf{k}}) df_{\mathbf{k}}^0 / dE_{\mathbf{k}} \cdot \mathbf{k}_x, \\ (\phi^+)^{-1} &= - \frac{2\pi S}{M} \left(\frac{NS}{2} - \sum_{\lambda} n_{\lambda} \right) \frac{N\tau_0}{4\pi^2} \frac{m^2}{\hbar^3 e s} \frac{\kappa_0^5}{k_f^3} \left(\frac{T}{\theta} \right)^5 J^2(0) \int_0^{\Theta/T} \frac{x^5 dx}{(e^x - 1)(1 - e^{-x})}. \end{aligned} \quad (23)$$

We can similarly determine $f_{\mathbf{k}}^-$ and obtain

$$f_{\mathbf{k}}^- = \frac{3\pi N v_0 m^3 S \kappa_0^5 J^2(0)}{2\hbar^2 e^2 s M k_f^6} \left(\frac{NS}{2} - \sum_{\lambda} n_{\lambda} \right) \left(\frac{T}{\theta} \right)^5 \int_0^{\Theta/T} \frac{x^5 dx}{(e^x - 1)(1 - e^{-x})}. \quad (24)$$

With Eqs. (15) and (24) we finally arrive at

$$\begin{aligned} \rho &= \rho_{\text{spin wave}} (1 + \delta), \\ \delta &= \frac{2^{2/3}}{\zeta(2)} \frac{mS}{M} \frac{E_f}{k_B \theta} \left(\frac{\theta}{\theta} \right)^2 \left(\frac{T}{\theta} \right)^3 \left(\frac{M(T)}{M(0)} \right)^2 \int_0^{\Theta/T} \frac{x^5 dx}{(e^x - 1)(1 - e^{-x})}, \end{aligned} \quad (25)$$

where $M(T)$ is the magnetization at T , $k_B \theta = \hbar^2 \kappa_0^2 / 2\mu$.

We now estimate the magnitude of ρ_{phonon} relative to $\rho_{\text{spin wave}}$, that is Eq. (26). δ is determined by the experimental value^{(13), (14)} of the coefficient of the term $T^{3/2}$ in $M(T)$ and is listed in Table I with other constants. Using these values we obtain values of δ . From Table II we see that the lattice vibration considered

here has only a small contribution to resistivity.

Table I

	S	$E_f(\text{eV})^{2)}$	$\theta(^{\circ}\text{K})$	$\Theta(^{\circ}\text{K})$
Ni	1/2	3.14	8670*	456
Gd	7/2	4.4	433**	152§

* θ is determined from the coefficient of $T^{3/2}$ term in $M(T)$ given in Ref. 14).

** Determined from Kurti's data¹⁵⁾ on the magnetic part of the specific heat of Gd at very low temperatures which has temperature dependence $T^{3/2}$ and is to be attributed to spin waves. If one uses the magnetic data¹³⁾ one has $\theta=467^{\circ}\text{K}$.

§ This is estimated from Kurti's data¹⁵⁾ on the lattice part of specific heat which has temperature dependence T^3 at very low temperatures and is to be ascribed to phonon.

Table II Values of δ .

Ni		Gd	
$T(^{\circ}\text{K})$	δ	$T(^{\circ}\text{K})$	δ
0	0.0000	0	0.0000
22.80	0.0020	7.6	0.0005
35.07	0.0072	11.69	0.0017
45.60	0.0150	15.2	0.0037
57	0.0255	19	0.0062
76	0.0420	25.33	0.0102
91.2	0.0514	30.4	0.0123
114	0.0585	38	0.0139 ₃
152	0.0594	50.66	0.0138 ₉
228	0.0493	76	0.0110
304	0.0391	101.3	0.0082

§ 3. Discussions

Electrical resistivities due to the spin-disorders in spin orientation and in its location have the temperature dependences of type T^2 and T^5 , respectively, in the low temperature region. This type of difference can be ascribed mainly to the difference in the energy spectra between that of magnon and that of phonon. Then if one has the metallic antiferromagnets in which the s - d interaction is effective, one can easily expect that the spin-disorders in orientation and in location give the resistivities of type T^4 and T^5 , respectively. This is because the energy spectrum of spin wave in antiferromagnetics is proportional to k . Hence, in an antiferromagnet, the lattice vibration may have effects of much more importance than in a ferromagnet.*

* In the case of inelastic scattering of slow neutrons by ferro- or antiferromagnet, one may expect at least qualitatively that lattice vibration plays a more important role in antiferromagnet than it does in ferromagnet (magneto-vibrational effect ¹⁶⁾).

Now from the view-point of the s - d transition³⁾ one can obtain the anomalous resistivity proportional to $\exp(-\theta_E/T)$. θ_E is related to such minimum wave vector q_{\min} as it enables an s - d transition to occur.* Further²⁾, this view-point does not seem to be able to explain satisfactorily the abrupt change in the resistivity at the Curie point, and to explain why the rare earth metal, whose unfilled shell does not seem to constitute a band, does have an anomalous resistivity.

In § 2-(b) we have confined ourselves to the discussion of spin-disorder in spin location and neglected the effect of the deformation of electronic wave functions by the presence of lattice waves, that is, the deformation effects of $J(\kappa)$. Indeed, this type of effect must exist but the resulting resistivity might be expected to be of the same order of magnitude of or even smaller than the resistivity due to the usual electron-phonon interaction. This latter type of resistivity, when determined from Potter's experimental data¹²⁾, is of order of the magnitude given in § 2-(b). Thus one can conclude that the electrical resistivity of ferromagnet is almost due to the spin-disorder in spin orientation and is given by Eq. (15). In Ni Eq.(16) agrees fairly well with the experimental results¹⁷⁾ both in the order of magnitude and in temperature dependence. As was shown in Potter's experiment, in Ni and Fe the electrical resistivity deviates very much from the one given by Grüneisen formula and this deviation, namely the anomalous part of resistivity, is proportional to $T^n (2 \leq n \leq 2.2)$ up to the room temperatures.

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* It is suggested by Wilson that if the energy surfaces are not spheres it is possible for s - and d -surfaces to cut one another in some directions in \mathbf{k} -space, and in the neighbourhood of these directions long lattice waves may be able to participate in the s - d transition. q_{\min} is then zero. These circumstances, however, seem to occur only in a very special case, and in general there is a non-zero value of wave vector for the phonon which is required to produce an s - d transition.

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Note added in proof:

Recently White and Woods (Phil. Trans. Roy. Soc. (London) **A251** (1959), 273) reported the detailed data about the electrical conductivity for twenty transition elements and showed that ρ is proportional to T^2 at very low temperatures in Mn, Fe, Co, Ni etc. They ascribed T^2 -term to Baber's mechanism (Proc. Roy. Soc. **158** (1937), 383), but his mechanism only is not sufficient to explain the abrupt change of ρ at Curie point.

Consideration on Non-Orthogonality Catastrophe in the Heitler-London Theory. I

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We here prove generally and rigorously that the non-orthogonality catastrophe does never arise in the Heitler-London theory. Our treatment may be regarded as a mathematical justification of Van-Vleck's objection against Inglis' criticism on the Heitler-London theory.

There are two well-known approaches to problems of the electronic structures of molecules and solids: the Heitler-London method and the molecular orbital method. Although the former is sometimes attractive from the point of view of the chemical intuition, the latter is almost exclusively used for quantitative discussions. The reason is that molecular orbitals are mutually orthogonal, hence the MO method has important mathematical advantages over the HL method in which non-orthogonal atomic orbitals are used.

In order to obtain an expectation value of any physical quantity represented by a one-body or two-body operator (H_1 or H_2), one should calculate

$$\text{Tr}(\rho_1 H_1) \text{ or } \text{Tr}(\rho_2 H_2)$$

where ρ_1 and ρ_2 are the one-body and the two-body reduced density matrices, respectively. Here we consider a case in which these density matrices are given by

$$\begin{aligned} \langle q' | \rho_1 | q'' \rangle &= N \int dq_2 \cdots dq_N \cdot \Psi(q' q_2 \cdots q_N) \cdot \Psi^*(q'' q_2 \cdots q_N) / (\Psi | \Psi), \\ \langle q'_1 q'_2 | \rho_2 | q''_1 q''_2 \rangle &= \frac{N(N-1)}{2} \int dq_3 \cdots dq_N \cdot \Psi(q'_1 q'_2 q_3 \cdots q_N) \cdot \Psi^*(q''_1 q''_2 q_3 \cdots q_N) / (\Psi | \Psi), \end{aligned} \quad (1)$$

$$\Psi(q_1 q_2 \cdots q_N) = A \left[\phi_1(\mathbf{r}_1) \phi_2(\mathbf{r}_2) \cdots \phi_N(\mathbf{r}_N) \cdot \theta(\sigma_1 \sigma_2 \cdots \sigma_N) \right].$$

In the above, A is the antisymmetrizer, θ is a spin function, \mathbf{r} and σ are space- and spin-coordinates, respectively, and $q = (\mathbf{r}, \sigma)$. The essential point in the HL method is to use non-orthogonal orbitals ϕ_1, \cdots, ϕ_N . The non-orthogonality makes it almost impossible to obtain ρ_1 and ρ_2 in the case of $N \gg 1$, because there appears an infinitely large number of permutations in the integrands of the numerators and

denominators on the right-hand side of Eq. (1). Consequently, one inevitably neglects differential overlaps $\psi_i(\mathbf{r})\psi_j(\mathbf{r})$ which are not large, if he wants to obtain any qualitative or quantitative result from the HL theory. However, the number of the terms neglected in the integrands is almost infinite, because of an extremely large number of higher permutations. Therefore one cannot readily conclude that errors involved in such a treatment are small, even if each of the terms neglected is very small. Indeed, the errors in the numerator and the denominator sometimes tend to infinity as $N \rightarrow \infty$, even if each of the overlaps neglected is small, no matter how small. This difficulty inherent in the Heiter-London or Heisenberg method was first pointed out by Inglis¹⁾.

Against him, Van Vleck published his opinion²⁾. He considered that this difficulty may be due to the fact that the denominator or the numerator by itself is not any observable physical quantity, and he expected that the ratio of these two quantities, which has a definite physical meaning, may not exhibit such a difficulty. In fact, he succeeded in justifying his expectation for special cases by neglecting certain terms.

The density matrices given by (1) may be expressed as

$$\left. \begin{aligned} \langle q' | \rho_1 | q'' \rangle &= \sum_{i,k} S_{ik}^k \cdot \psi_i(q') \psi_k^*(q'') \\ \langle q'_1 q'_2 | \rho_2 | q''_1 q''_2 \rangle &= \sum_{i \neq j, k \neq l} S_{ij}^{kl} \cdot \psi_i(q'_1) \psi_j(q'_2) \psi_k^*(q''_1) \psi_l^*(q''_2) \end{aligned} \right\} \quad (2)$$

In order to simplify mathematical treatments, we here define the following artificial wave functions:

$$\left. \begin{aligned} \Psi^i &\equiv A \left[\{ \dots \psi_{i-1}(\mathbf{r}_{i-1}) \bar{\psi}_1(\mathbf{r}_i) \psi_{i+1}(\mathbf{r}_{i+1}) \dots \} \theta(\sigma_1 \sigma_2 \dots \sigma_N) \right] \\ \Psi^{ij} &\equiv A \left[\{ \dots \psi_{i-1}(\mathbf{r}_{i-1}) \bar{\psi}_1(\mathbf{r}_i) \psi_{i+1}(\mathbf{r}_{i+1}) \dots \psi_{j-1}(\mathbf{r}_{j-1}) \bar{\psi}_2(\mathbf{r}_j) \psi_{j+1}(\mathbf{r}_{j+1}) \dots \} \theta(\sigma_1 \sigma_2 \dots \sigma_N) \right] \\ &\dots \text{ and so on} \end{aligned} \right\} \quad (3)$$

where $(\bar{\psi}_k | \psi_i) = 0$ for $i = 1, 2, \dots, N$ and $(\bar{\psi}_k | \bar{\psi}_l) = \delta_{kl}$. Then the coefficients S in Eq. (2) can be expressed as

$$\left. \begin{aligned} S_{ik}^k &= (\Psi^k | \Psi^i) / (\Psi | \Psi) \\ S_{ij}^{kl} &= (\Psi^{kl} | \Psi^{ij}) / (\Psi | \Psi) \end{aligned} \right\} \quad (4)$$

Our purpose is to show that S remains finite as $N \rightarrow \infty$ and is not much different from that given by neglecting the small overlap integrals (zeroth approximation), though numerators and denominators on the right-hand side of (4) sometimes have absurd values.

From the definition (3), we obtain the following equation,

$$(\Psi | \Psi) = \sum_{(k_1 \dots k_m)} (\psi_{k_1} | \psi_{j_1}) \dots (\psi_{k_m} | \psi_{j_m}) (\Psi^{k_1 \dots k_m} | \Psi^{j_1 \dots j_m}),$$

where the summation is taken over all values of $k_i = 1, \dots, N$ under the condition

that k_1, \dots, k_m should all be different. Substituting the above relation into Eq. (4), we obtain

$$\left. \begin{aligned} S_j^i &= T_{j/\sum_k}^i (\psi_k | \psi_j) T_j^k \\ S_{j_1 j_2}^{i_1 i_2} &= T_{j_1 j_2 / \sum_{(k_1 k_2)}}^{i_1 i_2} (\psi_{k_1} | \psi_{j_1}) (\psi_{k_2} | \psi_{j_2}) T_{j_1 j_2}^{k_1 k_2} \end{aligned} \right\} \quad (5)$$

where

$$T_{j_1 \dots j_m}^{i_1 \dots i_m} = \frac{(\Psi^{i_1 \dots i_m} | \Psi^{j_1 \dots j_m})}{(\Psi^{j_1 \dots j_m} | \Psi^{j_1 \dots j_m})}. \quad (6)$$

The T 's can be expanded as

$$T_{j_1 \dots j_m}^{i_1 \dots i_m} = \frac{\sum_{i(\neq i_1 \dots i_m)} (\psi_i | \psi_{j_{m+1}}) T_{j_1 \dots j_m j_{m+1}}^{i_1 \dots i_m i}}{\sum_{i(\neq j_1 \dots j_m)} (\psi_i | \psi_{j_{m+1}}) T_{j_1 \dots j_m j_{m+1}}^{j_1 \dots j_m i}}. \quad (7)$$

This equation is directly derived from the the relation

$$(\Psi^{i_1 \dots i_m} | \Psi^{j_1 \dots j_m}) = \sum_{i(\neq i_1 \dots i_m)} (\psi_i | \psi_{j_{m+1}}) (\Psi^{i_1 \dots i_m i} | \Psi^{j_1 \dots j_m j_{m+1}})$$

where j_{m+1} is arbitrary except that it should be different from $j_1 \dots j_m$. For these T 's we obtain the following inequalities which can be proved rigorously under certain conditions.

Theorem I. If $i_1 \dots i_m$ is merely a permutation of $j_1 \dots j_m$, then

$$|T_{j_1 \dots j_m}^{i_1 \dots i_m}| \leq 1. \quad (8)$$

(Proof)

In this case $(\Psi^{i_1 \dots i_m} | \Psi^{i_1 \dots i_m}) = (\Psi^{j_1 \dots j_m} | \Psi^{j_1 \dots j_m})$ holds, hence we obtain

$$|(\Psi^{i_1 \dots i_m} | \Psi^{j_1 \dots j_m})| \leq |(\Psi^{j_1 \dots j_m} | \Psi^{j_1 \dots j_m})|$$

which is an example of Schwartz's inequality. Therefore, the inequality (8) in this case is directly established from (6).

Theorem II. If

$$\sum_{\lambda(\neq \mu)} |(\psi_\lambda | \psi_\mu)| \leq 1/2 \quad (9)$$

is satisfied for all μ , then Eq. (8) holds for arbitrary values of $i_1 \dots i_m, j_1 \dots j_m$.

(Proof) Among $i_1 \dots i_n$ be found i_λ which is not equal to any of $j_1 \dots j_n$. If this is not the case, $i_1 \dots i_n$ is merely a permutation of $j_1 \dots j_n$, hence Theorem I, i. e., $|T_{j_1 \dots j_n}^{i_1 \dots i_n}| \leq 1$ reduces to a special case of Theorem I. Using such i_λ , we expand $T_{j_1 \dots j_n}^{i_1 \dots i_n}$ as follows,

$$T_{j_1 \dots j_n}^{i_1 \dots i_n} = \frac{\sum_{i(\neq i_1 \dots i_n)} (\psi_i | \psi_{i_\lambda}) T_{j_1 \dots j_n i_\lambda}^{i_1 \dots i_n i}}{\sum_{i(\neq j_1 \dots j_n)} (\psi_i | \psi_{i_\lambda}) T_{j_1 \dots j_n i_\lambda}^{j_1 \dots j_n i}}. \quad (7')$$

It should be noted that in the summation of the denominator i may coincide with i_λ and $(\phi_{i_\lambda} | \phi_{i_\lambda}) T_{j_1 \dots j_n i_\lambda}^{i_1 \dots i_n i_\lambda} = 1$. Now, let us assume Eq. (8) holds for $m = n + 1$. Then we get from (7')

$$|T_{j_1 \dots j_n}^{i_1 \dots i_n}| \leq \frac{\sum_{i(\neq i_\lambda)} |(\phi_i | \phi_{i_\lambda})|}{1 - \sum_{i(\neq i_\lambda)} |(\phi_i | \phi_{i_\lambda})|}.$$

Using Condition (9) in the above inequality, we obtain Eq. (8). For $m = N$ Eq. (8) has been proved, because $i_1 \dots i_N$ is a permutation of $j_1 \dots j_N$. Therefore, Theorem II has been established by induction.

Using (8), we can derive the following inequalities,

$$\left. \begin{aligned} \frac{1}{1 + \sum_{k(\neq i)} |(\phi_k | \phi_i)|} &\leq S_i^i \leq \frac{1}{1 - \sum_{k(\neq i)} |(\phi_k | \phi_i)|} \\ |S_j^i| &\leq \frac{\sum_{k(\neq i)} |(\phi_k | \phi_i)|}{[1 - \sum_{k(\neq j)} |(\phi_k | \phi_j)|] \cdot [1 - \sum_{l(\neq i)} |(\phi_l | \phi_i)|]} \\ \frac{1}{1 + \sum_{k, l(\neq i, j)} |(\phi_k | \phi_i)| \cdot |(\phi_l | \phi_j)|} &\leq S_{ij}^{ij} \leq \frac{1}{1 - \sum_{k, l(\neq i, j)} |(\phi_k | \phi_i)| \cdot |(\phi_l | \phi_j)|} \\ \frac{u((ij))}{1 \pm \sum_{kl(\neq i, j)} |(\phi_k | \phi_j)| \cdot |(\phi_l | \phi_i)|} &\leq S_{ij}^{ji} \leq \frac{u((ij))}{1 \mp \sum_{k, l(\neq i, j)} |(\phi_k | \phi_j)| \cdot |(\phi_l | \phi_i)|} \end{aligned} \right\} \quad (10)$$

where $u(P) = (-1)^P (P\theta | \theta)$, P denoting a permutation operator, and the double sign in the denominator depends on the sign of $u((ij))$.

$$\left. \begin{aligned} \left. \begin{aligned} |S_{ij}^{ih}| \\ |S_{ij}^{jh}| \end{aligned} \right\} &\leq \frac{\sum_{k(\neq i, h)} |(\phi_k | \phi_h)|}{[1 - \sum_{(k, l) \neq (i, j)} |(\phi_k | \phi_i)| \cdot |(\phi_l | \phi_j)|] [1 - \sum_{k(\neq i, j, h)} |(\phi_k | \phi_h)|]} \\ |S_{ij}^{gh}| &\leq \frac{[\sum_{k(\neq g, h)} |(\phi_k | \phi_g)|] [\sum_{l(\neq i, h)} |(\phi_l | \phi_h)|]}{[1 - \sum_{(kl) \neq (ij)} |(\phi_k | \phi_i)| \cdot |(\phi_l | \phi_j)|] [1 - \sum_{k(\neq i, j, g)} |(\phi_k | \phi_g)|] [1 - \sum_{l(\neq i, j, gh)} |(\phi_l | \phi_h)|]} \end{aligned} \right\} \quad (10')$$

From these inequalities we see that the exact values of S are not much different from the values obtained by neglecting all overlap integrals $(\phi_k | \phi_i)$ (the usual zeroth approximation) if $\sum_{k(\neq i)} |(\phi_k | \phi_i)|$ is sufficiently small compared with 1/2 or 1/4. Thus, we see that the non-orthogonality catastrophe in the sense pointed by Inglis never arises.

Next we consider a similar problem in the case of the electron pair bond (*epb*) theory. When ϕ_μ and $\phi_{\mu'}$ form an *epb*, $(\phi_\mu | \phi_{\mu'})$, whose sign is hereafter taken to be positive, is sometimes larger than 0.5. Then the usual zeroth approxi-

mation in which all overlap integrals are neglected will be too poor. In such a case, one adopts, as the zeroth approximation, the approximation in which large overlap integrals between bonding mates are taken into account but all the remaining (small) overlap integrals are neglected. Then we wish to prove that the non-orthogonality catastrophe never arises also in this case. Here we consider the case in which all atomic orbitals are real.

Theorem III. If Condition (9) is satisfied for ψ_μ having no bonding mate and

$$\left. \begin{aligned} \sum_{\lambda(\neq \mu, \mu')} |\langle \psi_\lambda | \psi_\mu \rangle| &\leq \frac{1}{2} (1 - \langle \psi_\mu | \psi_{\mu'} \rangle) \\ \langle \psi_\mu | \psi_{\mu'} \rangle &\geq \sum_{\lambda(\neq \mu, \mu')} |\langle \psi_\lambda | \psi_{\mu'} \rangle| \end{aligned} \right\} \quad (9')$$

is satisfied for any ψ_μ and $\psi_{\mu'}$ composing an *epb*, then Eq. (8) holds for arbitrary values of $i_1 \cdots i_m \ j_1 \cdots j_m$.

(Proof)

Let us return to the step in the proof of Theorem II where $|T_{j_1 \cdots j_{n+1}}^{i_1 \cdots i_{n+1}}| < 1$ was taken as an assumption of induction. It is sufficient to consider the case where we can find i_λ ($\lambda=1$ or 2 or $\cdots n$) which is not equal to any of $j_1 \cdots j_n$. If we find i_λ which satisfies the above condition and has no bonding mate, we expand the $T_{j_1 \cdots j_n}^{i_1 \cdots i_n}$ by means of (7'), using this i_λ . Then we can prove $|T_{j_1 \cdots j_n}^{i_1 \cdots i_n}| \leq 1$ in the same way as we have done in the case of Theorem II. If the i_λ inevitably has a bonding mate i'_λ , we proceed with the proof as follows.

Lemma.

$$T_{j_1 \cdots j_{n-1} \ j_n}^{j_1 \cdots j_{n-1} \ j'_n} > 0$$

where j_n and j'_n form an *epb*, and $j'_n \neq j_1, j_2, \cdots j_{n-1}$.

(Proof of Lemma)

The fact that all the ψ_i 's are real leads to the following result,

$$(\Psi^{j_1 \cdots j_{n-1} \ j'_n} | \Psi^{j_1 \cdots j_{n-1} \ j_n}) = (\Psi^{j_1 \cdots j_{n-1} \ j_n} | \Psi^{j_1 \cdots j_{n-1} \ j'_n}).$$

Thus we get

$$\begin{aligned} (\Psi^{j_1 \cdots j_{n-1} \ j'_n} | \Psi^{j_1 \cdots j_{n-1} \ j_n}) &= \sum_{i(\neq j_1 \cdots j_n)} (\psi_i | \psi_{j_n}) (\Psi^{j_1 \cdots j_{n-1} \ j_n i} | \Psi^{j_1 \cdots j_{n-1} \ j'_n j_n}) \\ &= \sum_{i(\neq j_1 \cdots j_n)} (\psi_i | \psi_{j_n}) (\Psi^{j_1 \cdots j_{n-1} \ j_n i} | \Psi^{j_1 \cdots j_{n-1} \ j'_n j'_n}). \end{aligned}$$

In the last transformation we have used the property

$$u(P \cdot (j_n \ j'_n)) = u(P)$$

which is derived directly from the fact that j_n and j'_n form an *epb*. Consequently we obtain

$$T_{j_1 \cdots j_{n-1} \ j_n}^{j_1 \cdots j_{n-1} \ j'_n} = C \sum_{i(\neq j_1 \cdots j_n)} (\psi_i | \psi_{j_n}) T_{j_1 \cdots j_n \ j'_n}^{j_1 \cdots j_n \ i}$$

where

$$C \equiv \frac{(\Psi^{j_1 \dots j_n j'_n} | \Psi^{j_1 \dots j_n j'_n})}{(\Psi^{j_1 \dots j_n} | \Psi^{j_1 \dots j_n})} > 0.$$

Therefore, by the assumption of induction

$$T_{j_1 \dots j_{n-1} j_n}^{j_1 \dots j_{n-1} j'_n} \geq C \{ (\psi_{j'_n} | \psi_{j_n}) - \sum_{i(\neq j_n j'_n)} (\psi_i | \psi_{j_n}) \} \\ > 0 \quad (\text{by Condition (9')}).$$

Now, we will return to the proof of our Theorem. From (7'), the Lemma, and the assumption of induction, we get

$$|T_{j_1 \dots j_n}^{i_1 \dots i_n}| = \frac{\sum_{i(\neq i_1 \dots i_n)} (\psi_i | \psi_{i_n}) T_{j_1 \dots j_n}^{i_1 \dots i_n}}{1 + (\psi_{i_n} | \psi_{i_n}) T_{j_1 \dots j_n}^{i_1 \dots i_n} + \sum_{i(\neq i_n)} (\psi_i | \psi_{i_n}) T_{j_1 \dots j_n}^{i_1 \dots i_n}} \\ \leq \frac{\sum_{i(\neq i_n)} |(\psi_i | \psi_{i_n})|}{1 - \sum_{i(\neq i_n)} |(\psi_i | \psi_{i_n})|}.$$

This inequality leads to Eq. (8) because of Condition (9'). (Q.E.D.)

Similar arguments to those given above lead to inequalities like (10). Here we give only an example:

$$\left[1 + \left\{ \frac{(\psi_\mu | \psi_{\mu'})^2 - \sum_{k(\neq \mu \mu')} (\psi_\mu | \psi_{\mu'}) (\psi_k | \psi_{\mu'})}{1 + \sum_{l(\neq \mu, \mu')} (\psi_l | \psi_{\mu'})} \right\} - \sum_{k(\neq \mu \mu')} |(\psi_k | \psi_{\mu'})| \right]^{-1} \\ > S_\mu^\mu \sim \\ \left[1 + \left\{ \frac{(\psi_\mu | \psi_{\mu'})^2 + \sum_{k(\neq \mu \mu')} (\psi_\mu | \psi_{\mu'}) (\psi_k | \psi_{\mu'})}{1 - \sum_{l(\neq \mu \mu')} (\psi_l | \psi_{\mu'})} \right\} + \sum_{k(\neq \mu \mu')} |(\psi_k | \psi_{\mu'})| \right]^{-1}.$$

(Here we may assume that

$$\psi_{\mu'}' \equiv 0$$

if ψ_μ has no bonding mate.) Then, in the zeroth approximation, we get

$$S_\mu^\mu \approx \frac{1}{1 + (\psi_\mu | \psi_{\mu'})^2}$$

which has been rigorously established if

$$\sum_{l(\neq \lambda \lambda')} |(\psi_l | \psi_\lambda)| \quad \text{and} \quad \sum_{l(\neq \lambda \lambda')} |(\psi_l | \psi_{\lambda'})|$$

are sufficiently small. Thus we see that the non-orthogonality catastrophe does not arise also in this case.

Finally we would like to express our sincere thanks to Prof. Kotani for his kind introduction and guidance on this problem. Further, we are very grateful to the members of Kotani Laboratory for their helpful discussions.

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Electromagnetic Structure of the Nucleon. II

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The 3π -state contributions to the isoscalar part of nucleon electromagnetic structure are explicitly calculated in the lowest order of perturbation theory (order g^6). The detailed calculational techniques, numerical evaluations and approximation methods are presented. The lowest order meson-structure correction to the isovector part (order g^4) also is calculated in detail.

§ 1. Introduction

In a previous paper,¹⁾ which is cited as I, we have investigated the electromagnetic structure of the nucleon. The present paper contains only the technical details of the investigation in I. In § 2, we summarize the general method of perturbation calculation presented by one of the authors (N.N.)²⁾. The four sections from § 3 to § 6 are devoted to the lowest order perturbation calculations of the 3π -state contributions. In § 3, the derivation of the final results, which are summarized in § 4, is illustrated in detail so that anyone who knows the Feynman-Dyson theory can check our calculations. Rough numerical evaluations and approximation methods are presented in § 5 and § 6, respectively. § 7 contains the final results of the lowest order meson-structure correction to the isovector part.

Notations are the same as those of I throughout this paper, unless otherwise indicated.

§ 2. General integral formula

In this section, we summarize the general integral formula presented in NI.²⁾

Consider an arbitrary Feynman graph G , which consists of α internal lines. They are numbered 1, 2, ..., α . Each internal line corresponds to a propagator. Each boson propagator has the form

$$1/\{(p_i + q_i)^2 + \bar{M}_i\}, \quad (2.1)$$

where p_i and q_i are respectively a momentum to be integrated and a constant one, and $M_i = m_i^2$. The upper bar denotes $-i\epsilon$, ($\epsilon \rightarrow +0$). Though q_i and M_i are con-

stants, we regard them as independent parameters without substituting their actual values for a while. Each fermion propagator has the form

$$(-i\gamma D_i + m_i) / \{ (p_i + q_i)^2 + \bar{M}_i \}, \quad (2.2)$$

where D_i is the following formal operator :*

$$D_i = \frac{1}{2} \int_{-\infty}^{M_i} dM_i \frac{\partial}{\partial q_i}. \quad (2.3)$$

Making use of the generalized Feynman identity,

$$\frac{1}{\prod_{i=1}^{\alpha} A_i} = (\alpha-1)! \int \frac{dx^{(\alpha-1)}}{(\sum_{i=1}^{\alpha} x_i A_i)^{\alpha}} \quad (2.4)$$

with
$$dx^{(\alpha-1)} \equiv \prod_{i=1}^{\alpha} dx_i \cdot \delta(1 - \sum_{i=1}^{\alpha} x_i), \quad (x_i \geq 0),$$

we can easily carry out the integrations over p_i , and the result is as follows :

$$(\pi^2 i)^n (\alpha - 2n - 1)! \int \frac{f(D) dx^{(\alpha-1)}}{U^2(x) \{ \bar{V}(q, x) \}^{\alpha-2n}}, \quad (2.5)$$

where n is the number of independent integration momenta, and we have assumed $\alpha > 2n$. If $\alpha \leq 2n$, it suffices to use the formula presented in § 4 of NI, but such a more general formula is not necessary in the calculation of the 3π -state contribution.

The notations used in (2.5) are as follows.

$$U(x) \equiv \sum x_{\nu_1} x_{\nu_2} \cdots x_{\nu_n}. \quad (2.6)$$

Here the summation goes over all possible sets $(\nu_1, \nu_2, \cdots, \nu_n)$ such that $p_{\nu_1}, p_{\nu_2}, \cdots, p_{\nu_n}$ are independent variables.

$$V(q, x) \equiv \sum_{i=1}^{\alpha} x_i (M_i + q_i^2) - (1/U) \sum_C U_C \cdot (\pm x_{c_1} q_{c_1} \pm x_{c_2} q_{c_2} \pm \cdots \pm x_{c_l} q_{c_l})^2, \quad (2.7)$$

where C is a closed circuit** belonging to the graph G , and c_1, c_2, \cdots, c_l are the lines belonging to C . The double signs correspond to the relative directions of q_{c_j} on the circuit C . Of course, for a fermion propagator the direction of q_i must coincide with that of the arrow.

$$U_C(x) \equiv \sum x_{\nu_1} x_{\nu_2} \cdots x_{\nu_{n-1}}. \quad (2.8)$$

The summation goes over all possible sets $(\nu_1, \nu_2, \cdots, \nu_{n-1})$ such that no one of $\nu_1, \nu_2, \cdots, \nu_{n-1}$ belongs to C , and that $p_{\nu_1}, p_{\nu_2}, \cdots, p_{\nu_{n-1}}$ are independent. The summation \sum_C in (2.7) means to sum over all possible closed circuits in G .

* This operator was first used by Karplus and Kroll.³⁾

** The terminology "closed loop" is not used, since it is usually used in a narrower sense.

The numerator in (2.5), $f(D)$, is a polynomial of D_i including γ - and τ -matrices. According to (2.3), we carry out the operations D_i , which yield the following results. To firstly operate D_i on a function of V is equivalent to a multiplication by a factor

$$Y_i(q, x) \equiv (1/2x_i) \partial V / \partial q_i = q_i - (1/U) \sum_{C(i)} U_{C(i)}(x_i q_i \pm \dots), \quad (2.9)$$

where $C(i)$ stands for a circuit containing the line i . The next operation of D_j yields an extra term since Y_i itself generally contains q_j , namely,

$$D_j D_i \frac{1}{V^s} = D_j \frac{Y_i}{V^s} = \frac{Y_i Y_j}{V^s} + \frac{1}{2(s-1)} \cdot \frac{X_{ij}}{V^{s-1}}, \quad (2.10)$$

where

$$X_{ij}(x) \equiv -(1/x_j) \partial Y_i / \partial q_j = (1/U) \sum_{C(i,j)} \pm U_{C(i,j)} \cdot \delta = X_{ji}(x). \quad (2.11)$$

Here $C(i, j)$ stands for a circuit containing both the lines i and j , and the double sign corresponds to the relative direction of i and j on $C(i, j)$ (+ for same, - for opposite). δ is the usual unit tensor. When the explicit indication of tensor suffices is necessary, we denote them by superscripts (*e.g.*, q_i^μ , Y_i^μ , $\delta^{\mu\nu}$, *etc.*). Since X_{ij} contains neither q nor M , further operations of D introduce no new type of function, and we can write down the result of the operation of $f(D)$ by using Y and X only.

When the operation of $f(D)$ has thus been finished, we may substitute the actual values of q_i and M_i . Then, of course, q_i is not uniquely determined by the external momenta. We may choose q_i suitably to the calculation. The functions V and Y_i can be written down directly in terms of the external momenta rather than q_i (NI § 3). The following calculations were independently checked by both formulae (in terms of q_i and directly in terms of the external momenta).

§ 3. Calculations

The lowest order graphs of the 3π -state contribution are shown in Fig. 1. In each graph the isovector part cancels when (a) and (b) are added. The three resultants are denoted by I, II, and III, respectively.

The matrix element of the vertex part is

$$\begin{aligned} I^\mu(q) &= i^3 \cdot e [ig(2\pi)^4]^0 [-i/(2\pi)^4]^0 (-1) \cdot (\pi^2 i)^3 2! \sum_{N=1}^{\text{III}} T_N \cdot F_N^\mu(q) \\ &= [2eg^6/(4\pi)^0] \sum_{N=1}^{\text{III}} T_N \cdot F_N^\mu(q), \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} T_I &\equiv \tau_k \tau_j \tau_i \cdot \text{Sp}(\tau_k \tau_j \tau_i) = -12, \\ T_{II} &= T_{III} = +12, \end{aligned} \quad (3.2)$$

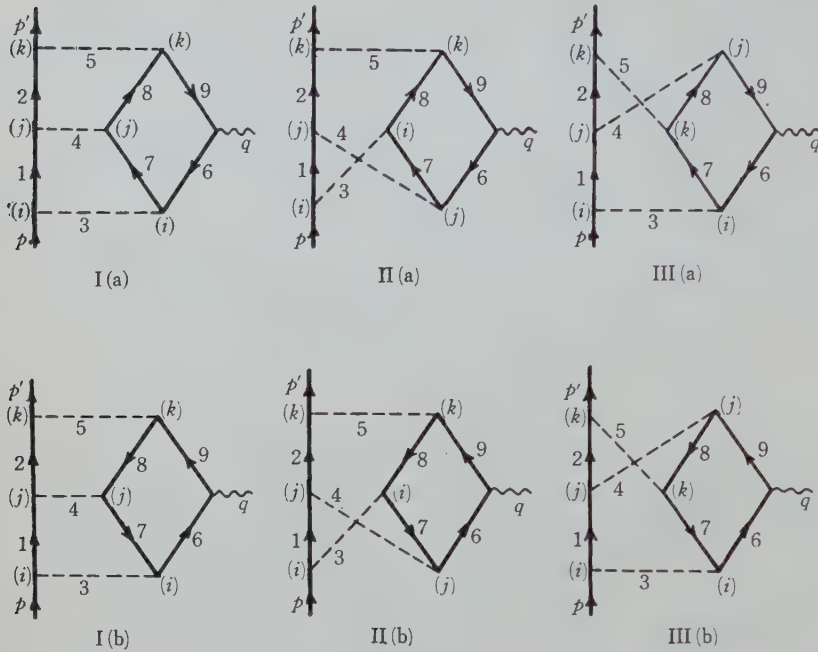


Fig. 1

and

$$F_N^\mu \equiv \int \frac{f^\mu(D)}{U_N^2 \bar{V}_N^3} dx^{(8)}, \quad (\alpha=9, n=3). \quad (3.3)$$

For simplicity, we omit the suffix N in the following, since N is irrelevant to the calculations up to (3.34).

$$\begin{aligned} f^\mu(D) &\equiv \bar{u}'(p') \gamma_5 (-i\gamma D_2 + M) \gamma_5 (-i\gamma D_1 + M) \gamma_5 u(p) \\ &\cdot \text{Sp} [(-i\gamma D_9 + M') \gamma_5 (-i\gamma D_8 + M') \gamma_5 (-i\gamma D_7 + M') \gamma_5 \\ &\cdot (-i\gamma D_6 + M') \gamma^\mu], \end{aligned} \quad (3.4)$$

where M' stands for the nucleon mass. We have discriminated the mass M' in the closed loop from M so as to be able to apply the result to the case of hyperon loop. Calculating the Spur part, we obtain

$$\begin{aligned} f^\mu(D) &= 4iM' \bar{u}'(p') (i\gamma D_2 + M) (-i\gamma D_1 + M) \gamma_5 u(p) \\ &\cdot \varepsilon^{\nu\sigma\tau\mu} [D_9^\nu D_8^\sigma D_7^\tau - D_9^\nu D_8^\sigma D_6^\tau + D_9^\nu D_7^\sigma D_6^\tau - D_8^\nu D_7^\sigma D_6^\tau]. \end{aligned} \quad (3.5)$$

It should be noticed that the highest degree term of D disappears and so no ultra-violet divergence occurs.

Next, we calculate the result of the operation of $f^\mu(D)$ on V^{-3} . Then we notice that

$$\varepsilon^{\nu\sigma\tau\mu} X_{ij}^{\nu\sigma} = 0, \quad (3.6)$$

and

$$\varepsilon^{\nu\sigma\tau\mu} Y_i^\nu Y_j^\sigma Y_k^\tau = 0. \quad (3.7)$$

(3.6) immediately follows from (2.11). (3.7) is obtained by noticing that Y_i is expressed in terms of the external momenta such as

$$Y_i = a_i p + b_i p'. \quad (3.8)$$

Here we must notice that (3.6) is a true identity, but (3.7) holds only after the substitutions of the actual values for q_i and therefore we must not use it until the completion of the operations of D .

The results of the operations of D are easily calculated as follows.

$$\varepsilon^{\nu\sigma\tau\mu} D_i^\nu D_j^\sigma D_k^\tau V^{-3} = 0.$$

$$D_l^\alpha \varepsilon^{\nu\sigma\tau\mu} D_i^\nu D_j^\sigma D_k^\tau V^{-3} = (1/4) \varepsilon^{\alpha\sigma\tau\mu} (X_{li} Y_j^\sigma Y_k^\tau + X_{lj} Y_k^\sigma Y_i^\tau + X_{lk} Y_i^\sigma Y_j^\tau) V^{-2},$$

$$D_2^\beta D_1^\alpha \varepsilon^{\nu\sigma\tau\mu} D_i^\nu D_j^\sigma D_k^\tau V^{-3}$$

$$\begin{aligned} &= (1/4) [Y_1^\alpha \varepsilon^{\beta\sigma\tau\mu} (X_{2i} Y_j^\sigma Y_k^\tau + X_{2j} Y_k^\sigma Y_i^\tau + X_{2k} Y_i^\sigma Y_j^\tau) \\ &\quad + Y_2^\beta \varepsilon^{\alpha\sigma\tau\mu} (X_{1i} Y_j^\sigma Y_k^\tau + X_{1j} Y_k^\sigma Y_i^\tau + X_{1k} Y_i^\sigma Y_j^\tau)] V^{-2} \\ &\quad + (1/8) \varepsilon^{\alpha\beta\tau\mu} [X_{1i} X_{2j} Y_k^\tau + X_{1j} X_{2k} Y_i^\tau + X_{1k} X_{2i} Y_j^\tau \\ &\quad - X_{1j} X_{2i} Y_k^\tau - X_{1k} X_{2j} Y_i^\tau - X_{1i} X_{2k} Y_j^\tau] V^{-1}, \end{aligned} \quad (3.9)$$

where for convenience we have redefined X_{ij} in the omission of δ in (2.11). The substitution of (3.9) in (3.5) yields

$$\begin{aligned} f^\mu(D) V^{-3} &= M' V^{-2} \cdot \bar{u}'(p') [(i\gamma Y_2 + M) \gamma^\alpha A_1^{\alpha\mu} + \gamma^\alpha A_2^{\alpha\mu} (i\gamma Y_1 - M)] \gamma_5 u(p) \\ &\quad + (1/2) i M' V^{-1} \bar{u}'(p') \gamma^\beta \gamma^\alpha B^{\alpha\beta\mu} \gamma_5 u(p), \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} A_l^{\alpha\mu} &\equiv \varepsilon^{\alpha\sigma\tau\mu} [X_{li} Y_8^\sigma Y_7^\tau + X_{li} Y_7^\sigma Y_9^\tau + X_{li} Y_9^\sigma Y_8^\tau \\ &\quad - X_{li} Y_8^\sigma Y_6^\tau - X_{li} Y_6^\sigma Y_9^\tau - X_{li} Y_9^\sigma Y_6^\tau \\ &\quad + X_{li} Y_7^\sigma Y_6^\tau + X_{li} Y_6^\sigma Y_9^\tau + X_{li} Y_9^\sigma Y_7^\tau \\ &\quad - X_{li} Y_7^\sigma Y_6^\tau - X_{li} Y_6^\sigma Y_8^\tau - X_{li} Y_8^\sigma Y_7^\tau], \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} B^{\alpha\beta\mu} &\equiv \varepsilon^{\alpha\beta\tau\mu} [X_{19} X_{28} Y_7^\tau + X_{18} X_{27} Y_9^\tau + X_{17} X_{29} Y_8^\tau \\ &\quad - X_{18} X_{29} Y_7^\tau - X_{17} X_{28} Y_9^\tau - X_{19} X_{27} Y_8^\tau \\ &\quad - X_{19} X_{28} Y_6^\tau - X_{18} X_{26} Y_9^\tau - X_{16} X_{29} Y_8^\tau \\ &\quad + X_{18} X_{29} Y_6^\tau + X_{16} X_{28} Y_9^\tau + X_{19} X_{26} Y_8^\tau \\ &\quad + X_{19} X_{27} Y_6^\tau + X_{17} X_{26} Y_9^\tau + X_{16} X_{29} Y_7^\tau] \end{aligned}$$

$$\begin{aligned}
& -X_{17}X_{29}Y_6^\tau - X_{16}X_{27}Y_9^\tau - X_{19}X_{26}Y_7^\tau \\
& -X_{18}X_{27}Y_6^\tau - X_{17}X_{26}Y_8^\tau - X_{16}X_{28}Y_7^\tau \\
& + X_{17}X_{28}Y_6^\tau + X_{16}X_{27}Y_8^\tau + X_{18}X_{26}Y_7^\tau].
\end{aligned} \tag{3.12}$$

It is convenient to express A_i and B in terms of the differences of Y_i . Namely, using the identity

$$\varepsilon^{\alpha\sigma\tau\mu}Y_9^\sigma Y_9^\tau = 0, \tag{3.13}$$

we can rewrite (3.11) as

$$\begin{aligned}
A_i^{\alpha\mu} = & \varepsilon^{\alpha\sigma\tau\mu}[(X_{i6} - X_{i9})(Y_7^\sigma - Y_9^\sigma)(Y_8^\tau - Y_9^\tau) \\
& + (X_{i7} - X_{i9})(Y_8^\sigma - Y_9^\sigma)(Y_6^\tau - Y_9^\tau) \\
& + (X_{i8} - X_{i9})(Y_6^\sigma - Y_9^\sigma)(Y_7^\tau - Y_9^\tau)],
\end{aligned} \tag{3.14}$$

and likewise (3.12) becomes

$$\begin{aligned}
B^{\alpha\beta\mu} = & \varepsilon^{\alpha\beta\tau\mu}[\{(X_{16} - X_{19})(X_{27} - X_{29}) - (X_{17} - X_{19})(X_{26} - X_{29})\}(Y_8^\tau - Y_9^\tau) \\
& + \{(X_{18} - X_{19})(X_{26} - X_{29}) - (X_{16} - X_{19})(X_{28} - X_{29})\}(Y_7^\tau - Y_9^\tau) \\
& + \{(X_{17} - X_{19})(X_{28} - X_{29}) - (X_{18} - X_{19})(X_{27} - X_{29})\}(Y_6^\tau - Y_9^\tau)].
\end{aligned} \tag{3.15}$$

Making use of

$$Y_6 - Y_9 = p' - p \tag{3.16}$$

and

$$X_{16} - X_{19} = X_{26} - X_{29} = 0, \tag{3.17}$$

which will be verified later by concrete calculations, we obtain

$$A_i^{\alpha\mu} = (p' - p)^\sigma \varepsilon^{\alpha\sigma\tau\mu}[-(X_{i7} - X_{i9})(Y_8 - Y_9)^\tau + (X_{i8} - X_{i9})(Y_7 - Y_9)^\tau], \tag{3.18}$$

$$B^{\alpha\beta\mu} = (p' - p)^\tau \varepsilon^{\alpha\beta\tau\mu}[(X_{17} - X_{19})(X_{28} - X_{29}) - (X_{18} - X_{19})(X_{27} - X_{29})], \tag{3.19}$$

which are rewritten as

$$A_i^{\alpha\mu} = p^\sigma p'^\tau \varepsilon^{\alpha\sigma\tau\mu} L_i, \tag{3.20}$$

$$B^{\alpha\beta\mu} = (p' - p)^\tau \varepsilon^{\alpha\beta\tau\mu} K, \tag{3.21}$$

with

$$L_i \equiv (X_{i7} - X_{i9})(a_8 + b_8 - a_9 - b_9) - (X_{i8} - X_{i9})(a_7 + b_7 - a_9 - b_9), \tag{3.22}$$

and

$$K \equiv (X_{17} - X_{19})(X_{28} - X_{29}) - (X_{18} - X_{19})(X_{27} - X_{29}), \tag{3.23}$$

where a_i and b_i are defined by (3.8).

By substituting (3.20) and (3.21) in (3.10), there appears the following factor in the first term:

$$\bar{u}'(p')(a_2 \cdot i\gamma p + b_2 \cdot i\gamma p' + M)\gamma^\alpha p^\sigma p'^\tau \varepsilon^{\alpha\sigma\tau\mu} \gamma_5 u(p). \tag{3.24}$$

By using an identity

$$\gamma^\alpha \varepsilon^{\alpha\sigma\tau\mu} = (\gamma^\tau \gamma^\mu \gamma^\sigma - \gamma^\sigma \gamma^\mu \gamma^\tau - \gamma^\tau \gamma^\sigma \gamma^\mu + \gamma^\sigma \gamma^\tau \gamma^\mu) \gamma_5 \quad (3.25)$$

and Dirac equations

$$i\gamma p \cdot u(p) = -Mu(p)$$

and

$$\bar{u}'(p') \cdot i\gamma p' = -M\bar{u}'(p'), \quad (3.26)$$

(3.24) becomes

$$-M(a_2 + b_2 - 1)\bar{u}'(p')[(pp' - M^2)\gamma^\mu - iM(p^\mu + p'^\mu)]u(p) \quad (3.27)$$

after some elementary algebraic manipulations. Likewise we obtain

$$\begin{aligned} \bar{u}'(p')\gamma^\alpha A_2^{\alpha\mu}(i\gamma Y_1 - M)\gamma_5 u(p) \\ = M(a_1 + b_1 - 1)\bar{u}'(p')[(pp' - M^2)\gamma^\mu - iM(p^\mu + p'^\mu)]u(p) \cdot L_2. \end{aligned} \quad (3.28)$$

From an identity

$$\gamma^\beta \gamma^\alpha \varepsilon^{\alpha\beta\tau\mu} = 2(\gamma^\tau \gamma^\mu - \delta^{\tau\mu})\gamma_5 \quad (3.29)$$

and Dirac equations (3.26), the last term of (3.10) can be rewritten as

$$\bar{u}'(p')\gamma^\beta \gamma^\alpha B^{\alpha\beta\mu}\gamma_5 u(p) = -2\bar{u}'(p')[-2iM\gamma^\mu + (p^\mu + p'^\mu)]u(p) \cdot K. \quad (3.30)$$

Substitution of (3.27), (3.28) and (3.30) into (3.10) yields

$$\begin{aligned} f^\mu(D) V^{-3} = -MM'V^{-2}P \cdot \bar{u}'(p')[(pp' - M^2)\gamma^\mu - iM(p^\mu + p'^\mu)]u(p) \\ - iM'V^{-1}K \cdot \bar{u}'(p')[-2iM\gamma^\mu + (p^\mu + p'^\mu)]u(p), \end{aligned} \quad (3.31)$$

where

$$P \equiv (a_2 + b_2 - 1)L_1 - (a_1 + b_1 - 1)L_2. \quad (3.32)$$

Further by utilizing

$$pp' = -q^2/2 - M^2$$

and

$$\bar{u}'(p')(p^\mu + p'^\mu)u(p) = i\bar{u}'(p')(-\sigma^{\mu\lambda}q^\lambda + 2M\gamma^\mu)u(p), \quad (3.33)$$

where

$$q = p' - p$$

and

$$\sigma^{\mu\nu} = (i/2)(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu),$$

(3.31) becomes

$$\begin{aligned} f^\mu(D) V^{-3} = (1/2)MM'q^2 \cdot PV^{-2} \cdot \bar{u}'(p')\gamma^\mu u(p) \\ + M'[M^2PV^{-2} - KV^{-1}] \cdot \bar{u}'(p')\sigma^{\mu\nu}q^\nu u(p). \end{aligned} \quad (3.34)$$

Thus the form of Γ^μ has been found. The remaining task is to calculate the functions U , V , P and K according to the general integral formula.

We will hereafter state the calculations on Graph I alone, since the other cases are similarly calculated.

The expressions for U and U_c are determined by the sets of independent integration variables as was stated in § 2. Therefore the external lines may be disregarded. It is convenient to deform Graph I(a) to Fig. 2 (Q -diagram) which is topologically equivalent to it. Since p_1 and p_3 are the same apart from an additive constant, x_1 and x_3 can appear only in the combination x_1+x_3 . Since p_1, p_2 and p_4 are not independent because they start from a common vertex, the term $x_1x_2x_4$ does not appear. Noticing such facts, we can easily construct the expression for U :

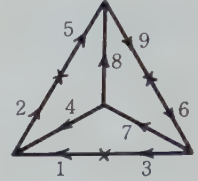


Fig. 2

$$\begin{aligned}
 U = & (x_1+x_3)(x_2+x_5)(x_6+x_9) + (x_1+x_3)(x_2+x_5)(x_7+x_8) \\
 & + (x_2+x_5)(x_6+x_9)(x_4+x_7) + (x_6+x_9)(x_1+x_3)(x_4+x_8) \\
 & + (x_1+x_3)(x_4+x_7)x_8 + (x_2+x_5)(x_4+x_8)x_7 + (x_6+x_9)(x_7+x_8)x_4 \\
 & + (x_1+x_3)x_4x_7 + (x_2+x_5)x_4x_8 + (x_6+x_9)x_7x_8.
 \end{aligned} \quad (3.35)$$

According to (2.7), we can easily write down also the expression for V :

$$\begin{aligned}
 V = & \sum_{i=1}^9 x_i (M_i + q_i^2) \\
 & - (1/U) [(x_4x_7+x_7x_8+x_8x_4)(x_3q_3+x_1q_1+x_2q_2+x_5q_5+x_9q_9+x_6q_6)^2 \\
 & + x_4(x_6+x_9)(x_3q_3+x_1q_1+x_2q_2+x_5q_5-x_8q_8-x_7q_7)^2 \\
 & + x_8(x_1+x_3)(x_2q_2+x_5q_5+x_9q_9+x_6q_6+x_7q_7+x_4q_4)^2 \\
 & + x_7(x_2+x_5)(x_3q_3+x_1q_1-x_4q_4+x_8q_8+x_9q_9+x_6q_6)^2 \\
 & + \{(x_2+x_5)x_8+x_8(x_6+x_9)+(x_6+x_9)(x_2+x_5)\}(x_3q_3+x_1q_1-x_4q_4-x_7q_7)^2 \\
 & + \{(x_1+x_3)x_7+x_7(x_6+x_9)+(x_6+x_9)(x_1+x_3)\}(x_2q_2+x_5q_5-x_8q_8+x_4q_4)^2 \\
 & + \{(x_1+x_3)x_4+x_4(x_2+x_5)+(x_2+x_5)(x_1+x_3)\}(x_7q_7+x_8q_8+x_9q_9+x_6q_6)^2],
 \end{aligned} \quad (3.36)$$

which is expressed in terms of the external momenta by representing q_i in terms of p and p' , e.g., by putting $q_1=q_2=p$, $q_5=q_9=p-p'$ and $q_3=q_4=q_6=q_7=q_8=0$. The result is given in the next section. From (3.36), Y_i and X_{ij} are immediately calculated according to (2.9) and (2.11), e.g.,

$$\begin{aligned}
 Y_1 = & q_1 - (1/U) [(x_4x_7+x_7x_8+x_8x_4)(x_3q_3+x_1q_1+x_2q_2+x_5q_5+x_9q_9+x_6q_6) \\
 & + x_4(x_6+x_9)(x_3q_3+x_1q_1+x_2q_2+x_5q_5-x_8q_8-x_7q_7) \\
 & + x_7(x_2+x_5)(x_3q_3+x_1q_1-x_4q_4+x_8q_8+x_9q_9+x_6q_6) \\
 & + \{(x_2+x_5)x_8+x_8(x_6+x_9)+(x_6+x_9)(x_2+x_5)\}(x_3q_3+x_1q_1-x_4q_4-x_7q_7)], \\
 X_{16} = & X_{19} = (1/U) [(x_4x_7+x_7x_8+x_8x_4) + x_7(x_2+x_5)].
 \end{aligned} \quad (3.37)$$

The expressions in terms of p and p' are as follows. In these calculations, we need to consider only $p=p'$ without loss of generality, since we need the sums a_i+b_i only. Namely, we can put $q_1=q_2=p$ and all other $q_i=0$.

$$\begin{aligned}
 a_1+b_1-1 &= -(1/U)[(x_1x_2+x_1x_4+x_2x_4+x_1x_5)(x_6+x_7+x_8+x_9) \\
 &\quad +x_1x_6x_8+x_1x_8x_9+x_1x_7x_8+x_2x_7x_8], \\
 a_2+b_2-1 &= -(1/U)[(x_1x_2+x_1x_4+x_2x_4+x_2x_5)(x_6+x_7+x_8+x_9) \\
 &\quad +x_2x_7x_9+x_2x_6x_7+x_1x_7x_8+x_2x_7x_8], \\
 a_6-a_9 &= -(b_6-b_9) = -1, \\
 a_7+b_7-a_9-b_9 &= -(a_1+b_1-1), \\
 a_8+b_8-a_9-b_9 &= -(a_2+b_2-1), \\
 X_{16}-X_{19} &= X_{26}-X_{29} = 0, \\
 X_{17}-X_{19} &= -(1/U)[(x_2+x_4+x_5)(x_6+x_7+x_8+x_9)+x_8(x_6+x_7+x_9)], \\
 X_{18}-X_{19} &= X_{27}-X_{29} = -(1/U)[x_4(x_6+x_7+x_8+x_9)+x_7x_8], \\
 X_{28}-X_{29} &= -(1/U)[(x_1+x_3+x_4)(x_6+x_7+x_8+x_9)+x_7(x_6+x_8+x_9)].
 \end{aligned} \tag{3.38}$$

Substituting (3.38) into (3.22), we find

$$\begin{aligned}
 L_1 &= -x_2(x_6+x_7+x_8+x_9)/U, \\
 L_2 &= x_1(x_6+x_7+x_8+x_9)/U.
 \end{aligned} \tag{3.39}$$

Therefore from (3.32) and (3.23) we obtain

$$\begin{aligned}
 P &= (1/U^2)[\{x_1^2(x_2+x_5)+x_2^2(x_1+x_3)+(x_1+x_2)^2x_4\}(x_6+x_7+x_8+x_9) \\
 &\quad +x_1^2x_6(x_6+x_9)+x_2^2x_7(x_6+x_9)+(x_1+x_2)^2x_7x_8\}(x_6+x_7+x_8+x_9), \\
 K &= (x_6+x_7+x_8+x_9)/U.
 \end{aligned} \tag{3.40}$$

Thus the calculation has been completed. The final results are obtained by collecting (3.1), (3.2), (3.3), (3.35), (3.36), (3.34) and (3.40).

§ 4. Results

We summarize the final results in the following.

$$\begin{aligned}
 F_1^s(q^2) &= \sum_{I \Pi \text{ III}} \frac{3}{16\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' \cdot (-q^2) \cdot \int \frac{\xi(x) \sum_6^9 x_i}{U^3(x) \bar{V}^2(q^2, x)} dx^{(8)}, \\
 F_2^s(q^2) &= \sum_{I \Pi \text{ III}} \frac{3}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' \left[-M^2 \int \frac{\xi(x) \sum_6^9 x_i}{U^3(x) \bar{V}^2(q^2, x)} dx^{(8)} + \right.
 \end{aligned}$$

$$+ \int \frac{\sum_6^9 x_i}{U^3(x) \bar{V}(q^2, x)} dx^{(8)} \quad (4.1)$$

with

$$dx^{(8)} = \partial(1 - \sum_1^9 x_i) \prod_1^9 dx_i, \quad x_i \geq 0, \\ V(q^2, x) = \hat{\xi}(x) M^2 + \sum_3^9 x_i \cdot \mu^2 + \sum_6^9 x_i \cdot M'^2 + \zeta(x) \cdot q^2. \quad (4.2)$$

The expressions for $U(x)$, $\hat{\xi}(x)$ and $\zeta(x)$ are as follows.

[Graph I]

$$U(x) = \{ (x_1 + x_3)(x_2 + x_5) + (x_1 + x_3)x_4 + x_4(x_2 + x_5) \} (x_6 + x_7 + x_8 + x_9) \\ + (x_1 + x_2 + x_3 + x_5)x_7x_8 + (x_1 + x_3 + x_4)x_8(x_6 + x_9) \\ + (x_2 + x_4 + x_5)x_7(x_6 + x_9) + x_7x_8(x_6 + x_9), \\ \hat{\xi}(x) = [1/U(x)] \cdot [\{x_1^2(x_2 + x_5) + x_2^2(x_1 + x_3) + (x_1 + x_2)^2x_4\} (x_6 + x_7 + x_8 + x_9) \\ + x_1^2x_8(x_6 + x_9) + x_2^2x_7(x_6 + x_9) + (x_1 + x_2)^2x_7x_8], \\ \zeta(x) = [1/U(x)] \cdot [x_3x_4x_5(x_6 + x_7 + x_8 + x_9) \\ + \{ (x_1 + x_3)(x_2 + x_5) + (x_1 + x_3)x_4 + x_4(x_2 + x_5) \} x_6x_9 \\ + (x_1 + x_3 + x_4)(x_5 + x_9)x_6x_8 + (x_2 + x_4 + x_5)(x_3 + x_6)x_7x_9 \\ + (x_4 + x_6)x_5x_6x_7 + (x_4 + x_7)x_3x_8x_9 + (x_3x_5 + x_6x_9)x_7x_8]. \quad (4.3)$$

[Graph II]

$$U(x) = \{ (x_1 + x_3)(x_2 + x_5) + (x_1 + x_3)x_4 + x_4(x_2 + x_5) \} (x_6 + x_7 + x_8 + x_9) \\ + (x_1 + x_2 + x_3 + x_5)x_7(x_6 + x_9) + (x_1 + x_3 + x_4)x_8(x_6 + x_9) \\ + (x_2 + x_4 + x_5)x_7x_8 + x_7x_8(x_6 + x_9), \\ \hat{\xi}(x) = [1/U(x)] \cdot [\{x_1^2(x_2 + x_5) + x_2^2(x_1 + x_3) + (x_1 + x_2)^2x_4\} (x_6 + x_7 + x_8 + x_9) \\ + x_1^2x_8(x_6 + x_9) + x_2^2x_7x_8 + (x_1 + x_2)^2x_7(x_6 + x_9)], \\ \zeta(x) = [1/U(x)] \cdot [x_3x_4x_5(x_6 + x_7 + x_8 + x_9) \\ + \{ (x_1 + x_3)(x_2 + x_5) + (x_1 + x_3)x_4 + x_4(x_2 + x_5) \} x_6x_9 \\ + (x_1 + x_3 + x_4)(x_5 + x_9)(x_6 + x_8) + (x_1 + x_2 + x_3 + x_5)(x_4 + x_6)x_7x_9 \\ + x_1x_2x_7x_9 + x_3x_5x_6x_7 + (x_4 + x_6)x_5x_7x_8 \\ + (x_3 + x_7)x_4x_8x_9 + x_6x_7x_8x_9]. \quad (4.4)$$

[Graph III]

These are obtained from Graph II by exchanges of parameters ($x_1 \leftrightarrow x_2$, $x_3 \leftrightarrow x_5$, $x_6 \leftrightarrow x_9$, $x_7 \leftrightarrow x_8$). Therefore the integrated values are equal to those of Graph II.

As for U and $\hat{\xi}$, the expressions are interchanged by the following exchanges:

Graph I \leftrightarrow Graph II for $x_8 \leftrightarrow x_8 + x_9$,

Graph I \leftrightarrow Graph III for $x_7 \leftrightarrow x_8 + x_9$.

It is of interest that in (4.1) $F_1^S(q^2)$ and the first term of $F_2^S(q^2)$ are obtained from the second term of $F_2^S(q^2)$ through the operation $\partial/\partial M^2$ apart from a constant factor.

From (4.1) μ^S and $\langle r^2 \rangle_{1,2}^S$ for no cut-off are immediately obtained.

$$Q^S = 0,$$

$$\begin{aligned} \langle r^2 \rangle_1^S &= \sum_{I \text{ III}} \frac{9}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' \int \frac{\hat{\xi}(x) \sum_6^9 x_i}{U^3(x) V^2(0, x)} dx^{(8)}, \\ \mu^S &= \sum_{I \text{ III}} \frac{3}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' \int \frac{\sum_3^5 x_i \cdot \mu^2 + \sum_6^9 x_i \cdot M'^2}{U^3(x) V^2(0, x)} \cdot \sum_6^9 x_i \cdot dx^{(8)}, \\ \mu^S \langle r^2 \rangle_2^S &= \sum_{I \text{ III}} \frac{9}{2\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' \int \frac{\sum_3^5 x_i \cdot \mu^2 + \sum_6^9 x_i \cdot M'^2 - \hat{\xi}(x) M^2}{U^3(x) \cdot V^3(0, x)} \\ &\quad \cdot \sum_6^9 x_i \cdot \zeta(x) dx^{(8)}. \end{aligned} \quad (4.5)$$

The spectral functions $\alpha_{1,2}^S(m^2)$ also are easily obtained from (4.1) by using the identity

$$\text{Im } 1/\bar{V}^n = [\pi/(n-1)!] \delta^{(n-1)}(-V), \quad (n \geq 1). \quad (4.6)$$

From $\alpha_{1,2}^S(m^2)$, the distribution functions are calculated by means of I(2.7).

$$\begin{aligned} 4\pi r^2 \rho_1^S(r) &= \sum_{I \text{ III}} \int \varphi_1(x) \kappa^4(x) [(1/2)\kappa(x)r^2 - r] e^{-\kappa(x)r} dx^{(8)}, \\ 4\pi r^2 \rho_2^S(r) &= \sum_{I \text{ III}} \int [\varphi_2(x) \kappa^2(x)r - 2M^2 \varphi_1(x) \kappa^3(x)r^2] e^{-\kappa(x)r} dx^{(8)}, \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} \varphi_1(x) &\equiv \frac{3}{16\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' \frac{\hat{\xi}(x) \sum_6^9 x_i}{U^3(x) V^2(0, x)}, \\ \varphi_2(x) &\equiv \frac{3}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' \frac{\sum_6^9 x_i}{U^3(x) V(0, x)}, \end{aligned} \quad (4.8)$$

and

$$\kappa(x) \equiv \sqrt{V(0, x)/\zeta(x)} \geq 3\mu.$$

Using (4.7), we can calculate r.m.s. radii, *etc.*, for cut-off. Namely, when $f(r)$ is a cut-off function, we have

$$\langle r^2 \rangle_{1 \text{ cut-off}}^S = 2 \cdot \int_0^\infty 4\pi r^4 \rho_1^S(r) f(r) dr, \text{ etc.}, \quad (4.9)$$

or more explicitly

$$\begin{aligned} Q_{\text{cut-off}}^S &= \sum_{I \text{ II III}} \int \varphi_1(x) \kappa^2(x) \{G_2(x) - G_1(x)\} dx^{(8)}, \\ \langle r^2 \rangle_{1 \text{ cut-off}}^S &= \sum_{I \text{ II III}} 12 \int \varphi_1(x) \{2G_4(x) - G_3(x)\} dx^{(8)}, \\ \mu_{\text{cut-off}}^S &= \sum_{I \text{ II III}} \left[\int \varphi_2(x) G_1(x) dx^{(8)} - 4M^2 \int \varphi_1(x) G_2(x) dx^{(8)} \right] \\ (\mu^S \langle r^2 \rangle_2^S)_{\text{cut-off}} &= \sum_{I \text{ II III}} 6 \left[\int \frac{\varphi_2(x)}{\kappa^2(x)} G_3(x) dx^{(8)} - 8M^2 \int \frac{\varphi_1(x)}{\kappa^2(x)} G_4(x) dx^{(8)} \right], \end{aligned} \quad (4.10)$$

with

$$G_n(x) \equiv (1/n!) \kappa^{n+1}(x) \int_0^\infty r^n e^{-\kappa(x)r} f(r) dr. \quad (4.11)$$

For the square cut-off

$$f(r) = \theta(r - r_c), \quad (4.12)$$

which is simple but effective, we can easily carry out the integration of (4.11):

$$G_n(x) = e^{-\kappa(x)r_c} \sum_{j=0}^n \{\kappa(x) r_c\}^j / j!. \quad (4.13)$$

§ 5. Numerical evaluations

The integration domain of the final results (4.1) is rather inconvenient for carrying out the parametric integrations. Therefore we transform it into the 8-dimensional unit cube. The choices of such transformations are of great arbitrariness, but we should choose a transformation such that the transformed integrand becomes simple without violating its symmetry. Actually, we have adopted the following transformation.

$$\begin{aligned} x_1 &= stuv \\ x_2 &= st(1-u)w \\ x_3 &= stu(1-v) \\ x_4 &= s(1-t) \\ x_5 &= st(1-u)(1-w) \end{aligned} \quad (5.1)$$

$$x_6 = r(1-x)z$$

$$x_7 = rxy$$

$$x_8 = rx(1-y)$$

$$x_9 = r(1-x)(1-z).$$

Then the δ -function in (4.2) becomes

$$\delta(1 - \sum_{i=1}^9 x_i) = \delta(1 - s - r), \quad (5.2)$$

so after the integration over r we have

$$r = 1 - s. \quad (5.3)$$

Substituting (5.3) in (5.1), we get the integrations over eight variables s, t, \dots, z in $[0, 1]$. The Jacobian J is easily calculated:

$$\begin{aligned} J &= s^4 t^3 u(1-u) \cdot r^3 x(1-x) \\ &= s^4 (1-s)^3 t^3 u(1-u) x(1-x). \end{aligned} \quad (5.4)$$

The results transformed by (5.1) are elementarily obtained. For example, for Graph I we have

$$\begin{aligned} U &= s^2(1-s)t[1-t+tu(1-u)] \\ &\quad + s(1-s)^2 x(1-x)[y(1-tu) + (1-y)(1-t+tu)] \\ &\quad + s(1-s)^2 t x^2 y(1-y) + (1-s)^3 x^2(1-x)y(1-y), \\ U \cdot \xi &= s^3(1-s)t^2[tu^2(1-u)v^2 + tu(1-u)^2 w^2 + (1-t)\{uv + (1-u)w\}^2] \\ &\quad + s^2(1-s)^2 t^2 x(1-x)[u^2 v^2(1-y) + (1-u)^2 w^2 y] \\ &\quad + s^2(1-s)^2 t^2 x^2 y(1-y)[uv + (1-u)w]^2, \text{ etc.} \end{aligned} \quad (5.5)$$

Thus U and ξ do not contain z . ξ alone contains z , but the z -integration can be easily carried out for (4.5). Though it is not so for (4.10), we have approximately carried out the z -integration by replacing ξ by $\int_0^1 \xi dz$.*

We must furthermore carry out the the remaining sevenfold integrals. But their straightforward evaluations are actually impossible. We therefore estimated them by a statistical method. The concrete programme of our calculation is as follows.

Taking ten points, 0.05, 0.15, 0.25, ..., 0.95, for each variable, we have 10^7 lattice points in the integration domain. We regard the set of the values of the integrand in these points, $\{X_n J_n\}$ ($n=1, 2, \dots, 10^7$), where X is the integrand of

* This is, of course, a rough approximation, but it is sufficient in the following very crude numerical evaluations.

(4.5) or (4.10) and J is the Jacobian (5.4), as a population universe, from which we take out N samples at random (e.g., $n=1, 2, \dots, N$). From the values of these samples we can estimate the value of the integral I in the following two ways.

(i) We regard the average of the samples itself as the average of the integrand, therefore, multiplying the integration volume 1, we have

$$I = \sum_{n=1}^N X_n J_n / N. \quad (5.6)$$

(ii) We consider the estimation of I in the original integral (4.5) or (4.10) by a similar statistical method. In this case, a sampling in equal probability is technically difficult. Therefore a random sampling has been performed after the transformation (5.1). Then to get a sampling in equal probability, we must weight the samples by J . Therefore I is estimated by the following weighted mean.

$$I = (1/8!) \sum_{n=1}^N X_n J_n / \sum_{n=1}^N J_n, \quad (5.7)$$

where $1/8!$ is the integration volume (cf. (2.4) with $A_4=1$).

The ratio of (5.6) and (5.7) is a factor

$$\nu \equiv 8! \sum_{n=1}^N J_n / N, \quad (5.8)$$

which is irrelevant to X . Though $\nu \rightarrow 1$ for $N \rightarrow \infty$, of course, $\nu \neq 1$ for a finite N . Hence there is an ambiguity in the estimation of I by a factor ν . When ν deviates from 1, which of (5.6) and (5.7) is more reliable is rather subjective question, but we think that (5.7) is more favorable since the possible inclination due to the special transformation (5.1) may be corrected to some extent by the denominator of (5.7).

Table I

n		No. 1~25	No. 26~50	No. 51~75	No. 76~100	No. 1~100
$\nu = 8! \sum J_n / N$		1.16	1.04	1.69	1.55	1.37
No cut-off	$\langle r^2 \rangle_1^S$	0.99	2.72	1.67	2.04	1.83
	μ^S	84	33	55	70	61
	$\mu^S \langle r^2 \rangle_2^S$	3.56	1.16	1.49	1.86	1.97
Cut-off $r_0=1/M$	Q^S	3.7	10.0	5.7	7.6	6.6
	$\langle r^2 \rangle_1^S$	0.93	1.73	1.64	1.94	1.58
	μ^S	20	5	10	12	12
	$\mu^S \langle r^2 \rangle_2^S$	2.72	0.74	0.89	1.16	1.33
Cut-off $r_0=2/M$	Q^S	0.7	0.6	1.4	1.5	1.1
	$\langle r^2 \rangle_1^S$	0.41	0.31	0.85	0.83	0.65
	μ^S	3.4	1.2	0.7	1.0	1.5
	$\mu^S \langle r^2 \rangle_2^S$	1.33	0.47	0.17	0.26	0.50

We have put $N=100$. Since the population universe is not of the normal distribution, the estimation of error is difficult. Hence we have divided these 100 samples into four sets, each of which contains 25 samples, and from the fluctuation of their averages we have estimated the magnitude of error.

The results are displayed in Table I, where $g^2/4\pi=14.6$, and $\langle r^2 \rangle_{1,2}^s$ and μ^s are written in units of 10^{-26}cm^2 and nuclear magneton, respectively. For simplicity, we show the values obtained by (5.7) only.

§ 6. Approximation methods

We have directly estimated the sevenfold integrals in last section, but this estimation is very crude, so it will be important to check the results by other approximate integration methods.

6-1. Inequalities

The expressions for U , $\hat{\xi}$ and ξ given in (4.3) and (4.4) are very complicated, but we can give them upper and lower limits by applying the general theory presented in NI⁽²⁾ and NII⁽⁴⁾.

i) U . Since the summation in (2.6) goes over the special sets $(\nu_1, \nu_2, \dots, \nu_n)$, U is not larger than the sum over all possible sets. Since the latter is a symmetrical expression, it reaches its maximum at $x_i=1/\alpha$ under the restrictions $\sum_{i=1}^n x_i=1$, $x_i \geq 0$. Applying this to the present case, we have

$$0 \leq U(x) \leq C_3 \cdot (1/9)^3 \doteq 0.12. \quad (6.1)$$

(6.1) is a rigorous inequality, but for practical purposes it is more useful to reduce the upper bound even at the sacrifice of rigor. U itself is not a symmetrical expression, but it is approximately so. Therefore it will become approximately maximum at $x_i=1/\alpha$. Thus we obtain

$$0 \leq U(x) \lesssim 50 \cdot (1/9)^3 \doteq 0.069. \quad (6.2)$$

Indeed, there is no U which exceeds (6.2) among the 100 samples chosen in last section. Hence this upper bound is sufficiently good.

ii) $\hat{\xi}$. In V , $\hat{\xi}$ consists of the coefficients of the internal nucleon mass and that of $-p^2 - p'^2$, namely,

$$\hat{\xi}(x) = x_1 + x_2 - \eta(x). \quad (6.3)$$

$U\eta$ consists of positive terms only (cf. NI § 3). Therefore we have

$$0 \leq \hat{\xi}(x) \leq x_1 + x_2, \quad (6.4)$$

which is very easily checked by using the concrete expressions (4.3) and (4.4).

iii) ξ . In order to derive the inequality for ξ , it is convenient to use Theorem 2 in NII, which was made use of in the proof of dispersion relations.

Consider a graph which is topologically equivalent to Graph I(a), but in which the masses of propagators are assigned as follows.

$$\begin{aligned} m_1 = m_2 = \mu/2, \quad m_3 = m_4 = m_5 = \mu, \\ m_7 = m_8 = \mu/2, \quad m_6 = m_9 = 3\mu/2. \end{aligned} \quad (6.5)$$

We specially choose the external momenta as

$$q = -2p = 2p' = (\mathbf{0}, 3\mu), \quad (6.6)$$

and the internal constant momenta q_i defined in (2.1) as

$$\begin{aligned} -q_1 = q_2 = q/6, \quad q_3 = q_4 = -q_5 = q/3, \\ q_7 = -q_8 = q/6, \quad q_6 = -q_9 = q/2. \end{aligned} \quad (6.7)$$

Then it is obvious from (6.6) and (6.7) that

$$-(\sum_i a_i q_i)^2 \geq 0 \quad (6.8)$$

for any real numbers a_i . Therefore the second term of (2.7) is non-negative definite. Hence we obtain

$$V(q, x) \geq \sum_{i=1}^9 x_i (m_i^2 + q_i^2) = 0 \quad (6.9)$$

from (6.5) ~ (6.7). In the present notations (6.9) is rewritten as

$$\begin{aligned} V = (x_1 + x_2) (\mu/2)^2 + (x_3 + x_4 + x_5) \mu^2 + (x_7 + x_8) (\mu/2)^2 \\ + (x_6 + x_9) (3\mu/2)^2 - \eta(x) (3\mu/2)^2 - \zeta(x) (3\mu)^2 \geq 0. \end{aligned} \quad (6.10)$$

Therefore, using $\eta(x) \geq 0$, we find

$$\begin{aligned} 0 \leq \zeta(x) \leq (1/36) (x_1 + x_2) + (1/9) (x_3 + x_4 + x_5) + (1/36) (x_7 + x_8) \\ + (1/4) (x_6 + x_9). \end{aligned} \quad (6.11)$$

Since this inequality is rigorous and fairly good as an upper bound, we have made use of it to check the numerical calculations of $\zeta(x)$. Using $\sum_1^9 x_i = 1$, we have a more rough inequality

$$0 \leq \zeta(x) \leq 1/4. \quad (6.12)$$

(6.11) holds also for Graphs II, III.

6-2. Approximate integration

Since the $\langle r^2 \rangle_1^s$ for no cut-off consists of positive terms only, we can give its lower bound by an approximate integration method.

We use the following approximations.

i) To replace $U(x)$ by a constant $\langle U \rangle$.

ii) To replace $\hat{\varepsilon}(x)$ by a linear expression $c \cdot (x_1 + x_2)$, where c is a constant between 0 and 1 (cf. (6.4)).

Then we have

$$\langle r^2 \rangle_1^s \approx \frac{27}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{MM'}{\langle U \rangle^3} c \cdot \int \frac{(x_1 + x_2) \sum_6^9 x_i}{\{ (x_1 + x_2) c M^2 + \sum_3^6 x_i \mu^2 + \sum_6^9 x_i M'^2 \}^2} dx^{(8)}. \quad (6.13)$$

If c is not too near zero, on account of $M^2 \gg \mu^2$ the second term of the denominator has effect only when $\varepsilon \equiv x_1 + x_2 + \sum_6^9 x_i$ is small, but then the numerator with the volume element is of order of ε^7 , therefore the contribution from such cases is negligible. Neglecting the second term, we have

$$\langle r^2 \rangle_1^s \approx \frac{27}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{MM'}{\langle U \rangle^3} c \cdot \frac{5!}{8!3!} \cdot \frac{1}{M'^4} \int_0^1 \frac{(1-t)^2 t^4}{\{ (1-t) c M^2 / M'^2 + t \}^2} dt, \quad (6.14)$$

or, when $M' = M$,

$$\langle r^2 \rangle_1^s \approx \frac{3}{896\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{1}{M^2 \langle U \rangle^3} \cdot c \int_0^1 \frac{(1-t)^2 t^4}{\{ (1-t) c + t \}^2} dt. \quad (6.15)$$

The numerical values of (6.15) for the various values of $\langle U \rangle$ and c are shown in Table II in units of 10^{-26}cm^2 . $\langle U \rangle = 0.069$ is due to (6.2), and $\langle U \rangle = 0.049$ is the weighted mean

$$\sum_n U_n J_n / \sum_n J_n$$

of the 100 samples in § 5. We estimate $c \approx 0.4$ from the numbers of the terms of the numerator and denominator of $\hat{\varepsilon}(x)$. In this method of estimation the contributions from very small U are omitted, and so the obtained value should be regarded as a lower bound. Thus we get

$$\langle r^2 \rangle_1^s \gtrsim 0.9 \times 10^{-26} \text{cm}^2, \quad (6.16)$$

which is consistent with the result in last section.

6-3. No-loop approximation

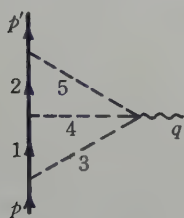


Fig. 3

If the closed-loop mass M' is regarded as sufficiently large, we may neglect higher order terms in the $1/M'$ -expansion. This graphically means that the closed loop is shrunk to one point, *i. e.*, Fig. 1 is replaced by Fig. 3. This approximation is, of course, very bad for small r , but probably good for large r . And actually, the form of the distribution function will be insensitive to higher order corrections to the closed loop part only in the region where this approximation is fairly good.

Table II

$c \backslash \langle U \rangle$	0.069	0.049
0.1	0.13	0.35
0.2	0.21	0.57
0.4	0.32	0.88
0.6	0.39	1.05

For the convenience of calculation, we use the spectral functions

$$\begin{aligned}\alpha_1^S(m^2) &= \sum_{\Pi \Pi\Pi} \frac{3}{16\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' m^2 \cdot I_1(m^2), \\ \alpha_2^S(m^2) &= \sum_{\Pi \Pi\Pi} \frac{3}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 MM' [-M^2 \cdot I_1(m^2) + I_2(m^2)],\end{aligned}\quad (6.17)$$

where

$$\begin{aligned}I_1(m^2) &= -(\partial/\partial M^2) I_2(m^2), \\ I_2(m^2) &= \int \frac{\sum_6^9 x_i}{U^3(x)} \delta(-V(-m^2, x)) dx^{(6)}.\end{aligned}\quad (6.18)$$

Now, I_1 and I_2 vanish for such m^2 that $V > 0$. Therefore in order that I_1 and I_2 are non-vanishing, it is necessary that at least

$$\sum_6^9 x_i \cdot M'^2 \leq \zeta(x) m^2 \leq m^2/4 \quad (6.19)$$

from (4.2) and (6.12). If M' is very large, from (6.19)

$$t \equiv \sum_6^9 x_i \quad (6.20)$$

must be very small for not so large m . Therefore neglecting higher order terms of t , we obtain the following approximations:

$$\begin{aligned}U &\approx U_0 t, \\ \hat{\zeta} &\approx \hat{\zeta}_0, \\ V &\approx V_0 + t M'^2,\end{aligned}\quad (6.21)$$

where U_0 , $\hat{\zeta}_0$ and V_0 denote the U , $\hat{\zeta}$ and V of Fig. 3, respectively,* or more concretely,

$$\begin{aligned}U_0 &= (x_1 + x_3)(x_2 + x_5) + (x_1 + x_3)x_4 + x_4(x_2 + x_5), \\ \hat{\zeta}_0 &= (1/U_0)[x_1^2(x_2 + x_5) + x_2^2(x_1 + x_3) + (x_1 + x_2)^2 x_4], \\ V_0 &= \hat{\zeta}_0 M^2 + \sum_3^5 x_i \cdot \mu^2 - \zeta_0 m^2\end{aligned}\quad (6.22)$$

with

$$\zeta_0 = x_3 x_4 x_5 / U_0.$$

Now, substituting (6.21) in (6.18) and multiplying by

$$1 = \int_0^1 \delta(t - \sum_6^9 x_i) dt,$$

* Properties such as (6.21) can be proved with wide generality.⁵⁾

we transform the integration variables as

$$\begin{aligned}x_i &= (1-t)y_i, \quad (i=1, 2, 3, 4, 5), \\x_j &= ty_j, \quad (j=6, 7, 8, 9).\end{aligned}\quad (6.23)$$

Carrying out the integrations over y_j , we obtain

$$I_2 \approx \frac{1}{6} \int_0^1 t(1-t)^4 dt \int \frac{1}{U_0^3(y_i)} \delta(-(1-t)V_0(y_i) - tM^2) dy^{(4)}. \quad (6.24)$$

Neglecting higher order terms of t again and rewriting y_i by x_i , we find

$$I_2 \approx \frac{1}{6M'^4} \int \frac{1}{U_0^3} (-V_0) \theta(-V_0) dx^{(4)}. \quad (6.25)$$

Therefore (since the three graphs coincide in this approximation) we get

$$\begin{aligned}\alpha_1^s(m^2) &\approx \frac{3}{32\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \cdot \frac{Mm^2}{M'^3} \int \frac{\hat{\zeta}_0}{U_0^3} \theta(-V_0) dx^{(4)}, \\ \alpha_2^s(m^2) &\approx \frac{3}{8\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \cdot \frac{M}{M'^3} \int \frac{1}{U_0^3} [-M^2 \hat{\zeta}_0 + (-V_0)] \theta(-V_0) dx^{(4)}.\end{aligned}\quad (6.26)$$

By using (6.11) the inequality (6.19) is improved as $tM'^2 \leq \zeta m^2 \leq (1/9 + 5/36 \cdot t)m^2$. Therefore since $t \leq 4/31$ when $m^2 = M'^2$, the approximate formulae (6.26) will be able to be used in $m^2 \lesssim M'^2$.

The distribution functions are easily calculated from (6.26) :

$$\begin{aligned}4\pi r^2 \rho_1^s(r) &\approx \frac{9}{8\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \cdot \frac{M}{M'^3} \cdot \frac{1}{r^3} \int \frac{\hat{\zeta}_0}{U_0^3} \left[\sum_{j=0}^3 (\kappa_0 r)^j / j! \right] \cdot e^{-\kappa_0 r} dx^{(4)}, \\ 4\pi r^2 \rho_2^s(r) &\approx \frac{3}{4\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \cdot \frac{M}{M'^3} \cdot \frac{1}{r^3} \int \frac{1}{U_0^3} [-M^2 \hat{\zeta}_0 r^2 (1 + \kappa_0 r) \\ &\quad + 2\hat{\zeta}_0 (3 + 3\kappa_0 r + \kappa_0^2 r^2)] e^{-\kappa_0 r} dx^{(4)},\end{aligned}\quad (6.27)$$

where

$$\kappa_0 \equiv \sqrt{(\hat{\zeta}_0 M^2 + \sum_3^5 x_i \cdot \mu^2) / \zeta_0} \geq 3\mu.$$

Thus the charge distribution function in this approximation becomes positive definite. (The negative part in Fig. 8 of I tends to a δ -function at the origin.) Both the distribution functions are divergent as $1/r^3$ near the origin.

§ 7. Isovector part

In this section, we consider the fourth order graphs, Fig. 4, of the isovector part, we here state only the final results. The calculation technique follows the method mentioned in § 2. We distinguish the closed-loop mass M' from M as before. For simplicity, we omit the superscript V throughout this section.

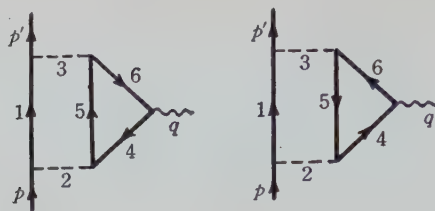


Fig. 4

The form factors are

$$\begin{aligned} F_1^{(4)}(q^2) &= -M^2(-\partial/\partial M^2)H^{(4)}(q^2) + (1/2)H^{(4)}(q^2), \\ F_2^{(4)}(q^2) &= M^2(-\partial/\partial M^2)H^{(4)}(q^2), \end{aligned} \quad (7.1)$$

where

$$\begin{aligned} H^{(4)}(q^2) &\equiv \frac{1}{\pi^2} \left(\frac{g^2}{4\pi} \right)^2 \int \left[\frac{K_1(M^2) - q^2 K_2}{U^5 \bar{V}(q^2; M^2)} \right. \\ &\quad \left. + \frac{U + 3x_5 \sum_1^3 x_i}{U^4} \{-\log \bar{V}(q^2; M^2)\} \right] dx^{(5)} + \text{const.}, \end{aligned} \quad (7.2)$$

with

$$\begin{aligned} V(q^2; M^2) &= (1/U) x_1^2 \sum_4^6 x_i M^2 + (x_2 + x_3) \mu^2 + \sum_4^6 x_i M'^2 + \zeta q^2, \\ U &= \left(\sum_1^3 x_i \right) \left(\sum_4^6 x_i \right) + x_5(x_4 + x_6), \\ \zeta &= (1/U) [x_2 x_3 \sum_4^6 x_i + x_4 x_6 \sum_1^3 x_i + x_5(x_2 x_6 + x_3 x_4 + x_4 x_6)], \\ K_1(M^2) &\equiv x_1^2 x_5^2 (x_4 + x_6) M^2 + (2x_5 + x_4 + x_6) U^2 M'^2, \\ K_2 &\equiv x_4 x_6 (x_1 + x_2 + x_3 + x_5) \left(U + x_5 \sum_1^3 x_i \right) \\ &\quad + x_5^2 \{ (x_1 + x_3) x_3 x_4 + (x_1 + x_2) x_2 x_6 \}. \end{aligned} \quad (7.3)$$

Unfortunately these form factors contain an internal divergence. Therefore we must renormalize them after Dyson's prescription. The subtraction terms are

$$F_{1,2}^{(R)}(q^2) = -(R_c + R_d) F_{1,2}^{(2)}(q^2), \quad (7.4)$$

where $F_{1,2}^{(2)}(q^2)$ are the lowest order form factors given in I § 3, namely,

$$\begin{aligned} F_1^{(2)}(q^2) &= -M^2(-\partial/\partial M^2)H^{(2)}(q^2) + (1/2)H^{(2)}(q^2), \\ F_2^{(2)}(q^2) &= M^2(-\partial/\partial M^2)H^{(2)}(q^2), \end{aligned} \quad (7.5)$$

with

$$H^{(2)}(q^2) \equiv \frac{1}{\pi} \frac{g^2}{4\pi} \int [-\log(x_1^2 M^2 + (x_2 + x_3) \mu^2 + x_2 x_3 \bar{q}^2)] dx^{(2)} + \text{const.}$$

$R_c + R_d$ in (7.4) is the contribution from the closed loop when the meson lines 2 and 3 are free, *i.e.*,

$$\begin{aligned} R_c &\equiv \frac{1}{\pi} \cdot \frac{g^2}{4\pi} \int_0^1 v dv \frac{v(1-v)^2 \mu^2 + (2-v) M'^2}{M'^2 - v(1-v) \mu^2} \\ &= \frac{1}{\pi} \frac{g^2}{4\pi} \left[\frac{2}{3} + \frac{3}{20} \frac{\mu^2}{M'^2} + \frac{13}{420} \left(\frac{\mu^2}{M'^2} \right)^2 + \dots \right], \end{aligned} \quad (7.6)$$

$$R_d \equiv \frac{1}{\pi} \frac{g^2}{4\pi} \lim_{\Lambda \rightarrow \infty} \int_0^\Lambda d\Lambda \int \frac{1 + 3x_5}{M'^2 - x_5(x_4 + x_6) \mu^2 + \Lambda} dx^{(2)}. \quad (7.7)$$

We transform the integration domain of (7.2) into 5-dimensional unit cube like as (5.1):

$$\begin{aligned} x_1 &= st \\ x_2 &= s(1-t)u \\ x_3 &= s(1-t)(1-u) \\ x_4 &= (1-s)vw \\ x_5 &= (1-s)(1-v) \\ x_6 &= (1-s)v(1-w), \end{aligned} \quad (7.8)$$

and

$$J = s^2(1-s)^2(1-t)v.$$

Correspondingly, we transform in (7.5) and (7.7) as

$$\begin{aligned} x_1 &= t & x_4 &= vw \\ x_2 &= (1-t)u & x_5 &= 1-v \\ x_3 &= (1-t)(1-u) & x_6 &= v(1-w). \end{aligned} \quad (7.9)$$

Then we combine $R_d \cdot F_{1,2}^{(2)}$ into a single factor by the Feynman technique. The renormalized results are as follows.

$$\begin{aligned} F_1^{(4)}(q^2) - F_1^{(R)}(q^2) &= -[F_2^{(4)}(q^2) - F_2^{(R)}(q^2)] \\ &+ \frac{1}{2\pi^2} \left(\frac{g^2}{4\pi} \right)^2 \cdot \int s^2(1-t)v \left[\frac{K_1^* - K_2^* q^2}{U^{*5} V} + \frac{v(1-v)}{U^{*4}} (-\log V) \right. \\ &+ \left. \frac{s(4-3v)}{1-s} \left\{ \frac{(-\log V)}{U^{*4}} - \frac{(-\log V')}{s^4} \right\} \right] d^5 \tau \\ &- R_c[F_1^{(2)}(q^2) + F_2^{(2)}(q^2)] + \text{const.}, \\ F_2^{(4)}(q^2) - F_2^{(R)}(q^2) &= \frac{1}{\pi^2} \left(\frac{g^2}{4\pi} \right)^2 M^2 \int s^4 t^2 (1-t)v \left[\frac{K_1^* - K_2^* q^2}{U^{*6} V^2} + \right. \end{aligned}$$

$$+\frac{v^2(1-v)}{U^{*6}V}+\frac{s(4-3v)}{1-s}\left\{\frac{1}{U^{*6}V}-\frac{1}{s^6V'}\right\}\Big]d^6\tau-R_c\cdot F_2^{(2)}(q^2), \quad (7\cdot10)$$

with

$$\begin{aligned} V &\equiv (1/U^*)s^2t^2M^2+s(1-t)\mu^2+(1-s)M'^2+\zeta\cdot q^2, \\ U^* &\equiv U/(1+s)=s+(1-s)v(1-v), \\ \zeta &= (1/U^*)[s^2(1-t)^2u(1-u)+s(1-s)v^2w(1-w) \\ &\quad +s(1-s)(1-t)v(1-v)\{u(1-w)+(1-u)w\} \\ &\quad + (1-s)^2v^2(1-v)w(1-w)], \\ K_1^* &\equiv K_1/(1-s)^3=s^2t^2v(1-v)^2M^2+(2-v)U^{*2}M'^2 \\ K_2^* &\equiv K_2/(1-s)^3=v^2w(1-w)\{s+(1-s)(1-v)\}\{U^*+s(1-v)\} \\ &\quad +s^2(1-t)v(1-v)^2[\{t+(1-t)(1-u)\}(1-u)w \\ &\quad +\{t+(1-t)u\}u(1-w)], \\ V' &\equiv st^2M^2+\{s(1-t)-(1-s)v(1-v)\}\mu^2+(1-s)M'^2 \\ &\quad +s(1-t)^2u(1-u)q^2, \\ d^6\tau &\equiv ds\,dt\,du\,dv\,dw, \quad (0\leq s, t, u, v, w\leq 1). \end{aligned} \quad (7\cdot11)$$

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Possibility of Super-Weak Interactions and the Stability of Matter

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There may exist some extremely weak interactions called here "super"-weak interactions, which are much weaker than usual weak interactions. As a possible example of such interactions we discuss "metastability" of matter.

During the past few years, we have learned a lot about the interesting nature of various interactions in the field of particle physics. Notably, we can classify known interactions into three groups, whose main features are summarized in Table I. Now, there arises a natural question: Do there exist any other types of interactions? We know at least one "super-weak" interaction from the earliest days of physics: the universal gravitation. Using a natural unit $\hbar=c=1$ together with m_π (pion mass)=1**, the gravitation constant is found to be

Table I

class of interactions	coupling constant $\hbar=c=m_\pi=1$	characteristics		
		CPT-theorem	charge independence and strangeness conservation	conservation laws
strong interactions	≥ 1 Example: pion-nucleon interaction constant (Ps(ps)) $g^2/4\pi=15$	C, P and T invariance	both satisfying	energy-momentum angular momentum electric charge baryonic number leptonic number
electromagnetic interactions	$\frac{e^2}{4\pi} = \frac{1}{137}$	C, P and T invariance	CI violating but strangeness conserving	
weak interactions	10^{-7} Example: β -decay constant $G_\beta = 1.4 \times 10^{-7}$	CP invariance =T invariance	both violating	
super-weak intetac- tions	?		?	

* On leave of absence from Osaka City University, Osaka, Japan.

** This is merely a matter of convenience.

$$r(=r m_{\pi}^2/\hbar c)=1.31 \times 10^{-41}. \quad (1)$$

This value is very much smaller than that of any other known coupling constants. Historically speaking, the β -decay of nuclei was the first example of weak interactions and many others have since been added to our inventory of weak interactions. We may perhaps expect that a similar situation would happen in the field of "super-weak" interactions. We can suppose that the strength of such super-weak couplings is somewhere between the β -decay constant and the gravitation constant.

As is well known, the stronger the interaction, the greater the number of symmetry properties it possesses. Therefore, super-weak interactions, if these exist, might violate time reversal invariance or even the CPT theorem and/or some of the other conservation laws.* We shall discuss the latter possibility in somewhat more detail. We know so far that the following five kinds of conservation laws are absolutely valid (or, more conservatively, to exceedingly high accuracy at least) for an isolated system:

- (I) Conservation of the total energy and momentum
- (II) " " " " angular momentum
- (III) " " " " electric charge
- (IV) " " " " sum of baryonic number, and
- (V) " " " " of leptonic number.

The last two expressions, (IV) and (V), may be considered as a statement of the stability of matter.^{1)†} Recalling the breakdown of parity conservation (and charge conjugation invariance) in weak interactions, we may now speculate that some of these conservation laws are violated in (some) super-weak interactions. In this paper we would like to pursue one special possibility:^{††} the "meta"-stability of matter.

We already know that non-radioactive elements must be stable at least during the age of the earth ($\gtrsim 1.5 \times 10^9 \text{yr} \approx 5 \times 10^{16} \text{sec}$) [We can instead take equally well the age of our Galaxy or a time scale of the order of the longest half-life of natural radioactive nuclei]:

$$(\text{lifetime of matter}) \equiv \tau_m \gtrsim 10^{17} \text{sec}.$$

* According to our "extrapolation," the gravitation interaction must violate many conservation laws, while our knowledge on gravitation forces contradicts this expectation. However, it should be noticed that our information on gravitation forces is limited to the "classical" region of large distance. Hence our proposal may not necessarily give rise to self-inconsistency. On the contrary, our proposal suggests that the gravitational interactions at small distance must show quite spectacular feature.

† Most conservatively one may replace (IV) and (V) by the stability of "stable nuclei" and "electrons".

†† Another possibility of charge non-conservation will be discussed in another paper by the present author (Prog. Theor. Phys., Suppl., to be published).

In fact we can find a much better lower limit for τ_m from the following argument. It is well known that there is heat generation in the interior of the earth, whose amount we can estimate from the temperature gradient in the earth's crust.²⁾ It is commonly accepted that the natural radioactive nuclei are responsible for it. We now suppose that a fraction ε of the earth's heat is due to spontaneous annihilation of nucleons whereby ~ 1 BeV of energy is released. Then we shall find

$$\tau_m \approx \frac{1}{\varepsilon} \times 10^{17} \text{yr} = \frac{3}{\varepsilon} \times 10^{24} \text{sec}. \quad (2)$$

Even if we take $\varepsilon \approx 10^{-4}$, we obtain $\tau_m \approx 10^{21} \text{yr} = 3 \times 10^{38} \text{sec}$.

The problem of the stability of matter has been checked by direct experiments using a large liquid scintillation counter.³⁾ The experimental results were as follows:

$$\left. \begin{array}{l} \text{ref. 3a) : } \tau_m \gtrsim 10^{22} \text{yr} = 3 \times 10^{29} \text{sec} \\ \text{ref. 3b) : } \tau_m \gtrsim 4 \times 10^{23} \text{yr} = 10^{31} \text{sec} \end{array} \right\} \quad (3)$$

It should be remarked that M. Goldhaber⁴⁾ has obtained theoretically a limit

$$\tau_m \gtrsim 10^{20} \text{yr}$$

from the spontaneous fission rate of Th^{232} .

Thus we find ourselves in a very interesting, but rather puzzling, situation. All information coming from astrophysics indicates that our Galaxy and its neighbours are in a very young stage of the "age" of the order of $10^9 \sim 10^{10} \text{yr}$. If all matter were created in some way at the time of birth of the galaxies, then we might naively expect that the death of galaxies (or stars) could be connected with some catastrophic end of the matter; for example, the supernova explosion might be associated with some annihilation processes of "stable" matter.* However, our experimentalists have guaranteed that the lifetime of matter is very much longer than the "age" of the galaxies.

We can, of course, take another point of view. The matter is immortal, but galaxies are "living" or mortal. We can observe only living galaxies so that, if the latter view were correct, there must be extremely many germs and corpses of galaxies in our universe, direct detection of which must be a very interesting problem. In this connection, many people have been inclined to accept the concept of steady state universe. In this theory (or one type of such theories) one sometimes postulates not only the continuous creation of stars and galaxies, but also the continuous creation of matter.⁵⁾ (Expanding steady state universe requires continuous creation of matter but not annihilation of it). This steady state theory

* In such occasion we might expect substantial production of antimatter. However, if we accept the supernova origin of the cosmic radiations, amount of antimatter should not be too much abundant in the outer parts of supernova explosion where the cosmic radiation is supposed to be born. This is because the primary cosmic radiations consist mostly of positively charged (i.e., normal and not anti-) nuclei.

seems to be rather ridiculous simply from the logical point of view: problems to be discussed are just replaced by (or hidden in) an "ill"-defined concept of "steadiness". Here we would rather like to accept an assumption of mortality of matter. So much about cosmological speculation, let us come back to our original problem.

When we talk about the "meta"-stability of matter, we may take either one of the next two possibilities:

- (a) Both conservation laws (IV) and (V) are no longer valid for some super-weak interactions;
- (b) Two conservation laws—(IV) and (V)—lose their separate validity for a super-weak interaction, but the fermion numbers must still be conserved, where the fermion number is defined by:

$$(\text{fermion number}) = (\text{number of protons and neutrons}) - (\text{number of electrons}) + \dots \quad (4)$$

(b) is a less drastic conjecture, as compared to (a).*

Examples of possible modes of matter annihilation are:

Case (a)

$$(2 \text{ nucleons}) \longrightarrow (\text{several pions and/or photons}) \quad (5)$$

Case (b)**

We have: *spontaneous decay*

$$(\text{proton}) \longrightarrow (2 \text{ positrons}) + (\text{electrons}) \quad (6)$$

electron capture

$$(\text{electron}) + (\text{proton}) \longrightarrow (\text{electron}) + (\text{positron}) \quad (7)$$

and

$$(\text{two nucleons}) \longrightarrow (\text{two leptons}) \quad (8)$$

and so on. If we assume a specific mode of matter annihilation and a corresponding super-weak interaction, we can estimate an upper limit of the coupling constant from the experimental result (3). Here is one point which must be kept in mind: the experimental checks³⁾ quoted above have been performed under special conditions so that some modes of matter annihilation, even though they exist with a rate

* In this connection we may remark that the annihilation of an odd number of fermions:

$$\left(\begin{array}{c} \text{a state with} \\ \text{odd number of fermions} \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{another state with} \\ \text{any number of bosons} \end{array} \right)$$

can happen with great sacrifice of violating the angular momentum conservation law (II).

** If a primary interaction gives rise to various reactions listed below, the effect of other stronger interactions affects them. Thus (6) would mean that we must also see

$$(\text{proton}) \longrightarrow (\text{electron}) + (\text{many pions}), \text{ etc.}$$

We do not go into such details hereafter.

somewhere between (2) and (3), might have escaped experimental detection. One of such examples (for the experiment of ref. 3b) may be given by the reaction (6) or its neutral counter mode :

(proton) → (positron) + (neutrino) + (antineutrino) (6')

Without further discussion of such details, we simply assume hereafter (3) or (2) as a possible lower limit of the lifetime of "matter".

Let us now examine a few theoretical possibilities. We begin with case (b). We choose a very special model (6) or (7), whose interaction Hamiltonian can be written as†

g / (sqrt(2)) * (e+ gamma_alpha (1 + gamma_5) p) (e- gamma_alpha (1 + gamma_5) e-). (9)

Notice that there is a possibility of violating the CPT theorem†† and thus we can construct such an interaction which allows only either one of the two annihilation modes, (6) and (7). It is quite easy to find an upper limit for the super-weak coupling constant g in Eq. (9) (See Table II).

Table II

mode of annihilation	spontaneous decay*		electron capture**	
	case ((α))	case ((β))	case ((γ))	case ((δ))
assumed lower limit of τ _m	10 ²⁵ sec	10 ³¹ sec	10 ²⁵ sec	10 ³¹ sec
upper limit of super-weak coupling constant g in units of*** ħ=c=m _π =1	10 ⁻²⁶	10 ⁻²⁹	10 ⁻¹⁸	10 ⁻²¹

We notice here the following very interesting numerical coincidence,⁶⁾

γ ≈ G_β⁶ (for ħ=c=m_π=1).

Furthermore, if we take case (a) or the process (8) the effective coupling constant turns out to be ≤10⁻²⁹ (i.e., same as case ((β)) in Table II).

The following important difference should be noticed: The spontaneous decay rate is (nearly) independent of nuclear states (i.e., whether the nucleons are free or bound in nuclei is irrelevant), while the electron capture rate is roughly proportio-

† This choice of V-A interaction is made here just because of the easiness in comparing with usual weak processes, (μ-decay and μ-capture by a proton). We do not insist on this choice.

†† Construction of a CPT-violating (phenomenological) interaction Hamiltonian is very easy if we give up the locality; see the preprint by R. Arnowitt and J. Feldman, A possible Theory of Beta Decay Interactions (1957).

* See Eq. (6)

** See Eq. (7)

*** See Table I.

nal to (the atomic number)⁴. This difference will serve to distinguish between these two modes of matter annihilation.

From such considerations we may propose the existence of super-weak interactions with extremely weak coupling strength g . Let us suppose τ_m were in fact equal to the experimental lower limit (3) and see what we can expect. Then the numerical coincidence, e.g., $g^2 \sim \gamma \sim G_N^6$, etc., is so attractive that we can hardly avoid a facinating conjecture about some inter-relations between super-weak interactions and gravitation or the usual weak interactions.⁶⁾

For example, we may bring up again a famous argument given by W. Heisenberg long ago⁷⁾: Using the dimensional argument, he has claimed that there must exist a constant r_0 with the dimension of length—the “universal length”—in the future theory of elementary particles. If we accept this philosophy, the interaction Hamiltonian (or if the Hamiltonian does not exist, some appropriate quantity which describes interactions) must be expanded in the power series of r_0 ⁸⁾:

$$H_{\text{int}}(r_0) = \sum_n (r_0)^n H^{(n)}. \quad (10)$$

If the energy involved is not too high (or more precisely, an appropriate momentum transfer is much less than $1/r_0$), such an expansion should be valid. We naturally make the following assignment: $H^{(0)}$ must contain at least all strong interactions, while all others $H^{(n)} (n > 0)$ represent various interactions of very tiny strength: the larger n , the weaker the interaction must be.

If there is an intermediate charged vector boson—whose second order effects are nothing but usual weak processes like β -decay, μ -decay and so on—, then the coupling constant f of this boson to fermion currents [$B_\alpha (\bar{\psi} \gamma_\alpha (1 + \gamma_5) p)$, $B_\alpha^* (\bar{\mu} \gamma_\alpha (1 + \gamma_5) \nu)$, etc.]—Yukawa type interaction!—is dimensionless. So that the only non-dimensionless constant we can have in our theory is the gravitation constant γ , and we may relate r_0 to $\sqrt{\gamma}$ ($= \sqrt{\gamma \hbar}/c$) $\sim 10^{-34}$ cm. In this case we can say that $H^{(0)}$ contains both strong and usual weak interactions, while $H^{(1)}$ and $H^{(2)}$ describe the matter annihilation and gravitation and thus $g \sim \sqrt{\gamma} \sim \gamma_0$. Unfortunately, we have found no link between usual weak interactions and super-weak or gravitation interactions.

On the other hand, if such an intermediate boson does not exist and (primary) weak interactions are of the Fermi type, we can take as r_0 the length:

$$\begin{aligned} \sqrt{G_F} &\approx 4 \times 10^{-4} \approx 4 \times 10^{-17} \text{ cm} \\ &\approx \frac{1}{400 \text{ BeV}} \end{aligned} \quad (11)$$

and the summation \sum_n in the right-hand side of (10) may be restricted to only even values for n

$$H_{\text{int}}(r_0) = \sum_{n'=0}^{\infty} (r_0)^{2n'} H^{(2n')}. \quad (12)$$

Here we can tentatively assume the following correspondence: Naturally we should take

n'	$H^{(2n')}$	Interpretation
0	$H^{(0)}$	strong interactions
1	$H^{(2)}$	weak //
6	$H^{(12)}$	gravitation

while the assignment of matter-destructing term depends on the choice of the various possibilities (case (a) or $((\alpha)) \dots ((\delta))$ of case (b)); e.g., the case $((\delta))$ would correspond to $n' \approx 3$. In this theory the strength of usual weak interaction is supposed to fix the strengths of all other weaker interactions, notably of the super-weak (matter-destructing) and gravitation interactions. Aesthetically, this type of philosophy would be a more appealing one.

In any case, we have seen a rather definite relationship between g and γ and/or G_3 . If the numerology we have tried in this paper carries some meaning—as we have just conjectured—, then we may choose either one of the cases (a) and (b)⁹⁾ (see Table II) and also we can expect that the true lifetime of matter should not be terribly different from the present experimental lower limit (3). A more precise determination of τ_m would therefore be very interesting.

Discussions given in this paper are without any doubt quite speculative. However, the problem of stability of matter and existence of other super-weak interactions than gravitation must be very important for the theory of cosmogony, as well as to the future of particle physics. We cannot, of course, exclude the possibility that some information on these problems would come from the field of astro- (or cosmo-) physics. Up to the present time we have not yet fully understood the meaning of—or the role played by—the gravitation interaction in the domain of particle physics. Existence of super-weak interactions, if established, may cast a new light on such a direction.

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- 9) In this case we may conclude that there are no asymmetries between positive and negative electric charges; the fact that the positive charges are carried by nuclei and the negative charges by electrons in (and near) our Galaxy, is reduced merely to a matter of fluctuation.

Pseudoscalar Coupling and S-Wave Pion-Nucleon and Kaon-Nucleon Scattering

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The pion-nucleon s -wave scattering is investigated in the case of the Yukawa interaction of pseudoscalar coupling, making use of the Chew-Low formalism. It is shown that the s -wave amplitude of the π - N scattering is strongly damped compared with the perturbation theoretic calculation. This is due to the fact that the second order matrix elements are almost cancelled by some of the fourth order ones which appear in the reduced Chew-Low equation as modified Born terms.

The photo-pion production near the threshold energy is also discussed on the same footing. In this case, this sort of cancellation does not occur and the amplitudes for this process remain unchanged. We calculate the charge ratio of photo-pion productions at the threshold energy and get a value 2.27 which is larger than the experimental one 1.87 ± 0.13 . In addition the low energy kaon-nucleon and the pion-hyperon scattering are treated in a similar manner.

§ 1. Introduction

While the static p -wave pion theory has attained great success¹⁾, phenomena connected with nucleon pairs, e. g., the s -wave scattering, have not been satisfactorily accounted for. Among the investigations, so far made, which were started with the pseudoscalar interaction, there are the perturbation calculations and the analysis by the Tamm-Dancoff (TD) approximation.²⁾ Discussions have been made making use of the effective s -wave Hamiltonian reduced by the Dyson-Foldy transformation.³⁾ In addition, the renormalization theories of the pseudoscalar coupling⁴⁾ have been developed.

The main difficulties are; (1) the phase shifts for the $1/2$ and the $3/2$ iso-spin states ($\delta_1 = 0.167k/\mu$ and $\delta_3 = -0.105k/\mu^{(5)}$)* are not split, because of the too large iso-spin even part compared with the iso-spin odd part, and (2) the coupling constant $g^2 = 15$, which is obtained from the pv -coupling constant by the equivalence theorem, is larger by about one order of magnitude.

It is also to be noticed that the value of $\delta_1 - \delta_3$ can be reproduced while the value of $\delta_1 + 2\delta_3$ cannot in terms of the dispersion relation.⁶⁾

In the photo-pion production near the threshold energy, Kroll-Ruderman (KR) and Deser-Thirring-Goldberger (DTG)⁴⁾ obtained $g^2 = 12^{**}$. This is consistent with

* The natural unit ($c = \hbar = 1$) is used throughout this paper.

** $g^2 = 25$ in their paper, but Bernardini and Goldwasser⁹⁾ got the above value later.

the p -wave theory. On the other hand, kaon interactions with nucleons and hyperons are favourable for the ps -coupling⁸⁾ and the s -wave amplitudes are predominant in the low energy K - N scattering. The experimental data of the scattering cross sections coincide, in magnitude, with the values obtained by the lowest order perturbation calculation with $g_k^2=1$ ⁹⁾ which is derived from the dispersion relation. The recent experimental value of the lifetime of $\pi^0 2\gamma$ decay also comes closer to the result of the perturbation theory¹⁰⁾. Thus, these phenomena suggest that the small s -wave amplitude of the π - N scattering seems to be due to an accidental situation.

As will be shown in § 4, the iso-spin even part is damped by the cancellation of the second order matrix elements with some of the fourth order ones which appear in the reduced Chew-Low (CL) equation. The iso-spin odd part is damped by the cancellation of the matrix element corresponding to a graph, in which the external pion lines are crossed, with the other matrix element in which the external pion lines are uncrossed. On the other hand, in the s -wave phenomena other than the π - N scattering there exists only one of the two types of graphes, i. e., either a crossed or an uncrossed one, and we get no cancellation at least with respect to the iso-spin odd part.

Since the energy denominators of the matrix elements for the s -wave scattering are larger than $2M$ (M =the nucleon mass) on account of the appearance of nucleon pairs, those corresponding to the crossed graphs are almost equal to the uncrossed ones. Therefore, the TD approximation which does not satisfy the crossing theorem is not reliable in this case. While in the TDG theory the scattering amplitude is renormalized as a whole, it is not possible to analyze the detailed mechanism of interactions. On the other hand, it is a sort of perturbation to obtain the effective s -wave Hamiltonian by the Dyson transformation. Here the effective s -wave potential is derived up to the second order in the coupling constant, and the Schroedinger equation is solved just as in the ordinary scattering problem. But there is no justification for neglecting other higher order effects. It seems, therefore, worth while to attack the problem in the CL formalism starting with the ps -coupling Hamiltonian, since this formalism made a success in the static p -wave theory.

The s -wave scattering arises through the virtual formation of nucleon pairs in the interior region of the nucleon, and it is likely that the some pions are exchanged among the virtual nucleons and anti-nucleons. This effect should, therefore, be taken into account and in fact it will be shown that this correction is really important.

In § 2 general consideration concerning the CL equation for the s -wave scattering will be obtained which is associated with the renormalized modified propagator and γ_5 vertex function. Following the consideration in § 2 the "modified Born terms" in CL equations including the pion correction will be calculated up to fourth order in the coupling constant (which is different from the ordinary fourth order perturbation calculation). The order of magnitude of the experimental s -wave cross section is obtained with $g^2=15$, but the phase shifts δ_1 and δ_3 are both negative.

§ 5 deals with the $\gamma\pi$ production near the threshold energy in the same method as in the πN scattering and shows that the matrix elements of the production are not damped by the cancellations in this case. The charge ratio of the production gives a large value than the experimental one 1.78 ± 0.13 . We shall show in § 6 that the low energy KN and πY^* scattering in the case of ps -coupling the s -wave amplitude will be predominant. The last section will be devoted to a summary and discussions concerning some remaining problems.

§ 2. General considerations

At first, we shall begin with physical considerations and preparations for dealing with the s -wave scattering. We take as the interaction Hamiltonian

$$H_I = (4\pi)^{1/2} i g_{(0)} \int \sum_{\alpha=1}^8 \bar{\psi}(x) \gamma_5 \tau_{\alpha} \psi(x) \phi_{\alpha}(x) dx \quad (2.1)$$

where $g_{(0)}$ is the unrenormalized and unrationalized coupling constant. $\psi(x)$ and $\phi_{\alpha}(x)$ are field operators for the nucleon and the pion, respectively, and τ_{α} is the α -component of the isospin operators. According to Low¹⁾, the transition amplitude** $T_p(q)$ satisfies the following equation,

$$T_p(q) = - \sum_n \left\{ \frac{T_p^+(n) T_q(n)}{\epsilon_n - E_q - \omega_q - i\epsilon} + \frac{T_q^+(n) T_p(n)}{\epsilon_n - E_p + \omega_q} \right\}, \quad (2.2)$$

where p and q represent the initial and the final states and n , the intermediate state, $\omega_p = (p^2 + \mu^2)^{1/2}$ (μ : meson mass) and $E_p = \sqrt{p^2 + M^2}$. In this case, we should take, for the state n , the scattering states $|N, n(N, \bar{N}), m\pi\rangle$ consisting of one nucleon, n -nucleon pairs and m pions and the states $|N, m\pi\rangle$ consisting of one nucleon and m pions. The former may be treated as the modified Born term V in the equation. Among the latter, the state $|N, \pi\rangle$ is related to the scattering amplitude and the states $|N, m\pi\rangle$ for $m \geq 2$ are disregarded (if they are not omitted, they should be included also in V). This means that the scattering is approximated by the repetition of the modified Born terms. Since the energy level of the $|N, N, \bar{N}\rangle$ state is higher than that of the $|N, 2\pi\rangle$ state, justification for this approximation may be open to question.

We put, in the static approximation, for the state $|N, \pi\rangle$,

$$T_p(q) = - \frac{2\pi}{(\omega_p \omega_q)^{1/2}} \sum_{I=1,3} P_I h_I(\omega_p), \quad (2.3)$$

with

$$P_1 = \frac{1}{3} \tau_j \tau_i, \quad P_3 = \delta_{ij} - \frac{1}{3} \tau_j \tau_i, \quad (2.4)$$

* Y means A and Σ

** Here T 's are used instead of Low's notations $\langle N_q, \pi_q | O(p) | N_p \rangle$ which give the transition amplitudes in the static limit.

where the P_i 's are the projection operators for the eigenstates of the total iso-spin, and τ_i and τ_j represent the iso-spins for the initial and the final nucleon. Then Eq. (2.2) is rewritten as

$$h_I(\omega) = V_I(\omega) + \frac{1}{\pi} \int \left\{ \frac{|h_I(\omega_p)|^2}{\omega_p - \omega - i\epsilon} + \sum_I A_{II'} \frac{|h_{I'}(\omega_p)|^2}{\omega_p + \omega} \right\} p v^2(p) d\omega_p, \quad (2.5)$$

with

$$A_{II'} = \frac{1}{3} \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix},$$

where $v(p)$ is a form factor.

While, in the case of p -wave scattering, it is a good approximation to take the Born term for $V(\omega)$, it will not be sufficient to take only the Born term in the s -wave scattering. The reason is that the nucleon pairs appear in the interior region of the nucleon and the coupling constant g is quite large ($g^2=15$), so the effect of the pion exchange among the virtual nucleons and anti-nucleons may be important.

Let us denote, by $V_A(\omega)$ and $V_B(\omega)$, the modified Born terms corresponding to the graphs (A) and (B) in Fig. 1. We divide them into the even part and the odd part with respect to the iso-spin. The crossing theorem allow to write down $V_A(\omega)$ and $V_B(\omega)$ as

$$\begin{aligned} V_A(\omega) &= g^2 [\delta_{ij} \varphi_e(\omega) + \frac{1}{2} [\tau_i, \tau_j] \varphi_o(\omega)], \\ V_B(\omega) &= g^2 [\delta_{ij} \varphi_e(-\omega) - \frac{1}{2} [\tau_i, \tau_j] \varphi_o(-\omega)], \end{aligned} \quad (2.6)$$

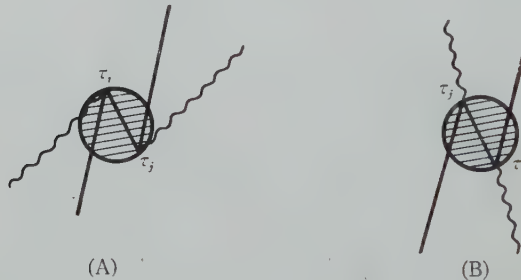


Fig. 1 The graphs for the modified Born terms $V_A(\omega)$ and $V_B(\omega)$.

and in the eigenstates of the iso-spin, they are reduced to

$$\begin{aligned} V_1(\omega) &= g^2 [\varphi_e(\omega) + \varphi_e(-\omega) - 2\varphi_o(\omega) + 2\varphi_o(-\omega)], \\ V_3(\omega) &= g^2 [\varphi_e(\omega) + \varphi_e(-\omega) + \varphi_o(\omega) - \varphi_o(-\omega)]. \end{aligned} \quad (2.7)$$

If $\omega=0$ in Eq. (2.7), $V_1=V_3$ as shown by Gell-Mann et al.¹¹⁾; the split of two phase shifts is due to the ω/M correction of $\varphi_o(\omega)$. The TD approximation which does not satisfy the crossing theorem is not reliable also with respect to this point. By expanding φ_e and φ_o in terms of ω and leaving the first and the second terms, we have

$$\varphi_{e,o}(\omega) = \alpha_{e,o} + \beta_{e,o} \omega / \mu, \quad (2.8)$$

then Eq. (2.7) becomes

$$\begin{aligned} V_1(\omega) &= 2g^2(\alpha_e - 2\beta_o \omega / \mu), \\ V_3(\omega) &= 2g^2(\alpha_e + \beta_o \omega / \mu). \end{aligned} \quad (2.7a)$$

As seen from Eq. (2.7), only the first term α_e of the iso-spin even part φ_e and the second term β_o of the iso-spin odd part φ_o contribute to the low energy scattering, so we can get sufficiently small scattering amplitude, only if α_e is small enough, though α_o is large. It will be shown in § 4 that in fact α_e becomes small if the pion correction is taken into account in $V(\omega)$

The higher order effects due to the repeated scattering will not be important as expected from the small experimental values of phase shifts. It will, however, be necessary to estimate the effect precisely. Following the CL effective range approximation, we put

$$(\mu/k) \tan \delta_I = g^2(\alpha_I + \beta_I \omega / \mu) / \text{Re } g_I(\omega), \quad (2.9)$$

with

$$g_I(\omega) \equiv V_I(\omega) / h_I(\omega), \quad (2.10)$$

where $V_I(\omega) = \alpha_I + \beta_I \omega / \mu$, $\text{Re } g_I(\omega)$ in Eq. (2.9) represents the effect of the repeated scattering. As the cross section of the s -wave scattering may be almost constant for energy variation, it will be allowed to approximate $h_I(\omega)$ in the integral in Eq. (2.5) by $V_I(\omega = \mu)$, and we have (α, β : real)

$$\begin{aligned} \text{Re } g_I(\omega) &= 1 - \frac{g^2}{\pi(\alpha_I + \beta_I \omega / \mu)} \int \left\{ P \frac{(\alpha_I + \beta_I)^2}{\omega_p - \omega} \right. \\ &\quad \left. + \sum_n A_{In} \frac{(\alpha_{In} + \beta_{In})^2}{\omega_p + \omega} \right\} p v^2(p) d\omega_p. \end{aligned} \quad (2.11)$$

After Substituting the experimental phase shifts into Eq. (2.9) and cutting off the integral in Eq. (2.11) at 7μ , the values in Table I are obtained for α_e and β_o . The last row shows the values obtained by ignoring the repeated scattering. These results show that the effect of the rescattering is not large and $V_I(\omega)$'s themselves must be much smaller than the values of the lowest order perturbation theory.

Table I. The effect of the rescattering. The last row shows the values obtained by disregarding the rescattering effect.

α_e	β_o	Reg_1 (1)	Reg_3 (1)
$-0.01 \pi/g^2$	$-0.01 \pi/g^2$	0.4	1.25
$-0.01/g^2$	$-0.045/g^2$	1	1

§ 3. Effective coupling constant for γ_5 interaction

In this section, we shall estimate the effective coupling constant taking into account the renormalized modified propagator and the γ_5 vertex function. The renormalized nucleon propagator is

$$S'_{FC}(p) = \left(1 + \frac{3g^2}{4\pi} f(p)\right)^{-1} S_F(p). \quad (3.1)$$

For the lowest self-energy type, $f(p)$ becomes

$$f(p) = \int_0^1 dx \left[\frac{(x-1)\gamma_\mu p_\mu + iM}{\gamma_\mu p_\mu - iM} \log \frac{\phi(p^2)}{\phi(-M^2)} + \frac{2M^2 x^2(1-x)}{\phi(-M^2)} \right], \quad (3.2)$$

with

$$\phi(p^2) = (1-x)\mu^2 + xM^2 + x(1-x)p^2.$$

Since $f(p) \approx 1$ for the nucleon in the negative energy state and Eq. (3.1) is written in the low energy limit as

$$S'_{FC}(p) = S_F(p) \left(1 + \frac{3g^2}{4\pi}\right)^{-1}, \quad (3.1a)$$

the nucleon pairs are suppressed¹²⁾ by the radiative correction. On the other hand, the renormalized vertex function,

$$\Gamma_{5C}(p', p) = i\gamma_5 \tau_\alpha [1 + Y(p', p)], \quad (3.3)$$

enhances the effect of nucleon pairs. The lowest order perturbation calculation gives $Y(p, p) \approx +g^2/4\pi$ (putting the external pion mass $\mu=0$), then

$$\Gamma_{5C}(p, p) = i\gamma_5 \tau_\alpha (1 + g^2/4\pi). \quad (3.3a)$$

As g^2 is large, Γ_{5C} cancels the effect of the pair-suppression due to S'_{FC} as pointed out by KR.⁴⁾

Eqs. (3.1a) and (3.3a) give the effective coupling constant for the ps -coupling at the zero energy

$$g_{eff}^2 = g^2 (1 + g^2/4\pi)^2 / (1 + 3g^2/4\pi) = 16, \quad (3.4)$$

where we have used $g^2=15$ obtained from $f^2=0.08$ by the equivalence theorem.

This value gives only an order of magnitude, because Eqs. (3.1a) and (3.3a) are obtained from the lowest order perturbation expansion. In what follows, therefore, we shall use $g_{eff}^2=15$, disregarding the effect of radiative corrections in non-zero energies, and simply write g^2 instead of g_{eff}^2 .

§ 4. Calculation of modified Born term V

As discussed in § 2, we shall show that the iso-spin even part α_e will become small by calculating the modified Born term up to 4th order of g . This situation

is supported also by the following fact. In the effective s -wave Hamiltonian obtained by Akiba and Sawada,³⁾ the ϕ^2 and the ϕ^4 terms of the iso-spin even parts have the opposite signs and these terms cancel each other if the cut-off parameter is chosen appropriately.

For $\omega \approx \mu$, the 2nd order $V_A^{(2)}(\omega)$ is

$$V_A^{(2)}(\omega) = -(\delta_{ij} - \frac{1}{2}[\tau_i, \tau_j])g^2/(2M + \omega). \quad (4.1)$$

After the renormalization of propagators and vertices, only the graphs shown in Fig. 2 remain in the fourth order. (i) ~ (iv) correspond to p -wave pion corrections and (v) ~ (ix) to s -wave ones. The graph (ix) should, however, be excluded from $V_A^{(4)}(\omega)$, for it is already included in the integration term in Eq. (2.5) as the repetition of the 2nd order because its intermediate state is $|N, \pi\rangle$ state. This graph gives the largest contribution in the 4th order perturbation terms, as easily seen from its energy denominator. Excluding this graph, $V_A(\omega)$ becomes smaller than that of the perturbation calculation. This situation is typical for the CL formalism.

Since the modified Born terms are strongly damped by the cancellation of the 2nd order and 4th order terms, we should take into account the recoil effect of nucleons in the fourth order calculation. Then we have

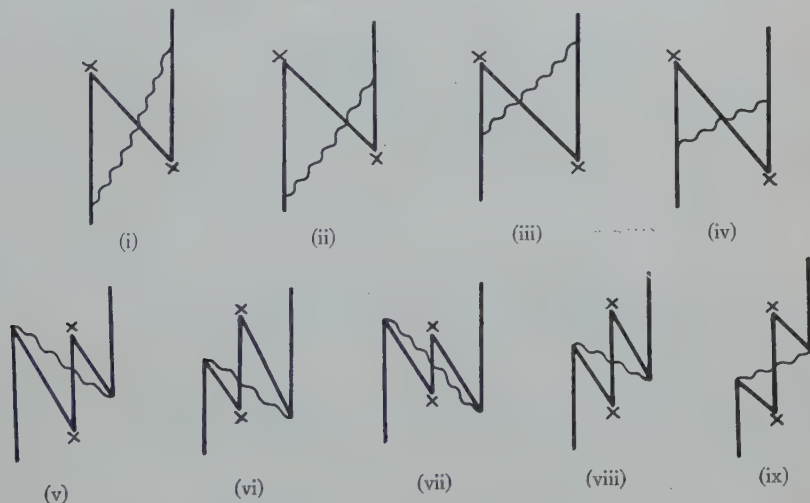


Fig. 2 The 4th order graphs contributing to $V_A^{(4)}(\omega)$ or $V_B^{(4)}(\omega)$; but the last graph (ix) should be excluded from $V_A^{(4)}(\omega)$. The mark \times denotes the vertex of the external pion.

$$V_A^{(4)}(\omega) = \left(3\delta_{ik} + \frac{1}{2}[\tau_i, \tau_j] \right) \frac{g^4}{\pi} \sum_{n=1}^8 I_n(\omega), \quad (4.2)$$

with

$$I_1(\omega) = - \int \frac{k'^3 v_p^2(k') d\omega'}{2E'(E'+M) (E'+\omega'-M)^2 (3E'+\omega'-M+\omega)}, \quad (4.3a)$$

$$I_2(\omega) = I_3(\omega) = - \int \frac{k'^3 v_p^2(k') d\omega'}{2E'(E'+M) (2E'+\omega) (E'+\omega'-M) (3E'+\omega'-M+\omega)}, \quad (4.3b)$$

$$I_4(\omega) = - \int \frac{k'^3 v_p^2(k') d\omega'}{2E'(E'+M) (2E'+\omega)^2 (3E'+\omega'-M+\omega)}, \quad (4.3c)$$

$$I_5(\omega) = I_6(\omega) = \int \frac{(E'+M) k' v_s^2(k') d\omega'}{2E'(2E'-\omega) (3E'+\omega'+M-\omega) (E'+\omega+M)}, \quad (4.3d)$$

$$I_7(\omega) = \int \frac{(E'+M) k' v_s^2(k') d\omega'}{2E' (E'+\omega'+M)^2 (3E'+\omega'+M-\omega)}, \quad (4.3e)$$

$$I_8(\omega) = \int \frac{(E'+M) k' v_s^2(k') d\omega'}{2E' (2E'-\omega)^2 (3E'+\omega'+M-\omega)}, \quad (4.3f)$$

where $v_p(k)$ and $v_s(k)$ denote the form factors for the p -wave and the s -wave pions and $E' = (M^2 + k'^2)^{1/2}$. From Eqs. (2.6), (4.1) and (4.2)

$$\begin{aligned} \varphi_e(\omega) &= -\frac{1}{2M+\omega} + \frac{3g^2}{\pi} \sum_{n=1}^8 I_n(\omega), \\ \varphi_o(\omega) &= \frac{1}{2M+\omega} + \frac{g^2}{\pi} \sum_{n=1}^8 I_n(\omega). \end{aligned} \quad (4.4)$$

Numerically integrating Eqs. (4.3a~f) and retaining up to the terms linear in ω , we get

$$\begin{aligned} \varphi_e(\omega) &= -\frac{1}{2M} \left[\left(1 - \frac{\omega}{2M} \right) - \frac{3g^2}{\pi} \left(0.067 + 0.015 \frac{\omega}{\mu} \right) \right], \\ \varphi_o(\omega) &= \frac{1}{2M} \left[\left(1 - \frac{\omega}{2M} \right) + \frac{g^2}{\pi} \left(0.067 + 0.015 \frac{\omega}{\mu} \right) \right]. \end{aligned} \quad (4.5)$$

In the above calculations, the cut-off parameters have been chosen at 6μ and 8μ respectively for the p -wave and the s -wave and we have used $g^2=15$. Eqs. (2.8) and (4.5) give

$$\alpha_e = -0.003 \quad \text{and} \quad \beta_o = -0.0005. \quad (4.6)$$

The value of α_e in Eq. (4.6) is almost equal to the value in Table I. But the value of β_o is so small that both δ_1 and δ_3 are negative. This result is sensitive to the cut-off parameter, since the very small quantities remain after the cancellation of the main terms are treated here. It is to be noticed, however, that if the cut-off parameters for the s -wave and the p -wave are varied in parallel, that is, if the s -wave cut-off value is kept about 2μ larger than the p -wave one e. g., 5μ and 7μ ,

the result is almost independent of the cut-off parameters. This reason comes from the fact that each term in Eqs. (4, a~c) has an opposite sign to the corresponding term in Eqs. (4.3d~f).

As for the energy dependence of the phase shifts, both δ_1 and δ_3 deviate slightly from linear variation and have tendency of splitting larger than the linear one. This consequence agrees qualitatively with the one obtained by the dispersion relation.¹³⁾

Despite the large coupling constant g , it has been shown that as far as the magnitude of the cross section is concerned, the consistent result is obtained. It may not be surprising that the sign of the phase shifts is not explained, because the terms of higher order than the 6th and the effect of the kaon and $|N, \pi\rangle$ states are neglected. As the 6th order and 8th order terms have the opposite signs to each other, their contribution may be small*, though it is difficult to calculate their values, and may not alter the qualitative consequence. It may, therefore, be possible to explain low energy phenomena connected with nucleon pairs in the frame of the present field theory, if the higher order and the kaon effects are taken into account correctly.

§ 5. Photo-pion productions near the threshold energy

We shall discuss $\gamma\pi$ productions near the threshold energy following the same method. Modified Born terms have two parts: the iso-spin vector part $V(\omega)$ and the iso-spin scalar part $U(\omega)$. We have

$$\begin{aligned} T_q^{(v)}(p) &\sim -\frac{2\pi i}{(\omega_p q_0)^{1/2}} \sigma \mathcal{E} V(\omega_p), \\ T_q^{(s)}(p) &\sim -\frac{2\pi i}{(\omega_p q_0)^{1/2}} \sigma \mathcal{E} U(\omega_p). \end{aligned} \quad (5.1)$$

Analogously to § 4, we put

$$\begin{aligned} V_A(\omega) &= \frac{eg}{2} \left[\delta_{3j} \phi_e(\omega) + \frac{1}{2} [\tau_3, \tau_j] \phi_0(\omega) \right], \\ V_B(\omega) &= -\frac{eg}{2} \left[\delta_{3j} \phi_e(-\omega) - \frac{1}{2} [\tau_3, \tau_j] \phi_0(\omega) \right], \end{aligned} \quad (5.2)$$

where the minus sign of V_B comes from the expectation value of γ_4 -matrix:

$$\langle \bar{u}_+ \gamma_5 u_- \times \bar{u}_- \gamma u_+ \rangle = -\langle \bar{u}_+ \gamma u_- \rangle \langle \bar{u}_- \gamma_5 u_+ \rangle,$$

u_{\pm} are nucleon spinors belonging to positive and negative energy states. In the eigenstates with respect to the iso-spin, we define

* In addition, by the uncertainty principle, the time between creation and annihilation of nucleon pairs will not be so long that two or more mesons are exchanged among nucleons and antinucleons.

$$V_1(\omega) = \frac{eg}{2} [\phi_e(\omega) - \phi_e(-\omega) - 2\phi_o(\omega) - 2\phi_o(-\omega)] = eg \left(-2\alpha_o + \beta_e \frac{\omega}{\mu} \right), \quad (5.3a)$$

$$V_3(\omega) = \frac{eg}{2} [\phi_e(\omega) - \phi_e(-\omega) + \phi_o(\omega) + \phi_o(-\omega)] = eg \left(\alpha_o + \beta_e \frac{\omega}{\mu} \right). \quad (5.3b)$$

Similarly, for the iso-spin scalar part which has only $I=\frac{1}{2}$ state, we have

$$U(\omega) = \frac{eg}{2} [\phi_s(\omega) - \phi_s(-\omega)] = eg \beta_s \frac{\omega}{\mu}. \quad (5.4)$$

Now let us calculate $V_A(\omega)$ and $U_A(\omega)$ up to the 4th order. We have for the electromagnetic interaction Hamiltonian

$$\begin{aligned} H_{e,m} = & - (4\pi)^{1/2} ie \int \bar{\psi}(x) \frac{1+\tau_3}{2} \gamma_\mu \psi(x) A_\mu(x) dx - (4\pi)^{1/2} e \int \{ (\phi_2(x) \nabla \phi_1(x) \\ & - \phi_1(x) \nabla \phi_2(x)) A(x) - (\pi_2(x) \phi_1(x) \\ & - \pi_1(x) \phi_2(x)) A_0(x) \} dx + \frac{4\pi e^2}{2} \int \{ \phi_1^2(x) + \phi_2^2(x) \} A^2(x) dx. \end{aligned} \quad (5.5)$$

In this case, each τ_i in Fig. 2 is replaced by $(1+\tau_3)/2$ and the graphs in Fig. 3 are added corresponding to the pion current. Near the threshold energy, the graph (i) in Fig. 3 is negligible and the graph (ii) should be omitted by the same reason stated for the graph (ix) in Fig. 2. Thus, the graph (iii) alone is considered which contributes only to the iso-spin even part ϕ_e of the iso-spin vector.

By putting $q_0 \approx \omega \approx \mu$ (q_0 is the incident photon energy),

$$\begin{aligned} \phi_e(\omega) &= -\frac{1}{2M+\omega} - \frac{3g^2}{\pi} \sum_{n=1}^9 J_n(\omega), \\ \phi_o(\omega) &= \frac{1}{2M+\omega} - \frac{g^2}{\pi} \sum_{n=1}^8 J_n(\omega), \\ \phi_s(\omega) &= -\phi_o(\omega), \end{aligned} \quad (5.6)$$

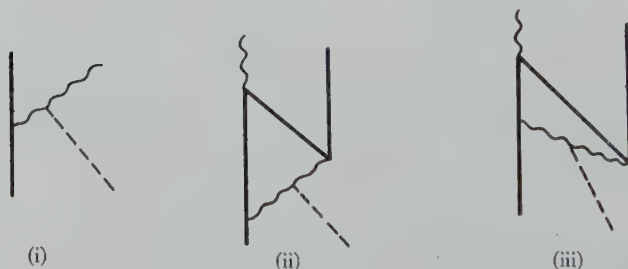


Fig. 3 The dotted line denotes the photon and the wavy line the pion. The graph (i) is negligible and the graph (ii) should be excluded from $V_A^{(4)}(\omega)$.

with

$$\begin{aligned}
 J_1(\omega) &= -\frac{1}{6} \int \frac{(2E' - M) k'^3 v_p^2(k') d\omega'}{E'^2(E' + M)(E' + \omega' - M)^2(3E' + \omega' - M + \omega)}, \\
 J_2(\omega) &= J_3(\omega) = -\frac{1}{6} \int \frac{(2E' - M) k'^3 v_p^2(k') d\omega'}{E'^2(E' + M)(2E' + \omega)(E' + \omega' - M)(3E' + \omega' - M + \omega)}, \\
 J_4(\omega) &= -\frac{1}{6} \int \frac{(2E' - M) k'^3 v_p^2(k') d\omega'}{E'^2(E' + M)(2E' + \omega)^2(3E' + \omega' - M + \omega)}, \\
 J_5(\omega) &= J_6(\omega) = \frac{1}{6} \int \frac{(E' + M)(2E' + M) k' v_s^2(k') d\omega'}{E'^2(2E' - \omega)(3E' + \omega' + M - \omega)(E' + \omega' + M)}, \\
 J_7(\omega) &= \frac{1}{6} \int \frac{(E' + M)(2E' + M) k' v_s^2(k') d\omega'}{E'^2(E' + \omega' + M)^2(3E' + \omega' + M - \omega)}, \\
 J_8(\omega) &= \frac{1}{6} \int \frac{(E' + M)(2E' + M) k' v_s^2(k') d\omega'}{E'^2(2E' - \omega)^2(3E' + \omega' + M - \omega)}, \\
 J_9(\omega) &= \frac{1}{2} \int \frac{k'^3 v_p^2(k') d\omega'}{E' \omega' (E' + \omega' + M)(2E' - \omega)(E' + \omega' + M - \omega)}. \quad (5.7)
 \end{aligned}$$

In performing the angular integral in (5.7), we have disregarded the angular dependence in the denominators. From Eqs. (5.6) and (5.7) we have

$$\begin{aligned}
 \phi_s(\omega) &= -\frac{1}{2M} \left(2.06 + 0.065 \frac{\omega}{\mu} \right), \\
 \phi_o(\omega) &= \frac{1}{2M} \left(0.621 - 0.125 \frac{\omega}{\mu} \right). \quad (5.8)
 \end{aligned}$$

As seen from Eq. (5.8) the strong damping does not occur, in the case of the γ - π production contrary to the case of the s -wave scattering.

Next we shall obtain the production ratio for the charge states at the threshold energy. Transition amplitudes are denoted by their final states, and are given by

$$\begin{aligned}
 T(\pi^+ n) &= \frac{\sqrt{2}}{3} (T_3 - T_1 - 3T_s), \\
 T(\pi^- p) &= \frac{\sqrt{2}}{3} (T_3 - T_1 + 3T_s), \\
 T(\pi^0 p) &= \frac{1}{3} (2T_3 + T_1 + 3T_s), \\
 T(\pi^0 n) &= \frac{1}{3} (2T_3 + T_1 - 3T_s), \quad (5.9)
 \end{aligned}$$

where T_1 and T_3 stand for the transition amplitudes for the $I = \frac{1}{2}$ and $\frac{3}{2}$ states of the iso-spin vector part and T_s for that of the iso-spin scalar part. As the very low energy pions are produced in the final states, the final state interactions may be negligible. Therefore, we have

$$\begin{aligned}
T(\pi^+ n) &= \sqrt{2} eg \left(\alpha_o - \beta_s \frac{\omega}{\mu} \right), \\
T(\pi^- p) &= \sqrt{2} eg \left(\alpha_o + \beta_s \frac{\omega}{\mu} \right), \\
T(\pi^0 p) &= eg (\beta_o + \beta_s) \frac{\omega}{\mu}, \\
T(\pi^0 n) &= eg (\beta_o - \beta_s) \frac{\omega}{\mu}.
\end{aligned} \tag{5.10}$$

The charge ratio of productions at $\omega = \mu$ is given by

$$\sigma(\pi^- p) / \sigma(\pi^+ n) = 2.27. \tag{5.11}$$

The experimental value is $1.87 \pm 0.13^{14)}$. In calculating Eq. (5.7), if the smaller cut-off parameters are chosen (e. g. 5μ and 7μ for the p - and s -waves), the value near 2 may be obtained. This ratio increases with the photon energy. It will be interesting to point out that in this approximation $\sigma(\pi^0 p) < \sigma(\pi^0 n)$ near the threshold. Though its experimental test is difficult, this result will be useful to check our approximation. Furthermore, it should be noticed that the cross section of π^0 production rises with $\omega^2(\omega^2 - \mu^2)^{1/2}$ in spite of the s -wave property of emitted pions, because the π^0 production amplitude is proportional to ω as seen from Eqs. (5.10).

KR proved that the vertex function of the electromagnetic interaction cancels exactly the damping factor coming from the modified propagator in Eq. (3.1) at the low energy limit. Then the effective coupling constant for the ps -coupling, in the case of the γ - π productions, becomes

$$g_{\gamma\pi} = g(1 + g^2/4\pi). \tag{5.12}$$

The coupling constant $g^2 = 14$ is obtained from Eq. (5.12) and the experimental value of $\sigma(\pi^+ n)$, and is consistent with the result of Eq. (3.4).

§ 6. Kaon-nucleon and pion-hyperon scatterings

In the first place, the K - N scattering will be discussed. Experimental data¹⁵⁾ show that up to 200 Mev for the kaon kinetic energy the s -wave amplitude is predominant and the scattering potential for the $I=1$ state is repulsive, its amplitude being larger than that for the $I=0$ state. The (YNK) interaction, therefore, is likely to be the ps -coupling. The interaction Hamiltonian is, provided that the parities of Λ and Σ are same,

$$\begin{aligned}
H_K &= (4\pi)^{1/2} ig_\Lambda \int \bar{\psi}_N(x) \gamma_5 \psi_\Lambda(x) \phi_K(x) dx \\
&\quad + (4\pi)^{1/2} ig_\Sigma \int \sum_i \bar{\psi}_N(x) \gamma_5 \tau_i \psi_{\Sigma i}(x) \phi_K(x) dx + h.c.
\end{aligned} \tag{6.1}$$

The modified Born terms can be written as

$$V(\omega) = \varphi_s(\omega) + \tau^{(1)} \cdot \tau^{(2)} \varphi_v(\omega). \quad (6.2)$$

The other graph obtained from crossing theorem does not exist in this case. By dividing into the eigenstates for $I=0$ and 1, we have

$$\begin{aligned} V_0(\omega) &= -\varphi_s(\omega) + 3\varphi_v(\omega), \\ V_1(\omega) &= \varphi_s(\omega) + \varphi_v(\omega). \end{aligned} \quad (6.3)$$

When $V(\omega)$ is calculated, the correction by kaons can be safely disregarded because of the smaller coupling constants $g_\Lambda^2 \approx g_\Sigma^2 \approx 1$. If we neglect the mass difference between Λ and Σ , we have

$$\begin{aligned} \varphi_s(\omega) &= -\frac{g_\Lambda^2}{M_N + M_Y - \omega} + \frac{3g_\Sigma^2}{\pi} \sum_{n=1}^8 I_n'(\omega), \\ \varphi_v(\omega) &= -\frac{g_\Sigma^2}{M_N + M_Y - \omega} + \frac{g_\Lambda^2 + 2g_\Sigma^2}{\pi} \sum_{n=1}^8 I_n'(\omega), \end{aligned} \quad (6.4)$$

where ω is the incident kaon energy and $I_n'(\omega)$'s are the integrals similar to Eqs. (4.3). Because of the large kaon mass, we avoid the series expansion in ω/M . At $\omega = m_K$

$$\begin{aligned} V_0(m_K) &= 0.30(g_\Lambda^2 - 3g_\Sigma^2) + 0.24(g_\Lambda^2 + g_\Sigma^2), \\ V_1(m_K) &= -0.30(g_\Lambda^2 + g_\Sigma^2) + 0.08(g_\Lambda^2 + 5g_\Sigma^2). \end{aligned} \quad (6.5)$$

The first terms represent the contributions from the 2nd order, and the second terms those from the 4th order. Also in this case the pion corrections for baryon pairs do not alter the magnitude of the Born terms very much, therefore, the s -wave amplitude is predominant in the low energy K - N scattering. As it is not obvious whether the scattering potential of the $I=0$ state is repulsive or attractive, the relation between g_Λ and g_Σ is not determined from Eq. (6.5). However, from the experimental fact that the $I=1$ state dominates over the $I=0$ state, though the former state is diminished by re-scatterings, we may conclude that

$$g_\Lambda^2 \approx 1.2g_\Sigma^2. \quad (6.6)$$

Lastly, we are interested in the π - Y scattering in comparison with the π - N scattering. If the global symmetry⁸⁾ exists with respect to observable masses of Λ and Σ and physical interactions with pions, the s -wave amplitude of the π - Y scattering may be small as the π - N scattering. It is, however, unlikely that the global symmetry holds for the observable quantities (even if their mass difference may be disregarded, two coupling constants with pions will not be same). Then it is anticipated that the s -wave amplitude will be appreciably large in the case of the low energy π - Y scattering, since the extreme damping by the cancellation will not occur.

This prediction will be tested by analysing the behaviour of the final state pion which is produced with the \bar{K} absorption by the nucleon. The pion energies in this case are 100~180 MeV. From our point of view, the s -wave amplitude will be large even in this energy region and if otherwise, the p -wave amplitude will be predominant there.

§ 7. Summary and discussions

We summarize here the results of the preceding sections:

(1) For the modified Born terms we have $\alpha_e = \beta_o = -0.01\pi/g^2$. As the effect of re-scatterings, $Reg_1=0.4$ and $Reg_3=1.25$ are obtained semi-phenomenologically from the phase shifts with use of the effective range approximation. The effect of rescatterings is not important.

(2) The effective ps -coupling constant is $g_{e\pi}^2=16$ in the lowest order corrections and the coupling constant $g^2 \approx 15$ is obtained both cases of the π - N scattering and the γ - π production.

(3) Of the s -wave amplitudes of the π - N scattering, the iso-spin odd part is damped by the cancellation of the matrix element of the crossed graph with that of the uncrossed graph. The iso-spin even part is damped by the cancellation of the 2nd order with the 4th order of the modified Born terms. In other phenomena, e. g., the γ - π production, the K - N and the π - Y scattering, those cancellations do not occur.

(4) The magnitude of the experimental cross section is obtained with $g^2=15$, but the signs of the phase shifts are both negative. The cut-off parameters should be chosen at 5μ and 7μ for the p -wave and the s -wave pions by considering both the s -wave scattering and the γ - π production.

(5) As for the production ratio of the charge states, we have $\sigma(\pi^- p)/\sigma(\pi^+ n) = 2.27$ and $\sigma(\pi^0 p) < \sigma(\pi^0 n)$.

(6) In the low energy K - N and π - Y scatterings for the ps -coupling, the s -wave amplitudes are predominant contrary to the case of π - N scatterings. From K - N scatterings we get $g_A^2 \approx 1.2g_\Sigma^2$.

The above results have been derived in the approximation retaining up to the 4th order terms in the coupling constant. The 6th order and the 8th order terms have the opposite signs and, therefore, will not give large contributions. However, as we discuss very small quantities remaining after the cancellation of main terms in the π - N scattering, the higher order terms cannot be completely disregarded. Moreover, these higher terms contain the effect of π - π scattering through the nucleon loops. We have no precise knowledge at present as to these effects. The split of the phase shifts δ_1 and δ_3 will occur by correctly taking into account those higher order effects. (The cut-off parameters may be different from the values chosen in (4) in order to make the 6th and 8th order effect small and it may be difficult to retain the consistency with the γ - π production.)

Our approximation will be good as the first approximation in the case of the threshold $\gamma\text{-}\pi$ production and the low energy $K\text{-}N$ scattering; therefore it may be useful to analyse these phenomena more precisely in order to check our approximation. In addition, it will be interesting to know whether the s -wave amplitude of the $\pi\text{-}Y$ scattering is large.

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Strong Interactions and Baryon Mass Levels

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Using a cut-off version of the Hamiltonian of d'Espagnat and Prentki for the strong interactions and assuming the baryons form a mass degenerate system in the absence of interaction, it is shown that the observed baryon mass spectrum is consistent with values of the coupling constants suggested by the dynamical processes, such as π meson-nucleon and K^+ meson-nucleon scattering. The π meson-baryon interaction has been taken to be of the $(\sigma \cdot \mathbf{p})$ type and that of the K meson-baryon interaction to be of the γ_5 type, and the Rayleigh-Ritz method using a simple one-meson trial wave function is used to calculate the masses.

§ 1. Introduction

The calculation of the self energies of elementary particles, arising through their mutual interaction, has generally been regarded as a hazardous and unrewarding pursuit, as high order terms in any perturbation scheme might be expected to be of importance. Recently, however, there have been indications that self energies calculated from low order perturbation theory may give, in certain cases, reasonable and meaningful results. For example, Gell-Mann¹⁾ has shown that if the baryons form a mass degenerate system in the absence of strong interactions, then for a scheme of global symmetry in which all the baryon-meson coupling constants are identical, the relation $\frac{1}{4}(M_\Lambda + 3M_\Sigma) = \frac{1}{2}(M_N + M_\Xi)$, between the masses of the baryons, is found in lowest order perturbation theory. This relation is in fact satisfied to within 0.3 π meson mass units*, by the observed masses. In another connection Salam and Tiomno²⁾ have examined an interesting empirical relation between the mass of the proton and that of the electron and the π -nucleon and electric coupling constants. They show that it may be possible to understand this relation by using perturbation theory to calculate the proton and electron masses, assuming a common bare mass for both particles.

The masses of the baryons in the global symmetry scheme of Gell-Mann have formed the subject of studies by Kleitman³⁾ and Katsumori⁴⁾. Kleitman investigated a certain class of Lagrangians due to Schwinger⁵⁾, for which the baryons are mass degenerate in absence of interaction and for which the π -meson inter-

* The mass of the π meson is used as the unit of mass throughout.

actions preserve the symmetry among the baryons—the mass splitting thus being attributed to the presence of the K meson field. Using fourth order perturbation theory with a suitable cut-off he found that no agreement with the observed masses could be obtained, if the K -baryon interaction was scalar, but that for a γ_5 interaction agreement could be found if $g_K^2/g_\pi^2 \sim 1$. This coupling constant relation applies to the unrenormalised coupling constants, and he suggested that renormalisation might account for the observed large difference in magnitude between the π and K meson coupling constants. The second order perturbation calculations of Katsumori⁴⁾ and later of Katsumori and Shimoura⁵⁾ have shown that in the global symmetry scheme agreement may still be obtained with the observed masses for a coupling scheme in which the relative parity of the nucleon and Ξ hyperon is odd. In this case, in contrast to that investigated by Kleitman, the required K -baryon coupling constant (assumed to be of the same value g_K for all the baryons) turns out to be of a reasonable value $g_K^2 \sim 1$, in harmony with the requirement of $K^+ - N$ scattering.

The fact that mass differences of the correct order may be obtained in low order perturbation theory applied to conventional Hamiltonians, with a common cut-off for all interactions, and using coupling constants of the correct order to explain scattering processes, is too striking to be dismissed as a coincidence, and gives confidence in future investigations. In this paper, initially no assumptions as to the equality of any of the coupling constants are made, but the limitations placed on these constants by the requirement that, within a certain calculation scheme, the observed masses must be obtained, is investigated. The usual charge independent Hamiltonian of d'Espagnat and Prentki⁷⁾ is used, all the baryons being assumed to have the same relative parity, and a common momentum cut-off K_m is applied. The Rayleigh-Ritz method is used to find the baryon masses, using a trial function of the one meson or one meson-one pair variety. A π meson-baryon interaction of the $(\sigma \cdot \nabla)$ type is taken (it being considered that the S wave π meson nucleon scattering provides evidence for the effective suppression of pairs in the π meson nucleon interactions), and the K -meson baryon interaction is of the γ_5 type, recoil is included.*

§ 2. The mass equations

Denoting the state vectors of the bare particle states by $|N\rangle, |N\pi\rangle, |N\bar{\Sigma}N\bar{K}\rangle$, etc., with amplitudes $C(\mathbf{k})\dots$, the one meson trial functions for the real baryons are

$$|N\rangle = Z_N^{-1/2} \left[|N\rangle + \int d\mathbf{k} \{ C_1^N(\mathbf{k}) |N\pi\rangle + C_2^N(\mathbf{k}) |\Sigma K\rangle \right]$$

* A short account of the results for a scalar K meson baryon interaction has already been published⁸⁾.

$$\begin{aligned}
& + C_3^N(\mathbf{k}) | \Lambda K \rangle + C_4^N(\mathbf{k}) | N \bar{\Sigma} N \bar{K} \rangle + C_5^N(\mathbf{k}) | N \bar{\Lambda} N \bar{K} \rangle \} \Big] \\
| \Lambda \rangle = & Z_{\Lambda}^{-1/2} \Big[| \Lambda \rangle + \int d\mathbf{k} \{ C_1^{\Lambda}(\mathbf{k}) | \Sigma \pi \rangle + C_2^{\Lambda}(\mathbf{k}) | N \bar{K} \rangle + C_3^{\Lambda}(\mathbf{k}) | \Xi K \rangle \\
& + C_4^{\Lambda}(\mathbf{k}) | \Lambda \bar{N} \Lambda K \rangle + C_5^{\Lambda}(\mathbf{k}) | \Lambda \bar{\Xi} \Lambda \bar{K} \rangle \} \Big] \\
| \Sigma \rangle = & Z_{\Sigma}^{-1/2} \Big[| \Sigma \rangle + \int d\mathbf{k} \{ C_1^{\Sigma}(\mathbf{k}) | \Sigma \pi \rangle + C_2^{\Sigma}(\mathbf{k}) | \Lambda \pi \rangle \\
& + C_3^{\Sigma}(\mathbf{k}) | N \bar{K} \rangle + C_4^{\Sigma}(\mathbf{k}) | \Xi K \rangle + C_5^{\Sigma}(\mathbf{k}) | \Sigma \bar{N} \Sigma K \rangle + C_6^{\Sigma}(\mathbf{k}) | \Sigma \bar{\Xi} \Sigma \bar{K} \rangle \} \Big] \\
| \Xi \rangle = & Z_{\Xi}^{-1/2} \Big[| \Xi \rangle + \int d\mathbf{k} \{ C_1^{\Xi}(\mathbf{k}) | \pi \Xi \rangle + C_2^{\Xi}(\mathbf{k}) | \Lambda \bar{K} \rangle \\
& + C_3^{\Xi}(\mathbf{k}) | \Sigma \bar{K} \rangle + C_4^{\Xi}(\mathbf{k}) | \Xi \bar{\Xi} \Lambda K \rangle + C_5^{\Xi}(\mathbf{k}) | \Xi \bar{\Xi} \Sigma K \rangle \} \Big] \quad (1)
\end{aligned}$$

$\int d\mathbf{k}$ symbolizes integration and summation over all variables and $Z_N^{-1/2}$ is a normalisation factor determined so that $\langle N | N \rangle = 1$, Z_{Λ} , Z_{Σ} , Z_{Ξ} being defined similarly. The amplitudes C_i are found by inserting the expressions¹⁾ into I, where

$$I = \langle \mathbf{B} | H - \varepsilon | \mathbf{B} \rangle,$$

H being the Hamiltonian of d'Espagnat and Prentki⁷⁾ and ε is the eigenvalue, (the mass of the baryon B), and requiring that $\partial I = 0$ for independent variations of the C_i . Using the amplitudes thus determined and denoting π meson coupling constants by f ($\sigma \cdot \nabla$ interaction) and K meson coupling constants by g (γ_5 interaction), it is found that*

$$\varepsilon_i = M_0 + \lambda_i F_{\pi}(\varepsilon_i) + \mu_i F_K(\varepsilon_i), \quad i=1, 2, 3, 4 \quad (2)$$

where M_0 is the common bare mass of the baryons and $\varepsilon, \lambda, \mu$ are given by

i	ε_i	λ_i	μ_i
1	M_N	$3f^2_{N\pi}$	$3g^2_{N\Sigma} + g^2_{N\Lambda}$
2	M_{Λ}	$3f^2_{\Lambda\pi}$	$2(g^2_{N\Lambda} + g^2_{\Xi\Lambda})$
3	M_{Σ}	$2f^2_{\Sigma\pi} + f^2_{\Lambda\pi}$	$2(g^2_{N\Sigma} + g^2_{\Xi\Sigma})$
4	M_{Ξ}	$3f^2_{\Xi\pi}$	$3g^2_{\Xi\Sigma} + g^2_{\Xi\Lambda}$

(3)

* The vacuum subtraction has been performed by standard means.¹⁰⁾

The $F_\pi(\varepsilon_i)$ and $F_K(\varepsilon_i)$ are the contributions to the masses from the π and K interactions, respectively.

$$F_\pi(\varepsilon) = \frac{1}{2\pi} \int_0^{K_m} \left(\frac{M_0 + E}{E} \right) \frac{k^4}{\omega} \frac{dk}{(\varepsilon - \omega - E)}; \quad \omega = \sqrt{1 + k^2}, E = \sqrt{M_0^2 + k^2}$$

$$F_K(\varepsilon) = \frac{1}{2\pi} \int_0^{K_m} \frac{k^2}{WE} \left[\frac{E - M_0}{(\varepsilon - W - E)} - \frac{E + M_0}{(\varepsilon - W - E - 2M_0)} \right] dk \quad (4)$$

$$W_\pi(k) = \sqrt{b^2 + k^2}$$

where b is the K meson mass.

To connect the unrenormalised coupling constants f_i, g_i with the renormalised constants \bar{f}_i, \bar{g}_i , the Watson-Lepore definition is employed, so that for example

$$\bar{f}_{N\pi} = \text{Lt}_{k \rightarrow 0} \frac{\langle N(0) | H' \pi^*(k) | N(-k) \rangle}{\langle N(0) | H' \pi^*(k) | N(-k) \rangle} f_{N\pi} \quad (a)$$

$$\bar{g}_{N\Lambda} = \text{Lt}_{k \rightarrow 0} \frac{\langle N(0) | H' K^*(k) | \Lambda(-k) \rangle}{\langle N(0) | H' K^*(k) | \Lambda(-k) \rangle} g_{N\Lambda} \quad (b)$$

and so on.* In calculating (5), the expressions (1) are used together with the variationally determined C_i so that the coupling constants \bar{g}_i, \bar{f}_i are consistent with the mass equations (2).

§ 3. Determination of coupling constants

Using the observed masses of the baryons, $M_N = 6.75$, $M_\Lambda = 8.02$, $M_\Sigma = 8.58$ and $M_{\Xi} = 9.50$, with $b = 3.55$, in equations (2), (3) and (4) four relations are obtained between eight unrenormalised coupling constants f_i, g_i . An additional equation is provided by 5(a) using the known value of $\bar{f}_{N\pi}, \bar{f}_{N\pi}^2 = 0.08$, so that for given M_0 and K_m , three more relations are needed before the unrenormalised (and hence the remaining renormalised) coupling constants can be found. These further relations were obtained by the introduction of parameters x, y, z , such that

$$g_{N\Lambda} = x g_{N\Sigma}, \quad g_{\Xi\Lambda} = y g_{\Xi\Sigma}, \quad f_{\Lambda\pi} = z f_{\Sigma\pi}. \quad (6)$$

The parameters x, y, z may be given any value, but if it is now required that solutions of the equation system (2), (5a) and (6) must be real (which is necessary for H to be Hermitian), the values that x, y, z and M_0 may take turn out to be *severely restricted*. These acceptable real solutions may then be used to calculate the renormalised $\bar{g}_{N\Lambda}, \bar{g}_{N\Sigma}$, which are known, within limits, from studies of $K^+ - N$ scattering.

Following this calculation scheme, numerical solutions were obtained for two

* The relations depend on the relative signs of the coupling constants in this work they are all taken to be positive.

values of K_m (6.6 and 7.8) and for a range of M_0 , 9(1)20 and for ranges of the parameters x, y, z . From each acceptable solution in addition to $\bar{g}_{N\Lambda}, \bar{g}_{N\Sigma}$ the values of $\bar{f}_{\Lambda\pi}, \bar{f}_{\Sigma\pi}$ were computed. Table I illustrates a typical result for $K_m=7.8$ at $M_0=12$ and $x=y=\sqrt{3}$ and a range of z . (This choice of x leads to a small amount of scattering in the $T=0$ state for K^+-N collisions⁹⁾ in harmony with observation. $y=x$ is suggested by symmetry considerations), Table II illustrates corresponding results at $K_m=6.6$ for $M_0=11$.

Having obtained sets of results in this way specific interaction schemes, suggested by symmetry or other considerations, may be examined. A solution cannot be found for the case $x=y=1$, thus ruling out complete symmetry between the K -baryon coupling constants. On the other hand, it is possible (unlike the scalar case⁸⁾) to find a solution for the global symmetry scheme ($x=\sqrt{3}, y=0.5, z=1$) and $f_{N\pi}^2=f_{\Lambda\pi}^2=f_{\Sigma\pi}^2=f_{\Xi\pi}^2=0.13$. The predicted values of $\bar{f}_{\Lambda\pi}^2$ and $\bar{f}_{\Sigma\pi}^2$ are 0.065 and 0.066, not perhaps too small to be in disagreement with hyperfragment data; but the computed $\bar{g}_{\Lambda N}^2, \bar{g}_{\Sigma N}^2$ of 1.28 and 0.43 are rather small, the calculations of Ceolin and Taffara⁹⁾ for scattering suggesting that $\bar{g}_{\Lambda N}^2=4, \bar{g}_{\Sigma N}^2=1.3$. This work on scattering employed a similar cut-off Hamiltonian and a similar Tamm-Dancoff wave-function, so that the coupling constants obtained in both cases should be consistent.

Among the schemes investigated, one is particularly satisfying. If $x=y=\sqrt{3}$ and $K_m=7.8$, a solution is found that gives agreement with K^+-N scattering $\bar{g}_{N\Lambda}^2=3.8, \bar{g}_{\Sigma N}^2=1.1$ and shows symmetry between the NK and ΞK constants.

$$g_{N\Lambda}=g_{\Xi\Lambda}=g_{\Lambda}$$

$$g_{N\Sigma}=g_{\Xi\Sigma}=g_{\Sigma}$$

$$g_{\Lambda}=\sqrt{3} g_{\Sigma}$$

In addition the predicted values of $\bar{f}_{\Lambda\pi}, \bar{f}_{\Sigma\pi}$ are $\bar{f}_{\Lambda\pi}^2=\bar{f}_{N\pi}^2=0.08$; $\bar{f}_{\Sigma N}^2=\frac{1}{2}\bar{f}_{\Lambda\pi}^2$, values consistent with Λ binding energies and the non-existence of bound Σ^- systems. It should perhaps be noted that the requirement $\bar{g}_{N\Lambda}^2=4$ and $x=\sqrt{3}$ determines the system completely if M_0, K_m, y are specified, but in practice real solutions are not found for this case unless y is close to $\sqrt{3}$. Although in theory M_0 is not completely determined, it is found that in practice a solution with $\bar{g}_{\Lambda N}^2=4$ and $x=y=\sqrt{3}$ can *only* be obtained for one value of M_0 (for a given K_m).

The results are insensitive to the value of K_m , in the sense that at $K_m=6.6$ the same coupling constants may be obtained as at $K_m=7.8$, but at a smaller value of M_0 . Inspection of the results show that they depend roughly on the single parameter $\left(\frac{K_m}{M_0}\right)$, rather than on the two parameters K_m, M_0 . This dependence is exact in perturbation theory⁸⁾.

§ 4. Conclusion

Taken in conjunction with the previous work using perturbation theory,^{8), 4), 7), 8)} the

present work provides strong evidence that the d'Espagnat and Prentki strong interaction Hamiltonian, with coupling constants of a magnitude consistent with the dynamical processes, is capable of describing the mass splitting of the baryon system, and that perturbation or simple variational methods are adequate calculational tools, in the present stage when coupling constants are imperfectly known. At the moment only the renormalised π meson-nucleon constant can be taken as reasonably well known, so that this leaves seven further renormalised constants, together with (M_0/K_m) to be limited or determined from equations (2) and (5). However, the equations turn out to be so stringent in their limitations that knowing only two more renormalised constants (for example, $\bar{g}_{\Lambda N}$, $\bar{g}_{\Sigma N}$), it seems likely that all the other constants can be predicted, including, of course, the relatively inaccessible Ξ couplings.

Table I. Unrenormalised and renormalised coupling constants for $K_m=7.8$, $M_0=12$, $x=y=\sqrt{3}$. The first two columns represent unphysical solutions

z	1.0	1.5	2.0	2.4	2.8	3.2	5.0
$f^2_{N\pi}$.21	.19	.18	.17	.17	.16	.15
$f^2_{\Lambda\pi}$.04	.09	.15	.22	.28	.35	.70
$f^2_{\Sigma\pi}$.04	.04	.04	.04	.04	.03	.03
$f^2_{\Xi\pi}$	-.08	.03	.04	.10	.16	.23	.55
$g^2_{N\Lambda}$	23.6	20.6	18.4	17.1	16.1	15.2	12.7
$g^2_{N\Sigma}$	7.9	6.8	6.2	5.7	5.7	5.1	4.2
$g^2_{\Xi\Lambda}$	-26.7	-13.6	1.8	15.7	30.6	46.3	119.1
$g^2_{\Xi\Sigma}$	-8.9	-4.6	.6	5.2	10.2	15.5	39.7
$\bar{f}^2_{\Lambda\pi}$	—	—	.06	.07	.08	.19	.13
$\bar{f}^2_{\Sigma\pi}$	—	—	.03	.03	.04	.05	.12
$\bar{g}^2_{N\Lambda}$	—	—	3.9	3.8	3.8	3.9	4.4
$\bar{g}^2_{N\Sigma}$	—	—	1.1	1.1	1.0	1.0	1.1

Table II. Unrenormalised and renormalised coupling constants for $K_m=6.6$, $M_0=11$, $x=y=\sqrt{3}$. The first column represents an unphysical solution.

z	1.5	2.0	2.8	3.0
$f^2_{N\pi}$.16	.16	.16	.16
$f^2_{\Lambda\pi}$.09	.16	.29	.32
$f^2_{\Sigma\pi}$.04	.04	.04	.04
$f^2_{\Xi\pi}$.017	.007	.18	.21
$g^2_{N\Lambda}$	8.9	8.3	7.7	7.6
$g^2_{N\Sigma}$	3.0	2.8	2.6	2.5
$g^2_{\Xi\Lambda}$	-4.1	7.7	30.2	36.2
$g^2_{\Xi\Sigma}$	-1.4	2.6	10.0	12.1
$\bar{f}^2_{\Lambda\pi}$	—	.07	.09	.10
$\bar{f}^2_{\Sigma\pi}$	—	.03	.04	.04
$\bar{g}^2_{N\Lambda}$	—	2.6	2.4	2.4
$\bar{g}^2_{N\Sigma}$	—	.8	.7	.7

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Applicability Conditions of the Hydrodynamical Model of Multiple Production of Particles from the Point of View of Quantum Field Theory

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In the framework of quantum field theory, it is attempted to investigate whether the hydrodynamical description is applicable to the meson cloud produced in extremely high energy collision of nucleons as considered in Landau's theory of the multiple production of particles. The applicability conditions of the hydrodynamical model consist of local equilibrium and conditions for the possibilities of defining the local system in the meson cloud, which are prepared by the methods based on quantum statistical mechanics of irreversible processes. These conditions are examined by comparison of the correlation lengths and the relaxation times of the meson fluid with a characteristic length and time, in which the thermodynamical parameters, the temperature for example, of the fluid decrease or increase by an appreciable amount on a macroscopic scale. From such examinations, it may be concluded that the hydrodynamical model holds almost everywhere except in the front part of the cloud after the whole cloud spreads over a region whose size is the order of the correlation length. It is, however, emphasized that the interactions in the initial cloud directly after collision and in the front part of the expanding cloud can never be described by any statistical law or hydrodynamics. The fact that the front particles are never in any thermal equilibrium suggests that they remember some features of initial high energy interactions in the very small cloud. In other words, it is inferred that the distributions (for example, K/π ratio and the momentum or angular distribution) of the front particles may inform us about the interactions at very small distances. On the other hand the influences of initial interactions on the remaining cloud are only taken into account through the initial boundary conditions for the hydrodynamical equation. In addition to the above discussions, it is pointed out that the assumption of the perfect fluid used by Landau is not so good; it turns out that one can expect an increment of the number of particles through the final interactions. Finally it is discussed whether these characteristics may be consistent with the recent experiments.

§ 1. Introduction

The statistical laws in thermal equilibrium or the hydrodynamical equations are often used in theories of the multiple production of particles in nucleon-nucleon collisions. Such theories are to be grounded in the statistical mechanics of irreversible processes as presented in the previous paper¹⁾. Following its prescriptions, it should be confirmed that interactions in the system relax most disturbances and revive thermal equilibrium in a reasonably short time. Furthermore, in the hydrodyna-

mical model, one must define a small cell which may be regarded as a point in the continuous meson fluid in question. In every such cell, it must be shown that good relaxation phenomena occur, in other words, local equilibrium holds.

A few years ago Blokhintsev²⁾ pointed out by discussions using the uncertainty principle that the momentum density could not be defined, in a small cell, compatible with the hydrodynamical description of the meson cloud. We however, think that such a small cell is not an isolated system as treated in Blokhintsev's criticism, but must be considered to have furious interactions with the surrounding cloud of high density and high temperature. Following statistical mechanics, it is natural that one should determine the smallest size of the above-mentioned cells by the longest of the various correlation lengths in the presence of interactions with the surrounding cells and self-interactions. Otherwise, the physical quantities in a cell change with some correlation to surrounding cells, so that an individual cell can not be considered as a local system in the fluid. If the correlation lengths are much smaller than the linear dimensions of the cloud, it becomes possible to define many local systems in the cloud and, consequently, to use the notion of the mass flow or the local velocity formulated in I. Local equilibrium can be expected for systems with sufficiently short relaxation times. Strictly speaking, the correlation length and the relaxation times should be compared with the characteristic length and time, in which the thermodynamical parameters, the temperature for example, of the meson fluid vary by an appreciable amount on a macroscopic scale. Moreover, in order to examine the assumption of the perfect fluid used by Landau, we must show the smallness of the transport coefficients, for example, the heat conductivity, the coefficients of shear and bulk viscosities, of the meson fluid. This can be performed by estimating quantities like the Reynolds number. As will be seen later, the transport coefficients can also be used as a measure of fluctuations of the transported quantities associated with them.

To summarize the above arguments, we must examine the following three applicability conditions of Landau's model: (i) the correlation lengths of the cloud must be much smaller than the linear dimensions of the system and the characteristic length for a macroscopic change of the temperature, (ii) the relaxation time of the cloud must be much shorter than the characteristic time for a macroscopic change of the temperature, and (iii) the transport coefficients must be small. Furthermore Landau has assumed $3p = \varepsilon$ for the equation of state of the meson fluid. Then the applicability conditions of Landau's model should be supplemented by examining such an equation of state. It is the purpose of the present paper to perform these examinations.

In Appendix A it is shown that the correlation lengths are as small as $(1/T)$ in the meson fluid with temperature T .* Since T decreases from a high initial value to a low final one, this guarantees in part the validity of defining a small

* The units $\hbar = c = k$ (Boltzmann constant) = 1 are used through the present note.

cell in the meson fluid after the size of the system has exceeded the correlation lengths. The exceptional case occurs in the earlier stage of expansion where the Lorentz contraction brings about too flat an initial shape of the cloud, whose thickness ($\sim (1/T^2)$) is smaller than the correlation length ($\sim (1/T)$). This can be easily seen without detailed calculations. Most of the remaining discussions in this note will be devoted to estimations of the relaxation times and transport coefficients and to examination of the conditions mentioned above.

In § 2, by replacing the Heisenberg equation of the meson field with a Langevin-like equation, we formulate the semi-phenomenological interaction Hamiltonian representing the furious interactions in the meson cloud. The fluctuation-dissipation theorem is used to characterize the interaction Hamiltonian. In § 3, the temperature dependences of the various transport coefficients, and the relaxation times associated with them, are determined. In § 4 we discuss whether the temperature dependences of the relaxation times and examination of other conditions permit us to use a hydrodynamical description of the meson cloud consistent with the space-time variations in temperature obtained from Landau's model. In § 5, discussions are presented of the information to be obtained from extremely high energy phenomena and the consistency of the results in this paper with some recent experiments. Appendix A is concerned with the estimation of the correlation length and the validity of defining the mass flow or the local velocity. In Appendix B, the method of the Green's function of one meson in a medium is presented. In Appendix C the detailed calculation of the relaxation times and the transport coefficients is explained.

§ 2. Fluctuation-dissipation theorem and interaction Hamiltonian

We now consider an appropriate interaction Hamiltonian to represent furious interactions in a meson cloud of high density and high temperature. In the perturbation theory with an elementary interaction (such as $i\phi^4$, for example), the calculations to the lowest order are only justified for a dilute meson gas. Consequently, it is of convenience for practical calculations to derive a semi-phenomenological Hamiltonian from the exact one, for the purpose of treating furious interactions in a compact form. We shall make use of the fluctuation-dissipation theorem on the analogy to the theory of the Brownian motion.

The interacting meson field* ϕ obeys the operator equation

$$(\square - m^2)\phi = F \quad (2.1)$$

in the Heisenberg representation, m being the meson mass. Here F consists of the absorption and creation operators of one and more mesons, nucleon pairs and other particles. From analogous discussions of the vacuum field theory, it may be expected that the operator ϕ or F can be divided into two parts, one representing

* For simplicity we deal exclusively with the neutral field.

asymptotically one clothed meson in the medium of the meson cloud and the other corresponding to the remaining part. We may denote the one clothed meson part by $II \cdot \phi$ and the remaining part by f , respectively, where II is a c -number operator. Naturally, in the asymptotic sense the operator f is only associated with two and more mesons and other particles, and the difference of the numbers of creation operators and annihilation operators is more than 1. The field equation (2.1) is rewritten as

$$(\square - m^2 - II)\phi = f. \quad (2.2)$$

The c -number operator II can be connected with the Green function, for one clothed meson in the medium of the meson cloud, defined by

$$\begin{aligned} G(\mathbf{x}, t; \mathbf{x}', t') &= \langle T(\phi(\mathbf{x}, t)\phi(\mathbf{x}', t')) \rangle \\ &= T_r\{\rho T(\phi(\mathbf{x}, t)\phi(\mathbf{x}', t'))\}, \end{aligned} \quad (2.3)$$

where ρ is the density matrix of the system and T stands for Wick's chronological operator. From (2.2) and (2.3), we easily find that the Green function G satisfies the equation

$$(\square - m^2 - II)G = 1, \quad (2.4)$$

because of $T_r\{\rho T(\phi f)\} = 0$. Here the symbol 1 means the delta function $\delta(\mathbf{x} - \mathbf{x}') \times \delta(t - t')$. The c -number function G or II is to be calculated from the exact Hamiltonian including the elementary interactions. The real part of II is nothing but the effective increment of meson mass, while its imaginary part implies the dissipation of one clothed meson in probability.

Since the effects of f in the exact equation are expected to become random on a rough time scale, neglecting the interval of order (the correlation length, the light velocity) because it contains furious changes in the meson cloud of high density and high temperature, one may replace the definite operator f by a fluctuating external source (or sink) function f which represents the random interaction between the cloud and the large surrounding heat bath or the cloud itself. The equation (2.2) with such a random f is considered to be analogous to the Langevin equation in the theory of Brownian motion. That is to say, such a Langevin-like equation should be regarded as an asymptotic one valid only when one disregards the fine interactions during each time interval whose width is the correlation time $(\tilde{\epsilon}_0/1)$, $\tilde{\epsilon}_0$ being the correlation length and 1 the light velocity. The fluctuating source f has a statistical character given by the fluctuation-dissipation theorem. The equation of form (2.2) with the external source f can be derived from the interaction Hamiltonian

$$\mathcal{H} = \int f(\mathbf{x}, t)\phi(\mathbf{x})d^3\mathbf{x} \quad (2.5)$$

in the Schrödinger representation. This takes the place of the Hamiltonian representing the fluctuating interactions between the local system and the surrounding

fluid, or the fluctuating self-interactions. Thus the power (or the dissipation of energy per unit time) due to the action of f becomes

$$\int \left\langle f(\mathbf{x}, t) \frac{\partial \phi(\mathbf{x}, t)}{\partial t} \right\rangle d^3 \mathbf{x}. \quad (2.6)$$

This fact permits us to regard $\partial \phi / \partial t$ and f as the thermodynamical flow and driving force, respectively. Hence the ratio of the Fourier transform of f to that of $\partial \phi / \partial t$ can be treated as the "impedance" of the system, so that the fluctuation-dissipation theorem³⁾ can be described as follows;

$$\langle \bar{f}(\mathbf{k}, \omega) \bar{f}^*(\mathbf{k}', \omega') \rangle = \coth \frac{|\omega|}{2T} (\text{Im } \bar{\Pi}_{k\omega}) \delta(\mathbf{k} - \mathbf{k}') \delta(\omega - \omega'), \quad (2.7)$$

where $\bar{\Pi}$ is the Fourier transform of Π and $\bar{f}(\mathbf{k}, \omega)$ that of $f(\mathbf{x}, t)$;

$$\bar{f}(\mathbf{k}, \omega) = \frac{1}{\sqrt{(2\pi)^4}} \iint e^{-i\mathbf{k} \cdot \mathbf{x} + i\omega t} f(\mathbf{x}, t) d^3 \mathbf{x} dt.$$

This theorem holds as far as f can be regarded as a random function, i.e., the oscillations in the interval $t < \xi_0$ can be disregarded. Hence the theorem (2.7) must be used in the frequency range $|\omega| \lesssim (1/\xi_0)$. In other words, the right-hand side of (2.7) must be multiplied by a cut-off factor for the range $|\omega| \gtrsim (1/\xi_0)$. Since $\xi_0 \sim (1/T)$ as seen in Appendix A, the frequency range of (2.7) becomes the interval from $(-T)$ to $(+T)$, so that we can always approximate the factor $\coth(|\omega|/2T)$ by $(2T/|\omega|)$. Thus one gets

$$\langle \bar{f}(\mathbf{k}, \omega) \bar{f}^*(\mathbf{k}', \omega') \rangle \simeq \frac{2\zeta T}{|\omega|} \delta(\mathbf{k} - \mathbf{k}') \delta(\omega - \omega'), \quad (2.8)$$

in which we have put $\zeta = \text{Im } \bar{\Pi}_{k\omega}$. On the other hand, we are only concerned with free mesons in the range $|\omega| \lesssim m$, so that $\zeta \simeq 0$ for $|\omega| \lesssim m$.

In the present formalism based on (2.7) or (2.8), the parameter ζ is only a phenomenological one unless we calculate it by means of the Green function within the framework of the exact Hamiltonian. Of course, we know the way to obtain the parameter ζ from the exact Hamiltonian. Although it is very difficult to calculate ζ exactly, it may be of some significance to obtain ζ from the exact Hamiltonian by conventional perturbation theory. Thus we get the rough formula (see Appendix B)

$$\begin{aligned} \zeta &\simeq 2\pi^2 g_s^2 T^2 & (\text{ps-coupling}) \\ &\simeq \pi^2 (g_v/m)^2 T^4 & (\text{pv-coupling}) \end{aligned} \quad (2.9)$$

to the lowest order of the coupling constant (g_s or g_v) for the meson-nucleon system. Here we have used the approximation $T \gg m$ and M , M being the nucleon mass. At first sight one may distinguish the types of elementary interaction by the T -dependence of ζ or the autocorrelation function $\langle f f^* \rangle$. Nevertheless, such a difference

might be washed out due to the damping effect that may multiply ζ by a factor of the form $[1 + A(g_v/m)^2 T^2]^{-1}$ which brings the T -dependence of ζ in the case of the pv -coupling to that in the case of the ps -coupling for high temperatures. Moreover, we may get the real part of H of order T^2 , so that the meson wave would propagate with effective mass $\sim T$.

It is true that the theory is relatively simple if one calculates the various quantities by making use of the interaction Hamiltonian (2.5), but it may be more convenient to formulate the theory by introducing an effective Hamiltonian in parallelism to the familiar form $\lambda \phi^4$ of elementary interaction. For this purpose we replace f in (2.5) by $\lambda \langle \phi^2 \rangle \phi \chi$, where $\langle \phi^2 \rangle \simeq 4\pi T^2 (T \gg m)$ is a density-like quantity of the meson fluid and χ represents the fluctuating potential. If we normalize the autocorrelation function of χ as follows;

$$\langle \bar{\chi}(\mathbf{k}, \omega) \bar{\chi}^*(\mathbf{k}', \omega') \rangle = \delta(\mathbf{k} - \mathbf{k}') \delta(\omega - \omega'), \quad (2.10)$$

then the dimensionless quantity (λT^2) becomes

$$(\lambda T^2)^2 \simeq \zeta T^{-2} / 32\pi^3, \quad (2.11)^*$$

(λT^2) is nothing but the (dimensionless) effective coupling constant of interactions between the meson field and the heat bath (i.e. the meson fluid as a medium). It is noted that (λT^2) is a slowly varying function, as $\ln(T/m)$, of T , but (λT^2) vanishes for $T \lesssim m$. The interaction Hamiltonian has then the form

$$\mathcal{H} = \int \lambda \langle \phi^2 \rangle \phi^2(\mathbf{x}) \chi(\mathbf{x}, t) d^3\mathbf{x} \quad (2.12)$$

in the Schrödinger representation.

§ 3. Relaxation phenomena in the meson cloud

We have shown in the first section that one can take a small cell, whose size is $(1/T)$ at least, as a local system of the meson cloud in question. We now talk about the relaxation phenomena in a small cell in its rest coordinate system, in which one can use the non-covariant expression obtained in I for the various quantities. It is of practical convenience to calculate the various transport coefficients and the relaxation times associated with them in a large system (having the volume V) in which all thermodynamical parameters, i.e., temperature, pressure and so on, are everywhere just the same slowly varying functions with the same constant gradients as those in the local system at a given point. In what follows, it is assumed that $T \gg m$.

The heat conductivity κ is defined by Fourier's law

* We have replaced the factor $(1/\omega)$ in the right-hand side of (2.8) by $(1/T)$ without an appreciable change of order, because such a factor appears in the integrals of form $\int_0^T d\omega \frac{1}{\omega} e^{-\omega/T} \dots$ in the calculation of various quantities.

$$\mathbf{q} = -\kappa \frac{1}{T} \nabla T, \quad (3.1)$$

where \mathbf{q} is the heat flow. In the previous paper I we have obtained the formula

$$\kappa = \int_V K(\mathbf{x} - \mathbf{x}') d^3 \mathbf{x}', \quad (3.2)$$

where

$$K(\mathbf{x} - \mathbf{x}') = \frac{1}{T} \int_0^\infty \langle \{g_1(\mathbf{x}', 0), g_1(\mathbf{x}, t)\} \rangle_0 dt. \quad (3.3)$$

Here the operator $g_1(\mathbf{x}, t)$ is a component of the momentum density operator

$$g_i(\mathbf{x}, t) = \phi^{(+)}(\mathbf{x}, t) \frac{\vec{P}_i - \vec{P}_i}{2i} \frac{\vec{P}_i - \vec{P}_i}{-2i} \phi^{(-)}(\mathbf{x}, t) \quad (3.4)$$

and the symbol $\langle \{A, B\} \rangle_0$ stands for

$$\langle \{A, B\} \rangle_0 = T_r \{ \rho_0 \frac{1}{2} (AB + BA) \} - \text{vacuum term},$$

where ρ_0 is the density matrix in thermal equilibrium with temperature T . The operator $(1/\sqrt{2})\phi^{(\pm)}$ are the plus and minus frequency parts of ϕ . As is easily shown, the function K is a function of the difference $\mathbf{x} - \mathbf{x}'$, so that the quantity κ is independent of \mathbf{x} . Hence we rewrite (3.2) as

$$\kappa = \frac{1}{VT} \int_0^\infty \langle \{ \phi_1(0), \phi_1(t) \} \rangle_0 dt, \quad (3.5)$$

where ϕ is a component of the total momentum operator

$$\phi_i(t) = \int_V g_i(\mathbf{x}, t) d^3 \mathbf{x}$$

of the meson cloud. Since the integrand of (3.5) is the response function for an unit pulse, its damping time is nothing but the relaxation time for disturbing the temperature, i.e., the relaxation time associated with the heat conduction.

Concerning the shear and bulk viscosities, we have the well-known phenomenological equation

$$t_{ik} = p_s \delta_{ik} - \eta_{(s)} [(\vec{P}_i u_k + \vec{P}_k u_i) - \frac{2}{3} \delta_{ik} (\nabla \cdot \mathbf{u})] - \eta_{(v)} \delta_{ik} (\nabla \cdot \mathbf{u}) \quad (3.6)$$

among the stress t_{ik} , the static pressure p_s , and the deformation velocity* $\vec{P}_i u_k$, where $\eta_{(s)}$ and $\eta_{(v)}$ are the coefficients of shear and bulk viscosities, respectively. The formulas for $\eta_{(s)}$ and $\eta_{(v)}$ are given by

$$\eta_{(s)} = \frac{1}{2VT} \int_0^\infty \langle \{ \mathcal{J}_{12}(0), \mathcal{J}_{12}(t) \} \rangle_0 dt, \quad (3.7)$$

* u_k is the local velocity.

$$\gamma_{(v)} = \frac{1}{3VT} \int_0^\infty \langle \{ \mathcal{J}_{11}(0), \mathcal{J}_{11}(t) \} \rangle_0 dt, \quad (3.8)$$

where the operator

$$\mathcal{J}_{ik}(t) = \int_V T_{ik}(\mathbf{x}, t) d^3\mathbf{x} \quad (3.9)$$

is a component of the total stress tensor operator. Here T_{ik} is connected with the meson field as follows;

$$T_{ik}(\mathbf{x}, t) = \phi^{(+)}(\mathbf{x}, t) \frac{\vec{p}_i - \vec{p}_i}{2i} \frac{\vec{p}_k - \vec{p}_k}{2i} \phi^{(-)}(\mathbf{x}, t). \quad (3.10)$$

We shall show details of the calculation of κ exclusively and write only the results for $\gamma_{(s)}$ and $\gamma_{(v)}$. Following the perturbation theory, the integrand of (3.5) can be developed in the series

$$\begin{aligned} \langle \{ \phi_1(0), \phi_1(t) \} \rangle_0 &= \sum_{n=1}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ &\times \langle \{ \phi_1(0), [[\cdots [[\phi_1(0), \mathcal{H}_1(t_1)], \mathcal{H}_1(t_2)], \cdots], \mathcal{H}_1(t_n)] \} \rangle_0, \end{aligned} \quad (3.11)$$

where $\mathcal{H}_1(t)$ is the interaction Hamiltonian in the interaction representation. Taking the randomness of $\mathcal{H}_1(t)$ into account, the series (3.11) can be easily summed up and becomes the simple expression⁴⁾

$$\langle \{ \phi_1(0), \phi_1(t) \} \rangle_0 = \langle [\phi_1(0)]^2 \rangle_0 \exp \left\{ - \int_0^t dt' (t-t') \mathcal{F}(t') \right\}, \quad (3.12)$$

where

$$\mathcal{F}(t') = \frac{1}{\langle [\phi_1(0)]^2 \rangle_0} \langle \{ [\phi_1(0), \mathcal{H}_1(t')], [\mathcal{H}_1(0), \phi_1(0)] \} \rangle_0. \quad (3.13)$$

Strictly speaking, the exponent of (3.12) must be supplemented with the infinite series consisting of higher order terms with respect to the even powers of \mathcal{H}_1 . However, one may expect the effects of these terms to be some modification of the effective coupling constant like, for example, the damping effects.

In terms of the Fourier transform $\overline{\mathcal{F}}(\omega)$ of $\mathcal{F}(t')$, the time-integral in the exponent of (3.12) can be rewritten as

$$\frac{1}{\sqrt{2\pi}} \int d\omega \overline{\mathcal{F}}(\omega) \mathcal{A}(\omega, t), \quad (3.14)$$

in which

$$\mathcal{A}(\omega, t) = \frac{1}{(i\omega)^2} [e^{-i\omega t} - 1 + i\omega t] \quad (3.15)$$

$$\overline{\mathcal{F}}(\omega) = \frac{1}{\langle [\phi_1(0)]^2 \rangle_0} \langle \{ [\phi_1(0), \overline{\mathcal{H}}_1(\omega)], [\mathcal{H}_1(0), \phi_1(0)] \} \rangle_0. \quad (3.16)$$

Here $\overline{\mathcal{H}}_1(\omega)$ is the Fourier transform of $\mathcal{H}_1(t')$. Let us define a characteristic time σ of the interaction by

$$\frac{1}{\sigma^2} = \frac{2}{\pi} \mathcal{F}(0) = \frac{-2}{\pi} \cdot \frac{\langle [\phi_1(0), \mathcal{H}_1(0)]^2 \rangle_0}{\langle [\phi_1(0)]^2 \rangle_0}. \quad (3.17)$$

Since the time σ characterizes the initial behavior of (3.14), we can proceed to evaluation of the integral (3.14) in the following way. If $\sigma \Delta\omega \gg 1$ for the spectral width $\Delta\omega$ of $\overline{\mathcal{F}}(\omega)$, then one gets the asymptotic formula $\Delta(\omega, t) \simeq \pi t \delta(\omega)$ because the integral (3.14) contains a number of oscillations and, consequently,

$$\int d\omega \overline{\mathcal{F}}(\omega) \Delta(\omega, t) \simeq t \left(\sqrt{\frac{\pi}{2}} \overline{\mathcal{F}}(0) \right).$$

In this case the Lorentzian type of relaxation occurs in the relaxation time

$$\tau = \left(\sqrt{\frac{\pi}{2}} \overline{\mathcal{F}}(0) \right)^{-1}. \quad (3.18)$$

Inversely, if $\sigma \Delta\omega \ll 1$, then we have

$$\frac{1}{\sqrt{2\pi}} \int d\omega \overline{\mathcal{F}}(\omega) \Delta(\omega, t) \simeq \frac{t^2}{2} \mathcal{F}(0) = \frac{\pi}{4\sigma^2} t^2,$$

because $\Delta(\omega, t) \simeq t^2/2$. Hence this is just the Gaussian type of relaxation, whose relaxation time is nothing but the characteristic time σ of the interaction defined by (3.17).

After some calculations (see Appendix C) one gets the formula for the T -dependence of σ and τ as follows;

$$\left. \begin{aligned} \sigma &\simeq \frac{\pi}{\sqrt{10}(\lambda T^2)} \cdot \frac{1}{T}, \\ \tau &\simeq \frac{1}{5(\lambda T^2)^2} \cdot \frac{1}{T}, \end{aligned} \right\} \quad (3.19)$$

where we have used the interaction Hamiltonian (2.12) and the fluctuation-dissipation theorem (2.10) and (2.11). The spectral width $\Delta\omega$ is of order $(1/\xi_0) \sim T$ as is expected from the spectral intensity of the autocorrelation function (see discussions given under (2.7)). Thus we have

$$\sigma \Delta\omega \simeq \frac{\pi}{\sqrt{10}} (\lambda T^2)^{-1}. \quad (3.20)$$

This is used as a criterion to judge whether the relaxation phenomena is Lorentzian or Gaussian, according as $\sigma \Delta\omega \gg 1$ or $\ll 1$. Because of $(\pi/\sqrt{10}) \simeq 1$, this criterion depends critically on the numerical value of the effective coupling constant (λT^2) .

If $(\lambda T^2) \ll 1$, that is, we are concerned with the case of weak coupling; then the relaxation is Lorentzian and the relaxation time is naturally τ . In this case there holds the relation

$$\tau \gg \sigma \gg \hat{\epsilon}_0.$$

If $(\lambda T^2) \gg 1$ in the case of strong coupling, we have the Gaussian type of relaxation and the relaxation time $\sigma \ll \hat{\epsilon}_0$. In this case, however, the effective coupling constant λT^2 would be reduced to a value of order 1 due to the strong damping effect. This is very plausible. In fact, if σ were much smaller than $\hat{\epsilon}_0$, the different regions with a size of order σ would attain thermal equilibrium independently of each other in local system. This is inconsistent with the notion of the correlation length. Thus we may as well consider σ to be of order $\hat{\epsilon}_0$ due to the strong damping effect on the effective coupling constant (λT^2) :

Although the calculations which lead to the formula (2.9) are very rough, one may estimate order of magnitude by using (2.9) with a value of g_s or g_o consistent with one obtained from the low energy meson physics. Thus it is reasonable to put the T -dependence of the relaxation time τ_0 (τ or σ) in the form

$$\tau_0 \simeq a/T, \quad a \text{ being of order } 1. \quad (3.21)$$

This value of τ_0 means that the mechanism of relaxation depends critically on the numerical values of a (that is, $(\lambda T^2)^{-2}$ or $(\lambda T^2)^{-1}$) and is perhaps intermediate between Lorentzian and Gaussian. At the end of § 2, we have remarked that the effective coupling constant (λT^2) will tend to zero as T approaches to m . Thus, the Lorentzian type of relaxation occurs for $T \simeq m$, as expected in dilute meson gases. Further it is noted that $(\lambda T^2)^2$ may depend on T as $\ln(T/m)$.

For the relaxation times associated with the shear viscosity, we have the formulas

$$\begin{aligned} \sigma' &\simeq \frac{\pi}{\sqrt{10}(\lambda T^2)} \cdot \frac{1}{T}, & (\text{Gaussian}), \\ \tau' &\simeq \frac{1}{5(\lambda T^2)^2} \cdot \frac{1}{T}. & (\text{Lorentzian}). \end{aligned} \quad (3.22)$$

The relaxation times associated with the bulk viscosity are given by the formulas

$$\begin{aligned} \sigma'' &\simeq \frac{\pi}{\sqrt{10}(\lambda T^2)} \cdot \frac{1}{T}, & (\text{Gaussian}), \\ \tau'' &\simeq \frac{1}{5(\lambda T^2)^2} \cdot \frac{1}{T}. & (\text{Lorentzian}). \end{aligned} \quad (3.23)$$

As is easily understood, one may use (3.21) as the common formula for the relaxation times in the above three cases.

Now we can readily evaluate the time-integral in the formula for the heat conductivity κ as follows;

$$\kappa = \frac{1}{VT} \langle [\phi_1(0)]^2 \rangle_0 \tau_0 \simeq \frac{4a}{\pi^2} T^3 \quad (3.24)$$

for either type (Gaussian or Lorentzian).^{*} Here we have used the expression

$$\langle [\phi_1(0)]^2 \rangle_0 \simeq \frac{4}{\pi^2} T^3 V \quad (A.17)$$

(see Appendix C). Similarly we get the formulas

$$\left. \begin{aligned} \eta_{(s)} &= \frac{1}{2VT} \langle [\mathcal{J}_{12}(0)]^2 \rangle_0 \tau_0 \simeq \frac{2a}{5\pi^2} T^3, \\ \eta_{(v)} &= \frac{1}{3VT} \langle [\mathcal{J}_{11}(0)]^2 \rangle_0 \tau_0 \simeq \frac{4a}{5\pi^2} T^3 \end{aligned} \right\} \quad (3.25)$$

for the coefficients of shear and bulk viscosities, respectively. These expressions for the transport coefficients show that their values can also be used as a measure of magnitude of fluctuations of the related quantities.

§ 4. Discussions on applicability conditions of Landau's model

Now we examine the applicability conditions of Landau's model presented in § 1. They are to be satisfied by the temperature of the meson cloud in question. Here we first formulate these conditions in the form of inequalities among the several quantities, such as the linear dimensions d of the system, the characteristic length x_0 and time t_0 for the macroscopic changes of T , the Reynolds number R , the correlation length $\hat{\xi}_0$, the relaxation time τ_0 and so on. We shall use the solutions⁵⁾ obtained by Landau and others as the functions representing the dependences of T on space and time. It is noted that such solutions contain a single space variable and a time variable.

The characteristic length x_0 and time t_0 are defined by the relations

$$\frac{1}{x_0} = \frac{1}{T} \sqrt{(\mathcal{A}_{\mu\nu} \mathcal{F}_\nu T)^2}, \quad (4.1a)$$

$$\frac{1}{t_0} = \frac{1}{T} |DT|, \quad (4.1b)$$

respectively. Here we have used the abbreviations

$$\mathcal{A}_{\mu\nu} = \partial_{\mu\nu} + U_\mu U_\nu, \quad D = U_\mu \mathcal{F}_\mu,$$

where $U_\mu = (\mathbf{u}/\sqrt{1-u^2}, i/\sqrt{1-u^2})$ and $\mathcal{F}_\mu = (\mathcal{F}, 1-i\mathcal{F}_i)$ are the local four velocity and the four vector of differentiation, respectively, \mathbf{u} being the local velocity. Because of the projection character of $\mathcal{A}_{\mu\nu}$ onto the space-like direction, $\mathcal{A}_{\mu\nu} \mathcal{F}_\nu$ means the

* Note that $\int_0^\infty e^{-t/\tau_0} dt = \tau_0$ and $\int_0^\infty e^{-\pi t^2/4\tau_0^2} dt = \tau_0$.

differential operator with respect to the space variable x' in the local rest system, while D is the invariant differential operator with respect to the time variable t' in the local rest system. Thus we can rewrite (4.1a) and (4.1b) as follows;

$$\frac{1}{x_0} = \left| \frac{1}{\sqrt{1-u^2}} \frac{1}{T} \left[\frac{\partial T}{\partial x} + u \frac{\partial T}{\partial t} \right] \right|, \quad (4.1a')$$

$$\frac{1}{t_0} = \left| \frac{1}{\sqrt{1-u^2}} \frac{1}{T} \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right] \right|, \quad (4.1b')$$

in terms of the coordinate variables x and t in the center-of-mass system.

To examine the assumption of the perfect fluid, we must obtain the Reynolds number R for viscosity and the number K for heat conduction analogous to R . For this purpose let us divide the energy-momentum density tensor $T_{\mu\nu}$ into the perfect fluid part and the part due to irreversible processes as follows;

$$T_{\mu\nu} = T_{\mu\nu}^{(p)} + T_{\mu\nu}^{(i)}, \quad (4.2)$$

where

$$T_{\mu\nu}^{(p)} = p_s \delta_{\mu\nu} + (p_s + \varepsilon) U_\mu U_\nu, \quad (4.3)$$

$$T_{\mu\nu}^{(i)} = U_\mu q_\nu + U_\nu q_\mu - P_{\mu\nu}.$$

Here ε , q_μ and $P_{\mu\nu}$ are the invariant energy density, the heat flow and the viscous stress tensor, respectively. The part $T_{\mu\nu}^{(i)}$ is moreover decomposed in the way

$$T_{\mu\nu}^{(i)} = T_{\mu\nu}^{(h)} + T_{\mu\nu}^{(v)}, \quad (4.4)$$

where $T_{\mu\nu}^{(h)}$ and $T_{\mu\nu}^{(v)}$ correspond to the heat conduction and the viscosity, respectively, as follows;

$$T_{\mu\nu}^{(h)} = U_\mu q_\nu + U_\nu q_\mu, \quad T_{\mu\nu}^{(v)} = -P_{\mu\nu}. \quad (4.5)$$

Here we can use the phenomenological equations

$$q_\mu = -\kappa \left[\frac{1}{T} \mathcal{A}_{\mu\nu} \nabla_\nu T + D U_\mu \right] \quad (4.6)$$

$$P_{\mu\nu} = \gamma_{(v)} \mathcal{A}_{\mu\nu} (\nabla_\alpha U_\alpha) + \gamma_{(s)} [\mathcal{A}_{\mu\alpha} \mathcal{A}_{\nu\beta} (\nabla_\alpha U_\beta + \nabla_\beta U_\alpha) - \frac{2}{3} \mathcal{A}_{\mu\nu} (\nabla_\alpha U_\alpha)] \quad (4.7)$$

for q_μ and $P_{\mu\nu}$. However, it is possible to discard the shear viscosity in $P_{\mu\nu}$ because the quantities $\nabla_1 U_2$ and $\nabla_2 U_1$ vanish in the one-dimensional motion of the fluid in Landau's model. By making use of the above tensors, we can define the invariant R and K by the following ratio:

$$R = \frac{\mathcal{A}_{\mu\nu} T_{\nu\mu}^{(p)}}{|\mathcal{A}_{\mu\nu} T_{\nu\mu}^{(v)}|} = \frac{\varepsilon}{3\gamma_{(v)} |\nabla_\alpha U_\alpha|}, \quad (4.8)$$

$$K = \frac{\mathcal{A}_{\mu\nu} T_{\nu\mu}^{(p)}}{\sqrt{(\mathcal{A}_{\mu\nu} T_{\nu\mu}^{(h)} U_\mu)^2}} = \frac{\varepsilon}{\kappa \sqrt{[(1/T) \mathcal{A}_{\mu\nu} \nabla_\mu T + D U_\mu]^2}}. \quad (4.9)$$

Here it is noted that $J_{\mu\nu} T_{\nu\mu}^{(p)} = 3p_s = \varepsilon = U_\mu T_{\mu\nu}^{(p)} U_\nu$ due to the equation of state, $3p_s = \varepsilon$.

We interpret briefly the behaviors of the solutions⁵⁾ of the hydrodynamical equation for T in Landau's model. The space-time distribution of T consists of the simple wave occupying the front part and the remaining wave occupying the back region, which is called the non-trivial region. In the non-trivial region, the dependence of T on x and t is well described by the approximate solution

$$\ln\left(\frac{T}{T_0}\right) = -\frac{1}{3} \left[\ln\left(\frac{t+x}{J}\right) + \ln\left(\frac{t-x}{J}\right) - \sqrt{\ln\left(\frac{t+x}{J}\right) \ln\left(\frac{t-x}{J}\right)} \right] \quad (4.9)$$

and

$$u = \frac{x}{t},$$

where J is the initial thickness of the cloud given by

$$J = 2.5/T_0^2, \quad (4.10)^*$$

T_0 being the initial temperature of the system assumed by Fermi and Landau. Instead of x and t , it is more convenient to use the variables T and α defined by the equations

$$\left. \begin{aligned} \frac{t+x}{J} &= \left(\frac{T}{T_0}\right)^{-\frac{3}{1+\alpha-\sqrt{\alpha}}}, \\ \frac{t-x}{J} &= \left(\frac{t+x}{J}\right)^\alpha = \left(\frac{T}{T_0}\right)^{-\frac{3\alpha}{1+\alpha-\sqrt{\alpha}}}. \end{aligned} \right\} \quad (4.11)$$

The variable α runs over the interval from $\alpha=1$ to $\alpha\sim 0$ (but $\neq 0$), according as the point restricted to the surface $T=\text{const.}$ passes through the non-trivial region from its central part to its front part. In the region of the simple wave, we have the exact solutions

$$\left. \begin{aligned} \frac{T}{T_0} &= \left(\frac{t-x}{t+x} \cdot \frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^{1/2\sqrt{3}}, \\ u &= \frac{t+\sqrt{3}x}{\sqrt{3}t+x}. \end{aligned} \right\} \quad (4.12)$$

and

Here it is convenient to introduce the variables T and β , defined by

$$\left. \begin{aligned} \frac{t+x}{J(\sqrt{3}-1)} &= \left(\frac{T}{T_0}\right)^{\frac{2\sqrt{3}}{\beta-1}}, \\ \frac{t-x}{J(\sqrt{3}+1)} &= \left(\frac{t+x}{J(\sqrt{3}-1)}\right)^\beta = \left(\frac{T}{T_0}\right)^{\frac{2\sqrt{3}\beta}{\beta-1}}, \end{aligned} \right\} \quad (4.13)$$

* In what follows, we shall use the unit $m=1$.

where β varies from $\beta = 7 - 4\sqrt{3}$ to $\beta = -\infty$ according as the point restricted to the surface $T = \text{const.}$ moves from the back boundary of the simple wave region to the wave front.

Now we can formulate the conditions (i), (ii) and (iii) presented in § 1 in the following inequalities:

$$(i) \quad d \gg \hat{\xi}_0 \quad \text{and} \quad x_0 \gg \hat{\xi}_0, \quad (4.14a)$$

$$(ii) \quad t_0 \gg \tau_0, \quad (4.14b)$$

$$(iii) \quad R \gg 1 \quad \text{and} \quad K \gg 1. \quad (4.14c)$$

These conditions will be examined in the following subsections.

4.1. Definition of the local system

In this subsection we discuss the conditions (4.14a) for the possibility of defining a local system. As is discussed in § 1, the first condition of (4.14a) becomes most serious in the initial cloud directly after collision because of its flatness due to Lorentz contraction. Using $\hat{\xi}_0 = (a'/T)$ obtained in Appendix A, (4.14a) becomes

$$T_0 \leq 2.5/a'. \quad (4.15)$$

For several values of a' (which is of order 1), (4.15) becomes the following inequalities

$$\left. \begin{aligned} T_0 &\leq 2.5 \quad \text{or} \quad E_{\text{lab.}} \leq 4 \text{ BeV} && \text{if } a' = 1, \\ T_0 &\leq 2 \quad \text{or} \quad E_{\text{lab.}} \leq 100 \text{ BeV} && \text{if } a' = 0.5, \\ T_0 &\leq 10 \quad \text{or} \quad E_{\text{lab.}} \leq 1000 \text{ BeV} && \text{if } a' = 0.25. \end{aligned} \right\} \quad (4.16)$$

At any rate it is clear that (4.16) is not satisfied at extremely high energies. Thus it is hardly acceptable to regard the initial cloud produced in extremely high energy collisions as a sort of fluid. On the other hand, the assumption that the initial cloud is in thermal equilibrium as a whole is, of course, not self-consistent because of finiteness (\leq light velocity) of the transmission velocity of disturbances. The features as a fluid will appear only after its thickness exceeds the correlation length $\hat{\xi}_0 \simeq (a'/T)$. In the initial period before some fluid features appear in the cloud, the interactions in the cloud are governed by another law apart from hydrodynamics. The results of such interactions are to be taken into account as the initial boundary conditions for the hydrodynamical equation to describe the subsequent expansion of the cloud. In other words the initial boundary conditions ought to be accepted as partial reflection of high energy interactions in the initial cloud.

The total energy E of a local system with volume V is

$$E = \frac{6.49}{2\pi^2} T^4 V,$$

while the mean square deviation is

$$\Delta E = \frac{2\sqrt{3}}{\pi} T^{5/2} V^{1/2}.$$

Thus we get the fractional fluctuation

$$\frac{\Delta E}{E} = \frac{4\pi\sqrt{3}}{6.49} T^{-3/2} V^{-1/2}.$$

For $V = (A/T)^3$, the condition, $\Delta E/E \ll 1$, of small fluctuation becomes

$$A \gg \left(\frac{4\pi\sqrt{3}}{6.49} \right)^{3/2} \simeq 1.$$

This is automatically satisfied by the local system whose size is much larger than ξ_0 . The condition $\Delta E/E \ll 1$ is nothing but that considered by Blokhintsev.

In the course of expansion, (4.15) will be satisfied as the system spreads. There we must examine the additional condition, that is, the second of (4.14a). From (4.1a'), (4.9), (4.11), (4.12) and (4.13), one can easily obtain the characteristic length x_0 for the macroscopic change of T as follows;

$$x_0 = \frac{15\sqrt{\alpha}}{1-\alpha} \frac{1}{T_0^2} \left(\frac{T_0}{T} \right)^{\frac{3(\alpha+1)}{2(1+\alpha-\sqrt{\alpha})}} \quad (\text{non-trivial region}), \quad (4.17a)$$

$$= \frac{5}{T_0^2} \left(\frac{T_0}{T} \right)^{\frac{V\sqrt{3}(\beta+1)}{\beta-1}} \quad (\text{simple wave region}). \quad (4.17b)$$

Thus we get the condition, $x_0 \gg \xi_0$, in the forms

$$\frac{x_0}{\xi_0} = \frac{15\sqrt{\alpha}}{(1-\alpha)a'T_0} \left(\frac{T_0}{T} \right)^{\frac{1-\alpha+2\sqrt{\alpha}}{2(1+\alpha-\sqrt{\alpha})}} \gg 1 \quad (\text{non-trivial region}), \quad (4.18a)$$

$$= \frac{5}{a'T_0} \left(\frac{T_0}{T} \right)^{\beta \frac{(V\sqrt{3}+1)+(V\sqrt{3}-1)}{1-\beta}} \gg 1 \quad (\text{simple wave region}). \quad (4.18b)$$

We first discuss the condition (4.18a) in the non-trivial region. The upper limit T_1 of temperature allowed by (3.18a) is expressed by

$$\left(\frac{T_1}{T_0} \right) = \left[\frac{15\sqrt{\alpha}}{(1-\alpha)a'T_0} \right]^{\frac{2(1+\alpha-\sqrt{\alpha})}{1+\alpha+2\sqrt{\alpha}}}. \quad (4.19)$$

The power in the right-hand side of (4.19) is always a positive number less than 1, so that T_1 increases with increasing T_0 (increasing incident energy). In fact, (4.19) becomes, for several values of α ,

$$\left. \begin{aligned} T_1/T_0 &= \infty & \text{for } \alpha &= 1, \\ T_1/T_0 &= [51.96/a'T_0]^{0.508} & \text{for } \alpha &= \frac{3}{4}, \\ T_1/T_0 &= [10/a'T_0]^{0.667} & \text{for } \alpha &= \frac{1}{4}, \\ T_1/T_0 &= [5.62/a'T_0]^{0.875} & \text{for } \alpha &= \frac{1}{9}. \end{aligned} \right\} \quad (4.20)$$

From this we can see that T_1 decreases as the observation point moves from the central part ($\alpha=1$) to the front ($\alpha \simeq 0$) in the non-trivial region. But, even at the front ($\alpha \simeq 0$), one can always find a value of T smaller than T_1 . This fact may guarantee the validity, in the non-trivial region, of the hydrodynamical description applied to the expansion of the meson cloud produced in extremely high energy collisions.

The condition $x_0 \gg \xi_0$ becomes more severe in the simple wave region. Since the power of (T_0/T) in the right-hand side of (4.18b) is not always positive, we must write (4.18b) as

$$\left. \begin{aligned} \left(\frac{5}{a'T_0} \right)^{\frac{1-\beta}{\beta(\sqrt{3}+1)+(\sqrt{3}-1)}} &\geq \frac{T}{T_0} \text{ for } 7-4\sqrt{3} > \beta > -2+\sqrt{3}, \\ \left(\frac{5}{a'T_0} \right) &\geq 1 \quad \text{for } \beta = -2+\sqrt{3}, \\ \left(\frac{5}{a'T_0} \right)^{\frac{\beta-1}{\beta(\sqrt{3}+1)+(\sqrt{3}-1)}} &\leq \frac{T}{T_0} \text{ for } -2+\sqrt{3} > \beta, \end{aligned} \right\} \quad (4.21)$$

where all the powers are positive. Although we may find a value of T allowed by the first condition of (4.21) for the range $(7-4\sqrt{3} > \beta > -2+\sqrt{3})$, one never finds T satisfying the last condition of (4.21) for the range $(\beta \leq -2+\sqrt{3})$, for very high values of T_0 . Consequently, it is concluded that the hydrodynamical description of the meson cloud breaks down in the neighbourhood of the wave front.

4.2. Local equilibrium

Here we examine the condition $t_0 \gg \tau_0$ (4.14b) for local equilibrium. The examination is quite similar to discussions given in the preceding subsection for the condition $x_0 \gg \xi_0$. The characteristic time t_0 defined by (4.1b') is given by the formula

$$t_0 = \frac{15\sqrt{\alpha}}{(4\sqrt{\alpha}-\alpha-1)T_0^2} \left(\frac{T_0}{T} \right)^{\frac{3(\alpha+1)}{2(1+\alpha-\sqrt{\alpha})}} \quad (\text{non-trivial region}), \quad (4.22a)$$

$$= \frac{5\sqrt{3}}{T_0^2} \left(\frac{T_0}{T} \right)^{\frac{\sqrt{3}(\beta+1)}{1-\beta}} \quad (\text{simple wave region}). \quad (4.22b)$$

Thus the condition (4.14b) becomes

$$\frac{t_0}{\tau_0} = \frac{15\sqrt{\alpha}}{(4\sqrt{\alpha}-\alpha-1)aT_0} \left(\frac{T_0}{T} \right)^{\frac{1+\alpha+2\sqrt{\alpha}}{2(1+\alpha-\sqrt{\alpha})}} \quad (\text{non-trivial region}), \quad (4.23a)$$

$$= \frac{5\sqrt{3}}{aT_0} \left(\frac{T_0}{T} \right)^{\frac{(\sqrt{3}+1)\beta+(\sqrt{3}-1)}{1-\beta}} \quad (\text{simple wave region}). \quad (4.23b)$$

As is easily seen, the temperature dependence of (t_0/τ_0) is just the same as that of x_0/ξ_0 , that is,

$$\frac{t_0}{x_0} = \left(\frac{\tau_0}{\xi_0} \right) \left(\frac{1-\alpha}{4\sqrt{\alpha-\alpha-1}} \right) \quad (\text{non-trivial region}), \quad (4.24a)^*$$

$$= \left(\frac{\tau_0}{\xi_0} \right) \sqrt{3} \quad (\text{simple wave region}). \quad (4.24b)^*$$

This means that the condition $t_0 \gg \tau_0$ can hardly be satisfied in the same region in which the hydrodynamical description has already broken down due to the condition $x_0 \gg \xi_0$. Since the physical content of the condition $t_0 \gg \tau_0$ is that particles in a local system are in local equilibrium, the front particles free from the condition $t_0 \gg \tau_0$ are, of course, not in thermal equilibrium. Consequently, it is concluded that the front particles remember the high energy initial interactions in a very small region and that the distributions of the front particles give us some knowledge about interactions at very small distances.

To illustrate the α -dependence of the condition $t_0 \gg \tau_0$ in the nontrivial region, it is convenient to define the upper limit T_1' of T allowed from (4.23) by

$$\frac{T_1'}{T_0} = \left[\frac{15\sqrt{\alpha}}{(4\sqrt{\alpha-\alpha-1})aT_0} \right]^{\frac{2(1+\alpha-V\alpha)}{1+\alpha+2V\alpha}}. \quad (4.25)$$

For several values of α , one gets

$$\left. \begin{aligned} T_1'/T_0 &= (7.5/aT_0)^{0.5} & \text{for } \alpha=1, \\ T_1'/T_0 &= (7.6/aT_0)^{0.508} & \text{for } \alpha=\frac{3}{4}, \\ T_1'/T_0 &= (10/aT_0)^{0.667} & \text{for } \alpha=\frac{1}{4}, \\ T_1'/T_0 &= (22.5/aT_0)^{0.875} & \text{for } \alpha=\frac{1}{9}. \end{aligned} \right\} \quad (4.26)$$

Although the α dependence of T_1' is inverse to that of T_1 (see (4.20)), the validity of the hydrodynamical description is not altered in the non-trivial region.

The Table I contains the values of (T_1'/T_0) in the non-trivial region for $a=1$, 0.5 and for $T_0=10 \sim 100$. In the simple wave region, one gets, for reasonable values of a , the upper limits of temperature:

Table. I

		$a=1$			$a=0.5$			
E_{lab}	T_0	α	1	1/4	1/9	1	1/4	1/9
10^{12} ev	10		0.87	1	>1	>1	>1	>1
10^{14} ev	25		0.55	0.54	0.91	0.78	0.87	>1
10^{15} ev	50		0.34	0.34	0.50	0.55	0.55	0.91
10^{16} ev	100		0.27	0.22	0.27	0.34	0.34	0.50

* Note that $(\tau_0/\xi_0) = (a/a') \simeq 1$.

$$\begin{aligned}
 &\text{if } a=1, \quad T=8.66 \quad \text{for } \beta=7-4\sqrt{3}, \\
 &\quad \quad \quad T_0=8.66 \quad \text{for } \beta=-2+\sqrt{3}, \\
 &\text{if } a=0.5, \quad T=17.32 \quad \text{for } \beta=7-4\sqrt{3}, \\
 &\quad \quad \quad T_0=17.32 \quad \text{for } \beta=-2+\sqrt{3},
 \end{aligned}$$

4.3. Assumption of the perfect fluid

It is the purpose of this subsection to examine the assumption of the perfect fluid used by Landau. To do this, we must estimate the Reynolds number R and the number K defined by (4.8) and (4.9), respectively.

By using the formulas⁵⁾

$$\varepsilon = \frac{6.49}{2\pi^2} T^4 \quad (4.27)$$

for the invariant energy density ε and (3.25) for $\eta_{(v)}$, one easily obtains the formula

$$R = \frac{3.38}{aT_0} \left(\frac{T_0}{T} \right)^{\frac{1+\alpha+2\sqrt{\alpha}}{2(1+\alpha-\sqrt{\alpha})}} \quad (4.28)^*$$

for the Reynolds number in the non-trivial region. For several values of α , (4.28) becomes

$$\begin{aligned}
 R &= \frac{3.38}{aT_0} \left(\frac{T_0}{T} \right)^2 \quad \text{for } \alpha=1, \\
 &= \frac{3.38}{aT_0} \left(\frac{T_0}{T} \right)^{1.97} \quad \text{for } \alpha=\frac{3}{4}, \\
 &= \frac{3.38}{aT_0} \left(\frac{T_0}{T} \right)^{1.5} \quad \text{for } \alpha=\frac{1}{4}, \\
 &= \frac{3.38}{aT_0} \left(\frac{T_0}{T} \right)^{1.14} \quad \text{for } \alpha=\frac{1}{9}.
 \end{aligned} \quad (4.29)$$

The formula (4.28) is closely related with the ratio (t_0/τ_0) as follows;

$$R = 10.14 \left(\frac{4\sqrt{\alpha} - \alpha - 1}{15\sqrt{\alpha}} \right) \left(\frac{t_0}{\tau_0} \right), \quad (4.30)$$

where the coefficient of (t_0/τ_0) varies from 0.5 to 0.2 as α decreases from 1 to $\frac{1}{9}$. In the simple wave region we have the formula

$$R = 0.45 \left(\frac{t_0}{\tau_0} \right) \quad (4.31)$$

for the Reynolds number. In every region the condition $R \gg 1$ is somewhat more

* It is to be noted that this formula is obtained under the assumption $T \gg m$.

severe than the condition $(t_0/\tau_0) \gg 1$. In fact, we obtain the value $R=1 \sim 10$, in the non-trivial region, for the values $T_0 \simeq 100$ and $(T_0/T) \simeq 10$. In the simple wave region, R becomes smaller than in the non-trivial region. Thus we may infer that the assumption of the perfect fluid is not so good as supposed by Landau.

In the non-trivial region, it is easily proved that

$$DU_\mu = 0.$$

Hence one gets the formula

$$K = \varepsilon x_0 / \kappa = 2.43 (\dot{\xi}_0 / \tau_0) (x_0 / \dot{\xi}_0), \quad (4.32)$$

where (4.1a), (3.24) and (4.27) have been used. Apart from the factor $(\dot{\xi}_0 / \tau_0)$ of order 1, the examination of the condition $K \gg 1$ is almost that of the condition $(x_0 / \dot{\xi}_0) \gg 1$. Although we may have somewhat larger values of K than R , it is hardly acceptable to neglect the irreversible process due to heat conduction. In the simple wave region, we can show that the heat flow vanishes, that is,

$$q_\mu = -\kappa \left(\frac{1}{T} \Delta_{\mu\nu} \nabla_\nu T + DU_\mu \right) = 0.$$

This means that the motion of the fluid is adiabatic. Thus we obtain an infinite K , but it is, of course, impossible to regard the meson cloud as a perfect fluid due to the smallness of R .

4.4. Production of entropy

In the preceding subsection, we have obtained rather small values for R and K , so that entropy must be produced by the final interactions described by the hydrodynamical equation. Such a production of entropy results in an increment of the number of particles produced in the final interactions. Here we shall calculate only the production of entropy in the non-trivial region.

Now the thermodynamical equation of the entropy balance is expressed in the form

$$\nabla_\mu S_\mu^{(irr)} = \frac{1}{T} \left[\kappa \left(\frac{1}{T} \Delta_{\mu\nu} \nabla_\nu T + DU_\mu \right)^2 + \eta_{(v)} (\nabla_\alpha U_\alpha)^2 \right], \quad (4.33)$$

where $S_\mu^{(irr)}$ is the entropy density four-current produced in irreversible processes. Denoting the total produced entropy by Σ_{irr} , we get

$$\begin{aligned} \Sigma_{irr} &= \iint (\nabla_\mu S_\mu^{(irr)}) d^3x dt \\ &= \iint (\nabla_\mu S_\mu^{(irr)}) dx dt. \end{aligned} \quad (4.32)^*$$

Dividing Σ_{irr} into the heat part $\Sigma_{irr}^{(h)}$ and the viscosity part $\Sigma_{irr}^{(v)}$, and transforming

* Note that $\int d^3x \dots = \left(\frac{1}{m} \right)^2 \int d\dot{x} \dots$ in the one-dimensional motion of the fluid and $m=1$.

the integration variable x and t to another pair of variables T and α , we have the integrals

$$\Sigma_{\text{irr}}^{(h)} = \iint \frac{1}{T} \kappa \left(\frac{1}{T} \mathcal{A}_{\mu\nu} \mathcal{P}_\nu T + D U_\mu \right)^2 \frac{\partial(x, t)}{\partial(T, \alpha)} dT d\alpha, \quad (4.33a)$$

$$\Sigma_{\text{irr}}^{(v)} = \iint \frac{1}{T} \eta_{(v)} (\mathcal{P}_\alpha U_\alpha)^2 \frac{\partial(x, t)}{\partial(T, \alpha)} dT d\alpha, \quad (4.33b)$$

where $\partial(x, t)/\partial(T, \alpha)$ is the Jacobian of the transformation from (x, t) to (T, α) and is given by

$$\frac{\partial(x, t)}{\partial(T, \alpha)} = \frac{9\mathcal{A}^2}{2T_0} (1 + \alpha - \sqrt{\alpha})^{-2} \left(\frac{T}{T_0} \right)^{-1 - \frac{3(1+\alpha)}{1+\alpha-\sqrt{\alpha}}} \ln \left(\frac{T_0}{T} \right). \quad (4.34)$$

The intervals of the integrations are the range from 1 to $\alpha_0 (\simeq 0)$ for α and the range from the initial value T_i to the final one T_f for T . We can use $\alpha_0 \simeq \beta_0 = 7 - 4\sqrt{3} = 0.072$ to a good approximation, because the equations defining α and β approach each other in the front region as time goes on. $T_f = 1$ will be used as an extrapolation, while the upper limit* T_1' of temperature, allowed by the condition $t_0 \gg \tau_0$, will be identified with T_i . (Here we disregard the slight dependence of T_1' on α .)

In the non-trivial region, we have the formulas

$$\left. \begin{aligned} \left(\frac{1}{T} \mathcal{A}_{\mu\nu} \mathcal{P}_\nu T + D U_\mu \right)^2 &= x_0^{-2}, \\ \mathcal{P}_\alpha U_\alpha &= \frac{1}{\mathcal{A}} \left(\frac{T}{T_0} \right)^{\frac{3(\alpha+1)}{2(1+\alpha-\sqrt{\alpha})}}. \end{aligned} \right\} \quad (4.35)$$

Substituting (3.24), (3.25), (4.34), (4.35) and (4.17a) into (4.33a) and (4.33b), one obtains

$$\Sigma_{\text{irr}}^{(h)} \simeq \frac{1.14 a T_0^2}{\pi^2} \left(\frac{T_i}{T_0} \right)^2 \left[1 + \ln \left(\frac{T_0}{T_i} \right)^2 \right], \quad (4.36a)$$

$$\Sigma_{\text{irr}}^{(v)} \simeq \frac{3.96 a T_0^2}{\pi^2} \left(\frac{T_i}{T_0} \right)^2 \left[1 + \ln \left(\frac{T_0}{T_i} \right)^2 \right]. \quad (4.36b)$$

It is important that, roughly speaking, the entropy produced in irreversible processes is proportional to T_i^2 , or T_0^2 for constant (T_i/T_0) . Because the total entropy given by Fermi or Landau is proportional to T_0 , it is possible that Σ_{irr} exceeds the original entropy of the fluid part of the meson cloud. The entropy production means that the initial energy of the fluid part dissipates into new degrees of freedom, in other words, into new produced particles. Consequently we may expect an increment in the number of particles due to the final interactions.

* The choice $T_i = T_1'$ is not significant unless $T_1' \leq T_0$. When $T_1' \geq T_0$, one should use T_0 as T_i .

4.5. Equation of state

One of the most important assumptions used by Landau is that the equation of state can be put in the form

$$3p = \varepsilon. \quad (4.37)$$

As has been shown in I, the form of the equation holds exactly only when particles in meson cloud interact with each other through the first kind of interactions, i. e., interactions having dimensionless coupling constants. The second kind* of interactions does not necessarily lead to the equation $3p = \varepsilon$, but, in general, to the equation

$$3p = \varepsilon + \lambda \langle \Xi \rangle, \quad (4.38)$$

where $\lambda \Xi$ is the interaction Hamiltonian apart from a numerical factor. Here, suppose that λ is the coupling constant with the dimension $[L']$. Then the additional term $\lambda \langle \Xi \rangle$ becomes

$$\lambda \langle \Xi \rangle = \begin{cases} \varepsilon \lambda T' \times (\text{the power series of } (\lambda T')) & \text{for } l = \text{even}, \\ \varepsilon \lambda^2 T'^2 \times (\text{the power series of } (\lambda^2 T'^2)) & \text{for } l = \text{odd} \end{cases} \quad (4.39)$$

at extremely high temperatures, from the view-point of perturbation theory. At first sight we feel as if the violation of the equation $3p = \varepsilon$ is serious. However, there is a possibility that the effects of the above power series may appear as damping effects. It is very plausible, though it is difficult to derive a definite conclusion from the exact calculations. If so, (4.39) may be reduced to

$$\lambda \langle \Xi \rangle = \begin{cases} \frac{\varepsilon A (\lambda T')}{1 + B (\lambda T')} & \text{for } l = \text{even}, \\ \frac{\varepsilon A' (\lambda^2 T'^2)}{1 + B' (\lambda^2 T'^2)} & \text{for } l = \text{odd}, \end{cases} \quad (4.40)$$

where A , A' , B and B' are numerical factors of the order 1. The interaction part of ε would have the same dependence on T . Thus, since $\lambda \langle \Xi \rangle$ attains the same T -dependence as ε , that is,

$$\lambda \langle \Xi \rangle \propto \varepsilon$$

at extremely high temperatures, the equation of state (4.38) has the form

$$3p = (1 + C) \varepsilon \quad (4.41)$$

* In I the authors talked as if there were always essential differences between derivative coupling and non-derivative coupling in the second kind of interactions. This is not necessarily so, because such differences would vanish at extremely high energies. The differences appear in the case of moderate temperatures and moderate densities.

Furthermore it may be noted that the additional term in (4.38) vanishes exactly in the case of pv-coupling of the nucleon-neutral meson system. This is a direct result due to the equivalence theorem.

Here C is a constant factor of the order 1. Consequently, this modification of the equation of state would result in some changes of the discussions about the hydrodynamical motion of the fluid, for example, the change of the sound velocity and the different power law of ε from T^4 .

§ 5. Concluding remarks

In the last section, we have investigated the consistency of Landau's model applied to the meson cloud which is produced in high energy nucleon-nucleon collision, with the applicability conditions derived from the statistical mechanics of irreversible processes. The results obtained are summarized in the following way: (i) The interactions in the initial cloud can not be described by any hydrodynamical equation. The interactions are, in part, reflected in the initial boundary condition for the subsequent hydrodynamical expansion. (Such a boundary condition might be different from that assumed by Landau.) (ii) After the cloud spreads over a region whose size is the order of the correlation length, the hydrodynamical description of the cloud is valid almost everywhere except in the front part of the cloud. Here it must be emphasized that the front particles are never in thermal equilibrium and consequently they remember the initial interactions in the very small region. The front particles would be subject to quite different distribution laws from those of the fluid particles, which are given by hydrodynamics or statistical mechanics. (iii) The assumption of the perfect fluid is not so good as expected by Landau, so that the number of particles increases as a result of the irreversible motion in the fluid part of the cloud. (iv) The equation of state, $3p=\varepsilon$, holds exactly if the interactions are the first kind (having dimensionless coupling constant), while it is necessary to modify this equation as $3p=\text{const.} \times \varepsilon$ for the second kind of interactions.

Although the above analysis is based on the solutions obtained by using Landau's assumption, the above conclusions can be applied to interpret the qualitative behavior of the meson clouds in question. Consequently, we are inclined to imagine the situation for the multiple production of particles in such a way that the produced particles will be clearly divided into two parts, one of which is the very high energy particles occupying the front part and the other of which is the fluid particles. These two parts will also be separated from each other in the experimental data, because there must be clear-cut differences between the two parts in the distributions of the particle number (for example, $K\text{-}\pi$ ratio), of the momentum and of other quantities. Particularly, the existence of irreversible processes results in the slowing-down of the speed and the increment of the number of the fluid particles. In other words, the irreversible processes strengthen the tendency to separate the above two parts from each other.

It seems that such considerations are consistent with the recent experiments⁶⁾ of cosmic ray performed by the Japanese group and the Bristol group. In these experimental data one can find that the energy spectrum of the γ -ray number

obtained from high energy jets ($\simeq 10^{14}$ e.v.) falls rapidly down and is cut off at the energies 10^{12} e. v. $\sim 10^{13}$ e.v.. The γ -rays in question are produced in the decay of neutral pions, so that the energy spectrum of neutral pions has perhaps the same form as that of the γ -rays. Thus, it may be plausible to regard these neutral pions as the fluid particles in our imagination mentioned above. It seems that similar evidence is found in the observation of muons in cosmic rays. The gradient of the number-energy curve changes critically from a large value to a somewhat small value at a definite energy. The muons with energies below this critical value may be the fluid particles, while the muons with energies higher than the critical value may be considered to be the front particles. Although the experimental evidence is not yet established, our considerations may play a role in suggesting how to analyse the extremely high energy phenomena. The group of particles with higher energies would inform us about the interactions at very short distances.

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Appendix A. Correlation length and mass flow

There are correlation lengths associated with the various quantities, such as the momentum density, the stress density and so on. As an example, we shall obtain the correlation length associated with the momentum density. This correlation length is defined by the width of the non-vanishing region of the function

$$\mathcal{G}(\mathbf{x}; \mathbf{x}') = \langle \{g_i(\mathbf{x}, 0), g_i(\mathbf{x}', 0)\} \rangle_0, \quad (\text{A} \cdot 1)$$

where $g_i(\mathbf{x}, 0)$ is a component of the momentum density operator (3.4). Here $\phi(\mathbf{x}, 0)$ is expressed by the following Fourier transform;

$$\phi(\mathbf{x}, 0) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 \mathbf{k}}{(2\sqrt{k^2 + m^2})^{1/2}} (a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} + a_{\mathbf{k}}^* e^{-i\mathbf{k} \cdot \mathbf{x}}), \quad (\text{A} \cdot 2)$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^*$ are the well-known annihilation and creation operators. Substituting (A.2) into (A.1) and using

$$\left. \begin{aligned} \langle a_{\mathbf{k}}^* a_{\mathbf{k}'} \rangle &= n_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}'), \\ \langle a_{\mathbf{k}} a_{\mathbf{k}'}^* \rangle &= (n_{\mathbf{k}} + 1) \delta(\mathbf{k} - \mathbf{k}'), \end{aligned} \right\} \quad (\text{A} \cdot 3)$$

the function $\mathcal{G}(\mathbf{x}, \mathbf{x}')$ is written as a sum of products of the following functions or their derivatives:

$$\begin{aligned} A(\mathbf{x} - \mathbf{x}') &= \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}}{\sqrt{k^2 + m^2}} (n_{\mathbf{k}} + \frac{1}{2}) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}, \\ B(\mathbf{x} - \mathbf{x}') &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{k} \sqrt{k^2 + m^2} (n_{\mathbf{k}} + \frac{1}{2}) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}, \end{aligned} \quad (\text{A} \cdot 4)$$

where

$$n_k = \frac{1}{e^{1/\sqrt{k^2 + m^2}/T} - 1}. \quad (\text{A}\cdot 5)$$

Hence \mathcal{G} consists of products of two integrals involving n_k , and products of an integral containing n_k in its integrand and an n_k -independent integral. The product of two integrals without n_k is cancelled out by the vacuum term. When $T \gg m$, the spatial variation of \mathcal{G} is mainly determined by the integral containing n_k , which vanishes unless $|\mathbf{x} - \mathbf{x}'| \lesssim (1/T)$. This means that the correlation length is the order of $(1/T)$. Thus we shall write the correlation length ξ_0 in the form

$$\xi_0 = a'/T, \quad a' \text{ being of the order 1.} \quad (\text{A}\cdot 6)$$

It is easily found that the correlation lengths associated with the other quantities are of the same order ξ_0 .

As is mentioned in § 1 and § 2, the hydrodynamical description is to be considered as the asymptotic form in which each region of the order ξ_0 is regarded as a point in the fluid. Thus, we can treat the local system, whose size is of the order ξ_0 , as if it moves like a mass point. This fact permits us to define a meaningful mass flow. In fact, the criterion formulated in the footnote on page 599 in I is satisfied in our case, because the right-hand side of this equation is proportional to the delta function $\delta^{(4)}(x - x')$, when x and x' are space-like, as far as each interval of the order ξ_0 is regarded as a point. Hence the mass flow has a definite meaning, so that the local velocity can be constructed from the mass flow as in I.

Appendix B. Green's function of one meson in a medium

Here we shall discuss briefly the Green's function of one meson in a medium. The Green's function $G(x, x')$ of one meson is defined by

$$G(x, x') = T_r \{ \rho T(\phi(x), \phi(x')) \} \quad (\text{A}\cdot 7)$$

in a medium represented by the density matrix ρ . The function G contains the one meson propagator in vacuum as a contribution of the vacuum term of ρ . If we introduce artificially an external source $J(x)$ of mesons into the Hamiltonian, $\phi(x)$ obeys the field equation

$$(\square - m^2)\phi(x) = J(x) + ig_s \bar{\psi}(x) \gamma_5 \psi(x) \quad (\text{A}\cdot 8)$$

for the ps -coupling system of meson and nucleon. With the help of $J(x)$, the function $G(x, x')$ can also be defined by

$$G(x, x') = \lim_{J \rightarrow 0} \frac{\delta \langle \phi(x) \rangle}{\delta J(x')}. \quad (\text{A}\cdot 9)$$

Thus the function $G(x, x')$ obeys the equation

$$(\square - m^2)G(x, x') = \delta(x - x') + \lim_{J \rightarrow 0} ig_s \frac{\delta}{\delta J(x)} \text{tr}(\gamma_5 K(x, x')),$$

where tr stands for the trace with respect to Dirac's indices and $K(x, x')$ is the one nucleon Green's function in a medium defined by

$$K(x, x') = T_r \{ \rho(T\phi(x)\bar{\phi}(x)) \}. \quad (\text{A} \cdot 10)$$

K contains also the one nucleon propagator in vacuum. If we assume the existence of the inverse function $K^{-1}(x, x')$ of $K(x, x')$, the last term can be rewritten as

$$\lim_{J \rightarrow 0} ig_s \frac{\delta}{\delta J(x')} \text{tr}(\gamma_5 K(x, x')) = \int \Pi(x, x'') G(x'', x') d^4 x'',$$

where

$$\Pi(x, x'') = -ig_s \text{tr} \left(\int \gamma_5 K(x, \xi) \Gamma_5(\xi, \eta; x'') K(\eta, x) d^4 \xi d^4 \eta \right). \quad (\text{A} \cdot 11)$$

Here $\Gamma_5(\xi, \eta; x'')$ is the vertex part in a medium and is defined by

$$\Gamma_5(\xi, \eta; x'') = \lim_{J \rightarrow 0} \frac{\delta K^{-1}(\xi, \eta)}{\delta \langle \phi(x'') \rangle}. \quad (\text{A} \cdot 12)$$

The c -number operator (A.11) is nothing but Π , which appeared in (2.4).

As has been seen in the above formulation, the calculation of Π can be conducted in a way quite similar to the vacuum field theory. Of course, Π contains the vacuum self-energy of one meson, to be ascribed to renormalization. However, Π has additional terms which correspond to the effective mass in a medium and to the imaginary part representing dissipation. The calculations in perturbation theory lead to the value (2.9) to the lowest order. In the case of pv -coupling, we can formulate the theory as mentioned here.

Appendix C. Relaxation times and transport coefficients

Here we shall interpret in detail the calculations of the relaxation times and the transport coefficients. For example, we calculate the heat conductivity and the relaxation times associated with it.

The heat conductivity κ is obtained from the formula

$$\kappa = \frac{1}{VT} \langle [\phi_1(0)]^2 \rangle_0 \tau_0, \quad (\text{A} \cdot 13)$$

where the relaxation time τ_0 (τ or σ) is defined by

$$\frac{1}{\sigma^2} = \frac{2}{\pi} \frac{-\langle [\phi_1(0), \mathcal{H}_1(0)]^2 \rangle_0}{\langle [\phi_1(0)]^2 \rangle_0} \quad (\text{Gaussian}), \quad (\text{A} \cdot 14)$$

or

$$\frac{1}{\tau} = \sqrt{\frac{\pi}{2}} \frac{\langle \{ [\phi_1(0), \bar{\mathcal{H}}_1(0)], [\mathcal{H}_1(0), \phi_1(0)] \} \rangle_0}{\langle [\phi_1(0)]^2 \rangle_0} \quad (\text{Lorentzian}). \quad (\text{A} \cdot 15)$$

Here $\bar{\mathcal{H}}_1(\omega)$ is the Fourier transform of $\mathcal{H}_1(t)$. From the definition (2.12) and one under (3.5), we get

$$\phi_1(0) = \int \frac{d^3 \mathbf{k}}{2} [a_{\mathbf{k}} a_{\mathbf{k}}^* + a_{\mathbf{k}}^* a_{\mathbf{k}}] k_1, \quad (\text{A} \cdot 16a)$$

$$\begin{aligned} \mathcal{H}_1(0) &= \frac{\lambda \langle \phi^2 \rangle_0}{(2\pi)^{3/2}} \iiint \frac{d^3 \mathbf{k}}{\sqrt{2\varepsilon_{\mathbf{k}}}} \frac{d^3 \mathbf{k}'}{\sqrt{2\varepsilon_{\mathbf{k}'}}} d\omega \\ &\times [a_{\mathbf{k}} a_{\mathbf{k}'} \chi^*(\mathbf{k} + \mathbf{k}', \omega) + a_{\mathbf{k}}^* a_{\mathbf{k}'}^* \chi^*(-\mathbf{k} - \mathbf{k}', \omega) \\ &+ a_{\mathbf{k}} a_{\mathbf{k}'}^* \chi^*(\mathbf{k} - \mathbf{k}', \omega) + a_{\mathbf{k}}^* a_{\mathbf{k}'} \chi^*(\mathbf{k}' - \mathbf{k}, \omega)], \end{aligned} \quad (\text{A} \cdot 16b)$$

$$\begin{aligned} \bar{\mathcal{H}}_1(0) &= \frac{\lambda \langle \phi^2 \rangle_0}{(2\pi)^2} \iint \frac{d^3 \mathbf{k}}{\sqrt{2\varepsilon_{\mathbf{k}}}} \frac{d^3 \mathbf{k}'}{\sqrt{2\varepsilon_{\mathbf{k}'}}} \\ &\times [a_{\mathbf{k}} a_{\mathbf{k}'} \chi(-\mathbf{k} - \mathbf{k}', -\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'}) + a_{\mathbf{k}}^* a_{\mathbf{k}'}^* \chi(\mathbf{k} + \mathbf{k}', \varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}'}) \\ &+ a_{\mathbf{k}} a_{\mathbf{k}'}^* \chi(\mathbf{k}' - \mathbf{k}, \varepsilon_{\mathbf{k}'} - \varepsilon_{\mathbf{k}}) + a_{\mathbf{k}}^* a_{\mathbf{k}'} \chi(\mathbf{k} - \mathbf{k}', \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'})], \end{aligned} \quad (\text{A} \cdot 16c)$$

where $\varepsilon_{\mathbf{k}} = \sqrt{k^2 + m^2}$. In what follows, we use the approximation $T \gg m$.

The mean square deviation $\langle [\phi_1(0)]^2 \rangle_0$ is easily obtained in the form

$$\langle [\phi_1(0)]^2 \rangle_0 \simeq \frac{4}{\pi^2} T^5 V, \quad (\text{A} \cdot 17)$$

where the relation $[\delta(\mathbf{k})]_{\mathbf{k}=0} = (V/(2\pi)^3)$ has been used. After some calculations, we get

$$-\langle [\phi_1(0), \mathcal{H}_1(0)]^2 \rangle_0 \simeq \frac{20}{\pi^3} (\lambda T^2)^2 T^7 V. \quad (\text{A} \cdot 18)$$

Thus we obtain the Gaussian relaxation time

$$\sigma \simeq \frac{\pi}{\sqrt{10}(\lambda T^2)} \cdot \frac{1}{T}. \quad (\text{A} \cdot 19)$$

Similar calculations lead to

$$\langle \{ [\phi_1(0), \bar{\mathcal{H}}_1(0)] [\mathcal{H}_1(0), \phi_1(0)] \} \rangle_0 \simeq \sqrt{\frac{2}{\pi}} \frac{20(\lambda T^2)^2}{\pi^2} T^8 V, \quad (\text{A} \cdot 20)$$

so that one gets

$$\tau = \frac{1}{5(\lambda T^2)^2} \cdot \frac{1}{T}. \quad (\text{A} \cdot 21)$$

The heat conductivity κ is easily obtained from (A.13) and (A.19) or (A.21).

The calculations of $\eta_{(s)}$ and $\eta_{(v)}$ are quite similar to the above. We write only the mean square deviations of the stress, that is,

$$\left. \begin{aligned} \langle [\mathcal{J}_{12}(0)]^2 \rangle_0 &\simeq \frac{4}{5\pi^2} T^5 V, \\ \langle [\mathcal{J}_{11}(0)]^2 \rangle_0 &\simeq \frac{12}{5\pi^2} T^5 V. \end{aligned} \right\} \quad (\text{A} \cdot 22)$$

Here it is noted that the quantity $\mathcal{J}_{ik}(0)$ is the fluctuating part of the stress.

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A Note on the Electromagnetic Response of Superconductors

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Based on the general formalism developed previously by the present author, it is shown how diamagnetism, longitudinal, and transverse conductivities of a superconductor are related with each other. In this way one can see the interrelation existing among recent works of Mattis and Bardeen, of Blatt and Matsubara, and of Tinkham and Ferrell. It is found that the infinite static conductivity still remains to be an unsolved problem.

§ 1. Introduction

Applying Kubo's formalism¹⁾, the present author²⁾ has obtained a general expression for the linear response of the system of conduction electrons exposed to a space and time dependent electromagnetic disturbance. We assume that at the remote past the system is field-free and in thermal equilibrium represented by the statistical operator $U_0 = \exp(-\beta H) / \text{Tr}(\exp(-\beta H))$. Note that H is the Hamiltonian of the field-free system. The field is then switched on and there appears an electric current in the system. The latter is obtained by applying first order perturbation theory to the electromagnetic disturbance. When Fourier analyzed with respect to space coordinates, the average current density was given in the form

$$J_\mu(\mathbf{k}, t) = \left(-\frac{ne^2}{mc} \delta_{\mu\nu} + c S_{\mu\nu}(\mathbf{k}, 0) \right) A_\nu(\mathbf{k}, t) + c^2 \int_{-\infty}^t dt' S_{\mu\nu}(\mathbf{k}, t-t') E_\nu(\mathbf{k}, t'). \quad (1.1)$$

In terms of the current operator \mathbf{j} of the field-free system and its Heisenberg representation, the kernel is expressed as

$$S_{\mu\nu}(\mathbf{k}, t) = \frac{1}{c^2} \int_0^\beta d\lambda \text{Tr} U_0 j_\nu(-\mathbf{k}) j_\mu(\mathbf{k}, t + i\hbar\lambda). \quad (1.2)$$

For simplicity we assume that the field-free system is isotropic so that

$$S_{\mu\nu}(\mathbf{k}, t) = S_0(k, t) \delta_{\mu\nu} + S_2(k, t) \frac{k_\mu k_\nu}{k^2}. \quad (1.3)$$

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The vector potential in (1.1) can be replaced by the magnetic field with use of the Buckingham identity³⁾

$$S_0(k, 0) + S_2(k, 0) = ne^2/mc^2 \quad (1.4)$$

which results from (1.2). We thus obtain

$$J_\mu(\mathbf{k}, t) = -cS_2(k, 0) \left((1/k^2) i\mathbf{k} \times \mathbf{B}(\mathbf{k}, t) \right)_\mu + c^2 \int_{-\infty}^t dt' S_{\mu\nu}(\mathbf{k}, t-t') \mathbf{E}(\mathbf{k}, t'). \quad (1.5)$$

In the previous paper²⁾, the author regarded the first term of (1.5) as the magnetic and the second term as the electric parts of the current. As pointed out recently by Blatt and Matsubara⁴⁾, there is an ambiguity in interpreting (1.5) in such a manner, though any final conclusion of physical significance should not depend upon our particular interpretation. The ambiguity arises from the fact that the transverse part of the electric field is connected with the magnetic field by the Maxwell equation. For instance, remembering Kubo's initial condition that fields vanish at $t = -\infty$, we may express (1.5) entirely in terms of the electric field by making use of

$$\mathbf{B}(\mathbf{k}, t) = -c \int_{-\infty}^t dt' i\mathbf{k} \times \mathbf{E}(\mathbf{k}, t'). \quad (1.6)$$

As we shall see below, the transverse conductivity is defined precisely in this way. Blatt and Matsubara have, however, proceeded another way and arrived at the decomposition

$$\mathbf{J}(\mathbf{k}, t) = \mathbf{J}_e(\mathbf{k}, t) + i c \mathbf{k} \times \mathbf{M}(\mathbf{k}, t). \quad (1.7)$$

Here the electric part is given as

$$\mathbf{J}_e(\mathbf{k}, t) = \int_{-\infty}^t dt' \phi(k, t-t') \mathbf{E}(\mathbf{k}, t') \quad (1.8)$$

with

$$\phi(k, t) = c^2 (S_0(k, t) + S_2(k, t)) \quad (1.9)$$

and the magnetic moment as

$$\mathbf{M}(\mathbf{k}, t) = \frac{1}{k^3} \int_{-\infty}^t dt' \frac{\partial S_2(k, t-t')}{\partial t} \mathbf{B}(\mathbf{k}, t'). \quad (1.10)$$

As Blatt and Matsubara have already noticed, the decomposition (1.7) is not very adequate in the case of a superconductor. In the present note, we shall discuss this point in more detail. In connexion with this, we shall show how the longitudinal and transverse conductivities are defined in the present formalism.

§ 2. The longitudinal conductivity and diamagnetism

Let us first take a purely longitudinal field. We thus assume

$$\mathbf{E}(\mathbf{k}, t) = -i\mathbf{k}\phi(\mathbf{k}) \exp(\alpha t - i\omega t), \quad \mathbf{B} \equiv 0,$$

where the factor $e^{\alpha t}$, $\alpha \rightarrow +0$ is attached to satisfy Kubo's initial condition at $t = -\infty$, then (1.5) gives

$$\mathbf{J}(\mathbf{k}, t) = \sigma_l(k, \omega) \mathbf{E}(\mathbf{k}, t) \quad (2.1)$$

where the longitudinal conductivity is defined by

$$\sigma_l(k, \omega) = \int_0^\infty dt \phi(k, t) \exp(-\alpha t + i\omega t). \quad (2.2)$$

From (1.4) and (1.9), the Fourier inversion of (2.2) leads to the sum rule

$$\frac{2}{\pi} \int_0^\infty d\omega \Re \sigma_l(k, \omega) = \frac{ne^2}{m} \quad (2.3)$$

and also to the Kramers-Kronig dispersion relation.

Second, if we apply equilibrium statistical mechanics to the system in a steady magnetic field, we find the diamagnetic current

$$J_\mu(\mathbf{k}) = \left(-\frac{ne^2}{mc} \delta_{\mu\nu} + c S_{\mu\nu}(\mathbf{k}, 0) \right) A_\nu(\mathbf{k}) \quad (2.4)$$

which is equivalent to the moment

$$\mathbf{M}(\mathbf{k}) = - (1/k^2) S_2(k, 0) \mathbf{B}(\mathbf{k}). \quad (2.5)$$

In the case of a free Fermi gas of electrons, we have the well-known Landau diamagnetism,

$$S_2(k, 0) = \frac{e^2}{mc^2} \left(\frac{k_F}{12\pi^2} \right) k^2 + \dots, \quad (2.6)$$

where k_F is the wave number at the Fermi surface. On the other hand, the superconductor is characterized by

$$S_2(k, 0) = \begin{cases} \frac{ne^2}{mc^2} & (k \text{ small}) \\ \frac{3\pi}{4\xi_0 k} \left(\frac{ne^2}{mc^2} \right) & (k \text{ large}) \end{cases} \quad (2.7)$$

where ξ_0 is the coherence length.

Our time dependent formalism should, of course, lead to the same result (2.4) when we assume the adiabatic switching on of the magnetic field

$$\mathbf{B}(\mathbf{k}, t) = \mathbf{B}(\mathbf{k}) e^{\alpha t}, \quad \mathbf{E}(\mathbf{k}, t) = -(\alpha/c) i\mathbf{k} \times \mathbf{B}(\mathbf{k}, t). \quad (2.8)$$

Let us first examine the decomposition of Blatt and Matsubara, (1.7). The magnetic part (1.10) gives (2.5) if

$$\lim_{t \rightarrow \infty} S_2(k, t) = 0 \quad (2.9)$$

as can be seen by inserting (2.8) into (1.10). The electric part of (1.7) is proportional to

$$\lim_{\alpha \rightarrow +0} \alpha \int_0^{\infty} dt e^{-\alpha t} \phi(k, t).$$

This vanishes for normal metals, but does not in the case of a superconductor where we have an infinite longitudinal conductivity in the static limit. In other words, the electric part of Blatt and Matsubara does make a contribution to the current in the superconductor placed in a steady magnetic field for a long time. This is why their decomposition is not very adequate in discussing superconductors.

Let us therefore go back to the original expression (1.1) and insert (2.8) directly in this expression. Since the first term is already in the form of (2.4), we need only to assume that the second term vanishes, i.e.,

$$\int_0^{\infty} dt S_0(k, t) = \text{finite.} \quad (2.10)$$

Note that the condition is satisfied even in the case of the free Fermi gas of electrons as far as $k \neq 0$.

From the definition of the longitudinal conductivity (2.2) with (1.9), we have now to attribute the infinite longitudinal conductivity in the static limit to a divergent behaviour of $S_2(k, t)$ at $t = \infty$;

$$\lim_{\alpha \rightarrow +0} \int_0^{\infty} dt e^{-\alpha t} S_2(k, t) = \infty. \quad (2.11)$$

This is in contrast to the condition of Blatt and Matsubara (2.9).

§ 3. The transverse conductivity

In connexion with the integral (2.10), we shall discuss the transverse conductivity, which appears in the analysis of infra-red absorption data. In defining this quantity, we follow what is usually done by experimentalists. Making use of (1.6), they express the transverse electromagnetic wave in terms of the electric field alone and write the current in the form

$$\mathbf{J}(\mathbf{k}, t) = \sigma_{tr}(k, \omega) \mathbf{E}(\mathbf{k}, t) \quad (3.1)$$

which is the definition of the transverse conductivity.

In our formalism, inserting (1.6) into (1.5) and remembering that we are dealing with the transverse field, we obtain

$$\mathbf{J}(\mathbf{k}, t) = \int_{-\infty}^t dt' \phi(k, t-t') \mathbf{E}(\mathbf{k}, t'). \quad (3.2)$$

Here the response function ϕ is given by

$$\phi(k, t) = c^2 (S_0(k, t) + S_2(k, 0)). \quad (3.3)$$

In particular, if we take

$$\mathbf{E}(\mathbf{k}, t) = \mathbf{E}(\mathbf{k}) \exp(\alpha t - i\omega t), \quad \mathbf{k} \cdot \mathbf{E} = 0,$$

(3.2) reduces to (3.1) with

$$\sigma_{tr}(k, \omega) = \int_0^\infty dt \psi(k, t) e^{-\alpha t + i\omega t}. \quad (3.4)$$

Again, from (1.4), the Fourier inversion leads to the sum rule

$$\frac{2}{\pi} \int_0^\infty d\omega \Re \sigma_{tr}(k, \omega) = \frac{ne^2}{m} \quad (3.5)$$

and also to the Kramers-Kronig dispersion relation. The analysis of infra-red absorption data made by Tinkham and Ferrell⁽⁵⁾ is based just on these two properties of the transverse conductivity. They have pointed out that the sum rule is related with the penetration law of the steady magnetic field. As shown below, this is rather trivial in our formalism.

Inserting (3.3) into (3.4), we obtain

$$\sigma_{tr}(k, \omega) = \sigma_{tr}^{(0)}(k, \omega) + c^2 S_2(k, 0) \left(\pi \delta(\omega) + \frac{iP}{\omega} \right). \quad (3.6)$$

Here P indicates to take the principal part of an integral over ω , and

$$\sigma_{tr}^{(0)}(k, \omega) = c^2 \int_0^\infty dt S_0(k, t) \exp(-\alpha t + i\omega t). \quad (3.7)$$

If we take this $\sigma_{tr}^{(0)}$ alone, then we do have the sum rule deficiency as

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty d\omega \Re \sigma_{tr}^{(0)}(k, \omega) \\ = c^2 S_0(k, 0) = \frac{ne^2}{m} - c^2 S_2(k, 0). \end{aligned} \quad (3.8)$$

It is to be noticed that the deficiency is just equal to the coefficient of the diamagnetic moment (2.5). For instance, $\sigma_{tr}^{(0)}$ of the free Fermi gas of electrons has been calculated by Lindhard⁽⁶⁾. Inserting his result into the left-hand side of (3.8), we can explicitly confirm that the deficiency in this case indeed corresponds to the Landau diamagnetism (2.6).

As discussed by Tinkham and Ferrell, the deficiency in the case of a superconductor is obtained empirically from infra-red absorption data. Because of the existence of an energy gap, a certain area under the $\sigma_{tr}^{(0)}$ v.s. ω curve is removed in a superconductor. For k large, the sum rule deficiency was found to be

$$S_2(k, 0) = 3/2 (ne^2/mc^2) (\omega_g/v_0 k) \quad (3.9)$$

where v_0 is the electron velocity at the Fermi surface and ω_g an effective energy gap. Comparing (3.9) with the Pippard limit in (2.7), we find the coherence length $\xi_0 = (\pi/2) (v_0/\omega_g)$.

From the theoretical point of view, it is to be noticed that $S_0(k, 0)$ in (3.8)

is the kernel which appears in the theory of the Meissner effect when the transverse of the vector potential is adopted. More generally, let us take a time dependent vector potential in the transverse gauge.

Inserting

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{k} \cdot \mathbf{A} = 0$$

into (1.1), we obtain

$$\begin{aligned} \mathbf{J}(\mathbf{k}, t) = & -\frac{ne^2}{mc} \mathbf{A}(\mathbf{k}, t) \\ & - c \int_{-\infty}^t dt' \frac{\partial S_0(\mathbf{k}, t-t')}{\partial t} \mathbf{A}(\mathbf{k}, t'). \end{aligned} \quad (3.11)$$

Assuming

$$\mathbf{A}(\mathbf{k}, t) = \mathbf{A}(\mathbf{k}) \exp(\alpha t - i\omega t),$$

we have

$$\mathbf{J}(\mathbf{k}, t) = -\left(\frac{ne^2}{mc} + c \int_0^\infty dt e^{-\alpha t + i\omega t} \frac{\partial S_0(\mathbf{k}, t)}{\partial t} \right) \mathbf{A}(\mathbf{k}, t).$$

In the static limit $\omega \rightarrow 0$, remembering (2.10), we obtain

$$\mathbf{J}(\mathbf{k}) = -\left(\frac{ne^2}{mc} - c S_0(\mathbf{k}, 0) \right) \mathbf{A}(\mathbf{k})$$

as mentioned above.

The transverse conductivity in this scheme is given as

$$\sigma_{tr}(\mathbf{k}, \omega) = \frac{1}{\alpha - i\omega} \left(\frac{ne^2}{m} + \int_0^\infty dt \exp(-\alpha t + i\omega t) \frac{\partial S_0(\mathbf{k}, t)}{\partial t} \right) \quad (3.12)$$

which is equivalent to our previous expression (3.6) with (3.7), as far as the theory is gauge invariant. The calculation of (3.12) for a superconductor has been carried out by Mattis and Bardeen⁷⁾, and the result is in good agreement with infra-red absorption data.

§ 4. Conclusions

On the basis of the general expression (1.1), we have seen how diamagnetism, longitudinal and transverse conductivities are generally related with each other. In this way we have also seen the present status of the theory of the electromagnetic response of a superconductor. So far we have found no satisfactory theory of the infinite longitudinal conductivity (2.11). This should be distinguished from the delta singularity in the transverse conductivity (3.6). The latter singularity appears even in connexion with the Landau diamagnetism of the normal Fermi gas, as

mentioned before.

The problem of the infinite conductivity is rather complicated, because we have to prove (2·11) in the presence of random impurities and also the problem is concerned with the gauge invariance of the theory.

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Polaron State at a Finite Temperature

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We extend the treatment of Feynman for the polaron problem at 0°K to the case at a finite temperature. A variational principle is applied to the free energy of polaron state.

§ 1. Introduction

A slow electron in a polar crystal interacts strongly with the longitudinal optical modes of lattice vibrations, and behaves itself as a quasi-particle which is the so-called "polaron". The properties of a polaron have been investigated by various authors, but do not yet seem to be made perfectly clear. Of all problems concerning a polaron, its self-energy and mobility are fundamentally important.

To compute the self-energy of polaron, Feynman's¹⁾ method seems to be most excellent, as it was shown by Shultz²⁾ that Feynman's method gives lower self-energy than any other methods which have been ever published. In the present paper we shall reform the Feynman's treatment so as to be applicable to the case of finite temperatures, that is, Feynman's variational principle of the self-energy of polaron is replaced by the variation of the free energy. Numerical calculation is carried out for a certain value of coupling constant and its result is compared with that obtained by Fulton⁵⁾ and by Yokota.⁶⁾

§ 2. Static property of polaron state at finite temperatures

In this paper our units are chosen just as electron mass, Planck constant and optical phonon energy are unities. We shall consider the density matrix of the system which consists of one electron and longitudinal lattice vibrations, and average the part of the lattice vibrations by path-integral method. Such a calculation is made by Abe³⁾ and the action of the polaron at finite temperatures is given by

$$S = -\frac{1}{2} \int_0^\beta (\dot{X}(t))^2 dt + 2^{-3/2} \alpha \left\{ \frac{e^\beta}{e^\beta - 1} \iint_0^\beta |X(t) - X(s)|^{-1} e^{-|t-s|} dt ds \right. \\ \left. + \frac{1}{e^\beta - 1} \iint_0^\beta |X(t) - X(s)|^{-1} e^{t-s} dt ds \right\}, \quad (1)$$

where $\beta = 1/kT$ and α is the coupling constant between an electron and lattice

vibrations. The above relation is derived more simply by the path-integral method in Appendix.* Reflecting upon Eq. (1) we make a choice of trial action S_0 at finite temperatures as follows.

$$S_0 = -\frac{1}{2} \int_0^\beta (\dot{X}(t))^2 dt - \frac{C}{2} \left\{ \frac{e^{\beta\omega}}{e^{\beta\omega}-1} \int_0^\beta |X(t) - X(s)|^2 \right. \\ \left. \times e^{-\omega|\ell-s|} dt ds + \frac{1}{e^{\beta\omega}-1} \int_0^\beta \int_0^\beta |X(t) - X(s)|^2 e^{\omega|\ell-s|} dt ds \right\}, \quad (2)$$

which is a natural extension of Feynman's trial action at 0°K. Here ω and C are variational parameters, respectively. The density matrix of the polaron is given by

$$\rho(X, X'; \beta) = \int_0^\beta \exp(S) \mathfrak{D}X(t) \overset{\widehat{X'}}{\underset{\widehat{X}}{\uparrow}}, \quad (3)$$

where $\mathfrak{D}X(t) \overset{\widehat{X'}}{\underset{\widehat{X}}{\uparrow}}$ denotes the path integration under the boundary conditions $X(0)=X$ and $X(\beta)=X'$.

The free energy F is expressed by the partition function Z as

$$F = -\frac{1}{\beta} \log Z, \quad Z = \int_{-\infty}^{+\infty} \rho(X, X) dX.$$

If the calculation of density matrix is performed by the same procedure as Feynman did, we obtain

$$\rho(X, X'; \beta) > \exp(\langle S - S_0 \rangle_{X, X'}) \rho_{s_0}(X, X'; \beta), \quad (4a)$$

where

$$\langle S - S_0 \rangle_{X, X'} = \int_0^\beta (S - S_0) \exp(S_0) \mathfrak{D}X(t) \overset{\widehat{X'}}{\underset{\widehat{X}}{\uparrow}} / \rho_{s_0}(X, X'; \beta) \\ \rho_{s_0}(X, X'; \beta) = \int_0^\beta \exp(S_0) \mathfrak{D}X(t) \overset{\widehat{X'}}{\underset{\widehat{X}}{\uparrow}}. \quad (4b)$$

Thus we get the variational principle that the trial free energy which is obtained from the quantity on the right-hand side of Eq. (4a) should be minimized with respect to the parameters ω and C . This is a natural extension of Feynman's variation principle at 0°K.

* This derivation was communicated by Dr. T. Yokota.

We shall now calculate the right side of Eq. (4a),

$$\rho_{tr}(\mathbf{X}, \mathbf{X}'; \beta) = \rho_{s_0}(\mathbf{X}, \mathbf{X}'; \beta) \exp \langle S - S_0 \rangle_{\mathbf{X}, \mathbf{X}'}. \quad (5)$$

The equation of motion determined by the action S_0 is

$$\begin{aligned} \ddot{\mathbf{X}}(t) &= 2C \left\{ \frac{e^{\beta\omega}}{e^{\beta\omega} - 1} \int_0^\beta (\mathbf{X}(t) - \mathbf{X}(s)) e^{-\omega|t-s|} ds + \frac{1}{e^{\beta\omega} - 1} \int_0^\beta (\mathbf{X}(t) - \mathbf{X}(s)) e^{\omega|t-s|} ds \right\} \\ &= \frac{4C}{\omega} \mathbf{X}(t) - 2C \int_0^\beta \left(\frac{e^{\beta\omega}}{e^{\beta\omega} - 1} e^{-\omega|t-s|} + \frac{1}{e^{\beta\omega} - 1} e^{\omega|t-s|} \right) \mathbf{X}(s) ds. \end{aligned} \quad (6a)$$

By the solution of Eq. (6a) under conditions $\mathbf{X}(\beta) = \mathbf{X}'$ and $\mathbf{X}(0) = \mathbf{X}$, the density matrix $\rho_{s_0}(\mathbf{X}, \mathbf{X}'; \beta)$ is represented as follows.

$$\rho_{s_0}(\mathbf{X}, \mathbf{X}'; \beta) = \exp(-\frac{1}{2}(\mathbf{X}(\beta) \cdot \dot{\mathbf{X}}(\beta) - \mathbf{X}(0) \cdot \dot{\mathbf{X}}(0))), \quad (6b)$$

where we adopted such an approximation that the integration over all paths is replaced by one path contributing predominantly to the density matrix of Eq. (4b). Let us now consider the system with the Lagrangian L'

$$L' = \frac{1}{2}(\dot{\mathbf{r}}^2 + M\dot{\mathbf{R}}^2 - \kappa(\mathbf{r} - \mathbf{R})^2), \quad (7)$$

where M has a dimension of mass and κ has a dimension of force constant. If the density matrix ρ' of the Lagrangian L' is calculated by the path-integral method and is integrated with respect to the coordinate \mathbf{R} , we obtain

$$\begin{aligned} \rho'(\mathbf{r}, \mathbf{r}'; \beta) &= \left(2 \sinh \frac{\beta\omega'}{2} \right)^{-3} \int \exp \left[- \int_0^\beta \left(\frac{1}{2} \dot{\mathbf{r}}^2 + \frac{\kappa}{2} \mathbf{r}^2 \right) dt \right. \\ &\quad \left. + \frac{M}{4} \omega'^3 \int_0^\beta \left[\frac{e^{\beta\omega'}}{e^{\beta\omega'} - 1} e^{-\omega'|t-s|} + \frac{1}{e^{\beta\omega'} - 1} e^{\omega'|t-s|} \right] \right. \\ &\quad \left. \times \mathbf{r}(t) \mathbf{r}(s) dt ds \right] \mathfrak{D} \mathbf{r}(t), \end{aligned} \quad (8)$$

where $\omega' = \sqrt{\kappa/M}$. From the action of the above equation the equation of motion for $\mathbf{r}(t)$ is derived.

$$\ddot{\mathbf{r}}(t) = \kappa \mathbf{r}(t) - \frac{M\omega'^3}{2} \int_0^\beta \left\{ \frac{e^{\beta\omega'}}{e^{\beta\omega'} - 1} e^{-\omega'|t-s|} + \frac{1}{e^{\beta\omega'} - 1} e^{\omega'|t-s|} \right\} \mathbf{r}(s) ds. \quad (9a)$$

Using the solution of this equation under the boundary conditions $\mathbf{r}(\beta) = \mathbf{r}'$ and $\mathbf{r}(0) = \mathbf{r}$, we obtain

$$\rho'(\mathbf{r}, \mathbf{r}'; \beta) = \exp(-\frac{1}{2}(\mathbf{r}(\beta) \cdot \dot{\mathbf{r}}(\beta) - \mathbf{r}(0) \cdot \dot{\mathbf{r}}(0))), \quad (9b)$$

where numerical factor $(2 \sinh \beta\omega'/2)^{-3}$ is omitted because this is not important. If we put $\omega' = \omega$ and $M\omega'^3 = 4C$, Eq. (9a) is identical to Eq. (6a) and if we

further put $\mathbf{r}'=\mathbf{X}'$ and $\mathbf{r}=\mathbf{X}$, Eq. (9b) is the same as Eq. (6b). Therefore, we can use the solution of Eq. (9a) to calculate $\rho_{s_0}(\mathbf{X}, \mathbf{X}'; \beta)$ in Eq. (4b) instead of the solution of Eq. (6a), that is, the path integral of action S_0 is replaced by the path integral of Lagrangian L' .

When we calculate $\langle S-S_0 \rangle_{\mathbf{X}, \mathbf{X}'}$ in Eq. (4b), the first term $-\int_0^\beta (\dot{\mathbf{X}})^2 dt$ of Eqs. (1) and (2) are cancelled out with each other. Therefore the symbols $\langle S \rangle_{\mathbf{X}, \mathbf{X}'}$ and $\langle S_0 \rangle_{\mathbf{X}, \mathbf{X}'}$ are used to denote averages of the second terms in Eqs. (1) and (2), respectively. In order to calculate $\langle S \rangle_{\mathbf{X}, \mathbf{X}'}$, it is necessary to evaluate $\langle |\mathbf{X}_\tau - \mathbf{X}_\sigma|^{-1} \rangle_{\mathbf{X}, \mathbf{X}'}$ which is represented by the relation

$$|\mathbf{X}_\tau - \mathbf{X}_\sigma|^{-1} = \int d^3\mathbf{K} \frac{1}{2\pi^2 K^2} \exp(i\mathbf{K} \cdot (\mathbf{X}_\tau - \mathbf{X}_\sigma)), \quad (10a)$$

as follows.

$$\begin{aligned} \langle \exp(i\mathbf{K} \cdot (\mathbf{X}_\tau - \mathbf{X}_\sigma)) \rangle_{\mathbf{X}, \mathbf{X}'} &= \left[\int \exp(S_0) \exp\left(\int f(t) \cdot \mathbf{X}(t) dt\right) \right. \\ &\quad \times \mathfrak{D}\mathbf{X}(t) \left. \begin{array}{c} \widehat{\mathbf{X}}' \\ \uparrow \\ \mathbf{X} \end{array} \right] / \rho_{s_0}(\mathbf{X}, \mathbf{X}'; \beta), \\ f(t) &\equiv i\mathbf{K}(\delta(t-\tau) - \delta(t-\sigma)). \end{aligned} \quad (10b)$$

Since $\langle S_0 \rangle_{\mathbf{X}, \mathbf{X}'}$ is obtained by $\langle (\mathbf{X}_\tau - \mathbf{X}_\sigma)^2 \rangle_{\mathbf{X}, \mathbf{X}'}$, $\langle S_0 \rangle_{\mathbf{X}, \mathbf{X}'}$ is derived from the expansion in power series of \mathbf{K} of Eq. (10b). As is mentioned above, the path-integral of the action S_0 can be replaced with that of the Lagrangian L' . Under the similar circumstances the path-integral of Eq. (10b) is replaced with that of the following Lagrangian,

$$L = \frac{1}{2} \dot{\mathbf{r}}^2 + \frac{1}{2} M \dot{\mathbf{R}}^2 - \frac{\kappa}{2} (\mathbf{r} - \mathbf{R})^2 + \mathbf{f}(t) \cdot \mathbf{r}.$$

Now let us introduce new variables,

$$\mathbf{q} = \mathbf{r} - \mathbf{R}, \quad \mathbf{Q} = M\mathbf{R} + \mathbf{r} / M + 1 \quad (11)$$

then the Lagrangian L is written as

$$L = \frac{1}{2} (M+1) \dot{\mathbf{Q}}^2 + \frac{1}{2} (M/M+1) \dot{\mathbf{q}}^2 - \frac{\kappa}{2} \mathbf{q}^2 + (M/M+1) \mathbf{f}(t) \cdot \mathbf{q} + \mathbf{Q} \cdot \mathbf{f}(t). \quad (12)$$

If the constant ν is defined by the relation,

$$\nu^2 = \kappa(M+1)/M = 4C/\omega + \omega^2,$$

the equations of motion for \mathbf{q} and \mathbf{Q} become

$$\ddot{\mathbf{q}} = \nu \mathbf{q} - \mathbf{f}(t), \quad (13a)$$

$$\ddot{\mathbf{Q}} = -\mathbf{f}(t)/(M+1). \quad (13b)$$

The solution of Eq. (13a) under the boundary conditions $\mathbf{q}(0) = \mathbf{q}_1$, and $\mathbf{q}(\beta) = \mathbf{q}_2$ is

$$\begin{aligned}
 \mathbf{q}(t) = & (\mathbf{q}_1 \sinh \nu(\beta - t) + \mathbf{q}_2 \sinh \nu t) / \sinh \nu \beta \\
 & + i\mathbf{K} \sinh \nu t (\sinh \nu(\beta - \tau) - \sinh \nu(\beta - \sigma)) / \nu \sinh \nu \beta \\
 & - i\mathbf{K} / \nu \begin{cases} \sinh \nu(t - \tau) - \sinh \nu(t - \sigma) & t > \tau \\ -\sinh \nu(t - \sigma) & \sigma < t < \tau \\ 0 & t < \sigma. \end{cases} \quad (14)
 \end{aligned}$$

In the case $\sigma > \tau$, the last term in the right-hand side of Eq. (14) must be replaced by

$$\begin{cases} \sinh \nu(t - \tau) - \sinh \nu(t - \sigma) & t > \sigma \\ \sinh \nu(t - \tau) & \tau < t < \sigma \\ 0 & t < \tau. \end{cases}$$

Under the conditions $\mathbf{Q}(0) = \mathbf{Q}_1$, and $\mathbf{Q}(\beta) = \mathbf{Q}_2$, the solution of Eq. (13b) is

$$\begin{aligned}
 \mathbf{Q}(t) = & \mathbf{Q}_1 + \{ (\mathbf{Q}_2 - \mathbf{Q}_1) - i\mathbf{K}\omega^2(\tau - \sigma)/\nu^2 \} / \beta \\
 & - i\mathbf{K}\omega^2/\nu^2 \left\{ \begin{bmatrix} t - \tau \\ 0 \end{bmatrix}_\tau - \begin{bmatrix} t - \sigma \\ 0 \end{bmatrix}_\sigma \right\},
 \end{aligned}$$

where the symbol $\begin{bmatrix} f \\ 0 \end{bmatrix}_\tau$ denotes $\begin{cases} f & t > \tau \\ 0 & t < \tau. \end{cases}$ Using these solutions, we obtain

$$\begin{aligned}
 & \langle \exp(i\mathbf{K} \cdot (\mathbf{X}_\tau - \mathbf{X}_\sigma)) \rangle_{(\mathbf{Q}_1, \mathbf{Q}_2; \beta)} \\
 & = \exp \left(-M(\mathbf{q}(\beta) \cdot \dot{\mathbf{q}}(\beta) - \mathbf{q}(0) \cdot \dot{\mathbf{q}}(0)) / 2(M+1) \right) \\
 & + \frac{M}{2(M+1)} \int_0^\beta \mathbf{f}(t) \cdot \mathbf{q}(t) dt \exp \left(-\frac{(M+1)}{2} \right. \\
 & \times (\mathbf{Q}(\beta) \cdot \dot{\mathbf{Q}}(\beta) - \mathbf{Q}(0) \cdot \dot{\mathbf{Q}}(0)) + \frac{1}{2} \int_0^\beta \mathbf{f}(t) \cdot \mathbf{Q}(t) dt \Big). \quad (15)
 \end{aligned}$$

In order to connect the variables in the Lagrangian L with that in the action S , we replace the variables \mathbf{Q}_1 , \mathbf{q}_1 and \mathbf{Q}_2 , \mathbf{q}_2 in the above equation with \mathbf{r}_1 , \mathbf{R}_1 and \mathbf{r}_1 , \mathbf{R}_2 by the use of Eq. (11) and then integrate Eq. (15) with respect to variable \mathbf{R}_1 , under the condition $\mathbf{R}_2 = \mathbf{R}_1$. Using Eqs. (15), (10a), and the fact that $\langle \exp i\mathbf{K} \cdot (\mathbf{X}_2 - \mathbf{X}_1) \rangle$ becomes unity in the case of \mathbf{K} tending to 0, we obtain

$$\begin{aligned}
 \langle S \rangle_{\mathbf{r}, \mathbf{r}} = & 2^{-3/2} \alpha \int \frac{d\mathbf{K}}{2\pi^2 K^2} \iint_0^\beta d\tau d\sigma \left(\frac{e^\beta}{e^\beta - 1} e^{-|\tau - \sigma|} \right. \\
 & + \frac{1}{e^\beta - 1} e^{|\tau - \sigma|} \Big) \exp \left(-K^2 \left[\frac{2C}{\nu^3 \omega} \left\{ 1 - e^{-\nu|\tau - \sigma|} \right. \right. \right. \right. \\
 & \left. \left. \left. + \left(1 - \coth \frac{\nu}{2} \beta \right) (\cosh \nu|\tau - \sigma| - 1) \right\} \right] \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\omega^2}{2\nu^2} |\tau - \sigma| (1 - |\tau - \sigma|/\beta) \Big] \\
& = \pi^{-1/2} \alpha \nu \beta \frac{e^\beta}{e^\beta - 1} \int_0^\beta d\tau e^{-\tau} \\
& \quad \times \left[\omega^2 \tau (1 - \tau/\beta) + (\nu^2 - \omega^2)/\nu \{1 - e^{-\nu\tau} \right. \\
& \quad \left. + \left(1 - \coth \frac{\nu}{2} \beta\right) (\cosh \nu\tau - 1) \} \right]^{-1/2} \\
& = \langle S \rangle_{X,X}.
\end{aligned} \tag{16}$$

The last relation of the above equation is valid, since $\langle S \rangle_{rr}$ is not dependent on the boundary condition. $\langle S_0 \rangle_{rr}$ is derived from the expansion in power series of \mathbf{K} of Eq. (16).

$$\begin{aligned}
-\langle S_0 \rangle_{r,r} &= 3C\beta/\nu^2 \left[\frac{e^{\beta\omega} + 1}{e^{\beta\omega} - 1} - 2/\omega\beta + (\nu^2 - \omega^2) \coth \frac{\nu}{2} \beta / \nu\omega + \frac{\nu^2 - \omega^2}{\nu} \left\{ \frac{e^{(\nu-\omega)\beta} - 1}{2(\nu-\omega)} \right. \right. \\
& \quad \left. \left. \times \left(1 - \coth \frac{\nu}{2} \beta \right) + \frac{e^{-(\nu+\omega)\beta} - 1}{2(\nu+\omega)} \left(1 + \coth \frac{\nu}{2} \beta \right) \right\} \right] \\
&= \frac{3C\beta}{\nu\omega} \left(\coth \frac{\nu}{2} \beta - \frac{2}{\nu\beta} \right) = -\langle S_0 \rangle_{X,X}.
\end{aligned} \tag{17}$$

As $\langle S_0 \rangle_{rr}$ is independent of \mathbf{r} , the last relation is valid. Since $\langle S \rangle_{XX}$ and $\langle S_0 \rangle_{XX}$ are not dependent on \mathbf{X} , we shall omit their suffices. The diagonal part of Eq. (5) becomes

$$\rho_{tr}(\mathbf{X}, \mathbf{X}; \beta) = \rho_{s_0}(\mathbf{X}, \mathbf{X}; \beta) \exp(\langle S - S_0 \rangle). \tag{18}$$

The trial free energy F_{tr} is given by

$$F_{tr} = -(\log Z_{s_0} + \langle S - S_0 \rangle) / \beta,$$

where Z_{s_0} is the partition function derived from the density matrix $\rho_{s_0}(\mathbf{X}_1, \mathbf{X}_1; \beta)$ and is easily derived from $\langle S_0 \rangle$ in the following manner. From the definitions of $\log Z_{s_0}$ and $\langle S_0 \rangle$, it is valid that

$$\partial \log Z_{s_0} / \partial C = \langle S_0 \rangle / C. \tag{19}$$

When $C=0$, Z_{s_0} is the partition function of free electron which is $(2\pi\beta)^{-3/2}$. If Eq. (19) is integrated by C under this condition, we obtain

$$\log Z_{s_0} = -\frac{3}{2} \log(2\pi\beta) + \int_0^C dC \frac{\langle S_0 \rangle}{C}.$$

Transforming the variable C in the integral into ν by the use of the relation $\nu^2 = 4C/\omega + \omega^2$, we obtain

$$\begin{aligned}\log Z_{s_0} &= -\frac{3}{2} \log(2\pi\beta) - \frac{3}{2} \beta \int_0^{\nu} \left(\coth \frac{\beta}{2} x - \frac{2}{\beta x} \right) dx \\ &= -\frac{3}{2} \log(2\pi\beta) + 3 \log \left(\frac{\nu}{\omega} \right) - 3 \log \frac{\sinh(\nu/2)\beta}{\sinh(\omega/2)\beta}.\end{aligned}\quad (20)$$

If the variational parameters C and ω are determined in such a way that the value of the free energy F_{tr} is minimum, then the average energy of polaron state is given by

$$E = -\frac{\partial \log Z_{s_0}}{\partial \beta} - \frac{\partial \langle S \rangle}{\partial \beta} + \frac{\partial \langle S_0 \rangle}{\partial \beta}.\quad (21)$$

Therefore, the expression for the self-energy part of average energy E_s at a finite temperature is given by

$$E_s = \bar{E} - 3/2\beta,\quad (22)$$

where the last term is the energy of free electron. When β tends to ∞ , the free energy $-(1/\beta)(\log Z_{s_0} + \langle S \rangle - \langle S_0 \rangle)$, becomes identical to Feynman's expression for self energy.

The calculation of E is straightforward, but some notices are needed. An example is given below. Pay your attention to the following expression, which appears in the evaluation of $\partial \langle S \rangle / \partial \beta$.

$$\frac{e^\beta}{e^\beta - 1} \beta \frac{\alpha \nu}{\omega \pi^{1/2}} \frac{\partial}{\partial \beta} \left[\int_0^\beta d\tau e^{-\tau} [j(\beta; \tau)]^{-1/2} \right],$$

where

$$j(\beta; x) = x(1 - (x/\beta)) + \frac{\nu^2 - \omega^2}{\nu \omega} \left\{ 1 - e^{-\nu x} + \left(1 - \coth \frac{\nu}{2} \beta \right) (\cosh \nu x - 1) \right\}.$$

Differentiation of the upper limit of the integral seems to lead to divergency of the expression, because of $j(\beta, \beta) = 0$. But this divergence is apparent. This fact is understood in the following manner. If we use $j(\beta; \tau) = j(\beta; \beta - \tau)$ and transform the variable into $y^2 = \tau$, we obtain

$$\begin{aligned}\int_0^\beta d\tau e^{-\tau} [j(\beta; \tau)]^{-1/2} &= \int_0^{\beta/2} d\tau [e^{-\tau} + e^{-(\beta-\tau)}] [j(\beta; \tau)]^{-1/2} \\ &= 2 \int_0^{Y\beta/2} y dy (e^{-y^2} + e^{-(\beta-y^2)}) [j(\beta; y^2)]^{-1/2}.\end{aligned}$$

From this equation, we obtain

$$\begin{aligned} \frac{\partial}{\partial \beta} \int_0^\beta d\tau e^{-\tau} [j(\beta; \tau)]^{-1/2} = & - \int_0^{V\beta/2} y [e^{-y^2} + e^{-(\beta-y^2)}] [j(\beta; y^2)]^{-3/2} \frac{\partial j(\beta; y^2)}{\partial \beta} dy \\ & + e^{-\beta/2} \left[j\left(\beta; \frac{\beta}{2}\right) \right]^{-1/2} - 2 \int_0^{V\beta/2} y e^{-(\beta-y^2)} [j(\beta; y^2)]^{-1/2} dy. \end{aligned}$$

This expression evidently has no divergent terms. Furthermore, it can be easily shown by elementary calculations that

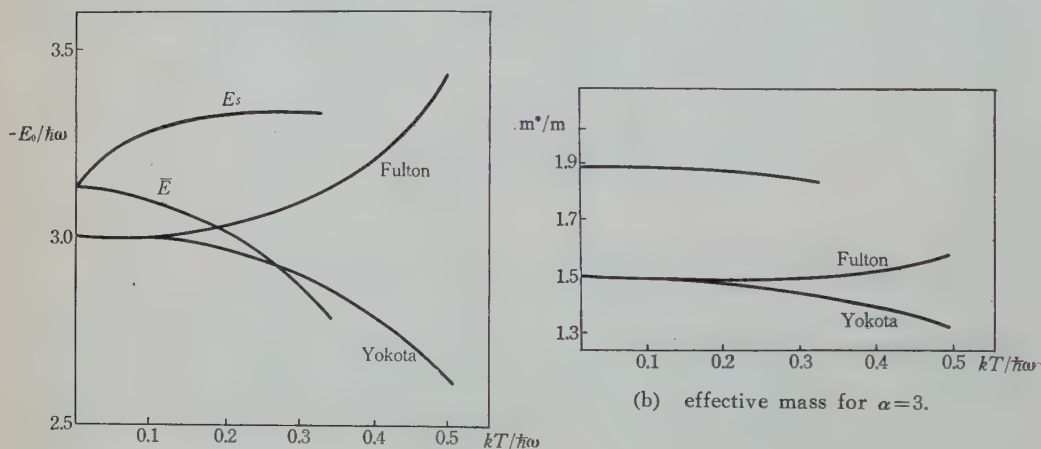
$$\lim_{\beta \rightarrow \infty} \left[\beta \frac{\partial}{\partial \beta} \left\{ \int_0^\beta d\tau e^{-\tau} [j(\beta; \tau)]^{-1/2} \right\} \right] \rightarrow 0.$$

The following can also be shown by using Eqs. (16), (17), and the relation just given above, though the calculation is somewhat tedious.

$$\lim_{\beta \rightarrow \infty} \bar{E}(\beta) = \lim_{\beta \rightarrow \infty} F_{tr}(\beta),$$

which should hold as mentioned before.

§3. Results and discussion



(a) Self energy and average energy for $\alpha=3$.
Curve \bar{E} from (21) and curve E_s from (22).

Fig. 1

We carried out the calculations of the average-energy and the effective mass in the region of low temperatures for the special case of $\alpha=3$. In Fig. 1, we compare our result with the results by Fulton⁴⁾ and Yokota⁵⁾. Their results are expressed in the analytic form, for small α , as follows.

$$\text{Fulton; } -E_0/\hbar\omega = (\bar{n}+1)\alpha, \quad m^*/m = 1 + (\bar{n}+1)\alpha/6.$$

$$\text{Yokota; } -E_0/\hbar\omega = \frac{\alpha}{\sqrt{2\bar{n}+1}}, \quad m^*/m = 1 + \frac{\alpha}{6(2\bar{n}+1)^{3/2}}.$$

The temperature dependence of the energy is contrary to each other. In the present calculation, the effective mass of polaron has been determined by the formula $m^*/m = v^2/\omega^2$. Fulton's calculation for excited states is equivalent to Gurari's⁽⁶⁾ variational calculation for the self-energy at 0°K. His treatment neglects the entropy part in the free energy of polaron. Yokota calculated the free energy of polaron state by the use of the Hartree approximation. Variational parameter appearing in his calculation were determined in such a way that the free energy is minimum. The present results for temperature dependence of the average energy and the effective mass agree qualitatively with the results obtained by Yokota. The temperature dependences may be qualitatively understood as follows. At finite temperatures, there is more gain in entropy when phonon is free from electron that it crowds together around the electron. This effect leads to a decrease of the average energy and the effective mass. The qualitative difference between our results and that obtained by Fulton is due to that Fulton neglected the effect of entropy in polaron state. More detailed numerical value will be given elsewhere including the case for other coupling constant.

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Appendix

We derive the expression of free energy F by the use of path-integral method. This derivation is equal to that carried out by Dr. T. Yokota, to whom the present author is indebted.

We consider the system having the following Hamiltonian,

$$H = P^2/2m + \frac{1}{2} \sum_w \hbar\omega_w (b_w b_w^* + b_w^* b_w) - \sum_w i I_w (b_w^* e^{-i\mathbf{w}\cdot\mathbf{r}} - b_w e^{i\mathbf{w}\cdot\mathbf{r}}),$$

$$e^{-\beta F} = T_r e^{-\beta H} = T_r e^{-\beta \sum_w (\hbar\omega_w/2) (b_w b_w^* + b_w^* b_w)}$$

$$\times \exp_{(+)} \left\{ - \int_0^\beta P^2/2m \cdot ds + \sum_w i I_w \int_0^\beta (b_w^* e^{-i\mathbf{w}\cdot\mathbf{r} + \hbar\omega_w s} - b_w e^{i\mathbf{w}\cdot\mathbf{r} - \hbar\omega_w s}) ds \right\},$$

where the symbol $\exp_{(+)}$ means $\exp(\lambda(A+B)) = \exp(\lambda A) \exp_{(+)} \left(\int_0^\lambda d\mu B(\mu) \right)$. If we apply the path-integration to the electronic system, we get

$$e^{-\beta F} = \text{Tr}_{(\text{lattice})} e^{-\beta \sum_w (\hbar\omega_w/2) (2n_w + 1)} \int \exp_{(+)} \left(- \int_0^\beta \frac{m}{2\hbar^2} \dot{\mathbf{r}}_s^2 ds + \sum_w i I_w \int_0^\beta (b_w^* e^{-i\mathbf{w}\cdot\mathbf{r}_s + \hbar\omega_w s} - b_w e^{i\mathbf{w}\cdot\mathbf{r}_s - \hbar\omega_w s}) ds \right) \mathfrak{D}(\mathbf{r}),$$

where $\text{Tr}_{(\text{lattice})}$ means that trace with respect to lattice system must be taken. Taking care of non-commutability of variables b_w^* and b_w , we obtain

$$e^{-\beta F} = \text{Tr}_{(\text{lattice})} e^{-\beta \sum_w (\hbar\omega_w/2) (2n_w+1)} \int \mathfrak{D}(\mathbf{r}) \exp\left(\sum_w b_w^* \int_0^\beta \dot{\gamma}_w^*(s) ds\right) \exp\left(\sum_w b_w \int_0^\beta \dot{\gamma}_w(s) ds\right) \\ \times \exp\left(-\int_0^\beta \frac{m}{2\hbar^2} \dot{\mathbf{r}}_s^2 ds + \sum_w \int_0^\beta ds \int_0^s ds' \dot{\gamma}_w(s) \dot{\gamma}_w^*(s')\right),$$

where

$$\dot{\gamma}_w^*(s) = i I_w e^{-i\mathbf{w} \cdot \mathbf{r} + \hbar\omega_w s}, \quad \dot{\gamma}_w(s) = -i I_w e^{i\mathbf{w} \cdot \mathbf{r} - \hbar\omega_w s}.$$

By the use of the formula

$$\text{Tr} \exp(-\beta \hbar \omega_w (n_w + \frac{1}{2})) \exp(b_w^* \Gamma_w^*) \exp(b_w \Gamma_w) \\ = \exp(\bar{n}_w \Gamma_w^* \Gamma_w) / 2 \sinh \frac{\beta \hbar \omega_w}{2}, \quad \bar{n}_w = \frac{1}{e^{\beta \hbar \omega_w} - 1},$$

we have

$$e^{-\beta F} = \prod_w \left(\frac{1}{2 \sinh(\beta \hbar \omega_w/2)} \right) \int \mathfrak{D}(\mathbf{r}) \\ \times \exp\left(-\int_0^\beta \frac{m}{2\hbar^2} \dot{\mathbf{r}}_s^2 ds + \sum_w \bar{n}_w I_w^2 \int_0^\beta ds \int_0^\beta ds' \dot{\gamma}_w(s) \dot{\gamma}_w^*(s') \right. \\ \left. + \sum_w I_w^2 \int_0^\beta ds \int_0^s ds' \dot{\gamma}_w(s) \dot{\gamma}_w(s') \right) \\ = \prod_w \left(\frac{1}{2 \sinh(\beta \hbar \omega_w/2)} \right) \int \mathfrak{D}(\mathbf{r}) \exp\left(-\int_0^\beta \frac{m}{2\hbar^2} \dot{\mathbf{r}}_s^2 ds \right. \\ \left. + \sum_w (\bar{n}_w + 1) I_w^2 \int_0^\beta ds \int_0^s ds' \dot{\gamma}_w(s) \dot{\gamma}_w^*(s') \right. \\ \left. + \sum_w \bar{n}_w I_w^2 \int_0^\beta ds \int_0^s ds' \dot{\gamma}_w^*(s) \dot{\gamma}_w(s') \right).$$

Using this result and omitting the part for a free energy of phonon, we can easily derive Eq. (1) for the polaron.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures and tables.

On the Collective Energy Loss Mechanism of Electrons Passing through Solids

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April 28, 1959

It is well known that a fast electron passing through a solid can excite a plasma oscillation and loses its energy. This collective loss mechanism has been analysed by several authors.¹⁾ Here we propose another approach to this problem in the limiting case of a dense electron gas; that is, we make use of the results of the high-density-limit theory of Sawada et al.²⁾ to calculate the transition probability of an incident fast electron being scattered by exciting the electron gas from its ground state Ψ_0 to the one-plasmon state $\Psi_{pi}(q)$, which, following Sawada, may be written as $A_q^+ \Psi_0$ with the operator A_q defined as follows:

$$A_q \equiv N_q \left[\sum_{\substack{p < p_F \\ |p+q| > p_F}} + \sum_{\substack{p > p_F \\ |p+q| < p_F}} \right] \cdot \frac{a_p^+ a_{p+q}}{\hbar\omega - E_{p+q}^0 + E_p^0}, \quad (1)$$

where

$\hbar\omega$: the plasmon energy determined by the dispersion relation,

$E_p^0 = p^2/2m$, m being the electron mass,

a_p, a_p^+ ; the annihilation and creation operators of the solid electrons,

p_F : Fermi momentum,

and

$$\{|N_q|^2\}^{-1} \equiv \left[\sum_{\substack{p < p_F \\ |p+q| > p_F}} - \sum_{\substack{p > p_F \\ |p+q| < p_F}} \right] \cdot (\hbar\omega - E_{p+q}^0 + E_p^0)^{-2}. \quad (2)$$

Using these formulae we can calculate the transition probability in question. If we denote our initial and final state respectively by Ψ_i and Ψ_f and the Coulomb interaction between the fast electron and the electrons in the gas by H_{int} , the transition matrix element can be expressed as

$$\begin{aligned} & (\Psi_f | H_{int} | \Psi_i) \\ &= \frac{4\pi\hbar^2 e^2}{Vq^2} \sum_p (\Psi_0 | A_q a_{p+q}^+ a_p | \Psi_0) \\ &= \frac{4\pi\hbar^2 e^2}{Vq^2} \sum_p (\Psi_0 | [A_q, a_{p+q}^+ a_p]_- | \Psi_0), \end{aligned} \quad (3)$$

since $A_q \Psi_0 = 0$. We have here neglected the exchange effect between the incident electron and the electrons in the solid. Making use of the dispersion relation

and approximating that

$$(\Psi_0|a_p^+ a_p|\Psi_0) = \begin{cases} 1 & \text{for } p \leq p_F, \\ 0 & \text{for } p > p_F, \end{cases}$$

we get from (3)

$$(\Psi_f|H_{\text{int}}|\Psi_i) = N_q. \quad (4)$$

Now (4) is found to be of the following form suitable for comparison with the results of the other methods:

$$|(\Psi_f|H_{\text{int}}|\Psi_i)|^2 = G^{-1} \left(\frac{2\pi e^2 \hbar^3 \omega_p}{Vq^2} \right), \quad (5)$$

where V is the normalization volume, $\omega_p = (4\pi Ne^2/mV)^{1/2}$, N being the total number of electrons in the solid, and

$$G = \frac{1}{N} \times \sum_{p < p_F} \frac{\hbar^3 \omega_p^3 (\hbar\omega - \mathbf{p} \cdot \mathbf{q}/m)}{[(\hbar\omega - \mathbf{p} \cdot \mathbf{q}/m)^2 - (q^2/2m)^2]^2}. \quad (6)$$

The same result was obtained by Ferrel³⁾ in the classical limit using the time-dependent self-consistent field method. This result (5) is different from that of the Bohm-Pines theory by factor G . In the B-P theory

$$G^{-1} = \begin{cases} 1 & \text{for } q \leq q_c, \\ 0 & \text{for } q > q_c, \end{cases}$$

where q_c is the critical momentum of the plasmons, while our G^{-1} drops from 1 to 0 rapidly but continuously as q approaches to q_c . It is to be noted that the damping factor G is essential to explain the experimental fact that the scattering angle of the electron beam has a certain critical value.⁴⁾

The author would like to express his sincere gratitude to Prof. H. Kanazawa for his kind introduction to this field.

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Pion Production in Pion-Nucleon Collision with Assumption of Strong Pion-Pion Interaction

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For the theoretical explanation of a pion-production in pion-nucleon collisions, several models have been proposed. These are the statistical model by Fermi, the strong pion-pion interaction model by Kovacs and others¹⁾, the isobaric model by Sternheimer and Lindenbaum,²⁾ etc. However, the explanation of the experimental results does not yet seem to be satisfactory all over the observed phenomena.

In 1956, we took up the strong pion-pion interaction model and analyzed the pion production experiment at 1.4 Bev. At that time, this interaction was only hypothetical, and no evidence of it had

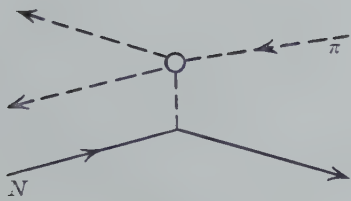


Fig. 1

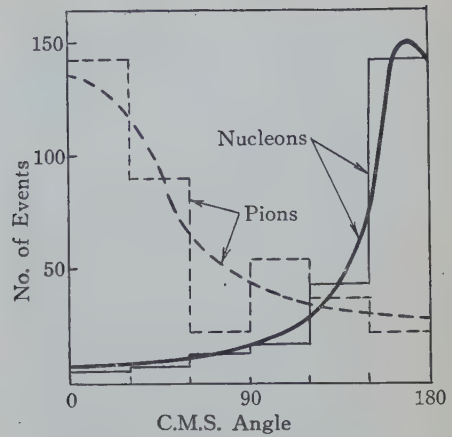


Fig. 2. Angular Distributions

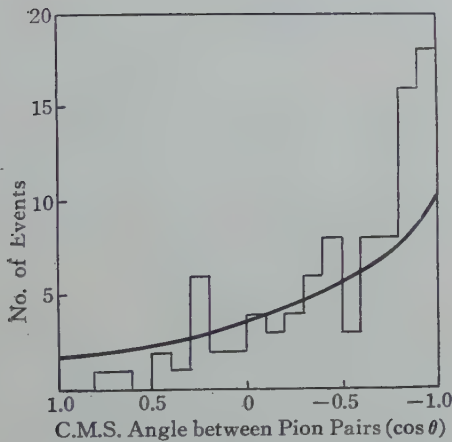


Fig. 3. Angular Correlation

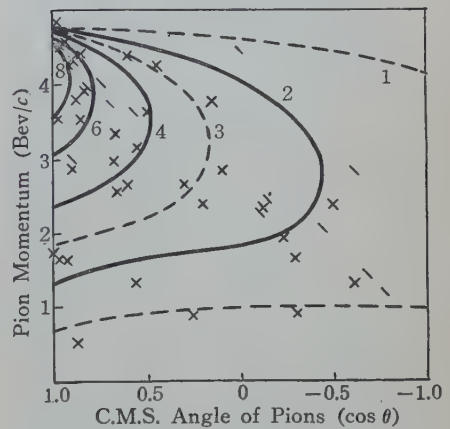


Fig. 4. Momentum-Angular Correlation

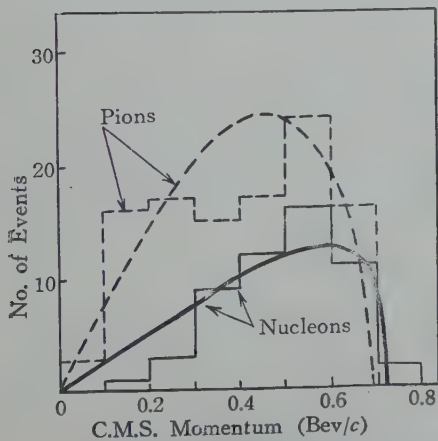


Fig. 5. Momentum Distributions

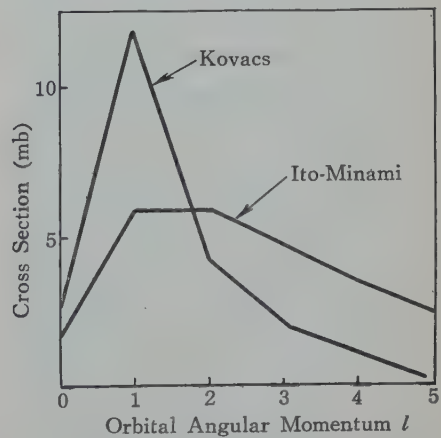


Fig. 6. Contributions from Partial Waves

been given. As the existence of this interaction has recently been remarked, we report here the results of the previous analysis.

It was pointed out by Ito and Minami³⁾ that the strong pion-pion interaction model by Kovacs is a favourable one in that it has the following two natures required from experiments: (1) large angle deflection of nucleons is improbable, and (2) [the matrix elements are not strongly dependent to the direction of emitted pions. Assuming the simplest type of pion-pion interaction, we examined to what extent it could explain the experimental results.

Using the Hamiltonian

$$H = ig\bar{\psi}\gamma_5\tau_\alpha\psi\phi_\alpha + \lambda(\phi_\alpha\phi_\alpha)^2$$

and perturbational calculation for the process of Fig. 1, we obtained the following results for the incident pion at 1.4 Bev.

a) *Angular and Momentum Distribution of Emitted Particles.*

Angular distributions of nucleons and pions are shown in Fig. 2, in comparison with the experimental results by Eisberg et al.⁴⁾ Fig. 3 shows the angular correlation between two pions in comparison with the experimental results by Walker et al.⁵⁾ Momentum distributions of nucleons and pions are given in Fig. 5, and momentum-angular correlation of emitted pions is given in Fig. 4, with the experimental results by Eisberg et al. In Fig. 4, curve $i(i=1, 2, \dots)$ is given by the relation $F(k, \theta) = (i/10)F_{\max}$ where $F(k, \theta) = d^2\sigma/dk \cdot d\cos\theta$. F_{\max} occurs at $\theta \sim 0$, $k \sim 4.3 \mu/c$.

b) *Branching Ratio.*

For the branching ratio

$$R = \frac{\sigma(p + \pi^- \rightarrow n + \pi^+ + \pi^-)}{\sigma(p + \pi^- \rightarrow p + \pi^- + \pi^0)}$$

the Kovacs model gives $R=8$, which is too large compared with the experimental value, $R_{\text{exp}} \sim 1$. This discrepancy is an unfavourable point in the Kovacs model, but it is to be noticed that the recent experiments show a rather high value of R_{exp} , for example, $R_{\text{exp}} \sim 7$ at 1.85 Bev (by Whitten and Block⁶⁾) and $R_{\text{exp}} \sim 2.5$ at 800 Mev (by McCormick and Baggett⁷⁾).

c) *Contributions from Separate Partial Waves.*

In Fig. 6 these contributions are compared with the results of the phenomenological analysis by Ito and Minami⁸⁾. In the Kovacs model, the P -wave contribution is quite dominant and it leads to the difficulty in explaining the magnitude of total cross section.

We have only tried the simplest type of pion-pion interaction, therefore we cannot conclude anything definite about the evidence of strong pion-pion interaction. But it may be noticed that the kinematical side in one pion production process is explained considerably by this model, and we may expect to obtain more desirable results by more advanced treatment of this type of interaction which, for example, takes into account the effect of meson clouds around the nucleon.

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Branching Ratios for the K^- Capture from Hydrogen at Low Energy

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Data on K^- -proton interactions are now available in some detail from the bubble chamber experiments.¹⁾ For K^- capture at rest from hydrogen, the data has increased but being still compatible with the branching ratios $4:2:2:\frac{1}{2}$ for the $\Sigma^- : \Sigma^+ : \Sigma^0 : \Lambda^0$ reaction ratios.

Several attempts²⁾ have so far been made to explain the behaviour of the above ratios and it turned out that the hypothesis of the global symmetry was incompatible with the above ratios under the assumption that the K meson-baryon coupling is weak and $\Sigma' - \Lambda^0$ mass difference can be neglected.

In this letter, the author points out that rather natural explanation of the branching ratios for the $K^- + p \rightarrow \Sigma^{\pm 0}$

$+\pi^{\mp 0}$ may be possible for appropriate magnitude of the pion-baryon coupling constants, at the same time the reason why the global symmetry hypothesis is not compatible with the above reaction ratios is made clear. It will be noticed that this conclusion is insensitive to the $\Sigma' - \Lambda^0$ mass difference and also to the magnitude of the K -meson baryon coupling constants.

To see this, we first direct our attentions to the reactions at zero energy. It is known from the phenomenological analysis¹⁾ that the phase difference between the reaction amplitudes $M_1(T=1)$ and $M_0(T=0)$ is shown to be about $\pm 70^\circ$ in order to fit the ratios for $K^- + p \rightarrow \Sigma^{\pm 0} + \pi^{\mp 0}$ reactions. We assume in the following that K meson is pseudoscalar, relative parity of hyperon and nucleon is even and the capture occurs from the s -state. We can prove the following relation under the assumption of the smallness of the channel for $\Sigma' + \pi \rightarrow \Lambda^0 + \pi$.

Phase of the Reaction Amplitude

$$\cong \delta_{\Lambda^0 N} + \delta_{\Sigma \pi} \quad (1)$$

where $\delta_{\Lambda^0 N}$ and $\delta_{\Sigma \pi}$ mean the real parts of the phase shifts for $K^- N$ and $\Sigma \pi$ scattering, respectively. Then, the above phase difference is entirely [due to the final s -wave $\Sigma' - \pi$ scattering phase shift, i.e. $\delta_{\Sigma \pi}(T=1) - \delta_{\Sigma \pi}(T=0) \cong \pm 70^\circ$, as $\delta_{\Lambda^0 N} = 0$ at zero energy]. This large phase difference cannot be explained if we assume the global symmetry. In fact, then,

$$\begin{aligned} & \delta_{\Sigma \pi}(T=1) - \delta_{\Sigma \pi}(T=0) \\ & \cong 1/3(\delta_{N \pi}(T=3/2) - \delta_{N \pi}(T=1/2)) \end{aligned} \quad (2)$$

which is shown to be about -7° , using

Orear's s -wave pion nucleon scattering formula. On the other hand, if we abandon the hypothesis of the global symmetry, it can be shown that the phase difference in general becomes several times larger than the corresponding s -wave pion nucleon phase shift difference.

The Hamiltonian for the s -wave pion Σ -hyperon interaction can be written as follows. ($\sim 0(\mu/M):M$ baryon mass)

$$H = \frac{g_{\Sigma\Sigma}^2}{2M} (\bar{\Psi}_{\Sigma} \cdot \Psi_{\Sigma}) (\varphi \cdot \varphi) + \frac{1}{2M} (g_{\Sigma\Lambda}^2 - g_{\Sigma\Sigma}^2) (\bar{\Psi}_{\Sigma} \cdot \varphi) (\Psi_{\Sigma} \cdot \varphi). \quad (3)$$

In contrast with the s -wave pion nucleon interaction, this Hamiltonian gives rise to the splitting between $\delta_{\Sigma\pi}(T=1)$ and $\delta_{\Sigma\pi}(T=0)$ in the lowest order expansion if $g_{\Sigma\Lambda}^2 \neq g_{\Sigma\Sigma}^2$, which means that the splitting between δ_1 and δ_0 can be roughly speaking, M/μ (μ is the pion mass) times larger than the corresponding s -wave pion nucleon phase splitting. In fact, perturbation or Tamm-Dancoff treatment yields the correct order of magnitude of the phase difference for the appropriate magnitude of the coupling constant ($|g_{\Sigma\Lambda}^2/4\pi + g_{\Sigma\Sigma}^2/4\pi| \approx 3 \sim 10$). It is to be noticed that to the lowest order in μ/M , the reaction $\Sigma^+ + \pi \rightarrow \Lambda^0 + \pi$ does not occur.

After the large phase difference is obtained, it is an easy matter to reproduce the observed branching ratios for $K^- + p \rightarrow \Sigma^{\pm 0} + \pi^{\mp 0}$. Finally, we comment on the K^- capture in flight. In this case, the ratio Σ^-/Σ^+ , based on the bubble chamber data, seems to increase rapidly to 7 at about 100 Mev and then levels off to a value about

unity. However, the emulsion data of comparable statistics do not support this rapid variation of the Σ^-/Σ^+ ratio. So, one may simply assume that the ratio Σ^-/Σ^+ smoothly changes to a value about unity, though we cannot of course draw any definite conclusion on this point from the data so far presented.

Then, the phase difference mentioned above must change to a value about 90° in order to fit the Σ^-/Σ^+ ratio if the magnitudes of the matrix elements M_1 and M_0 do not vanish. The change of this phase difference ($\approx 20^\circ$) is mainly induced by $\delta_{\Lambda N}$ phase shift difference because the final state pion energy changes little in this energy range. Rough estimate can be made by the zero range approximation developed by D. Jackson et al. with solution A_+ and B_+ .¹⁾³⁾ The result turns out to be about $\pm 30^\circ$ at 20 Mev which fairly well agrees with our expectation.

Details will be published in this journal.

The author wishes to express his sincere thanks to Dr. H. Miyazawa for his kind guidance and encouragement. He is also indebted to the Yomiuri-Yukawa Fellowship for financial aid.

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On a Variation Principle for Calculating the Electrical Conductivity

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June 24, 1959

We have recently proposed a new method of calculating the conductivity and given the formula for the electrical conductivity.¹⁾ Several authors have tried to develop and generalize the method so that it affords different kinds of kinetic coefficient.²⁾ We suggest here to formulate the problem in the variation principle. An expression, which is a functional of the state functions relating to the density matrix of the system, is optimized with regard to those state functions. The equation of the condition that the expression should be extremum is equivalent to the Schrödinger equation of the system in the presence of the electric field with appropriate boundary conditions, and the extremum value is just a component of the electrical conductivity tensor. In case that the problem can be described in terms of one-body picture, e. g., as in the one of the system having the impurity atoms as the scattering centers of electrons, it is more practical to express the variation principle in terms of the one-body picture. A perturbation approximation of this case is equivalent to the Kohler-Sondheimer principle of variation.³⁾

We consider the system described by the Hamiltonian H . For the sake of simplicity we assume the charge carriers

are of one kind with charge e . In presence of the electric field the part expressing the electrostatic potential of charge carriers is added to the Hamiltonian, which is written as

$$eE_\nu x_\nu \equiv eE_\nu \sum_{n=1}^N x_{n\nu} \quad (\nu = x, y, z), \quad (1)$$

we mean by $x_{n\nu}$ the ν -component of the coordinate of the n -th carrier, E_ν that of the applied electric field.

The Schrödinger equation for the density matrix of the system,

$$i \frac{\partial \rho(t)}{\partial t} = [H, \rho(t)] + [eE_\nu x_\nu, \rho(t)], \quad (2)$$

is investigated in the linear approximation in regard to the field and in the following we shall assume two kinds of boundary conditions: the one is

$$E_\nu(t) = E_\nu \cdot e^{st} \quad (t < 0) \quad (2a)$$

and

$$\lim_{t \rightarrow -\infty} \rho(t) = \rho_c, \quad (2a')$$

the other is

$$E_\nu(t) = E_\nu \cdot e^{-st} \quad (t > 0) \quad (2b)$$

and

$$\lim_{t \rightarrow \infty} \rho(t) = \rho_c, \quad (2b')$$

where ρ_c represents the canonical distribution.

In order to derive the solution, putting

$$\rho(t) = \rho_c + F^{(\pm)} e^{\pm st}, \quad (3)$$

and substituting this into (2), we get

$$\pm is F^{(\pm)} = [H, F^{(\pm)}] + [eE_\nu x_\nu, \rho_c], \quad (4)$$

respectively for (2a, a') and (2b, b').

Putting

$$F_\nu^{(\pm)} = E_\nu \rho_c \int_0^\beta d\lambda e^{\lambda H} \mathcal{P}_\nu^{(\pm)} e^{-\lambda H}, \quad (5)$$

we have the variation principle that the expression,

$$\begin{aligned} \sigma_{\mu\nu}(\Psi) = & \int_0^\beta d\lambda T_r \{ \Psi_\mu^{(-)} \rho_C e^{\lambda H} (s \Psi_\nu^{(+)} \\ & + i[H, \Psi_\nu^{(+)}]) e^{-\lambda H} \} \\ & + \int_0^\beta d\lambda T_r \{ \Psi_\nu^{(+)} \rho_C e^{\lambda H} J_\mu e^{-\lambda H} \} \\ & - \int_0^\beta d\lambda T_r \{ \Psi_\mu^{(-)} \rho_C e^{\lambda H} J_\nu e^{-\lambda H} \} \quad (6) \end{aligned}$$

should be extremum in regard to the state functions: $\Psi_\nu^{(+)}$ and $\Psi_\mu^{(-)}$, where J_μ and J_ν are the μ - and ν -components of the current density operator and β is the inverse of the absolute temperature times the Boltzmann constant. The extremum condition that the functional derivatives of $\sigma_{\mu\nu}(\Psi)$ in respect to $\Psi_\mu^{(-)}$ and $\Psi_\nu^{(+)}$ do vanish reduces to precisely to Eq. (4). When the solutions of Eq. (4) are substituted into (6), we obtain the formula for the electrical conductivity,

$$\sigma_{\mu\nu} = \int_0^\infty dt \int_0^\beta d\lambda T_r \{ \rho_C J_\nu J_\mu(t+i\lambda) \} \quad (7)$$

in the limit $s \rightarrow 0$.

In the case where it is possible to express the problem in the one-body picture, the expression (6) is reduced to the form,

$$\begin{aligned} \sigma_{\mu\nu}(\psi) = & \frac{i}{2} \lim_{\varepsilon \rightarrow 0} \int_0^\infty dt e^{-\varepsilon t} \text{tr} \{ f_c ([\psi_\mu^{(-)}(t) \\ & - \psi_\mu^{(-)}(-t), s \psi_\nu^{(+)} + i[h, \psi_\nu^{(+)}]) \\ & + [\psi_\nu^{(+)}(t) - \psi_\nu^{(+)}(-t), j_\mu] \\ & - [\psi_\mu^{(-)}(t) - \psi_\mu^{(-)}(-t), j_\nu] \} \quad (8) \end{aligned}$$

where the operators written in small letters and $\text{tr}(\dots)$ represent the oper-

ators and the trace in the one-body picture. These one-body operators are related to the operators in (6) as

$$H = \sum_{m,n} (m|h|n) a_m^* a_n, \quad (9a)$$

$$J_\nu = \sum_{m,n} (m|j_\nu|n) a_m^* a_n, \quad (9a)$$

$$\Psi_\nu^{(\pm)} = \sum_{m,n} (m|\psi_\nu^{(\pm)}|n) a_m^* a_n, \quad (9b)$$

$$(m|f_c|n) = T_r(\rho_C a_m^* a_n). \quad (9c)$$

And, further, $\psi_\nu^{(\pm)}(t)$ are the Heisenberg operators,

$$\psi_\nu^{(\pm)}(t) = e^{iHt} \psi_\nu^{(\pm)} e^{-iHt}.$$

Use is made of the second quantization scheme, that is a_m^* and a_n represent the creation and annihilation operators for the state n in the one-body problem of the charge carrier. The variation problem is formulated so that the functional, $\sigma_{\mu\nu}(\psi)$, of the operators $\psi_\mu^{(-)}$ and $\psi_\nu^{(+)}$ should be extremum in regard to these operators. The extremum value is expressed as

$$\begin{aligned} \sigma_{\mu\nu} = & \frac{i}{2} \lim_{\varepsilon \rightarrow 0} \lim_{s \rightarrow 0} \int_0^\infty dt \int_0^\infty dt' \\ & \times e^{-\varepsilon t - st'} \text{tr} \{ f_c [j_\nu(t) - j_\nu(-t), j_\mu(t')] \}. \quad (10) \end{aligned}$$

This is the formula for the electrical conductivity in terms of one-body picture.

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A Proposed Model for the Explanation of the Urbach Rule

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The role of electron-lattice interaction in the fundamental optical absorption in insulator has been discussed by many authors. In the case of band-to-band transition, it causes a temperature dependent shift of the absorption edge¹⁾. The line shape of the exciton absorption band in the central portion was discussed using the method of generating function.²⁾ In both cases, the electron-lattice interaction also causes indirect transition by making the electron system violate the momentum selection rule.³⁾ Most of the experimental results on the fundamental absorption of insulators can be explained qualitatively by means of these theories, with one very remarkable exception.

That is the so-called Urbach rule⁴⁾, according to which the absorption coefficient A in the long wave-length part of the fundamental absorption decays exponentially with the decrease of photon energy $h\nu$:

$$A(\nu) \propto \exp\{-\sigma(h\nu_0 - h\nu)/\kappa T\}, \quad (1)$$

where T is the absolute temperature and σ is a dimensionless constant of the order of unity. This rule is valid for many ionic crystals (silver-halides^{4),5)}, alkali halides⁶⁾, cadmium-sulfides⁷⁾, thalliumchloride⁸⁾, etc.) for the range of A as wide as several decades, and for moderate and higher temperatures.

Dexter⁹⁾ proposed an explanation by making use of the deformation potential. His idea consists in that the energy of the excited states changes by $\delta E = -E_1 \Delta$, while the probability distribution of the local dilation Δ is given by $\exp(-\Delta^2/C\kappa T)$ with an appropriate elastic constant C . As the result A is given by $f(\nu, T)\exp\{-(h\nu_0 - h\nu)^2/E_1^2 C\kappa T\}$ where $f(\nu)$ is a factor originating from the energy band width. By a suitable choice of parameters, he obtained a curve which resembles to (1) in a certain range of A . However, it seems difficult to fit it to (1) through the whole range especially in cases of KI and KBr where (1) is valid while A varies as large as 10^{5-6} times.

By an extension of Dexter's idea, we obtained a satisfactory explanation for the whole range. As we are concerned with the tail part of the absorption band, the important contribution comes from those parts of the crystal which deforms markedly, so that it is necessary to add a quadratic term to the energy change:

$$\delta E = E_1 \Delta + E_2 \Delta^2. \quad (2)$$

A similar term was taken by Koshino¹⁰⁾ in discussing the temperature dependence of mobilities in semiconductors. If we take into account only the second term

(E_2 , being assumed negative), and follow Dexter's procedure, we obtain the absorption constant

$$\exp\{-(h\nu_0 - h\nu)/|E_2|\kappa T\}$$

multiplied by a factor with slow ν -dependence, and the values of $|E_2|$ determined by comparison with experimental data are several eV, which seem to be reasonable values.

However, we encounter a new difficulty. The width of the central part of the excitation absorption band is due to the linear term $E_1\mathcal{A}$, and $|E_1|$ can be estimated to be several eV from the observed widths.²⁾ Then the linear term is not negligible to the quadratic even for the tail part where $-\delta E = h(\nu_0 - \nu) \sim 0.5$ eV (ν_0 : the absorption peak).

Consequently, it is necessary to assume that the mode of lattice vibration responsible for the tail part is different from that mode which contributes to the width of the central portion, the latter being probably the longitudinal acoustical mode as was discussed in (I). Then we have two possibilities for the former modes. The first is the transverse acoustical mode which is expected to have usually rather small linear term of the exciton-lattice interaction. In fact it vanishes exactly in the continuum approximation for cubic crystals. Moreover, there is some reason to believe that the quadratic term is rather large in case of alkali- and silver-halides, because in these crystals a positive hole has strong tendency of forming a molecular binding between a pair of neighbouring halide ions, and lattice deformation makes pairs of halide ions with certain relative direction to draw nearer

to each other.

The second possibility is the longitudinal optical mode, which is expected to have relatively small linear terms of exciton-lattice interaction if the effective masses of an electron and a hole are not very different, because the exciton is electrically neutral. As was noted in (I), however, this interaction has large matrix elements between different exciton energy bands. In other words, the polarization wave $\mathbf{P}(\mathbf{r})$ of ionic lattice causes electric field $\mathbf{F}(\mathbf{r})$, which in turn polarizes the exciton, the energy change of the latter being given by $\delta E = -\alpha \mathbf{F}^2/2 \propto -P^2$. Assuming that the exciton is in the hydrogen-like $1s$ state, and making use of the fact that polarizability of the hydrogen atom is given by $\alpha_H = (9/2) \cdot a_H^3$, we get the quadratic exciton-lattice interaction:

$$\begin{aligned} H_{eL} = \delta E = & -\frac{9}{4} \left(\frac{m}{\mu} \kappa_0 \alpha_H \right)^3 \\ & \times \frac{2\pi\hbar\omega}{V} \left(\frac{1}{\kappa_0} - \frac{1}{\kappa} \right) \\ & \times \left\{ \sum_w (b_w e^{i\mathbf{w}\cdot\mathbf{r}} + b_w^* e^{-i\mathbf{w}\cdot\mathbf{r}}) \right\}^2. \quad (3) \end{aligned}$$

μ and m are the exciton reduced mass and the true electron mass, respectively, κ and κ_0 are static and optical dielectric constants, $\hbar\omega$ the phonon energy and V the total volume.

We now make use of the general formula presented in (I) and note that the generating function for small values of t is important in determining the absorption coefficient at the tail part. In this limit, with H_{eL} given by (3), each term U_n of the expansion (I, 3.15) can be calculated by collecting all possible graphs, with the result:

$$\sum_{n=0}^{\infty} U_n(t) = \left(1 - \frac{it}{\hbar} B\right)^{-1/2}, \quad (4)$$

$$A(\nu) d\nu = \left(\frac{h\nu_0 - h\nu}{B(T)}\right)^{-1/2} \times \exp\left(-\frac{h\nu_0 - h\nu}{B(T)}\right) \frac{d(h\nu)}{\sqrt{\pi} B}. \quad (5)$$

(5) is normalized to unity, and B is given by

$$B(T) = 18\pi \left(\frac{1}{\kappa_0} - \frac{1}{\kappa}\right) \times \left(\frac{m}{\mu} \kappa_0 a_H\right)^3 / v_0 \cdot \frac{\hbar\omega}{2} \coth\left(\frac{\hbar\omega}{2\kappa T}\right), \quad (6)$$

and is proportional to T at high temperatures. (5) practically fits the experimental formula (1) in the whole range of ν in which (1) was ascertained by experiments, because the first factor of (5) has a much slower ν -dependence. In order to get the values $\sigma=0.79$ (determined by Martienssen⁶) from his data) both for KBr and KI, from Eq. (6) with known values of κ_0 , κ , ω and v_0 (the volume of a unit cell), we have to take $\mu/m=0.64$ and 0.63 , respectively, which are reasonable values. However, Martienssen's data on KBr at the lowest temperature ($T=20^\circ\text{K}$) shows that (1) is valid with T replaced by $T_0 \sim 60^\circ\text{K}$, while according to our theory $T_0=120^\circ\text{K}$ for longitudinal optical mode (see (6)) and $T_0 \sim 50^\circ\text{K}$ for transverse acoustic modes (averaged over propagation direction). It seems that in alkali halides, the linear interaction between an exciton and optical phonons is appreciable due to the large difference in effective masses, preventing quadratic term from contributing to the tail part, and transverse acoustic mode plays the

main rôle.

Although there still remains some difficulty in the general derivation of the rule, especially concerning the effect of width of the exciton energy band (or of the electron-and-hole-bands, in the case of band-to-band transition), it seems to be rather well established that the quadratic interaction of the exciton (or, the electron and the hole) with those modes of lattice vibrations which do not make appreciable contributions to the main part of the exciton (or band-to-band) absorption plays an essential rôle in causing the tail part described by Urbach's rule. The detail of this work will soon be reported in this journal.

The writer wishes to express his gratitude to Professor T. Muto for important remarks, especially on the low temperature behavior. He is also indebted to Professor D. L. Dexter, Professor F. Urbach and their collaborators for valuable discussions on this problem.

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Angular Distributions of (^{14}N , α) Reactions

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Recently, in the 28 MeV ^{14}N bombardment of ^{27}Al and ^{16}O , differential cross sections for the emission of protons and alpha particles with various energies were measured by Zucker et al.^{1,2)} over the angular range between 0° and 100° . According to them, protons are emitted isotropically while the angular distributions of alpha particles show definite forward peaks and something like diffraction minima for high energy groups. This fact cannot be explained simply by the compound nucleus formation process and it suggests the occurrence of some direct reactions. After the analogy of the stripping process, we consider in this note that a part of the nitrogen nucleus, ^{10}B , is captured, and the rest, alpha particle escapes.

To treat process $(\alpha + ^{10}\text{B}) + \text{target} \rightarrow \alpha + (^{10}\text{B} + \text{target})$ in the Born approximation, we make the following assumptions. First, adopting the alpha particle model for ^{14}N , we suppose that only the states in which the alpha particle and ^{10}B in the ^{14}N nucleus are separated beyond some extent, say a , contribute

to the process, and further, for convenience sake, the relative angular momentum between ^{10}B and alpha particle in ^{14}N is assumed to be zero. Second, we consider that the reaction is caused by the interaction between ^{10}B and the target nucleus when the former comes into contact with the surface of the latter. Finally, as the excitation energy E of the residual nucleus in the experiment is high, the cross sections obtained by assigning definite final states must be summed over all the final states in the appropriate energy interval between E and $E + dE$. For the case in which the final nucleus is in a so highly excited state that it can disintegrate into initial nucleus emitting ^{10}B , these summations have been worked out by Bhatia et al.;³⁾ we borrow their result and suppose it to be valid even for about 10 MeV excitation of the residual nucleus.

Then, the differential cross section of the alpha particles emitted in θ direction and energy interval E_α and $E_\alpha + dE_\alpha$, is simply proportional to

$$\left| \int_a^\infty R(r) j_0(kr) r^2 dr \right|^2,$$

with

$$k = k_i - k_f M_B / M_N,$$

where k_i , k_f is the momentum vector of the incident ^{14}N and outgoing alpha particle, and M_B , M_N are the mass of ^{10}B and ^{14}N respectively. $R(r)$ describes the radial part of the wave function of relative motion of the alpha particle and ^{10}B in ^{14}N , j_0 is the spherical bessel function of order zero.

The above considerations were applied

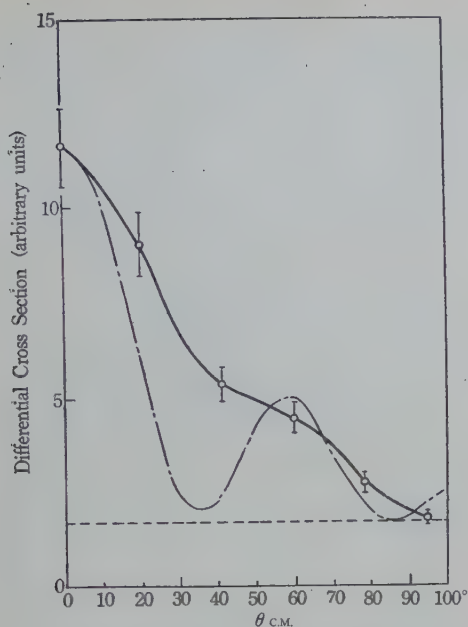


Fig. 1. Angular distribution of alpha particles from the reaction $^{16}\text{O}(^{14}\text{N}, \alpha)^{26}\text{Al}^*$. The residual excitation energy is 8 MeV. Solid line: experimental; dashed line: compound nucleus process (assumed); dot-dashed line: direct+compound nucleus process.

to the typical case of $^{16}\text{O}(^{14}\text{N}, \alpha)^{26}\text{Al}^*$ with 26 MeV ^{14}N and the residual excitation energy of 8 MeV.²⁾ This is shown in Fig. 1. Here we have subtracted an isotropic part from the experimental angular distribution as due to the compound nucleus process, and compared the rest with the calculation normalized at zero degree. Parameter a is taken as 5.3×10^{-13} cm, which corresponds to the sum of the radius of alpha particles and ^{10}B when we take $r_0 = 1.4 \times 10^{-13}$ cm. The forward peak may have resulted from the assumption of zero relative angular momentum between the alpha particle and ^{10}B ; but in order to say which value of the relative angular momentum is best, it

will be necessary to collect more data and clarify the choice of other parameters.

We should like to thank Professor S. Takagi and Mr. S. Yamasaki for their valuable discussions and careful reading of the manuscript.

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Energy Levels of Zr^{90}

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Ford's prediction,¹⁾ based on simple shell model considerations, of a 0^+ level as the first excited state of Zr^{90} was soon experimentally confirmed.²⁾ Recently Lazar et al.³⁾ have measured the low-lying energy levels of Zr^{90} and found them to be in agreement with the qualitative conjectures of Ford. However, they report that a quantitative calculation by Lane³⁾ based on a short-range interaction between the nucleons, gives poor agreement with the experimentally observed splitting of the $(g_{9/2})^2$ configuration. We have calculated the level-scheme for a more realistic shell model, by taking a finite range for the nuclear interaction, and obtained con-

siderably better agreement with the experimental results.

Following the two-nucleon model as suggested by Ford,¹⁾ we consider the low levels of Zr^{90} to arise from the interactions of the last two protons in the configurations $(p_{1/2})^2$, $J=0$; $(g_{9/2})^2$, $J=0, 2, 4, 6, 8$; and $(p_{1/2} g_{9/2})$, $J=4, 5$. From the data on Y^{89} we take the separation energy of the single particle levels to be 1.0 Mev.⁴⁾ We assume harmonic oscillator wave functions for the nucleons, and a central two-body internucleon potential of the gaussian shape viz., $(a+b\sigma_1 \cdot \sigma_2)\exp(-r^2/r_0^2)$ where a and b are parameters giving the exchange character of the potential, and are determined by comparison with the experimental data; r_0 denotes the range of the interaction and only enters the calculations in the combination, $\lambda = r_0/r_1$, where r_1 determines the range of the nucleon wave-function.⁵⁾

The energy levels are calculated for $\lambda=0, 0.5$, and 1.0 , and a and b are determined by fitting the experimental energy separations of the 2^+ , 4^+ and

6^+ levels except for $\lambda=0$. In this last case only one parameter is needed, and this is fixed by fitting the experimental separation of 2^+ and 6^+ levels. Table I lists the excitation energies of the levels relative to the ground state 0^+ .

We note that the separation of the two 0^+ levels and the splitting of the levels of $(g_{9/2})^2$ configuration are quite sensitive to the range parameter λ . It is clear that a satisfactory agreement with the experimental results can be obtained only for a rather large value of λ viz. $\lambda \approx 1.0$. The energy values reported in Table I do not include the effect of the repulsion of the 0^+ levels. A rough estimate shows this to be small, giving a shift of ≈ 0.1 Mev. in each of the two 0^+ levels, for $\lambda=1.0$. This, however, further improves the agreement between the predicted and the observed values; in particular, the discrepancy between the predicted and the observed separation of the 0^+ levels is now reduced to 0.38 Mev. We believe that in view of the simplicity of the model, the agreement between theory and experiment is quite satisfactory.

The values of the parameters a and b for $\lambda=1.0$ are found to be -17.5 Mev and 3.8 Mev respectively. It may be noted that the large value of the range and the relative strengths of the spin-independent and spin-dependent interactions are in qualitative agreement with the results obtained for nuclei near $A=40$.⁶⁾

We finally remark that while this model provides reasonably good agreement for a splitting of the energy levels within a given configuration, the relative separations of the levels of different

Table I. Calculated and experimental energy levels

Configuration	J	Calculated energy in Mev.			Experimental energy (Mev.)
		$\lambda=0$	$\lambda=0.5$	$\lambda=1.0$	
$(p_{1/2})^2$	0^+	0	0	0	0
$(p_{1/2} g_{9/2})$	4^-	2.50	3.39	2.24	?
	5^-	1.14	2.30	1.72	2.32
$(g_{9/2})^2$	0^+	-4.00	-0.92	1.18	1.76
	2^+	1.68	2.80	2.04	2.19
	4^+	2.53	3.60	2.93	3.08
	6^+	2.94	4.00	3.32	3.45
	8^+	3.20	4.51	3.38	3.59

configurations is not so well given. However, the calculation predicts the separation of the levels of the configuration ($p_{1/2} g_{9/2}$) to be ≈ 0.5 Mev., and this would place the as yet unobserved 4^- level at about 2.8 Mev., i.e. very close to the 4^+ level.

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Determination of Velocity Distribution of Electrons from Observed Spectral Distribution of Bremsstrahlung*

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The cross-section of collision between

a positive ion of charge Ze and an electron of energy x , emitting a photon of energy $y = h\nu$ in the process, is given by^{1,2)}

$$\begin{aligned} &\phi(x, y) d(y/x) \\ &= (A''/y) \ln \{ (\sqrt{x} + \sqrt{x-y}) / \sqrt{y} \}^2 \\ &\times f \cdot d(y/x) \end{aligned} \quad (1)$$

where $A'' = 8Z^2 e^6 / 3\hbar mc^3$,

m is the mass of electron and f a factor, introduced by Sommerfeld²⁾ to take into account the nonvalidity of Born approximation. To keep the mathematics simple, in what follows we shall assume Born approximation to be valid and hence $f=1$.

The spectral distribution of bremsstrahlung per unit volume is given by

$$\begin{aligned} F(y) dy &= \eta_+ dy \int_{x=y}^{x=\infty} y \phi(x, y) \\ &\times \frac{v^2}{x} n(v) v dv \end{aligned} \quad (2)$$

where $F(y) dy$ is the energy of bremsstrahlung photons per unit volume per unit time between the energies y and $y+dy$, η_+ is the number of ions per unit volume and $v^2 n(v) dv$ is the number of electrons per unit volume having velocities between v and $v+dv$.

Combining Eqs. (1) and (2) and remembering $x = (1/2)mv^2$, we obtain

$$F(y) = A \int_y^\infty \ln \left\{ \frac{\sqrt{x} + \sqrt{x-y}}{\sqrt{y}} \right\} N(x) dx \quad (3)$$

* Work sponsored by Rome Air Development Center.

where

$$A = 32 Z^2 e^6 \eta_+ / 3 \hbar m^3 c^3$$

$$N(x) = n(v) = n(\sqrt{2x/m}) \quad (3A)$$

$$\text{and } n(v) = N(mv^2/2). \quad (3B)$$

Eq. (3) may be solved by a method similar to that used by Montroll³⁾ for an analogous problem. The result is

$$N(x) = (2\pi A)^{-1} \int_{-\infty}^{+\infty} \frac{x^{-i u}}{I(u)} du \\ \times \int_0^{\infty} y^{i u - 2} F(y) dy \quad (4)$$

where

$$I(u) = \int_0^{\infty} e^z \ln \{ \sqrt{e^z} + \sqrt{e^z - 1} \} e^{-i u z} dz.$$

Another method of solution is based on the assumption that $N(x) \rightarrow 0$ as $x \rightarrow \infty$ and $N(x)$ is finite for $x=0$. Further, it is assumed that $F(y)$ is differentiable. Hence, upon differentiating (3) with respect to y , we get

$$-2yF'(y) = \int_y^{\infty} \sqrt{\frac{x}{x-y}} N(x) dx. \quad (5)$$

With the substitutions $x=1/t$ and $y=1/u$ we may bring (5) to the form

$$f(u) = \int_0^u \frac{G(t)}{\sqrt{u-t}} dt \quad (6)$$

where

$$f(u) = -2u^{-3/2} F'(u^{-1})$$

$$G(t) = t^{-2} N(t^{-1}).$$

Since (6) is an integral equation of the Abel type⁴⁾, we immediately have that

$$G(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \frac{f(u)}{\sqrt{t-u}} du. \quad (7)$$

In our original notation, the solution is given by

$$N(x) = -\frac{1}{\pi} \frac{d}{dx} \int_x^{\infty} 2x^{1/2} \frac{F'(y)}{\sqrt{y-x}} dy. \quad (8)$$

Eqs. (3) and (8) can be used to determine $N(x)$ from $F(y)$.

As pointed out earlier this treatment holds only when Born approximation is valid and hence should be used, only in regions of validity of Born's approximation. This limitation can be easily overcome by using Sommerfeld's²⁾ cross-section for bremsstrahlung, which, however, entails more laborious mathematics.

$F(y)$ cannot be experimentally determined for all values of y but extrapolation methods can be used with ample justification at some temperatures because $F(y) \rightarrow \text{constant}$ at low values of y and $F(y) \rightarrow 0$ at high values of y . The total number of ions per unit volume η_+ can be measured by standard methods since the number of electrons per unit volume $\eta_- = Z\eta_+$. Starting from an experimental value for n_- one can determine the velocity distribution, which in turn gives η_- by integration. Hence the method of successive approximation may be used to get more and more accurate velocity distribution and η_- .

$F(y)$ refers to emission per unit volume per unit time neglecting self absorption. However, the coefficient of absorption is related to $F(y)$ by Kirchoff's Law...

$$\frac{F(y)}{\chi(y)} = B(y) = \frac{2y^3}{c^2 h^2} \\ \times \{1 - \exp(-y/kT)\}^{-1} \\ (\text{Planck's Function}). \quad (9)$$

The net emission per unit area of surface per unit time $\phi(y)$ can be expressed in terms of $F(y)$ and $\chi(y)$ if the geometry of the radiating plasma is known. For a plasma enclosed between two parallel infinite planes at a distance l Allen and Hindmarsh⁵⁾ have shown that

$$\phi(y) = B(y) \left[1 - \exp\left(-\frac{l\chi(y)}{4\pi}\right) \cdot \left\{ 1 - \frac{l\chi(y)}{4\pi} \right\} - \left(\frac{l\chi(y)}{4\pi} \right)^2 E_1\left(\frac{l\chi(y)}{4\pi}\right) \right] \quad (10)$$

where

$$E_1(z) = \int_1^{\infty} \frac{\exp(-zt)}{t} dt.$$

It can be shown that

$$\phi(y) = \pi B(y) \quad \text{when} \quad \frac{l\chi(y)}{4\pi} \gg 1,$$

$$\text{and } \phi(y) = \frac{l}{2} F(y) \quad \text{when} \quad \frac{l\chi(y)}{4\pi} \ll 1.$$

Hence to get information about $F(y)$ and subsequently electron distribution one should work with plasmas of smaller dimensions (provided the radiation is of sufficient measurable intensity).

It may be repeated that the velocity distribution is given by $v^2 N(mv^2/2) dv$ from which the energy distribution can also be derived.

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Diamagnetism of a Dense Electron Gas

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G. Wentzel¹⁾ has investigated the effect of the Coulomb interaction on the electronic diamagnetic susceptibility, with use of the reduced Hamiltonian which is equivalent to the pair theory of Sawada et al.²⁾ He obtained the result that the usual diamagnetic susceptibility of free electron gas (Landau diamagnetism) is unaffected by the Coulomb interaction. His argument is based on the fact that the interaction Hamiltonian H_c , expressed in terms of pair operators, is unaffected by the unitary transformation which eliminates the magnetic interaction H' .

Wentzel's argument is, however, not rigorous because he takes only that part as the magnetic interaction H' , which creates or annihilates electron-hole pairs.

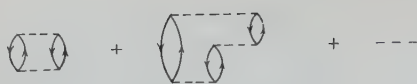


Fig. 1

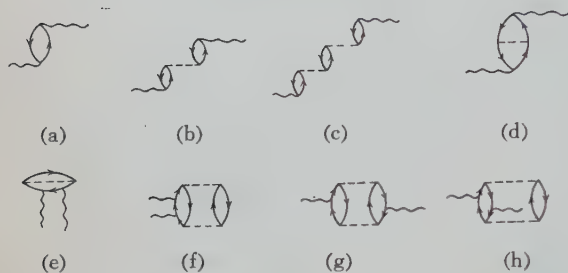


Fig. 2

Other parts, which scatter electrons or holes, do affect the total energy of free electron system.

Write the Hamiltonian and the energy of the system:

$$H = H_0 + H_e + H' + H''$$

$$E(A) = E_f(A) + E_{ex}(A) + E_{corr}(A)$$

$$\equiv E(0) + \Delta E(A)$$

where A means vector potential, H' and H'' are magnetic interaction of first and second order with respect to A , and $\Delta E(A)$ is the increase of energy by the magnetic field.

According to Gell-Mann and Brueckner³⁾, to the high density limit the correlation energy $E_{corr}(0)$ can be calculated by summing the terms which are represented graphically by Fig. 1. Effect of H' is shown in Fig. 2 and the contribution from H'' to the energy is only its diagonal part. In Fig. 2 the dotted line and the wavy line represent H_e and H' respectively.

Contribution from each graphs in Fig. 2 to the total energy $E(A)$ can be calculated by an elementary perturbation

method, at least to the first few terms. We write them E_a , E_b , etc. E_a gives, with the contribution from H'' , Landau diamagnetism. Contributions from the graphs of type (b), (c), are that part which was calculated by Wentzel and they really vanish, as is easily seen. As an example we consider graph (b). We obtain ($\hbar=1$)

$$E_b = \frac{1}{2} (e/mc)^2 \sum_k V(k) \times \sum_p \sum_{p'} [(p+k/2) A(k)] \times [(p'+k/2) A(-k)]$$

$$\times \frac{n_{p+k}^0 - n_p^0}{\varepsilon_{p-k}^0 - \varepsilon_p^0} \cdot \frac{n_{p'+k}^0 - n_{p'}^0}{\varepsilon_{p'+k}^0 - \varepsilon_{p'}^0} = 0,$$

where $A(k)$ and $V(k)$ are Fourier components of vector potential and Coulomb potential, $\varepsilon_p^0 = p^2/2m$ and n_p^0 is the occupation number of the electron of momentum p in the Fermi sphere.

E_d , E_e , ... are those parts which was dropped in Wentzel's calculation. Examining E_d and E_e , we get the result

$$\begin{aligned} \Delta E_{ex}(A) &= F_{ex}(A) - E_{ex}(0) \\ &= -\frac{k_0^2 e^4}{16\pi^3 c^2} \sum_q \sum_{\alpha, \beta=1}^3 A_\alpha(q) A_\beta(-q) \\ &\quad \times \left[\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2} \right]. \end{aligned}$$

Susceptibility χ is related to $\Delta E(A)$ by the relation

$$\begin{aligned} \Delta E(A) &= -(\chi/2) \sum_q \sum_{\alpha, \beta} A_\alpha(q) A_\beta(-q) \\ &\quad \times [\delta_{\alpha\beta} q^2 - q_\alpha q_\beta] \cdot [1 - 0(q^2)]. \end{aligned}$$

Thus we can conclude from the above result that the susceptibility diverges in the simple perturbational calculation.

Just as in the case of correlation energy²⁾³⁾ or electronic specific heat⁴⁾ of the dense electron gas, we get a finite result if we treat the effect of H_c exactly. We are performing such a calculation at present, utilizing the method of Green's function. Detailed discussion will be presented in near future.

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and R. Brout, Phys. Rev. **108** (1957), 507.
- 3) M. Gell-Mann and K. A. Brueckner, Phys.
Rev. **106** (1957), 364.
- 4) M. Gell-Mann, Phys. Rev. **106** (1957), 369.

Effects of Potential Well Parameters on the Polarization of Elastically Scattered Protons

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July 7, 1959

Polarization experiments with high energy nucleons are usually analysed by the use of an optical model potential of the form

$$V = V_c \rho + V_s (\hbar/\mu c)^2 1/r \cdot d\rho/dr \cdot \sigma \cdot \mathbf{l}.$$

The complex well depths V_c and V_s , together with the parameters characterizing ρ , the radial dependence of the potential, are called well parameters.

In a recent paper¹⁾ Sternheimer studied

the effects of V_{sl} (the imaginary part of V_s) on the calculated polarization of 150 Mev. protons elastically scattered by emulsion nuclei (Ag and Br). He fixed the values of all the other well parameters so that the best fit to the experimental results is obtained and, by varying V_{sl} , he calculated three polarization curves by the WKB method. Among the conclusions he obtained are:

- (i) P_{\max} , the maximum polarization inside the first diffraction peak, increases as $|V_{sl}|$ increases.
- (ii) P_{\max} appears at a larger scattering angle as $|V_{sl}|$ increases.
- (iii) The dip of the polarization curve at the first diffraction minimum becomes narrower and deeper as $|V_{sl}|$ increases.

Previous to this, the present author developed an approximate expression for polarization with use of the distorted wave Born approximation.²⁾ The purpose of this note is to check Sternheimer's conclusions by this method which, although inferior to the WKB method in accuracy, can handle the problem more easily. To this end, the pairs of curves corresponding to the following well parameters were compared.

Pair (i): Target carbon,

$$V_c = -(0 + 10i) \text{ Mev}, \quad V_{sr} = 1 \text{ Mev}, \\ V_{sl} = -0.5 \text{ and } -1 \text{ Mev}.$$

Pair (ii): Target carbon,

$$V_c = -(0 + 10i) \text{ Mev}, \quad V_{sr} = 1.5 \text{ Mev}, \\ V_{sl} = -0.5 \text{ and } -1 \text{ Mev}.$$

Pair (iii): Target carbon,

$$V_c = -(0 + 12i) \text{ Mev}, \quad V_{sr} = 0.75 \text{ Mev}, \\ V_{sl} = -0.33 \text{ and } -0.42 \text{ Mev}.$$

Pair (iv): Target calcium,

$$V_c = -(0 + 12i) \text{ Mev}, \quad V_{sr} = 0.75 \text{ Mev}, \\ V_{sl} = -0.33 \text{ and } -0.42 \text{ Mev}.$$

Pair (v) : Target calcium,

$$V_c = -(0 + 12i) \text{ Mev}, V_{sR} = 0.84 \text{ Mev},$$

$$V_{sI} = -0.33 \text{ and } -0.42 \text{ Mev}.$$

All the well parameters used here satisfy the condition

$$V_c R / \hbar v \ll 1$$

mentioned in ref. 2). They are also roughly in agreement with the estimates of Riesenfeld and Watson.³⁾ Note that for each member of the above pairs V_c and V_{sR} are the same and thus we can study how the effects of V_{sI} on P_{\max} change as the other well parameters are altered.

From (iii) we learn that P_{\max} increases as $|V_{sI}|$ increases but for (i), (ii), (iv) and (v) P_{\max} is found to decrease as $|V_{sI}|$ increases, which is contrary to Sternheimer's conclusion. As to the depth and width of the dip of the curve, hardly any significant changes are observed.

As already mentioned in ref. 2), the shape of the polarization curve depends on the well parameters in such a complicated manner that it is not possible to find any simple relation between P_{\max} and V_{sI} , or any other parameter. Thus we agree with Sternheimer in saying that all the well parameters must be changed if we are fitting experimental results.

Green Function Method in Quantum-Statistics

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July 17, 1959

In this short note we propose a simple method for computing the grand partition function which is a slightly refined one of Matsubara's method.¹⁾ Matsubara showed that the theorem, which is analogous to Wick's theorem in quantum field theory, is also valid in quantum statistics so that one can use the field theoretical techniques for the computation of the quantum statistical averages.¹⁾

On the basis of his consideration we can easily derive the following integral equation :

$$\begin{aligned} G_2^{\beta, \mu}(12; 34) = & G_1^{\beta, \mu}(1, 3) G_1^{\beta, \mu}(2, 4) \\ & + \theta G_1^{\beta, \mu}(1, 4) G_1^{\beta, \mu}(2, 3) \\ & + \int G_1^{\beta, \mu}(1, 5) G_1^{\beta, \mu}(2, 6) \Gamma^{\beta, \mu}(56; 78) \\ & \times G_2^{\beta, \mu}(78; 34) dx_5 dx_6 dx_7 dx_8, \end{aligned}$$

where $i (=1, 2, \dots)$ stands for the four-dimensional coordinate $x_i = (\mathbf{x}_i, t_i)$, where t_i is the variable corresponding to the inverse temperature, $dx_i = d\mathbf{x}_i dt_i$, $\theta = +1$ or -1 according to whether the particle field in question obeys Bose or Fermi statistics, and the ranges of integrations are $(0, \beta)$ for t_i and the whole space for \mathbf{x}_i . $G_n^{\beta, \mu}(1 \dots n; 1' \dots n')$ characterized by the parameters $\beta = (kT)^{-1}$ and μ , the chemical potential, is the quantum statistical analogue to the Green

- 1) R. M. Sternheimer, Phys. Rev. **112** (1958), 1785.
- 2) K. Nishimura, Phys. Rev. **110** (1958), 1166.
- 3) W. B. Riesenfeld and K. M. Watson, Phys. Rev. **102** (1956), 1157.

function of n particles in quantum field theory and is defined as

$$G_n^{\beta,\mu}(1\cdots n; 1'\cdots n') = \langle T \{ \psi(1) \cdots \psi(n) \bar{\psi}(1') \cdots \bar{\psi}(n') S(\beta) \} \rangle_0 / \langle S(\beta) \rangle_0,$$

where

$$\langle \cdots \rangle_0 = \text{Tr } e^{-\beta(H_0 - \mu N)} \cdots / \text{Tr } e^{-\beta(H_0 - \mu N)},$$

$$S(\beta) = P \exp \left(- \int_0^\beta H_1(t) dt \right),$$

$$H_1(t) = \exp[(H_0 - \mu N)t] H_1 \times \exp[- (H_0 - \mu N)t],$$

$$\psi(x) = e^{(H_0 - \mu N)t} \psi(x) e^{-(H_0 - \mu N)t},$$

$$\bar{\psi}(x) = e^{(H_0 - \mu N)t} \bar{\psi}(x) e^{-(H_0 - \mu N)t}.$$

$\psi(x)$, $\bar{\psi}(x)$ are the field operator and its hermite conjugate. We have assumed that the total Hamiltonian H can be written as $H_0 + H_1$, H_0 and H_1 being the free and interaction respectively. Further we will assume the interaction is the two-body one, that is,

$$H_1 = \frac{1}{2} \iint \psi^\dagger(x) \psi^\dagger(x') v(x-x') \times \psi(x) \psi(x') dx dx'.$$

$\Gamma^{\beta,\mu}(56; 78)$ is also the quantum statistical counterpart of the so-called irreducible 4-vertex part in quantum field theory.

Using $G_2^{\beta,\mu}$, the grand partition function $\Xi = \text{Tr } e^{-\beta(H - \mu N)}$ can be given as follows:

$$\log \Xi = \log \Xi_0 - \theta \frac{\beta}{2} \int_0^\theta \frac{dg}{g} \times \iint dx dx' v(x-x') \times \lim_{t_2 \rightarrow t_1=0} G_2^{\beta,\mu}(xt_1, x't_1; xt_2, x't_2),$$

where $\Xi_0 = \text{Tr } e^{-\beta(H_0 - \mu N)}$ and g is the

coupling constant. Taking account of the periodicity of $G_2^{\beta,\mu}$ we may also write

$$\Xi / \Xi_0 = \exp \left[- \theta \frac{\beta}{2} \int_0^\theta \frac{dg}{g} \times \iint dx dx' v(x-x') \times G_2^{\beta,\mu}(x\beta, x'\beta; x0, x'0) \right].$$

Thus the problem of the computation of Ξ is reduced to that of the determination of $\Gamma^{\beta,\mu}$ suitable for the system in question and of the solution of the integral equation for $G_2^{\beta,\mu}$.

In the case of the electron gas, we can easily get the result of Montroll and Ward²⁾ if we take the following approximations:

$$\begin{aligned} G_1^{\beta,\mu}(x, x') &\rightarrow G_{1,0}^{\beta,\mu}(x, x') \\ &= \langle T \{ \psi(x) \bar{\psi}(x') \} \rangle_0, \\ G_2^{\beta,\mu} &\rightarrow G_{1,0}^{\beta,\mu} G_{1,0}^{\beta,\mu}, \\ \Gamma^{\beta,\mu}(56; 78) &\rightarrow \delta^{(4)}(5-7) \\ &\times \delta^{(4)}(6-8) \mathcal{V}^{\beta,\mu}(5, 6), \end{aligned}$$

where $\mathcal{V}^{\beta,\mu}$ is determined by the following integral equation,

$$\begin{aligned} \mathcal{V}^{\beta,\mu}(1, 2) &= v(1, 2) \\ &+ \int v(1, 3) G_{1,0}^{\beta,\mu}(3, 4) G_{1,0}^{\beta,\mu}(4, 3) \\ &\times \mathcal{V}^{\beta,\mu}(4, 2) dx_3 dx_4, \\ v(1, 2) &= v(x_1 - x_2) \delta(t_1 - t_2). \end{aligned}$$

Further, if we do not take the second approximation and solve the integral equation by the binary collision approximation, we get the result which corresponds, in the classical limit, to the result of Abe, the so-called "water-melon" approximation.³⁾

Details of the above discussion and other results will be given in a forthcoming paper.

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- 2) E. W. Montroll and J. C. Ward, Phys. of Fluids **1** (1958), 55. See also R. Abe, Prog. Theor. Phys. **21** (1959), 941.
- 3) R. Abe, Prog. Theor. Phys. **21** (1959), 475. A. A. Vedenov and A. I. Larkin, JETP **36** (1959), 1133.

Errata

Molecular Processes induced by μ^- Mesons in Hydrogen Bubble Chamber. II

Yukio MIZUNO, Takeo IZUYAMA and Mikio SHIMIZU

Prog. Theor. Phys. **21** (1959), 479.

Following corrections should be made:

p. 479, 7th line from the bottom—" > (44 ev) " should be replaced by "> 44 ev)".

Our previous correction, *i.e.*

(44 ev) \rightarrow (-44 ev) which have appeared on this journal (**22** (1959), 312) should be neglected. p. 479, Table I—The reaction rate for $\Phi_I \rightarrow \Phi_P$ by Auger's process should be 10^5 sec^{-1} instead of 10^4 sec^{-1} .

$T-Nt$ Diagram for Fusion Researches

M. IMOTO, G. TOMINAGA and S. FUJII

Prog. Theor. Phys. **21** (1959), 939 (L)

	originally	should read
P. 939 right 5 from the bottom	and τ the reduced time $\tau \equiv Nt$	and the reduced time $\tau \equiv Nt$
P. 941 left 3 from the bottom	narrower	wider
P. 941 reference ¹⁾	Prog. Theor. Phys. 21 (195), ₈₈ No. 758	Prog. Theor. Phys. 21 (1959), 758.

The $\Sigma-A$ Relative Parity and the $\bar{K}-N$ Reaction

Yukihisa NOGAMI

Prog. Theor. Phys. **22** (1959), 25

The Eq. (2.12) should read

$$F_+ = 3.0, \quad F_+' = 0.4, \quad \text{Re } F_- = 6.0, \quad \text{Re } F_-' = 1.2. \quad (2.12)$$

2 $^3,^1\text{P}$ State of Helium Atom

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(Received May 27, 1959)

The fairly accurate eigenfunctions of 2 ^3P and 2 ^1P states of the He atom are determined by the variation method so as to assume the simplest analytical form. The polarization of the s-orbital is taken into account, and the configuration interaction is examined. The non-relativistic energy eigenvalues are $E_{\text{theo}}(^3\text{P}) = -2.132\,612\,64$ and $E_{\text{theo}}(^1\text{P}) = -2.123\,198\,66$ atomic units which include both the normal and specific mass correction. These theoretical values are in fairly good agreement with the observed values $E_{\text{obs}}(^3\text{P}) = -2.132\,968\,6 \pm 7 \times 10^{-7}$ and $E_{\text{obs}}(^1\text{P}) = -2.123\,637\,3 \pm 7 \times 10^{-7}$ atomic units.

Introduction

The publishers regret two weeks' delay in publication of Vol. 22, No. 5 which are to appear on 25th November. They also would like to inform that Vol. 22, No. 6 are now in due course.

one of the
1 for the
paper is to
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also some insight into the structure of wave functions of more complex many-body systems.

In the previous paper A¹⁾ six functions were examined. Three of these functions had a cusp in their radial part. Strictly speaking, such a function does not belong to the domain of the kinetic energy (operator), and a special caution was necessary for employing it as a wave function. A correction term arises from the cusp. Such a correction was not fully taken into account but only partly in A. In the present paper the calculation is repeated fully taking into account the cusp correction, and the new parameter values are re-determined. The polarization of the s-orbital is taken into account. This effect can be interpreted as the configuration interaction as was discussed in A. The non-relativistic energy eigenvalues are calculated taking into account the effect of the nuclear motion.

§ 1. Basic wave functions

We shall consider the deepest ^3P and ^1P states of the He atom. If we neglect the kinetic energy of the nucleus, the non-relativistic Hamiltonian is given by

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$$H_{\infty} = -\frac{1}{2}(\mathcal{A}_1 + \mathcal{A}_2) - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_{12}}, \quad (1.1)$$

where all quantities are measured in atomic units. The effect of the nuclear motion will be considered in the last section.

Our first task is to obtain the eigenfunctions of H_{∞} according to the variation principle. We assume a linear combination of basic functions as the variation function. We seek for the simplest analytical eigenfunction. This restricts the basic functions in a few number. In order to recover the deprived freedom due to this restriction we vary the effective charges contained in the basic functions as well as the coefficients of the linear combination. It must be better to vary all effective charges and coefficients independently and simultaneously, but this is extremely difficult for us having no electronic machine. Thus we should content ourselves with a substitute device. The motivation and the meaning of the device will be accounted for in the third section, and only the method and result of determining the basic functions are described in this section.

We assume the normalized wave function of the starting approximation in the following form:

$$\psi_0 = 2^{-1/2}(1 \mp P)F(\kappa, \mu; r_1, r_2)Z_{sp}(\omega_1, \omega_2) \quad (1.2)$$

where P denotes the operator which exchanges the coordinates of two electrons. $1-P$ corresponds to the triplet state, and $1+P$ to the singlet state. Z_{sp} is the normalized two-electron angular function of the P state corresponding to the sp -configuration:

$$Z_{sp}(\omega_1, \omega_2) = Y_{00}(\omega_1)Y_{10}(\omega_2) \quad (1.3)$$

where $Y_{lm}(\omega)$ is the normalized one-electron spherical surface harmonic corresponding to the azimuthal quantum number l and the orbital magnetic quantum number m , and ω stands for the spherical surface coordinates θ and φ . As the energy eigenvalue is independent of the orbital magnetic quantum number M_L of the total system we consider the case of $M_L=0$. The normalized radial function is of the hydrogen type:

$$F(\kappa, \mu; r_1, r_2) = N_F \hat{r}_2 e^{-(\kappa r_1 + \mu r_2)} \quad (1.4)$$

where N_F is the normalization constant:

$$N_F(\kappa, \mu) = (2\kappa)^{3/2}(2\mu)^{5/2}(2!4!)^{-1/2} \quad (1.5)$$

$Z_i = \kappa$ and $Z_a = 2\mu$ are the inner and outer effective charges respectively. They are determined by minimizing the expectation value of H_{∞} . If the variation parameters κ and μ are transformed into λ and x_0 according to the equations $\kappa = \lambda x_0$ and $\mu = \lambda(1 - x_0)$ the simultaneous equations for λ and x_0 can be easily solved in any desired accuracy. The explicit form, as the function of λ and x_0 , of the expectation value of H_{∞} will be given in the second section. The determined values of κ and μ are shown in

Table 1. We see that the value of the inner effective charge is a little smaller than 2 for the triplet and a little greater than 2 for the singlet while the value of the outer effective charge is a little greater than unity for the triplet and a little smaller than unity for the singlet. Such a phenomenon is generally seen in the hydrogen-like orbital of the $1s\ nl$ configuration ($l=n-1$) of the two-electron atom.²⁾ If we denote the effective charge by the primed and unprimed symbols respectively for the singlet and the triplet they can be written in the form :

$$Z_i=Z-\delta_i \quad Z_a=Z-1+\delta_a \quad (1.6a)$$

$$Z'_i=Z+\delta'_i \quad Z'_a=Z-1-\delta'_a. \quad (1.6b)$$

The notable characteristic of the deviation δ can be summarized in the following way. The deviation satisfies the inequalities :

$$0 < \delta_i < \delta_a \ll 1, \quad 0 < \delta'_i < \delta'_a \ll 1. \quad (1.7)$$

The sum of the inner and outer effective charges is given by

$$Z_i+Z_a=2Z-1+(\delta_a-\delta_i), \quad Z'_i+Z'_a=2Z-1-(\delta'_a-\delta'_i) \quad (1.8)$$

and the deviation of these sums from $2Z-1$ satisfies the inequalities :

Table 1 Parameter values of wave functions

	³ P	¹ P
κ	1.991 185 792	2.003 024 271
μ	0.544 574 887 8	0.482 362 881 4
μ_1	1.335 750 000	0.807 500 000
μ_2	1.975 000 000	1.437 000 000
μ_3	0.921 250 000	1.119 000 000
N_1	0.934 071 888	0.982 971 341
N_3	0.974 574 903	0.957 776 963
a_0	0.990 273 307	1.000 566 493
a_1	0.023 487 573 6	-0.007 028 260 02
a_2	0.013 873 997 3	-0.000 406 779 024
a_3	-0.016 443 103 5	-0.018 107 143 9

$$0 < \delta_a - \delta_i \ll 1, \quad 0 < \delta'_a - \delta'_i \ll 1. \quad (1.9)$$

For the outer effective charge, $1-\delta_a$ or $1+\delta'_a$ may be interpreted as representing the screening by the s-electron but it is the over-screening in the latter case. The same interpretation can be applied to δ_i but is not the case for δ'_i . The numerical values of δ_i , δ_a , δ'_i and δ'_a are shown in Table 2.*

In order to improve the accuracy of the eigenfunction we further consider two more functions of the sp-type and the third function of the pd-type :

* These values are calculated from the inner and outer effective charges which were determined by Ishizu²⁾.

Table 2 Deviation of effective charge

		ls2p ³ P		ls2p ¹ P		ls3d ³ D		ls3d ¹ D	
Z	Atom	δ_i	δ_a	δ_i'	δ_a'	δ_i	δ_a	δ_i'	δ_a'
2	He I	0.0088	0.0891	0.0030	0.0353	0.0000	0.0008	0.0000	0.0005
3	Li II	0.0206	0.1632	0.0064	0.0596	0.0001	0.0033	0.0001	0.0015
4	Be III	0.0273	0.1979	0.0082	0.0700	0.0003	0.0050	0.0002	0.0025
5	B IV	0.0313	0.2172	0.0091	0.0749	0.0004	0.0065	0.0002	0.0031
6	C V	0.0338	0.2294	0.0098	0.0776	0.0007	0.0074	0.0003	0.0036
7	N VI	0.0357	0.2378	0.0101	0.0791	0.0006	0.0085	0.0003	0.0041
8	O VII	0.0370	0.2437	0.0102	0.0803	0.0007	0.0092	0.0003	0.0040
9	F VIII	0.0382	0.2483	0.0103	0.0807	0.0008	0.0098	0.0003	0.0042
10	Ne IX	0.0390	0.2520	0.0106	0.0812	0.0007	0.0099	0.0003	0.0048
11	Na X	0.0396	0.2551	0.0103	0.0817	0.0010	0.0106	0.0003	0.0046
12	Mg XI	0.0402	0.2572	0.0106	0.0819	0.0007	0.0110	0.0002	0.0049

$$\phi_1 = 2^{-1/2} (1 \mp P) G(\kappa, \mu_1; r_2, r_1) Z_{sp}(\omega_1, \omega_2) \quad (1.10)$$

$$\phi_2 = 2^{-1/2} (1 \mp P) F(\kappa, \mu_2; r_1, r_2) Z_{sp}(\omega_1, \omega_2) \quad (1.11)$$

$$\phi_3 = 2^{-1/2} (1 \mp P) G(\kappa, \mu_3; r_1, r_2) Z_{pd}(\omega_1, \omega_2) \quad (1.12)$$

where $F(\kappa, \mu; r_1, r_2)$ is given by (1.4). The other normalized radial function is defined by

$$G(\kappa, \mu; r_1, r_2) = N_G r_{<} r_2 e^{-(\kappa r_1 + \mu r_2)} \quad (1.13)$$

where $r_{<}$ is the smaller one of r_1 and r_2 and the normalization constant N_G is given by

$$N_G(\kappa, \mu) = (2\kappa)^{5/2} (2\mu)^{5/2} \{4! N(\kappa, \mu)\}^{-1} \quad (1.14)$$

$$[N(\kappa, \mu)]^2 = 1 - y_0^5 \{1 + 5x_0 + (25/2)x_0^2 + (35/2)x_0^3\} \quad (1.15)$$

$$x_0 = \kappa(\kappa + \mu)^{-1}, \quad y_0 = 1 - x_0. \quad (1.16)$$

Z_{pd} is the two-electron angular function of the P state corresponding to the pd-configuration:

$$\begin{aligned} Z_{pd}(\omega_1, \omega_2) = (10)^{-1/2} [2Y_{10}(\omega_1)Y_{20}(\omega_2) - \sqrt{3}Y_{11}(\omega_1)Y_{2-1}(\omega_2) \\ - \sqrt{3}Y_{1-1}(\omega_1)Y_{21}(\omega_2)]. \end{aligned} \quad (1.17)$$

The parameters μ_1 , μ_2 , and μ_3 are determined in the following way.

We consider the linear combination $\alpha_0\phi_0 + \alpha_1\phi_1$ and minimize the expectation value of H_∞ by varying α_0 and α_1 . The minimum expectation value is then given by

$$E_1 = H_{00} - \Delta E_1 \quad (1.18)$$

where

$$\Delta E_1 = (\varepsilon^2 + \gamma^2)^{1/2} - \varepsilon \quad (1.19)$$

$$\varepsilon = \{H_{11} - 2c_{01}H_{01} + (2c_{01}^2 - 1)H_{00}\} \{2(1 - c_{01}^2)\}^{-1} \quad (1.20)$$

$$\gamma = (H_{01} - c_{01}H_{00}) (1 - c_{01}^2)^{-1/2} \quad (1.21)$$

$$H_{jk} = (\psi_j, H_\infty \psi_k) \quad (1.22)$$

$$c_{jk} = (\psi_j, \psi_k). \quad (1.23)$$

We determine the value of μ_1 by minimizing the value of E_1 , for the fixed values of κ and μ . The values of the parameters μ_2 and μ_3 are determined in the same way. Their numerical values together with those of $N_1 = N(\kappa, \mu_1)$ and $N_3 = N(\kappa, \mu_3)$ are shown in Table 1. The reason why we determine the parameter values in such a way will be accounted for in the third section. The values of the effective charge for ψ_1 and ψ_3 are different from those in A because the cusp correction is fully taken into account in the present paper. The cusp correction will be discussed in the next section.

§ 2. Matrix elements of the Hamiltonian

The radial function $G(\kappa, \mu; r_1, r_2)$ has a cusp at $r_1 = r_2$. Strictly speaking, such a function does not belong to the domain of the kinetic energy as an Hermitian operator. In order to avoid the difficulty we modify the radial function in the immediate vicinity of the cusp so that the derivative becomes continuous and differentiable. The modified function belongs to the domain of the kinetic energy, and we can apply all theorems of quantum mechanics to the evaluation of physical quantities expressed in terms of the above mentioned wave function.

We can choose the vicinity of the cusp as small as we wish. This is practically equivalent to considering the limit as the length of the vicinity approaches zero, that is, to modifying the definition of the matrix element of the Laplacian Δ in the following way. For the sake of simplicity we consider the one-dimensional case. If $\varphi(x)$ is continuous and $\psi(x)$ has a cusp we define the Hermitian inner product of φ and $\Delta\psi$ by

$$(\varphi, \Delta\psi) = \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \varphi^*(x) \Delta\psi_\varepsilon(x) dx \quad (2.1)$$

where $\psi_\varepsilon(x)$ is the modified function, and ε is the length of the vicinity of the cusp. We intentionally retain the notation $\Delta\psi$ in order to show the general form in the three dimensional case. If the cusp of $\psi(x)$ is at $x=a$ we have

$$(\varphi, \Delta\psi) = \int_{-\infty}^{\infty} \varphi^*(x) \Delta\psi(x) dx + \varphi^*(a) \{\psi'(a+0) - \psi'(a-0)\} \quad (2.2)$$

where the prime represents the derivative. The second term is a correction arising from the cusp at $r=a$, and it is proportional to the saltus of the derivative of

$\psi(x)$ at the cusp. We shall refer to such a term as the cusp correction. The finiteness of the cusp correction in (2.2) means that $\psi''_\epsilon(a)$ is inversely proportional to ϵ for the sufficiently small value of ϵ . It follows that the cusp correction of $\|\Delta\psi\|^2$ in the modified definition is infinite. Therefore $\|\Delta\psi\|^2$ is infinite even if $\Delta\psi$ is quadratically integrable. Consequently, ψ does not belong to the domain of the kinetic energy in the mathematical meaning.^{3a)} For the practical purpose we need not consider the mathematical limit but the sufficiently small ϵ . Thus we need not mind about the norm of $\Delta\psi$.

We denote the matrix element of the Hamiltonian with respect to the basic functions given in the first section as follows:

$$H_{ij} = (\psi_i, H_\infty \psi_j) \quad (2.3)$$

where the Hermitian inner product is defined by (2.1). The modified definition guarantees the Hermiticity of Δ and iV in the following meaning:

$$(\psi_i, \Delta\psi_j) = - (V\psi_i, V\psi_j) = (\Delta\psi_i, \psi_j). \quad (2.4)$$

This is because of taking into account the cusp correction. The matrix (H_{ij}) is consequently Hermitian. We can evaluate H_{ij} for $j=0,2$ according to the ordinary definition of the Hermitian inner product and H_{ij} for $i=0,2$ and $j=1,3$ can be obtained by equating it with H_{ji} while in other cases the matrix element includes the cusp correction. For example, we have

$$H_{11} = \iint_{-\infty}^{\infty} \psi_1^* H_\infty \psi_1 d(\mathbf{x}_1, \mathbf{x}_2) + \int_0^\infty r^3 [G(\kappa, \mu_1; r, r)]^2 dr \quad (2.5)$$

where the argument of the d-sign indicates the integration variables. The second integral is the cusp correction.

As will be seen later we should separately evaluate the kinetic and potential energy parts of H_{ij} for the scaling procedure. We denote the kinetic and potential parts respectively by H_{ij}^k and H_{ij}^p :

$$H_{ij} = H_{ij}^k + H_{ij}^p. \quad (2.6)$$

The non-diagonal elements are given by

$$H_{10}^k = -(c_{10}/2)(\kappa^2 + \mu^2) + \kappa I'_{10} + 2\mu I''_{10} \quad (2.7a)$$

$$H_{10}^p = -2(I'_{10} + I''_{10}) + J_{10} \mp K_{10} \quad (2.7b)$$

$$c_{10} = 2^8 \sqrt{3} N_1^{-1} x_0^4 (\mu\mu_1)^{5/2} \lambda_1^{-5} [1 - \xi_1^4 (1 + 3\gamma_1 + 5\gamma_1^2 + 5\gamma_1^3)] \quad (2.7c)$$

$$I'_{10} = 2^6 \sqrt{3} N_1^{-1} x_0^4 (\mu\mu_1)^{5/2} \lambda_1^{-4} [1 - \xi_1^4 \{1 + (8/3)\gamma_1 + (10/3)\gamma_1^2\}] \quad (2.7d)$$

$$I''_{10} = 2^6 \sqrt{3} N_1^{-1} x_0^4 (\mu\mu_1)^{5/2} \lambda_1^{-4} [1 - \gamma_1^4 \{1 + (8/3)\xi_1 + (10/3)\xi_1^2\}] \quad (2.7e)$$

$$J_{10} = 2^8 \kappa (\sqrt{3} N_1)^{-1} x_0^3 (\mu\mu_1)^{5/2} \lambda_1^{-5} [1 - \xi_1^3 (1 + 3\gamma_1 + 5\gamma_1^2 + 5\gamma_1^3)] \quad (2.7f)$$

$$K_{10} = 8(3\sqrt{3} N_1 \kappa)^{-1} (\mu + \mu_1)^{-3} (\mu\mu_1)^{5/2} [1 - \gamma_1^3 (1 + 3x_1 + 6x_1^2 + 10x_1^3)] \quad (2.7g)$$

$$H_{20}^k = (c_{20}/2) (\kappa^2 + \mu\mu_2) \quad (2.8a)$$

$$H_{20}^p = -c_{20} \{2\kappa + (\mu + \mu_2)/2\} + J_{20} \mp K_{20} \quad (2.8b)$$

$$c_{20} = 2^5 (\mu\mu_2)^{5/2} (\mu + \mu_2)^{-5} \quad (2.8c)$$

$$J_{20} = (c_{20}/4) (\mu + \mu_2) [1 - y_2^4 (1 + 2x_2)] \quad (2.8d)$$

$$K_{20} = (2^7 \kappa x_0^2/3) (\mu\mu_2)^{5/2} (\lambda + \lambda_2)^{-5} [1 + 5(\lambda/\lambda_2) + (\lambda/\lambda_2)^2] \quad (2.8e)$$

$$H_{30}^k = 0 \quad H_{30}^p = J_{30} \mp K_{30} \quad (2.9a)$$

$$J_{30} = \frac{2^5 (\mu\mu_3)^{5/2}}{6\sqrt{6} N_3 \kappa (\mu + \mu_3)^3} [1 - y_3^3 (1 + 3x_3 + 6x_3^2 + 10x_3^3)] \quad (2.9b)$$

$$K_{30} = 2^8 \sqrt{6} \kappa^4 (\mu\mu_3)^{5/2} (N_3 \lambda \lambda_3)^{-1} (\lambda + \lambda_3)^{-6} \quad (2.9c)$$

$$H_{13}^k = 0 \quad H_{13}^p = J_{13} \mp K_{13} \quad (2.10a)$$

$$J_{13} = \frac{5\sqrt{2} (2\kappa)^5 2^5 (\mu_1 \mu_3)^{5/2}}{12 N_1 N_3 \lambda_3^2 (\lambda_1 + \lambda_3)^7} \left[1 + 7 \left(\frac{\lambda_3}{\lambda_1} \right) + \left(\frac{\lambda_3}{\lambda_1} \right)^2 \right] \quad (2.10b)$$

$$K_{13} = \frac{\sqrt{2} (2\kappa)^5 2^5 (\mu_1 \mu_3)^{5/2}}{4 N_1 N_3 (\mu_1 + \mu_3)^2 (\lambda_1 + \lambda_3)^7} \left[1 + 7y_4 + \frac{56}{y_4^7} \left(\log \frac{1}{x_4} - \sum_{k=1}^8 \frac{y_4^k}{k} \right) \right] \quad (2.10c)$$

where the minus of the double signs corresponds to the triplet, the plus to the singlet, and

$$\begin{aligned} \lambda &= \kappa + \mu & \lambda_i &= \kappa + \mu_i & i &= 1, 2, 3 \\ x_0 &= \kappa/\lambda & x_j &= 2\kappa/(\lambda + \lambda_j) & j &= 1, 2, 3 \\ x_4 &= 2\kappa/(\lambda_1 + \lambda_3) & y_k &= 1 - x_k & k &= 1, 2, 3, 4 \\ \hat{\epsilon}_1 &= \lambda/(\lambda + \lambda_1) & \gamma_1 &= 1 - \hat{\epsilon}_1 \\ N_i &= N(\kappa, \mu_i) & i &= 1, 3. \end{aligned} \quad (2.11)$$

The normalization constant $N(\kappa, \mu)$ is defined by (1.15) and c_{ij} is the overlap integral defined by (1.23). J_{ij} and K_{ij} are respectively the direct and exchange integrals of the Coulomb interaction between electrons. H_{12} and H_{32} can be obtained from H_{10} and H_{30} respectively by replacing μ with μ_2 .

The diagonal elements H_{00} and H_{22} can be obtained from H_{20} by replacing μ_2 with μ and μ with μ_2 respectively. The remaining diagonal elements are given by

$$H_{11}^k = (\lambda_1/N_1)^2 S_1(\kappa/\lambda_1) - (\kappa^2 + \mu_1^2)/2 \quad (2.12a)$$

$$H_{11}^p = (\lambda_1/N_1^2) [U_1(\kappa/\lambda_1) \mp W_1(\kappa/\lambda_1)] \quad (2.12b)$$

$$S_1(x) = x^2 + (2/3)(1-x)^2 - (1-x)^5 \{ (2/3) + 2x + (7/2)x^2 + (25/6)x^3 - 5x^4 \} \quad (2.12c)$$

$$U_1(x) = -(1+x)/2 + (1-x)^5 \{ (1/2) + 3x + (25/3)x^2 + (25/2)x^3 \} \quad (2.12d)$$

$$W_1(x) = 30x^5(1-x)^5 \quad (2.12e)$$

$$H_{33}^k = (\lambda_3/N_3)^2 S_3(\kappa/\lambda_3) - (\kappa^2 + \mu_3^2)/2 \quad (2 \cdot 13a)$$

$$H_{33}^p = (\lambda_3/N_3^2) [U_3(\kappa/\lambda_3) \mp W_3(\kappa/\lambda_3)] \quad (2 \cdot 13b)$$

$$S_3(x) = S_1(x) + (1-x)^2 - (1-x)^5(1+3x+5x^2+5x^3) \quad (2 \cdot 13c)$$

$$U_3(x) = U_1(x) + 2^{-1}x^5(1-x)^3(8-7x) \\ + 28x^5(1-x)^{-4}[\log x^{-1} - \sum_{k=1}^8 k^{-1}(1-x)^k] \quad (2 \cdot 13d)$$

$$W_3(x) = Cx^5(1-x)^5$$

$$C = 6 + 9 \times 2^{12}(\log 2 - \sum_{k=1}^8 2^{-k}k^{-1}) = 20.634 \ 807 \ 02. \quad (2 \cdot 13e)$$

The numerical values of these elements and overlap integrals are shown in Tables 3, 4 and 5, and the numerical values of N_1 and N_3 are shown in Table 1. In order to facilitate the solution of the secular equation we transform the set of basic functions into an orthonormal set by the Gram-Schmidt method.^{3b)} We denote the orthonormal functions by χ_k ($k=0, 1, 2, 3$), and the matrix element with respect to them by

Table 3 The kinetic part of the matrix elements of the Hamiltonian in atomic units

ij	H_{ij}^k (3P)	H_{ij}^k (1P)
00	2.130 691 33	2.122 390 09
11	2.587 143 52	2.204 701 15
22	3.932 722 93	3.038 537 61
33	2.948 778 15	3.391 326 07
01	0.070 398 620 6	0.006 382 806 74
02	0.952 810 727	1.156 098 74
12	1.199 541 97	0.340 322 029
$H_{00}^k = H_{13}^k = H_{23}^k = 0$		

Table 4 The matrix elements of the Hamiltonian in atomic units

ij	H_{ij} (3P)	H_{ij} (1P)
00	-2.130 691 33	-2.122 390 09
11	-0.235 494 532	-0.189 875 303
22	-1.302 246 82	-1.652 745 67
33	0.468 439 887	0.867 328 720
01	-0.399 116 229	-0.152 357 278
02	-0.827 816 805	-1.038 796 04
03	0.041 428 709 2	0.054 935 496 0
12	-0.718 527 443	-0.378 366 002
13	0.035 232 567 4	0.105 804 954
23	0.066 353 780 1	0.166 442 092

Table 5 The overlap integrals

ij	c_{ij} (3P)	c_{ij} (1P)
01	0.163 282 412	0.079 087 124 1
02	0.378 072 770	0.491 406 778
12	0.552 248 592	0.278 786 343

Table 6 The matrix elements of the Hamiltonian in atomic units with respect to the orthonormal basic functions

ij	h_{ij} (3P)	h_{ij} (1P)
00	-2.130 691 33	-2.122 390 09
11	-0.166 400 307	-0.180 178 349
22	-0.743 663 324	-1.438 848 36
33	0.468 439 887	0.867 328 720
01	-0.051 908 453 8	0.015 545 142 4
02	0.004 543 071 11	0.000 501 170 904
03	0.041 428 709 2	0.054 935 496 0
12	-0.626 089 935	-0.312 318 946
13	0.028 855 243 4	0.101 779 066
23	0.046 539 169 1	0.137 333 973

$$h_{jk} = (\chi_j, H_\infty \chi_k). \quad (2.14)$$

These elements can be represented in terms of H_{jk} and c_{jk} . Their numerical values are shown in Table 6.

§ 3. Variation procedure

The wave function ψ_0 of the starting approximation represents the deepest state among the hydrogen type sp-configuration. All states of the same configuration can lower the energy expectation value when they are linearly combined with ψ_0 . A complete way to improve the radial function is to expand the one-electron radial function in the infinite series of the complete set of functions given by $e^{-n\kappa r}$ (for s-function) or $re^{-n\kappa r}$ (for p-function) ($n=1, 2, 3, \dots$). The sum of terms for $n=2, 3, 4, \dots$ in the series represents the correction to the starting approximation. It is practically impossible to take into account a considerable number of terms. Therefore we search for a single function which can effectively be used as a substitute for the correction series. The simplest way for this purpose is to add a function of the same type but with a different effective charge to the starting approximation. Then the radial behaviour of the new variation function becomes somewhat different from the starting approximation, namely, the concavity or convexity of the radial function is different inside and outside its maximum point from that of the single exponential function. It has been well known that such a modification improves

the radial function.⁴⁾ The wave function ψ_2 represents the correction term for the p-orbital in this kind. In order to select the most effective ψ_2 we determine the effective charge μ_2 of ψ_2 by minimizing the energy expectation value for the linear combination of ψ_0 and ψ_2 . The correction for the s-orbital of the same kind was found to have only a very small effect as was shown in A. Therefore we discard it in the present calculation.

Another way to improve the hydrogen-like wave functions of the sp-configuration is to take into account the polarization of the s-orbital caused by the p-orbital.⁵⁾ This modification can be taken into account by introducing the factor in the s-orbital as follows:⁶⁾

$$1 + \sum_{n=1}^{\infty} u_n(r_1, r_2) P_n(\cos \theta_{12}) \quad (3.1)$$

where θ_{12} is the angle between \mathbf{x}_1 and \mathbf{x}_2 . The most important term of this series is the one containing $P_1(\cos \theta_{12}) = \cos \theta_{12}$. If higher degree terms are omitted the correction factor reduces to $1 + u_1 \cos \theta_{12}$ and the wave function of the total system consists of two terms. The first belongs to the sp-configuration and the second represents the polarization correction. The correction contains $\cos \theta_{12} Z_{sp}(\omega_1, \omega_2)$ as its angular part where $Z_{sp}(\omega_1, \omega_2)$ is defined by (1.3). This angular part can be decomposed in two parts as follows:

$$\cos \theta_{12} Z_{sp}(\omega_1, \omega_2) = \frac{1}{3} [Z_{sp}(\omega_2, \omega_1) + \sqrt{2} Z_{pd}(\omega_1, \omega_2)] \quad (3.2)$$

where $Z_{pd}(\omega_1, \omega_2)$ is defined by (1.17). Therefore the polarization correction is the superposition of sp- and pd-configurations. The radial part of the correction is given by a product of $u_1(r_1, r_2)$ and $F(\kappa, \mu; r_1, r_2)$ which is defined by (1.4). We find by the numerical calculation that this radial part can be approximately represented by $G(\kappa, \mu'; r_1, r_2)$, which is defined by (1.13), if an appropriate value of μ' is assumed. For this reason the wave functions ψ_1 and ψ_3 are introduced to take into account the polarization effect. The method of determining μ_1 and μ_3 is devised to select the most effective ψ_1 and ψ_3 as in the case of ψ_2 . It may be considered that the effective charge for ψ_3 can be determined in another way, namely by minimizing the expectation value $(\psi_3, H_{\infty} \psi_3)$ independently of ψ_0 . However, if this procedure gives the better value the former method must determine the same value because the coefficient of ψ_0 in the former method must automatically vanish in such a case. In fact we find by the numerical calculation that the function ψ_3 determined by minimizing $(\psi_3, H_{\infty} \psi_3)$ is less effective in order to lower the energy expectation value when it is linearly combined with ψ_0 .

We next consider the linear combination of the basic functions and minimize the expectation value of H_{∞} by varying its coefficients. The solution of the secular and simultaneous equations for the minimum expectation value of H_{∞} and for the coefficients, respectively, can be facilitated by introducing an orthonormal set and the secular matrix of H_{∞} with respect to this set in every case of assuming two,

three and four basic functions in the same way as was done in the end of the preceding section.

To assume a linear combination of ψ_0 , ψ_1 and ψ_2 may be considered as a device to improve the radial part of the sp-configuration instead of taking into account the higher states of the same configuration because the radial form of the lowest basic function is arbitrary. If we denote this normalized linear combination by ψ_{sp} the normalized linear combination of the four basic functions is written as

$$\begin{aligned}\psi &= \sum_{k=0}^8 a_k \psi_k = \sum_{k=0}^8 b_k \chi_k \\ &= a_{sp} \psi_{sp} + a_{pd} \psi_{pd} \quad \|\psi\| = 1\end{aligned}\tag{3.3}$$

where ψ_{pd} is written in place of ψ_3 to manifest its configuration.

The minimum expectation values of H_∞ for various linear combinations are shown in Table 7, where ΔE denotes the depression of them from $(\psi_0, H_\infty \psi_0)$. It is to be noted that the minimum expectation value for $\{\psi_0, \psi_1\}$ and $\{\psi_0, \psi_1, \psi_2\}$ is lower than that for $\{\psi_0, \psi_3\}$ in the case of 3P . This means that the improvement in the radial function of the sp-configuration is more effective than the interaction of ψ_0 with the pd-configuration. On the contrary, the configuration interaction is more effective in the case of 1P . If we further compare the expectation value for $\{\psi_0, \psi_2\}$ with that for $\{\psi_0, \psi_3\}$ we see that the former is higher than the latter, that is, the configuration interaction is more effective in this case. These phenomena show that the quantitative effect of the configuration mixing is strongly dependent on the functional form of the radial part and different for the different states, that is, the mere configuration interaction has no definite quantitative meaning.

The approximate analytical expression of $u_1(r_1, r_2)$ of (3.1) was obtained by Bethe⁽⁶⁾⁽⁷⁾. He calculated the correction due to this term to the energy expectation value and obtained -2.1320 (3P) atomic units of the expectation value of H_∞ by omitting certain terms. The evaluation was carefully repeated by Mano et al.⁽⁸⁾ by taking into account all terms, and the expectation value of $-2.131\ 153\ 2$ was obtained. The result shows that the influence of the omitted terms cannot be neglected even in the accuracy of five figures. This value is surpassed by the expectation value for $\{\psi_0, \psi_2\}$ which is far simpler than the Bethe function and does not include the polarization effect. As was discussed in the beginning of this section, ψ_1 and ψ_3 can be considered as the polarization terms. In this meaning, it is important to take into account the polarization effect though it has no definite significance more than the configuration interaction does.

The numerical values, corresponding to the minimum root of the fourth degree secular equation, of the coefficients of the linear combination of ψ_0 , ψ_1 , ψ_2 and ψ_3 are shown in Table 1. We finally minimize the expectation value of H_∞ by the Hylleraas scaling procedure.⁽⁹⁾ Using the numerical values of a_k we have the values of the kinetic and potential parts of the expectation of H_∞ as follows (in atomic units) :

$$E^k = \sum_{i=0}^8 \sum_{j=0}^8 a_i a_j H_{ij}^k = 2.122 \ 663 \ 30 \ (^3P), \ 2.124 \ 987 \ 81 \ (^1P) \quad (3.4)$$

$$E^p = \sum_{i=0}^8 \sum_{j=0}^8 a_i a_j H_{ij}^p = -4.255 \ 548 \ 95 \ (^3P), \ -4.248 \ 483 \ 44 \ (^1P).$$

From these values we have

$$\delta = E^k + E^p/2 = -0.005 \ 111 \ 176 \ (^3P), \ 0.000 \ 746 \ 095 \ (^1P). \quad (3.5)$$

Hence the virial theorem is not satisfied and we can lower the expectation value of H_∞ by scaling. The minimum expectation value is then given by

$$\langle H_\infty \rangle_{\text{scal}} = \langle H_\infty \rangle_4 - \delta^2/E^k \quad (3.6)$$

where $\langle H_\infty \rangle_4$ is the minimum root of the fourth-degree secular equation. The numerical values of $\langle H_\infty \rangle_{\text{scal}}$ are shown in the last line of Table 7. We see that the triplet level is better improved than the singlet as is usually observed.

Table 7 The minimum expectation values of H_∞ in atomic units

$\langle H_\infty \rangle$ denotes the minimum expectation value of H_∞ with respect to the linear combinations of the basic functions shown in the first column. ΔE denotes the depression of $\langle H_\infty \rangle$ from $(\psi_0, H_\infty \psi_0)$.

basic function	³ P		¹ P	
	$\langle H_\infty \rangle$	ΔE	$\langle H_\infty \rangle$	ΔE
ψ_0	-2.130 691 33		-2.122 390 09	
ψ_0, ψ_1	-2.132 062 11	0.001 370 78	-2.122 514 50	0.000 124 41
ψ_0, ψ_2	-2.131 277 21	0.000 585 88	-2.122 427 27	0.000 037 18
ψ_0, ψ_3	-2.131 351 52	0.000 660 18	-2.123 399 18	0.001 009 09
ψ_0, ψ_1, ψ_2	-2.132 182 91	0.001 491 58	-2.122 528 72	0.000 138 63
$\psi_0, \psi_1, \psi_2, \psi_3$	-2.132 885 65	0.002 194 32	-2.123 495 63	0.001 105 54
scaling	-2.132 897 96	0.002 206 63	-2.123 495 89	0.001 105 80

§ 4. The effect of the nuclear motion

In this section we shall consider the effect of the nuclear motion. When we take into account the motion of the nucleus the non-relativistic Hamiltonian of the helium atom is given by

$$H_M = -\frac{1}{2M} \Delta^{(n)} - \frac{1}{2} (\Delta_1^{(e)} + \Delta_2^{(e)}) - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_{12}} \quad (4.1)$$

where M is the mass of the nucleus, $\Delta^{(n)}$ and $\Delta_i^{(e)}$ ($i=1,2$) denote the Laplacian of the nucleus and the electrons respectively, and all quantities are measured in atomic units. If we introduce the centre-of-mass and relative coordinates we have

$$H_M = -\frac{1}{2M_0} \Delta_0 - \frac{1}{2m} (\Delta_1 + \Delta_2) - \frac{2}{r_1} - \frac{2}{r_2} + \frac{1}{r_{12}} - \frac{1}{M} \nabla_1 \cdot \nabla_2 \quad (4.2)$$

where M_0 and m are the total and reduced mass

$$M_0 = M + 2 \quad m^{-1} = 1 + M^{-1}, \quad (4.3)$$

Δ_0 is the Laplacian of the centre of mass, and Δ_i and ∇_i ($i=1,2$) are the Laplacian and the nabla of the relative coordinates. The internal energy of the atom is given by

$$H = H_M + \Delta_0 / (2M_0). \quad (4.4)$$

If we measure the length in m^{-1} times atomic units we have

$$H = H_\infty - (1 + M)^{-1} H_\infty - M(1 + M)^{-2} \nabla_1 \cdot \nabla_2 \quad (4.5)$$

where H_∞ is given by the same expression as (1.1), only difference between them being in the unit of length. The expectation value of this Hamiltonian is given by

$$\langle H \rangle = \langle H_\infty \rangle - (1 + M)^{-1} \langle H_\infty \rangle - M(1 + M)^{-2} \langle \nabla_1 \nabla_2 \rangle. \quad (4.6)$$

The term with the factor $(1 + M)^{-1}$ is usually referred to as the normal mass correction and the last term as the specific mass correction.

We denote the matrix element of $\nabla_1 \cdot \nabla_2$ by

$$M_{jk} = (\psi_j, \nabla_1 \cdot \nabla_2 \psi_k) \quad (4.7)$$

where the definition of the Hermitian inner product is the same as before. Every element consists of the direct and exchange integrals. The matrix (M_{jk}) is Hermitian. The direct integrals of M_{00} , M_{01} , M_{02} , M_{11} , M_{12} , M_{22} and M_{33} all vanish, and only their exchange parts survive. On the contrary, the exchange integrals vanish and the direct integrals survive in the case of M_{03} , M_{13} and M_{23} . The specific mass correction which comes from

$$\langle \nabla_1 \nabla_2 \rangle_{\text{ex}} = \sum_{k=0}^3 a_k^2 M_{kk} + 2(a_0 a_1 M_{01} + a_0 a_2 M_{02} + a_1 a_2 M_{12}) \quad (4.8)$$

is referred to as the exchange mass correction and that which comes from

$$\langle \nabla_1 \nabla_2 \rangle_{\text{dir}} = 2(a_0 M_{03} + a_1 M_{13} + a_2 M_{23}) a_3 \quad (4.9)$$

is referred to as the polarization mass correction, because the former consists of exchange terms only and the latter can be considered as arising from the polarization of the s-orbital as was mentioned in the third section. The expectation value of $\nabla_1 \cdot \nabla_2$ is given by the sum of these two parts:

$$\langle \nabla_1 \nabla_2 \rangle = \langle \nabla_1 \nabla_2 \rangle_{\text{dir}} + \langle \nabla_1 \nabla_2 \rangle_{\text{ex}}. \quad (4.10)$$

The angular integrals of the non-vanishing matrix elements can be calculated according to the standard method.¹⁰⁾ They are given by

$$\iint Z_{sp}^*(\omega_2, \omega_1) \nabla_1 \cdot \nabla_2 Z_{sp}(\omega_1, \omega_2) d(\omega_1, \omega_2) = \frac{1}{3} \frac{\partial}{\partial r_1} \left(\frac{\partial}{\partial r_2} + \frac{2}{r_2} \right) \quad (4.11)$$

$$\iint Z_{pd}^*(\omega_1, \omega_2) \nabla_1 \cdot \nabla_2 Z_{sp}(\omega_1, \omega_2) d(\omega_1, \omega_2) = \frac{\sqrt{2}}{3} \frac{\partial}{\partial r_1} \left(\frac{\partial}{\partial r_2} - \frac{1}{r_2} \right) \quad (4.12)$$

$$\iint Z_{pd}^*(\omega_2, \omega_1) F_1 \cdot F_2 Z_{pd}(\omega_1, \omega_2) d(\omega_1, \omega_2) = \frac{1}{15} \left(\frac{\partial}{\partial r_1} - \frac{1}{r_1} \right) \left(\frac{\partial}{\partial r_2} + \frac{3}{r_2} \right). \quad (4.13)$$

The radial integration can be easily carried out in the elementary way. The non-diagonal elements are given by

$$M_{01} = \pm (4/\sqrt{3}) (\kappa/N_1) (\mu\mu_1)^{5/2} (\mu + \mu_1)^{-5} [3(\mu + \mu_1) \{1 - y_1^4(1 + 8x_1/3 + 10x_1^2/3)\} - 4\mu \{1 - y_1^5(1 + 10x_1/3 + 5x_1^2)\}] \quad (4.14)$$

$$M_{02} = \pm 2^6 \kappa x_0^4 (\mu\mu_2)^{5/2} \lambda_2^{-4} \quad (4.15)$$

$$M_{03} = (2^5/\sqrt{6}) (\kappa\mu/N_3) (\mu\mu_3)^{5/2} (\mu + \mu_3)^{-5} [1 - y_3^5(1 + 10x_3/3 + 5x_3^2)] \quad (4.15)$$

$$M_{13} = \frac{2^8 \sqrt{2} \kappa^5 (\mu_1 \mu_3)^{5/2}}{2N_1 N_3 \lambda_1^3 \lambda_3^5} \left[\frac{5\kappa\mu_1}{\lambda_1 \lambda_3} \{1 - \xi_4^4(1 + 4\gamma_4 + 9\gamma_4^2 + 14\gamma_4^3 + 14\gamma_4^4)\} + (5/3) (\mu_1/\lambda_1) \xi_4^4 \gamma_4^2 (1 + 6\gamma_4 + 21\gamma_4^2) - (5/3) (\kappa/\lambda_3) \gamma_4^6 (1 + 6\xi_4 + 21\xi_4^2) - \xi_4^3 \gamma_4^2 \{2(\kappa/\lambda_3) (1 + 5\gamma_4 + 15\gamma_4^2 + 35\gamma_4^3) + (2/3) (1 + 5\gamma_4 + 15\gamma_4^2)\} + 35\xi_4^3 \gamma_4^5 \right] \quad (4.16)$$

where λ_i , x_i , y_i and N_i are given by (2.11), and

$$\xi_4 = \lambda_1/(\lambda_1 + \lambda_3) \quad \gamma_4 = 1 - \xi_4. \quad (4.17)$$

The plus of the double signs corresponds to the triplet and the minus to the singlet. M_{12} and M_{23} can be obtained from M_{01} and M_{03} respectively by replacing μ with μ_2 . The diagonal elements M_{00} and M_{22} can be obtained from M_{02} by replacing μ_2 with μ and μ with μ_2 respectively. The remaining diagonal elements are given by

$$M_{11} = \pm (2/9) (\kappa/N_1)^2 (\kappa/\lambda_1)^3 (\mu_1/\lambda_1)^5 \{348 - 1207(\kappa/\lambda_1) + 975(\kappa/\lambda_1)^2\} \quad (4.18)$$

$$M_{33} = \pm (1/9) (\kappa/N_3)^2 (\kappa/\lambda_3)^4 (\mu_3/\lambda_3)^5 \{74 + 390(\kappa/\lambda_3)\}. \quad (4.19)$$

The numerical values of $\langle F_1 F_2 \rangle$, its exchange and polarization parts and M_{jk} are shown in Table 8. The mass correction for ${}^4\text{He}$ is numerically evaluated by making use of these numerical values where $M=7294.18$ atomic units for ${}^4\text{He}$ is adopted,⁽¹¹⁾⁽¹²⁾ and the result is shown in Table 9. We see that the magnitude of the exchange mass correction is large as compared with the polarization correction but the latter is by no means negligible. The calculated energy eigenvalues are compared with the observed values* in the same table where E denotes the eigenvalue of the non-relativistic internal energy including the mass correction for ${}^4\text{He}$ and it is assumed that⁽¹¹⁾ 1 atomic unit is equal to $219\,474.618\text{ cm}^{-1}$. We see that the agreement is improved as compared with the previous calculation⁽¹³⁾ and

* The observed values are derived from the observed value of ${}^4\text{He}^+ 1s\ 2S_{1/2}$ in Moore's table⁽¹⁴⁾ and the observed values of ${}^2S_{1/2}-{}^3P_1$ and ${}^2S_{1/2}-{}^1P_1$ measured by Herzberg⁽¹⁵⁾.

Table 8 The matrix elements and the expectation values of $\mathbf{r}_1 \cdot \mathbf{r}_2$ in atomic units

jk	M_{jk} (³ P)	M_{jk} (¹ P)
00	0.056 125 580 4	-0.037 010 763 6
11	-0.056 789 039 6	0.011 109 793 8
22	0.983 082 492	-0.644 767 025
33	0.109 312 427	-0.157 907 943
01	0.056 060 062 4	-0.122 059 030
02	0.234 895 882	-0.154 477 571
03	0.379 620 830	0.264 855 336
12	0.172 587 048	-0.084 718 951 9
13	0.087 755 053 2	-0.047 715 136 1
23	1.084 696 08	1.128 132 16
$\langle \mathbf{r}_1 \mathbf{r}_2 \rangle_{\text{ex}}$	0.064 401 303 5	-0.035 262 078 9
$\langle \mathbf{r}_1 \mathbf{r}_2 \rangle_{\text{dir}}$	-0.012 925 548 6	-0.009 592 507 01
$\langle \mathbf{r}_1 \mathbf{r}_2 \rangle$	0.051 475 754 9	-0.044 854 585 9

the present wave function is more accurate than the function in A. The further improvement may be obtained by adding another function of pd-configuration with the hydrogen-like radial part. In order to improve the singlet level the radial function with $(r_1 - r_2)^2 r_{>}^{-1}$ may be effective where $r_{>}$ denotes the larger one of r_1 and r_2 .

Finally we calculate the specific isotope shift of the level. We denote the specific shift by

$$(\delta P)_s = - \{ {}^3M(1 + {}^3M)^{-2} - {}^4M(1 + {}^4M)^{-2} \} \langle \mathbf{r}_1 \mathbf{r}_2 \rangle \quad (4.20)$$

where 3M and 4M are the nuclear mass of ³He and ⁴He respectively. If we adopt the value ${}^3M = 5495.79$ atomic units⁽¹¹⁾⁽¹²⁾ we have

$$(\delta {}^3P)_s = -506.510 \text{ mK (theor)}, \quad (\delta {}^1P)_s = 441.359 \text{ mK (theor)}$$

Table 9 The mass corrections and the energy eigenvalues for ⁴He in atomic units

	³ P	¹ P
mass correction		
normal	0.000 292 370 7	0.000 291 081 9
exchange	-0.000 008 826 712	0.000 004 832 949
polarization	0.000 001 771 549	0.000 001 314 729
specific	-0.000 007 055 162	0.000 006 147 678
total	0.000 285 32	0.000 297 23
$\langle H_{\infty} \rangle$	-2.132 897 96	-2.123 495 89
E_{calc}	-2.132 612 64	-2.123 198 66
E_{obs}	-2.132 968 6 (± 7)*	-2.123 637 3 (± 7)

* the error in the last figure

(mK = 10^{-3} cm $^{-1}$). The observed values¹³⁾ are

$$(\delta^3\text{P})_s = -642 \pm 5 \text{ mK (obs)}, \quad (\delta^3\text{P})_s = 461 \pm 5 \text{ mK (obs)}.$$

We see that the agreement between theoretical and observed values was better in the previous calculation¹³⁾ which was carried out by making use of the parameters in A. The true reason of this discrepancy is not known. It may partly be due to the inaccuracy of the observed values. It may be quite possible, however, that the accuracy of the wave function is somewhat different for the energy and for the specific mass correction.

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Dispersion Relations and High Energy Limits in Quantum Field Theory

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It is shown that the high energy limits in quantum field theory can be determined starting from Bogoliubov's and Symanzik's dispersion relations if the Boson propagator and pion-nucleon forward scattering amplitude have no zero on their complex energy planes. A possibility excluding zeros is discussed using Nambu's method in perturbation theory.

§ 1. Introduction and summary

The recent development of quantum field theory based on the causality principle has brought a success in achieving the formalism independent of detailed structure of interactions, the canonical formalism and perturbational treatments. The dispersion relations derived from the postulate that the theory is local made us possible to compare the theory and the experiment, and gave a clue to check the justification of the present quantum field theory based on the point model. For example, for pion-nucleon forward scattering, comparison of theory and experiment has been tried to use the dispersion relations derived by Goldberger et al.¹⁾ and several discussions have been made on this problem.²⁾ However, the concrete forms of dispersion relations should depend on the assumption how field quantities behave at high energies. Dispersion relations derived by Goldberger et al. have been constructed upon a presupposition that $\int_0^\infty \sigma_{\text{tot}}(\omega)/\omega^2 \cdot d\omega < \infty$, where $\sigma_{\text{tot}}(\omega)$ is the pion-nucleon total cross section. Thus, in order to compare theory with experiment definitely, using dispersion relations, it is necessary to establish the behavior of field quantities at high energy. The dispersion relations found by Bogoliubov et al.³⁾ and Symanzik⁴⁾ start from looser assumptions on the behavior of field quantities at high energy than those so far. Therefore, we must start our arguments from the dispersion relations given by Bogoliubov et al. and Symanzik. On the other hand, the inconsistency problem of the present quantum field theory discussed by Lee, Landau and others,⁵⁾ is intimately connected with the assumption on the high energy limit of the field quantities, and as this limit has not so far been established, we cannot rely upon their conclusions.⁶⁾ Namely, we can say that the problem of high energy limit in quantum field theory plays a central role both in the realistic problem of comparing theory with experiment and in the academic problem of consistency of theory. For the present, the discussion to determine the high energy limit in

quantum field theory has been carried on by Källén⁷⁾ (using the Born approximation for Boson propagator), Symanzik⁸⁾ ("tangent approximation" for pion-nucleon total cross section), and Arnowitt and Feldman⁹⁾ (only for lower limit of the pion-nucleon total cross section), though they seem to remain only tentative ones. Under these circumstances, we wish to propose a method to determine the high energy limit starting the argument from the dispersion relations of Bogoliubov et al. and Symanzik. The method is based upon an assumption of uniqueness of the "solutions" of the dispersion relations. The problem of manifold of solutions of the Low-type equation was pointed out firstly by Castellijo et al.¹⁰⁾ and thereafter many authors¹¹⁾, and this remains as a very mathematical, academic one. Nevertheless, it is still unsolved whether or not propagation functions and forward scattering amplitudes have their zeros on the complex energy plane, and in the present author's opinion it will perhaps be difficult to exclude zeros so far as we do not introduce dynamics into the theory in some way. For the present, it does not become clear what physical requirements are equivalent to the possibility of excluding zeros, but if one resorts to the perturbation expansion it will be possible.

It will be shown in § 4 that there is no zero in propagation function at any order of perturbation expansions applying Nambu's recent investigation¹²⁾ using parametric representation of Feynman diagrams. For pion-nucleon scattering we could not draw any conclusion. Anyhow, the limitation on the high energy behavior will be obtained for the Boson propagator in § 2, for pion-nucleon total cross section in § 3 provided that these quantities have no zero on the complex energy planes. Particularly it may be worthwhile to notice that we obtained the same conclusion with Symanzik⁸⁾ for pion-nucleon total cross section.

§ 2. Zeros of Boson propagator and determination of its high energy limit

The form of Boson propagator $\Delta_F'(k)$ is given by Bogoliubov, Medvedev and Polivanov³⁾ as follows:

$$\Delta_F'^{(n)}(k) = \frac{1}{m^2 + k^2} + \sum_{j=2}^n C_j^{(n)} (m^2 + k^2)^{j-2} + (m^2 + k^2)^{n-1} \int_{(3m)^2}^{\infty} \frac{\sigma^{(n)}(\lambda^2) d\lambda^2}{(m^2 - \lambda^2)^{n-1} (\lambda^2 + k^2)}, \quad (1)$$

where $C_j^{(n)}$ is a real unknown constant and the index (n) specifies $\Delta_F'^{(n)}(k)$ cannot increase faster than $(-k^2)^{n-2}$ as $-k^2 \rightarrow \infty$. The reason of writing as $C_j^{(n)}$ is that in general C_j depends on (n) . The same is true for $\sigma^{(n)}(\lambda^2)$, the imaginary part of $\Delta_F'^{(n)}(k)$, and m means the Boson mass.

Δ_F' in Källén-Lehmann's theory¹³⁾ is a theory assuming $n=1$ in Eq. (1). Now, according to Lehmann, Symanzik and Zimmermann,¹⁴⁾ writing the vacuum expectation value of product of Boson field operator $A(x)$ as

$$(\Omega, A(x) A(y) \Omega) = \int \Delta_F'(x - \xi) F(\xi - \eta) \overline{\Delta_F'}(\eta - y) d\xi d\eta \quad (2)$$

and by such a way introducing a positive definite function $F(-k^2)$, the formula given by LSZ [LSZ, Eq. (7)] remains valid even though $\mathcal{A}'_F(k)$ is given by Eq. (1) :

$$2\pi\sigma^{(n)}(-k^2) = |\mathcal{A}'^{(n)}_F(k^2)|^2 F^{(n)}(-k^2). \quad (3)$$

Substituting Eq. (1) into (3), and changing the notations ;

$$\begin{aligned} \kappa^2 = -k^2, \quad \frac{\kappa^2 - 9m^2}{8m^2} = t, \quad 8m^2 \sigma^{(n)}(\kappa^2) = \rho^{(n)}(t), \\ \frac{1}{2\pi} \frac{F^{(n)}(\kappa^2)}{8m^2} = k^{(n)}(t), \end{aligned} \quad (4)$$

Eq. (3) will then become as follows :

$$\begin{aligned} \rho^{(n)}(t) = k^{(n)}(t) \left\{ \pi^2 \rho^{(n)}(t)^2 + \left[\frac{1}{t+1} + \sum_{j=2}^n C_j^{(n)} (-8m^2)^{j-1} (t+1)^{j-2} \right. \right. \\ \left. \left. + (t+1)^{n-1} P \int_0^\infty \frac{\rho^{(n)}(t') dt'}{(t'+1)^{n-1} (t-t')} \right]^2 \right\}. \end{aligned} \quad (5)$$

Solving $k^{(n)}(t)$ from Eq. (5), we obtain

$$k^{(n)}(t) = \frac{1}{\pi} \operatorname{Im} \lim_{\varepsilon \rightarrow 0} \frac{1}{\frac{1}{t+1} + \sum_{j=2}^n C_j^{(n)} (-8m^2)^{j-1} (t+1)^{j-2} + (t+1)^{n-1} \int_0^\infty \frac{\rho^{(n)}(t') dt'}{(t'+1)^{n-1} (t-t' + i\varepsilon)}}.$$

Now, since $\rho^{(n)}(t)$ does not increase faster than t^{n-2} as $t \rightarrow \infty$, $k^{(n)}(t)$ given in Eqs. (3), (4) cannot decrease faster than $t^{-(n-2)}$. Namely, $k^{(n)}(t)$ will decrease like $t^{-(n-2)}$ when $\rho^{(n)}(t)$ increases as t^{n-2} as $t \rightarrow \infty$, that is, the stronger $\rho^{(n)}(t)$ increases as $t \rightarrow \infty$, the stronger $k^{(n)}(t)$ decreases. Therefore, the following integral is convergent if $n \geq 3$:

$$\int_0^\infty \frac{k^{(n)}(t)}{t+1} dt < \infty.$$

In other words, for $n \geq 3$ the following equation holds,

$$\int_0^\infty \frac{k^{(n)}(t)}{t+1} dt = \frac{1}{2\pi i} \int_C \frac{dt}{\left[1 + \sum_{j=2}^n C_j^{(n)} (-8m^2)^{j-1} (t+1)^{j-1} + (t+1)^n \int_0^\infty \frac{\rho^{(n)}(t') dt'}{(t'+1)^{n-1} (t-t')} \right]}. \quad (6)$$

Integral path C in the right-hand side of the above equation is shown in Fig. 1. Thus, the only contribution of the integral is from the poles in the internal region of this contour. Hence, if $\mathcal{A}'_F(t)$ has no zero in the complex t -plane, we have

$$\int_0^{\infty} \frac{k^{(n)}(t) dt}{t+1} = 0$$

for $n \geq 3$. (7)

However, since $k^{(n)}(t)$ is positively definite, Eq. (7) shows that the cases $n \geq 3$ should be excluded from our theory, that is, $\rho^{(n)}(t)$ should not increase as t or more strongly at $t \rightarrow \infty$. In other words, in the expression of the Boson propagator given by Bogoliubov et al., it is sufficient to take $n=3$.

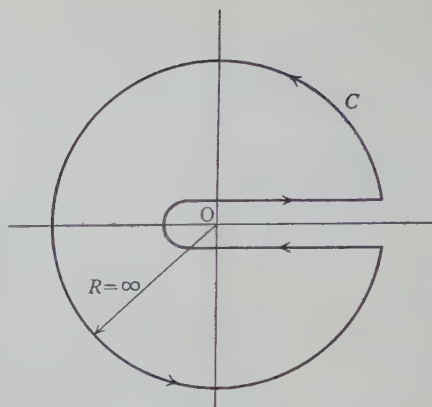


Fig. 1. Integral contour C in Eq. (6).

§ 3. The high energy limit of total cross section for pion-nucleon scattering

In this section we start from the expression for pion-nucleon forward scattering amplitude given by Symanzik (Eq. (15) of ref. 4) ;

$$\begin{aligned} T_{\omega}^{(n)} = & R^{(n)}(4M^2\omega^2; \mu^2) - \sum_{\nu=1}^n R^{(n)}(4M^2\omega_{\nu}^2; \mu^2) \prod_{\lambda \neq \nu}^n \frac{\omega^2 - \omega_{\lambda}^2}{\omega_{\nu}^2 - \omega_{\lambda}^2} \\ & + \sum_{\nu=1}^n \operatorname{Re} T_{\omega_{\nu}}^{(n)} \prod_{\lambda \neq \nu}^n \frac{\omega^2 - \omega_{\lambda}^2}{\omega_{\nu}^2 - \omega_{\lambda}^2} \\ & - g^2 \frac{\mu^3/(2M)}{(\mu^4/4M^2) - \omega^2} \prod_{\nu=1}^n \frac{\omega^2 - \omega_{\nu}^2}{(\mu^4/4M^2) - \omega_{\nu}^2} \\ & + \frac{2}{\pi} \prod_{\nu=1}^n (\omega^2 - \omega_{\nu}^2) P \int_{\mu}^{\infty} \frac{\omega' d\omega'}{\omega'^2 - (\omega + i\varepsilon)^2} \frac{\operatorname{Im} T_{\omega'}^{(n)}}{\prod_{\nu=1}^n (\omega'^2 - \omega_{\nu}^2)}, \end{aligned} \quad (8)$$

where $R^{(n)}(4M\omega^2; \mu^2)$ is an unknown real function which increases more slowly than ω^{2n} as $|\omega| \rightarrow \infty$, ω is the pion energy in the laboratory system, M, μ are nucleon and pion mass respectively, $\omega_{\nu} (> \mu)$ real parameter, and $\sigma_{\text{tot}}(\omega) = -\frac{1}{2} \cdot M^{-1}(\omega^2 - \mu^2)^{-1/2} \operatorname{Im} T_{\omega}$ is the total cross section.

Now, writing all the real terms except for the integral term in the right-hand side of Eq. (8) as $P^{(n)}(\omega)$, Eq. (8) becomes

$$T_{\omega}^{(n)} = P^{(n)}(\omega) + \frac{2}{\pi} \prod_{\nu=1}^n (\omega^2 - \omega_{\nu}^2) P \int_{\mu}^{\infty} \frac{\omega' d\omega'}{\omega'^2 - (\omega + i\varepsilon)^2} \frac{\operatorname{Im} T_{\omega'}^{(n)}}{\prod_{\nu=1}^n (\omega'^2 - \omega_{\nu}^2)}. \quad (9)$$

Here, it has been assumed $T_{\omega}^{(n)}$ could not increase faster than ω^{2n} as $|\omega| \rightarrow \infty$.

Following LSZ, constructing the analogous function $k^{(n)}(\omega)$ as follows,

$$k^{(n)}(\omega) = \lim_{\varepsilon \rightarrow 0} \frac{-1}{T_{\omega}^{(n)}}, \quad (10)$$

we obtain from Eq. (9)

$$k^{(n)}(\omega) = \frac{\text{Im } T_{\omega}^{(n)}}{\left[P^{(n)}(\omega) + \frac{2}{\pi} \prod_{\nu=1}^n (\omega^2 - \omega_{\nu}^2) P \int_{\mu}^{\infty} \frac{\omega' d\omega'}{\omega'^2 - \omega^2} \frac{\text{Im } T_{\omega'}^{(n)}}{\prod_{\nu=1}^n (\omega'^2 - \omega_{\nu}^2)} \right]^2 + [\text{Im } T_{\omega}^{(n)}]^2}.$$

This function is positively definite because $\text{Im } T_{\omega}^{(n)} \geq 0$ for forward scattering.

Now, the next integral will be convergent if $n \geq 1$:

$$\int_{\mu}^{\infty} k^{(n)}(\omega) d\omega < \infty.$$

Therefore, taking the integral path C in the complex z -plane as shown in Fig. 2, and taking into account that $k^{(n)}(\omega)$ is an odd function of real ω , the following equation is valid,

$$\int_{\mu}^{\infty} k^{(n)}(\omega) d\omega = -\frac{1}{4i} \int_C \frac{dz}{T_z^{(n)}} \quad \text{for } n \geq 1. \quad (11)$$

Hence, the integral of the right-hand side of Eq. (11) will vanish provided that $T_z^{(n)}$ has no zero in the complex z -plane. But the integrand in the left-hand side is a positively definite function; thus, it should be concluded that the cases $n \geq 1$ are excluded from the theory. Namely, the forward scattering amplitude T_{ω} should increase more slowly than ω^2 at $|\omega| \rightarrow \infty$, in other words, it is sufficient to take $n=1$ in the additive denominator $\prod_{\nu=1}^n (\omega'^2 - \omega_{\nu}^2)$ in the left-hand side integral in Eq. (8).

Therefore, the total cross section $\sigma_{\text{tot}}(\omega)$ obviously has the following feature:

$$\int_{\mu}^{\infty} \frac{\sigma_{\text{tot}}(\omega)}{\omega^2} d\omega < \infty. \quad (12)$$

This is identical with the conclusion that was obtained by Symanzik⁸⁾ using the "tangent approximation".

§ 4. Exclusion of zeros from Boson propagator in perturbation theory

Writing $\Sigma^*(k)$ for the proper self-energy part of Boson, $\Delta'_F(k)$ is defined by the following well-known integral equation

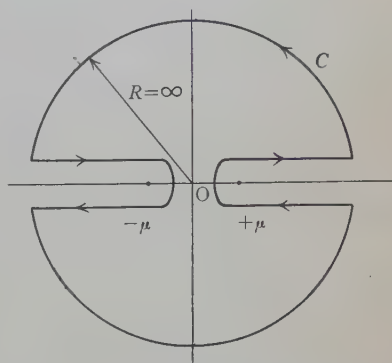


Fig. 2. Integral contour in Eq. (11)

$$\mathcal{A}'_F(k) = \mathcal{A}_F(k) + \mathcal{A}_F(k) \Sigma^*(k) \mathcal{A}'_F(k). \quad (13)$$

From this we rewrite

$$\begin{aligned} \mathcal{A}'_F{}^{-1}(k) &= (k^2 + m^2) + \Sigma^*(k) \\ &= (k^2 + m^2) + (k^2 + m^2) \mathcal{A}_F(k) \Sigma^*(k) \mathcal{A}_F(k) (k^2 + m^2). \end{aligned} \quad (14)$$

Thus, the existence of zeros in $\mathcal{A}'_F(k)$ corresponds to the existence of the singularities in the proper self-energy part $\Sigma^*(k)$. However, $\mathcal{A}_F \Sigma^* \mathcal{A}_F$ in the right-hand side of Eq. (14) is just the radiative correction of \mathcal{A}_F , therefore we can apply Nambu's Theorems⁽¹²⁾ to this function. By Nambu's theorem in any order of perturbation expansion

$$\mathcal{A}_F(k) \Sigma^*(k) \mathcal{A}_F(k) = \int_0^\infty \frac{\sigma(\alpha) d\alpha}{\alpha k^2 + m^2 - i\varepsilon}. \quad (15)$$

On the other hand, a parametric representation of $\mathcal{A}_F(k)$ is, apart from the numerical factors,

$$\mathcal{A}_F(k) = \int_0^\infty \frac{\partial(1 - \alpha_0) d\alpha_0}{\alpha_0 k^2 + m^2 - i\varepsilon}.$$

Therefore, a following relation must hold for a real number q (ref. 12) Theorem 5) :

$$\alpha_0 q^2 \geq \alpha q^2. \quad (16)$$

Since $\alpha_0 = 1$, we get

$$\alpha \leq 1. \quad (17)$$

Finding out the singularities of $\mathcal{A}_F \Sigma^* \mathcal{A}_F$ from Eq. (15)

$$-k^2 = m^2 / \alpha \geq m^2. \quad (18)$$

By the stability conditions on vacuum and one-particle state and by the pseudoscalar character of pion, Eq. (18) actually becomes to

$$-k^2 \geq (3m)^2. \quad (19)$$

Hence in the complex k^2 -plane with a cut from $(3m)^2$ to $+\infty$, $\mathcal{A}'_F{}^{-1}(k)$ is analytic in any order of perturbation expansion, namely, $\mathcal{A}'_F(k)$ has no zero on this plane.

The same argument as above cannot be applied to scattering amplitude, because it does not satisfy a closed integral equation as Eq. (13). Therefore, we cannot at present get any reason for excluding zeros from the complex energy plane for the pion-nucleon forward scattering amplitude.

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Nucleon-Nucleon Scattering in Momentum Space: OPEP as Example

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A practical formalism is developed for treating nucleon-nucleon scattering in momentum space. The non-adiabatic One-Pion-Exchange-Potential is used as an example.

§ 1. Introduction

One way of obtaining the non-adiabatic predictions of the pion theory of nuclear forces would be to solve that theory in momentum space. The purpose of this note is to write down a practical formalism for such solution of nucleon-nucleon scattering. We will first obtain the equations for the momentum-space scattering amplitudes, then show the relation of these amplitudes to the usual phase shifts. Finally, as an example, we will treat the second-order ("one pion exchange") potential.

§ 2. Scattering amplitude equations

The basic equation for this work is the momentum-space two-nucleon Schrödinger equation

$$(2E_k - 2E_p)\phi(\mathbf{p}) = \int d\mathbf{s} V(\mathbf{s}, \mathbf{p})\phi(\mathbf{s}) \quad (1)$$

with \mathbf{s} and \mathbf{p} as shown in Figure 1. The quantity $E_k = \frac{1}{2}E$ is the initial energy of either nucleon in the center-of-mass system, and $E_p = \frac{1}{2}H_0$.

(a) uncoupled states**

In order to obtain a one-dimensional equation in place of the three-dimensional equation (1), we expand the wave function and potential in the spin-angle functions $Y_{fls}^m(\Omega)^{1)}$

$$\phi(\mathbf{p}) = \sum_l g_l(p) Y_l(\Omega_p)$$

$$V_l(s, p) \equiv \int d\Omega_s d\Omega_p Y_l^*(\Omega_p) V(\mathbf{s}, \mathbf{p}) Y_l(\Omega_s) \quad (2)$$

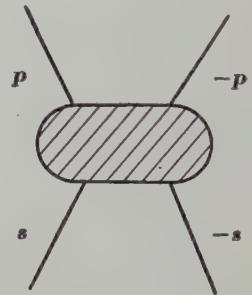


Fig. 1

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** $S=0$ and $S=1$, $l=J$.

In case* $V(\mathbf{s}, \mathbf{p}) = V(\mathbf{s}, -\mathbf{p})$, the above equation for V_l reduces to the simpler expression

$$V_l(s, p) = 2\pi \int_{-1}^{+1} dx [V(s, p, x) P_l(x)] \quad (2')$$

where $P_l(x)$ is a Legendre polynomial and $x \equiv (\mathbf{p} \cdot \mathbf{s}) / (|\mathbf{p}| |\mathbf{s}|)$. In either case, the Schrödinger equation becomes one dimensional:

$$(2E_k - 2E_p) g_l(p) = \int s^2 ds V_l(s, p) g_l(s). \quad (3)$$

The scattering solution can be written²⁾

$$g_l(p) = \delta(2E_k - 2E_p) + \frac{1}{2E_k - 2E_p} \int s^2 ds V_l(s, p) g_l(s).$$

An equation for a non-singular amplitude can be obtained by defining an $f_l(p)$ by²⁾

$$g_l(p) = \delta(2E_k - 2E_p) + P \frac{1}{2E_k - 2E_p} f_l(p). \quad (4)$$

The equation for $f_l(p)$ is seen to be

$$f_l(p) = \int_0^\infty s^2 ds V_l(s, p) g_l(s),$$

but since

$$g_l(s) = \delta(2E_k - 2E_s) + P \frac{1}{2E_k - 2E_s} f_l(s)$$

and $\delta(2E_k - 2E_s) = (E_k/2k) \delta(k - s)$, we can rewrite the equation for $f_l(p)$ as

$$f_l(p) = f_{l,n}(p) + P \int_0^\infty \frac{s^2 ds V_l(s, p) f_l(s)}{2E_k - 2E_s}, \quad (5)$$

$f_{l,n}(p)$ is the Born approximation amplitude and is given by

$$f_{l,n}(p) \equiv \frac{kE_k}{2} V_l(k, p). \quad (6)$$

(b) coupled states

For the $S=1$, $l=J \pm 1$ states of the two-nucleon system we must decompose the wave function and potential slightly differently:

$$\begin{aligned} \phi(\mathbf{p}) &= g_{J-1}(p) Y_{J-1}(\mathcal{Q}_p) + g_{J+1}(p) Y_{J+1}(\mathcal{Q}), \\ V_{l,l'}(s, p) &\equiv \int d\mathcal{Q}_s d\mathcal{Q}_p Y_l^*(\mathcal{Q}_p) V(\mathbf{s}, \mathbf{p}) Y_{l'}(\mathcal{Q}_s). \end{aligned} \quad (7)$$

* This case corresponds to a local potential in coordinate space: $V(\mathbf{r}, \mathbf{r}') = V(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}')$

Substitution of these into the Schrödinger equation yields the one dimensional

$$(2E_k - 2E_p)g_i(p) = \sum_{\nu} \int s^2 ds V_{i,\nu}(s, p) g_{\nu}(s),$$

or in matrix form

$$(2E_k - 2E_p)g(p) = \int s^2 ds \hat{V}(s, p) g(s).$$

As before, we write the scattering solution

$$g(p) = \delta(2E_k - 2E_p) + P \frac{1}{2E_k - 2E_p} \int_0^{\infty} s^2 ds V(s, p) g(s).$$

To obtain equations for non-singular amplitudes, we write the two independent solutions in the form

$$\begin{aligned} g_1(p) &= \begin{pmatrix} \delta(2E_k - 2E_p) + P \frac{1}{2E_k - 2E_p} u_1(p) \\ P \frac{1}{2E_k - 2E_p} w_1(p) \end{pmatrix}, \\ g_2(p) &= \begin{pmatrix} P \frac{1}{2E_k - 2E_p} u_2(p) \\ \delta(2E_k - 2E_p) + P \frac{1}{2E_k - 2E_p} w_2(p) \end{pmatrix}. \end{aligned} \quad (8)$$

The equations for the amplitudes u and w are then

$$\begin{pmatrix} u_{\lambda}(p) \\ w_{\lambda}(p) \end{pmatrix} = \int s^2 ds V(s, p) g_{\lambda}(s) : \lambda = 1, 2.$$

Substitution for $g_{\lambda}(s)$ then yields

$$u_{\lambda}(p) = u_{\lambda B}(p) + P \int_0^{\infty} \frac{s^2 ds}{2E_k - 2E_s} [V_{J-1, J-1}(s, p) u_{\lambda}(s) + V_{J-1, J+1}(s, p) w_{\lambda}(s)], \quad (9)$$

$$w_{\lambda}(p) = w_{\lambda B}(p) + P \int_0^{\infty} \frac{s^2 ds}{2E_k - 2E_s} [V_{J+1, J-1}(s, p) u_{\lambda}(s) + V_{J+1, J+1}(s, p) w_{\lambda}(s)],$$

where the Born approximation amplitudes are given by

$$u_{1B}(p) \equiv \frac{kE_k}{2} V_{J-1, J-1}(k, p),$$

$$w_{1B}(p) \equiv \frac{kE_k}{2} V_{J+1, J-1}(k, p),$$

$$u_{2B}(p) \equiv \frac{kE_k}{2} V_{J-1, J+1}(k, p),$$

$$w_{2B}(p) \equiv \frac{kE_k}{2} V_{J+1, J+1}(k, p). \quad (10)$$

§ 3. Connexion of scattering amplitudes with phase shifts

(a) uncoupled states

The phase shift is defined by the asymptotic coordinate-space wave function

$$\phi_l(r) \sim \sin\left(kr - \frac{l\pi}{2}\right) + \tan\delta_l \cos\left(kr - \frac{l\pi}{2}\right).$$

Transforming to momentum space in the neighborhood of the singularity²⁾

$$\phi_l(p) \sim \delta(k-p) - \frac{\tan\delta_l}{\pi} P \frac{1}{k-p} + R(p), \quad (11)$$

where $R(p)$ is a function which is non-singular at k . Now equation (4) shows that in the neighborhood of the singularity our scattering amplitude can be written

$$g_l(p) \sim \delta(k-p) + P \frac{1}{k-p} f_l(k) + R(p).$$

Comparing with (11), we can identify the phase shift

$$\tan\delta_l = -\pi f_l(k). \quad (12)$$

In Born approximation, then,

$$\tan\delta_{l,R} = -\frac{\pi k E_k}{2} V_l(k, k). \quad (13)$$

(b) coupled states

We define the coupled scattering parameters in the representation³⁾

$$\begin{aligned} \phi_1(r) &\sim \begin{pmatrix} \sin\left[kr - \frac{(J-1)\pi}{2}\right] + x_{J-1} \cos\left[kr - \frac{(J-1)\pi}{2}\right] \\ y_{J-1} \cos\left[kr - \frac{(J+1)\pi}{2}\right] \end{pmatrix}, \\ \phi_2(r) &\sim \begin{pmatrix} y_{J+1} \cos\left[kr - \frac{(J-1)\pi}{2}\right] \\ \sin\left[kr - \frac{(J+1)\pi}{2}\right] + x_{J+1} \cos\left[kr - \frac{(J+1)\pi}{2}\right] \end{pmatrix}. \end{aligned}$$

The x 's and y 's are related to the usual Blatt-Biedenharn⁴⁾ parameters by

$$\begin{aligned} x_{J\mp 1} &= (\cos^2 \epsilon) \tan\delta_{J\mp 1} + (\sin^2 \epsilon) \tan\delta_{J\pm 1}, \\ y_{J\mp 1} &= (\sin \epsilon) (\cos \epsilon) [\tan\delta_{J-1} - \tan\delta_{J+1}]. \end{aligned} \quad (14)$$

Transformed to momentum space in the neighborhood of the singularity, ϕ_1 and ϕ_2 become

$$\phi_1(p) \sim \begin{pmatrix} \delta(k-p) - \frac{x_{J-1}}{\pi} P \frac{1}{k-p} + R_1(p) \\ + \frac{y_{J-1}}{\pi} P \frac{1}{k-p} + R_2(p) \end{pmatrix},$$

$$\phi_2(p) \sim \begin{pmatrix} + \frac{y_{J+1}}{\pi} P \frac{1}{k-p} + R_3(p) \\ \delta(k-p) - \frac{x_{J+1}}{\pi} P \frac{1}{k-p} + R_4(p) \end{pmatrix}.$$

Comparing to (8), when written in the neighborhood of the singularity, we can identify

$$\begin{aligned} x_{J-1} &= -\pi u_1(k), \\ y_{J-1} &= +\pi v_1(k), \\ y_{J+1} &= +\pi u_2(k), \\ x_{J+1} &= -\pi v_2(k). \end{aligned} \quad (15)$$

In Born approximation, then

$$\begin{aligned} x_{J-1,B} &= -\frac{\pi k E_k}{2} V_{J-1,J-1}(k, k), \\ y_{J-1,B} &= +\frac{\pi k E_k}{2} V_{J+1,J-1}(k, k), \\ y_{J+1,B} &= +\frac{\pi k E_k}{2} V_{J-1,J+1}(k, k), \\ x_{J+1,B} &= -\frac{\pi k E_k}{2} V_{J+1,J+1}(k, k). \end{aligned} \quad (16)$$

§ 4. The one pion exchange potential (OPEP)

We assume the $PS(PS)$ interaction Hamiltonian:

$$H_I = G \int d\mathbf{r} \psi^\dagger(\mathbf{r}) (i\beta\gamma_5) \boldsymbol{\tau} \cdot \boldsymbol{\phi}(\mathbf{r}) \psi(\mathbf{r})$$

Then the one pion exchange potential corresponding to Figure 2 can be written⁵⁾

$$V(\mathbf{s}, \mathbf{p}) = -\left(\frac{G^2}{4\pi}\right) \times \frac{[(\boldsymbol{\sigma}_1 \cdot \mathbf{s})(E_p + M) - (\boldsymbol{\sigma}_1 \cdot \mathbf{p})(E_s + M)][(\boldsymbol{\sigma}_2 \cdot \mathbf{s})(E_p + M) - (\boldsymbol{\sigma}_2 \cdot \mathbf{p})(E_s + M)]}{8\pi^2 E_p E_s (E_p + M)(E_s + M)[\omega^2 - (E_p - E_s)^2]}, \quad (17)$$

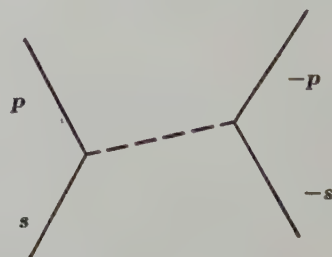


Fig. 2

where $E_p \equiv (p^2 + M^2)^{1/2}$ and $\omega^2 \equiv |\mathbf{s} - \mathbf{p}|^2 + \mu^2$. M is the nucleon mass and μ is the pion mass. Upon performing the integrations of equations (2) and (7), for this potential, one finds

$$\begin{aligned}
 {}^1V_i(s, p) &= -\left(\frac{G^2}{4\pi}\right) \frac{[(x_0 - \eta)Q_i(x_0) - \delta_{i,0}]}{2\pi E_p E_s}, \\
 {}^3V_J(s, p) &= -\left(\frac{G^2}{4\pi}\right) \frac{[(2J+1)\eta Q_J(x_0) - JQ_{J+1}(x_0) - (J+1)Q_{J-1}(x_0)]}{2\pi E_p E_s (2J+1)}, \quad (18) \\
 {}^3V_{J-1, J-1}(s, p) &= \left(\frac{G^2}{4\pi}\right) \frac{[Q_J(x_0) - \eta Q_{J-1}(x_0)]}{2\pi E_p E_s (2J+1)}, \\
 V_{J\pm 1, J\mp 1}(s, p) &= \left(\frac{G^2}{4\pi}\right) \\
 &\quad \times \frac{\sqrt{J(J+1)} \left[\frac{s(E_p + M)}{p(E_s + M)} Q_{J+1}(x_0) + \frac{p(E_s + M)}{s(E_p + M)} Q_{J-1}(x_0) - 2Q_J(x_0) \right]}{2\pi E_p E_s (2J+1)} \\
 V_{J+1, J+1}(s, p) &= \left(\frac{G^2}{4\pi}\right) \frac{[-Q_J(x_0) + \eta Q_{J+1}(x_0)]}{2\pi E_p E_s (2J+1)},
 \end{aligned}$$

where

$$\begin{aligned}
 x_0 &\equiv \frac{\mu^2 + 2(E_s E_p - M^2)}{2sp}, \\
 \eta &\equiv \frac{s(E_p + M)}{2p(E_s + M)} + \frac{p(E_s + M)}{2s(E_p + M)},
 \end{aligned}$$

and the Q_l are Legendre functions of the second kind.

§ 5. Born Approximation for the OPEP

To obtain the Born approximation parameters we evaluate the potentials⁵ at $s=p=k$ according to (13) and (16):

$$\begin{aligned}
 \tan^1 \delta_{i,B} &= \frac{k(G^2/4\pi)}{4E_k} [(x_0 - 1)Q_i(x_0) - \delta_{i,0}], \\
 \tan^3 \delta_{J,J,B} &= -\frac{k(G^2/4\pi)[JQ_{J-1}(x_0) + (J+1)Q_{J-1}(x_0) - (2J+1)Q_J(x_0)]}{4E_k(2J+1)}, \\
 x_{J-1,B} &= -\frac{k(G^2/4\pi)}{4E_k(2J+1)} [Q_J(x_0) - Q_{J-1}(x_0)], \\
 x_{J+1,B} &= -\frac{k(G^2/4\pi)}{4E_k(2J+1)} [Q_{J+1}(x_0) - Q_J(x_0)],
 \end{aligned}$$

$$y_B = -\frac{k(G^2/4\pi)\sqrt{J(J+1)}}{4E_k(2J+1)}[Q_{J+1}(x_0) + Q_{J-1}(x_0) - 2Q_J(x_0)].$$

Here x_0 has become $(1 + \mu^2/2k^2)$ and η has become unity. When the above quantities are small, they are approximately equal to their corresponding R -matrix amplitudes divided by $(2i)$. Since this is usually true when the Born approximation is valid, the above agrees with the result obtained by Cziffra, et al.⁶⁾

§ 6. Discussion

Solving the integral equations (5) and (9) for the amplitudes should be straightforward but somewhat tedious unless an electronic computer is used. In any case, Dyson et al.²⁾ have discussed at length several methods they found practical for accurately solving very similar equations for pion-nucleon scattering.

Without actually solving the equations, one can already see (section 5) that the long range component of the lowest-order velocity-dependent potential (other than the spin-orbit potential) is $(M/E_k - 1)$ times the static second order potential. When calculating in coordinate space, one can think of this as lowering the coupling constant as the scattering energy increases. The smallness of the velocity-dependent potential is seen in the factor being only $\sim (-.07)$ at 300 Mev.

Chew⁷⁾ and others⁸⁾ have used cut-offs on the virtual meson momentum. When dealing with the non-adiabatic potential, such a cut-off makes the potential integrals diverge. Instead, one can use a cut-off on the nucleon momentum.

§ 7. Acknowledgment

The author would like to thank Professor S. Machida, on leave from Rikkyo University, Tokyo, currently at the University of Pennsylvania, for several very interesting discussions.

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Note added in proof All equations from (17) on should also contain the isotopic spin factor $\tau_1 \cdot \tau_2$.

A Note on the Dispersion-Theoretic Approach to the One Additional Pion Production by Pion-Nucleon Collision

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A "natural" frame of reference to derive the dispersion relation for the one additional pion production by pion-nucleon collision is introduced. It is a natural extension of what is introduced in the elastic pion-nucleon scattering. Thus, the retarded matrix element on our problem may be derived to be analytic on the upper half plane of the complex laboratory pion energy fixing the other independent four variables. We then derive the dispersion relation connecting the dispersive parts and the absorptive parts. We shall use this relation in future as the integral equation for the scattering matrix element. Finally, the absorptive parts are investigated. Numerical calculation is reserved for another occasion.

§ 1. Introduction

The static theory of Chew and Low has proved successful at low energies for the pion-nucleon scattering and photopion production by nucleon.^{1),2)} Recently, several authors³⁾⁻⁶⁾ have applied this theory to the one additional pion production by pion-nucleon collision at moderate energies about 0.5 Bev, and have shown under some approximations that the total cross section and the charge ratios of the produced pions qualitatively agree with experiment.

Generally speaking, for such high energy collision as to produce an additional pion, however, the static theory in which the recoil effect of the nucleon is assumed to be neglected may not be suitable since the momentum of the incident pion becomes comparable with the nucleon mass. Indeed, as far as one assumes the interaction of only the p -wave pion with the core, which is characteristic in the static theory, one cannot explain the observed large total cross section for collisions in the Bev energy range. However, the possibility to explain this might still be left if one takes into account the recoil effect of the nucleon, since due to this effect the spacial range of the interaction between the incident pion and the nucleon may be considerably spread out. On the other hand, many authors⁷⁾⁻¹²⁾ have suggested the possibility of the π - π direct interaction to explain such experimental evidences in the Bev energy range as the above mentioned large total cross section, the forward peak of the faster of the outgoing two pions and the backward peak of the recoil nucleon in the center of mass system. It does not seem to be evident, however, that the latter two evidences also might not be explained without the assumption

of the π - π interaction, even if one takes into account the recoil of the nucleon. To investigate this, the dispersion-theoretic approach will be the most adequate. It will turn out to the relativistic extension of the static theory of Chew and Low, as it has been so in the elastic pion-nucleon scattering¹³⁾ and photopion production.¹⁴⁾ The dispersion relation of the inelastic scattering has already been investigated by Polkinghorne.¹⁵⁾ Recently, Screaton has extended the method introduced by Polkinghorne in the relativistically invariant way.¹⁶⁾ Their method, however, seems us to be somewhat complicated owing to the use of the properties of the triple pion current operators, $j_\alpha(x)$'s, and the introduction of the auxiliary scalar variables. Here, we shall easily derive the dispersion relation in the more similar way to that in the two-body problem by investigating the scattering matrix element expressed by the use of the double Heisenberg operators, fixing the three 4-momenta of the particles among the five relevant to our problem. In § 2, we shall note the kinematical relations to derive the dispersion relation in the relativistically invariant way, for future reference. In § 3, we shall construct the natural frame of reference and thus derive that the retarded matrix element in our problem is analytic on the upper half plane of the complex laboratory pion energy fixing the other independent four variables. In § 4, we then derive the dispersion relation connecting the dispersive parts and absorptive parts. We intend to use this relation in future as the integral equation for the scattering matrix element. In § 5, we shall carefully investigate the symmetrical properties between the various classes contributing to the absorptive part in order to simplify the future calculations, which will be still fairly complicated. Thus, we may have to use further approximations for the practical calculation so as to extract the characteristic feature in our problem. This point is reserved for another occasion.

§ 2. Kinematical consideration

We denote the 4-momenta of the incident pion and nucleon as q and p , and those of the outgoing two pions and the recoil nucleon as k^1, k^2 and p' respectively. We then investigate the properties of the scattering matrix when the three 4-momenta among the five are fixed and the residual two 4-momenta are varied under the condition of the total energy-momentum conservation and under the mass-shell conditions :

$$q^2 = (k^1)^2 = (k^2)^2 = -\mu^2, \quad p^2 = p'^2 = -m^2, \quad (2.1)$$

where we write the pion mass and nucleon mass as μ and m respectively.* For instance, we hereafter fix the three 4-momenta of the pions q, k^1 and k^2 and vary the two 4-momenta of the nucleons p and p' . For this case, it is convenient to write the scattering matrix by the standard procedure as follows :

* We use the natural units $c=1$ and $\hbar=1$ throughout.

$$\langle p', k^1, k^2 | S | p, q \rangle = (-i) (2\pi)^4 \delta(p' + k^1 + k^2 - p - q) (m^2/p_0 p'_0)^{1/2} \\ \times (8q_0 k_0^1 k_0^2)^{-1/2} \bar{u}(p') F u(p),^* \quad (2.2)$$

where the Feynman matrix element F is defined as

$$F = (-i) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle k^1 k^2(\text{out}) | T(f(z/2), \bar{f}(-z/2)) | q \rangle e^{-i(p' + p)/2 \cdot z} d^4 z. \quad (2.3)$$

Here, Heisenberg operator f is the nucleon source function defined by

$$(\gamma_\mu \partial / \partial x_\mu + m) \psi(x) = f(x), \quad (2.4)$$

$u(p)$ is the wave function of the nucleon p normalized as $\bar{u}(p)u(p) = 1$, $T(f(z/2), \bar{f}(-z/2))$ is the time ordered product of $f(z/2)$ and $\bar{f}(-z/2)$ defined by Wick. In the physically meaningful states, it is proved as usual that the Feynman matrix element F may be replaced by the following retarded matrix element $M^{(\text{out})}$:

$$M^{(\text{out})} = (-i) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle k^1 k^2(\text{out}) | \theta(z_0) [f(z/2), \bar{f}(-z/2)]_+ | q \rangle e^{-i(p' + p)/2 \cdot z} d^4 z, \quad (2.5)$$

where $\theta(z_0) = 0$ for $z_0 < 0$ and $= 1$ for $z_0 > 0$, and the bracket $[]_+$ stands for the anticommutator of $f(z/2)$ and $\bar{f}(-z/2)$.

We shall then derive the general form for the matrix element $M^{(\text{out})}$, which is the pseudoscalar function composed of p, p', q, k^1 and k^2 and of γ_μ -matrices. From the total energy-momentum conservation, the four independent 4-momenta may be taken as

$$P \equiv (p' + p)/2, \quad \kappa \equiv (p' - p)/2 = (q - k^1 - k^2)/2, \quad k \equiv (k^1 - k^2)/2 \quad \text{and} \quad q. \quad (2.6)$$

As the following conditions are derived from (2.1),

$$P^2 + \kappa^2 + m^2 = 0, \quad (2.7)$$

$$(P \cdot \kappa) = 0, \quad (2.8)$$

$$(q \cdot \kappa) = (3/4)\mu^2 + k^2 + \kappa^2, \quad (2.9)$$

$$(q \cdot k) = 2(k \cdot \kappa), \quad (2.10)$$

$$q^2 + \mu^2 = 0, \quad (2.11)$$

the five independent scalar products composed of P, κ, k and q may be chosen as follows:

$$\nu \equiv -(P \cdot q)/m, \quad (P \cdot k), \quad (\kappa \cdot k), \quad k^2 \quad \text{and} \quad \kappa^2, \quad (2.12)$$

where the first term ν is connected with the laboratory pion energy ω as

$$\omega = \nu + (3\mu^2)/(4m) + (k^2 + \kappa^2)/m. \quad (2.13)$$

* Here, the spin indices and the isospin indices are also included in the symbols p, p', q, k^1 and k^2 , for brevity.

Next, the independent pseudoscalar functions including γ_μ -matrices are written as

$$O_1 \equiv \gamma_5, \quad O_2 \equiv \gamma_5(i\gamma q), \quad O_3 \equiv \gamma_5(i\gamma k) \quad \text{and} \quad O_4 \equiv \gamma_5[(\gamma \cdot q), (\gamma \cdot k)]/2, \quad (2.14)$$

taking into account the relations $(i\gamma p + m)u(p) = 0$ and $\bar{u}(p')(i\gamma p' + m) = 0$. General form of the matrix element, thus, may be written as

$$M^{(\text{int})} = O_\lambda M_\lambda^{(\text{int})}(\nu, (P \cdot k), (\kappa \cdot k), k^2, \kappa^2). \quad (2.15)$$

Here, each $M_\lambda^{(\text{int})}$ includes the isotopic spin variables, which are easily separated out, introducing the following four independent isospin functions

$$Q_1 \equiv i\tau_q \delta_{k^1, k^2}, \quad Q_2 \equiv i\tau_{k^1} \delta_{q, k^2}, \quad Q_3 \equiv i\tau_{k^2} \delta_{q, k^1} \quad \text{and} \quad Q_4 \equiv \varepsilon_{k^1 k^2 q}, \quad (2.16)$$

where τ 's are Pauli's matrices, δ is Kronecker's δ -function and $\varepsilon_{k^1 k^2 q}$ Levi Civita's symbol. The matrix element $M^{(\text{int})}$ thus turns out to the following form:

$$M^{(\text{int})} = O_\lambda P_j M_{\lambda j}^{(\text{int})}(\nu, (P \cdot k), (\kappa \cdot k), k^2, \kappa^2). \quad (2.17)$$

§ 3. Construction of the natural frame of reference

In this section, we shall derive that each $M_{\lambda j}^{(\text{int})}$ is analytic on the upper half plane of the complex ν , fixing the other variables $(P \cdot k)$, $(\kappa \cdot k)$, k^2 and κ^2 . For this purpose, we shall construct the natural frame of reference in a similar manner to that of the elastic pion-nucleon scattering or the photopion production.¹⁷⁾⁻²⁰⁾

Now, since κ is a spacelike 4-vector, one can always choose the special Lorentz frame in which $\kappa_0 = 0$. In this way, if one takes the direction of κ as z -axis, the z -components of the \mathbf{P} , \mathbf{q} , \mathbf{k} are fixed by (2.8), (2.9) and $(\kappa \cdot k) = \text{constant}$, respectively, as follows;

$$P_z = 0, \quad (3.1)$$

$$q_z = (1/\kappa)(3\mu^2/4 + k^2 + \kappa^2) \quad (3.2)$$

$$k_z = (1/\kappa)(k \cdot \kappa). \quad (3.3)$$

We may now rewrite (2.7), (2.10) and (2.11) introducing the projections \mathbf{P}' , \mathbf{q}' , \mathbf{k}' of \mathbf{P} , \mathbf{q} , \mathbf{k} on the x - y plane, and using (3.1), (3.2) and (3.3), as follows:

$$\mathbf{P}'^2 + \kappa^2 + m^2 = P_0^2, \quad (3.4)$$

$$(\mathbf{q}' \cdot \mathbf{k}') + (k \cdot \kappa)(3\mu^2/4 + k^2 + \kappa^2)/\kappa^2 - q_0 k_0 = 2(k \cdot \kappa), \quad (3.5)$$

$$\mathbf{q}'^2 + (3\mu^2/4 + k^2 + \kappa^2)^2/\kappa^2 + \mu^2 = q_0^2. \quad (3.6)$$

Also, the following equation must be satisfied:

$$(\mathbf{P}' \cdot \mathbf{k}') - P_0 k_0 = (P \cdot k) = \text{constant}. \quad (3.7)$$

If $\mathbf{k}'^2 > k_0^2$, i.e. $\kappa^2 k^2 > (k \cdot \kappa)^2$, then one can always choose the special frame in which $k_0 = 0$ and take the direction of \mathbf{k}' as y -axis. The y -components of \mathbf{P} and \mathbf{q} are fixed by (3.7) and (3.5) respectively:

$$P_y = (k^2 - (k \cdot \kappa)^2 / \kappa^2)^{-1/2} (P \cdot k), \quad (3.8)$$

$$q_y = (k^2 - (k \cdot \kappa)^2 / \kappa^2)^{-1/2} \cdot (k \cdot \kappa) / \kappa^2 \cdot (-3\mu^2/4 - k^2 + \kappa^2). \quad (3.9)$$

We thus see the x -components of \mathbf{P} and \mathbf{q} are still unfixed by the mass shell conditions (2.7)–(2.11). So that one can vary P_x arbitrarily and may choose q_x as 0 so as to satisfy the relation $(\mathbf{P} \cdot \mathbf{q}) = \text{constant}$. Hence, from (3.8) and (3.9), $(\mathbf{P} \cdot \mathbf{q})$ is written as

$$(\mathbf{P} \cdot \mathbf{q}) = (\kappa^2 k^2 - (\kappa \cdot k)^2)^{-1} (-3/4) \mu^2 - k^2 + \kappa^2 (P \cdot k) (\kappa \cdot k). \quad (3.10)$$

We thus arrive at the following relation for \mathbf{P} and $P_0 \equiv E$ in this Lorentz frame:

$$\mathbf{P} = \sqrt{E^2 - M^2} \mathbf{e}_x + (k^2 - (k \cdot \kappa)^2 / \kappa^2)^{-1/2} (P \cdot k) \mathbf{e}_y, \quad (3.11)$$

where \mathbf{e}_x and \mathbf{e}_y are the unitvectors in the directions of x - and y -axis respectively, M^2 is written, by the use of (3.4), as

$$M^2 = \kappa^2 + m^2 + (k^2 - (\kappa \cdot k)^2 / \kappa^2)^{-1} (P \cdot k)^2. \quad (3.12)$$

Since P_y does not depend on E , the E dependence of the matrix element $M^{(\text{out})}$ is written as

$$M^{(\text{out})} \sim \int e^{-i(\sqrt{E^2 - M^2} x - Et)} \cdot f(\mathbf{x}, t) d\mathbf{x} dy dz dt, \quad (3.13)$$

where $f(\mathbf{x}, t)$ (which is independent of E) = 0 except $x_0 \geq \mathbf{x}^2$ and $x_0 \geq 0$. $M^{(\text{out})}$ thus has just the same form as in the natural frame in the elastic pion-nucleon scattering. The matrix element $M^{(\text{out})}$ may be derived to be analytic on the upper half plane of the complex E and also ν (which is linearly connected with E , see (2.12)) in the analogous way as in the elastic pion-nucleon scattering.

The foregoing derivation* was, however, restricted by the condition: $k^2 \kappa^2 > (k \cdot \kappa)^2$. In the case of $k^2 \kappa^2 < (k \cdot \kappa)^2$, i.e. $k'^2 < k_0^2$, on the contrary, one can always choose the special frame in which $k' = 0$. In this, P_0 and q_0 are completely fixed by (3.7) and (3.5) respectively. Therefore, in this case, the matrix element $M^{(\text{out})}$ may not be restricted by the micro-causality condition.

In this section, we investigated the analytic property of the retarded matrix element when the 4-momenta of three pions were fixed. However, there are several ways to fix the three 4-momenta among the five. So it may be derived that the scattering matrix element is analytic with the different variable (which may be the linear combination of ν , $(P \cdot k)$, $(k \cdot \kappa)$, k^2 and κ^2) for each way. It may then be possible to treat our scattering matrix element in the similar way as has been done by Mandelstam for the elastic pion-nucleon scattering.²¹⁾

§ 4. Dispersion relation

We first define as usual the dispersive part and the absorptive part of the

* Here, we gave only a heuristic derivation, the rigorous derivation will be fairly complicated owing to the large number of the independent variables.

matrix element $M^{(\text{out})}$ by the following equations :

$$D^{(\text{out})} = (-i/2) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle k^1 k^{2(\text{out})} | \mathcal{E}(z_0) [f(z/2), \bar{f}(-z/2)]_+ | q \rangle e^{-iPx} d^4 z, \quad (4.1)$$

$$A^{(\text{out})} = (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle k^1 k^{2(\text{out})} | [f(z/2), f(-z/2)]_+ | q \rangle e^{-iPx} d^4 z. \quad (4.2)$$

Then, $D^{(\text{out})}$ and $A^{(\text{out})}$ as well as $M^{(\text{out})}$ may be written in the invariant form :

$$D^{(\text{out})} = O_\lambda Q_j D_{\lambda j}^{(\text{out})}(\nu, (P \cdot k), (\kappa \cdot k), k^2, \kappa^2), \quad (4.3)$$

$$A^{(\text{out})} = O_\lambda Q_j A_{\lambda j}^{(\text{out})}(\nu, (P \cdot k), (\kappa \cdot k), k^2, \kappa^2). \quad (4.4)$$

Thus defined $D_{\lambda j}^{(\text{out})}$ and $A_{\lambda j}^{(\text{out})}$, however, may not be real numbers, so we introduce instead the matrix element $M^{(\text{out})}$ the more essential $M^{(+)}$ and $M^{(-)}$ in which the final state $\langle k^1 k^{2(\text{out})} |$ of the matrix element $M^{(\text{out})}$ is replaced by the following :

$$\langle k^1 k^{2(+)} | \equiv (\langle k^1 k^{2(\text{out})} | + \langle k^1 k^{2(\text{in})} |) / 2 \quad (4.5)$$

and

$$\langle k^1 k^{2(-)} | \equiv (\langle k^1 k^{2(\text{out})} | - \langle k^1 k^{2(\text{in})} |) / 2i, \quad (4.6)$$

respectively.

For those matrices $M^{(\pm)}$, the dispersive parts $D^{(\pm)}$ and the absorptive parts $A^{(\pm)}$ are defined by replacing $\langle k^1 k^{2(\text{out})} |$ in (4.1) and (4.2) by $\langle k^1 k^{2(\pm)} |$. And then the invariant forms for $D^{(\pm)}$ and $A^{(\pm)}$ may be written as

$$D^{(\pm)} = O_\lambda Q_j D_{\lambda j}^{(\pm)}(\nu, (P \cdot k), (\kappa \cdot k), k^2, \kappa^2), \quad (4.7)$$

$$A^{(\pm)} = O_\lambda Q_j A_{\lambda j}^{(\pm)}(\nu, (P \cdot k), (\kappa \cdot k), k^2, \kappa^2). \quad (4.8)$$

From the relation : $\langle k^1 k^{2(\text{out})} | = \langle k^1 k^{2(+)} | + i \langle k^1 k^{2(-)} |$, it follows that

$$M^{(\text{out})} = M^{(+)} + iM^{(-)}, \quad D^{(\text{out})} = D^{(+)} + iD^{(-)}, \quad A^{(\text{out})} = A^{(+)} + iA^{(-)},$$

$$D_{\lambda j}^{(\text{out})} = D_{\lambda j}^{(+)} + iD_{\lambda j}^{(-)} \quad \text{and} \quad A_{\lambda j}^{(\text{out})} = A_{\lambda j}^{(+)} + iA_{\lambda j}^{(-)}. \quad (4.9)$$

Thus introduced $D_{\lambda j}^{(\pm)}$ and $A_{\lambda j}^{(\pm)}$ will be proved to be all real numbers from the requirement of the time-reversal invariance in Appendix 1. We thus arrive at the dispersion relation for $M^{(\pm)}$:

$$D_{\lambda j}^{(\pm)}(\nu, (P \cdot k)) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{A_{\lambda j}^{(\pm)}(\nu', (P \cdot k))}{\nu' - \nu} d\nu', \quad (4.10)$$

where we omit the fixed variables $(\kappa \cdot k)$, k^2 and κ^2 for brevity. From (4.9), the dispersion relations for $M^{(\text{out})}$ thus are written as

$$D_{\lambda j}^{(\text{out})}(\nu, (P \cdot k)) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{A_{\lambda j}^{(\text{out})}(\nu', (P \cdot k))}{\nu' - \nu} d\nu', \quad (4.11)$$

where it may be noted that $D_{\lambda j}^{(\text{out})}$ and $A_{\lambda j}^{(\text{out})}$ are generally complex numbers. Finally, $A_{\lambda j}(\nu, (P \cdot k))$ and $A_{\lambda j}(-\nu, -(P \cdot k))$ are connected by the following equations from the requirement of the charge conjugation invariance:

$$A_{\lambda j}(-\nu, -(P \cdot k), (k \cdot \kappa), k^2, \kappa^2) = \pm A_{\lambda j}(\nu, (P \cdot k), (k \cdot \kappa), k^2, \kappa^2), \quad (4.12)$$

where the plus sign is to be taken for $\lambda=1, 2, 3$ and $j=4$, and for $\lambda=4$ and $j=1, 2, 3$, otherwise the minus sign is to be taken. (4.12) will be proved in Appendix 2. Using (4.12), (4.10) and (4.11) may be rewritten as

$$D_{\lambda j}(\nu, (P \cdot k)) = \frac{P}{\pi} \int_0^\infty \left(\frac{A_{\lambda j}(\nu', (P \cdot k))}{\nu' - \nu} \mp \frac{A_{\lambda j}(\nu', -(P \cdot k))}{\nu' + \nu} \right) d\nu'. \quad (4.13)$$

(4.10), (4.11) and (4.13) may be used to calculate the dispersive parts when the absorptive parts are known, as the latter might be more easily evaluated directly from (4.2) than the former. We then investigate the absorptive parts in the following section.

§ 5. Investigation of the absorptive parts

In this section, we will investigate the symmetric properties of the absorptive part, $A^{(+)}$. This will be divided into two parts, $A_I^{(+)}$ and $A_{II}^{(+)}$, which are connected with each other by the charge conjugation. $A_I^{(+)}$ and $A_{II}^{(+)}$ will also be divided into four classes, i.e. a, b, c, d and a_c, b_c, c_c, d_c , respectively, as will be illustrated in Fig. 1. However, by the use of the properties of the scattering matrix element when the signs of the 4-momenta of pions are changed, we will show the simple symmetric relations between a and d, b and c , respectively, (5.13–14), (and hence the similar relations for the conjugate classes). Thus, the absorptive part will be expressed by the only use of a and b (and these conjugate a_c and b_c), (5.19).

First, it may be stressed that the dispersion relations (4.10) for the matrix elements $M^{(\pm)}$ are the more essential than those (4.11) for the matrix element $M^{(\text{out})}$, because the latter are consequent upon the former in which the reality of $D_{\lambda j}^{(\pm)}$ and $A_{\lambda j}^{(\pm)}$ plays an important role. Now, the difference between $M^{(\text{out})}$ and $M^{(+)}$ is in the final states of two pions $\langle k^1 k^{2(\text{out})} |$ and $\langle k^1 k^{2(+)} |$ defined by (4.5). These are connected by the relation

$$\langle k^1 k^{2(\text{out})} | = \sum_n \langle k^1 k^{2(\text{in})} | 2S / (1+S) | n^{(\text{in})} \rangle \langle n^{(+)} |, \quad (5.1)$$

since

$$\langle k^1 k^{2(\text{out})} | = \langle k^1 k^{2(\text{in})} | S^* \quad (5.2)$$

Here, $2S/(1+S)$ equals just to $1 - \frac{1}{2}iK$ in which the K matrix is defined by

* Here, we choose the Heisenberg operators as to coincide with their interaction operators when time t approaches $-\infty$. In this choice the "in" state is equal to that of the Schroedinger representation.

Schwinger.²²⁾ So, if the pion-pion scattering phase shifts are known, $\langle k^1 k^{2(\text{out})} |$ may be evaluated by (5.1) from $\langle k^1 k^{2(+)} |$, for such a low energy pion production as the elastic pion-pion scattering is dominant. Naturally, for the vanishing pion-pion interaction, $\langle^{(+)}| = \langle^{\text{out}}|$. We then restrict our consideration to the absorptive part $A^{(+)}$ of the matrix element $M^{(+)}$. $A^{(+)}$ are divided into two parts,

$$A_I^{(+)} = (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle k^1 k^{2(+)} | f(z/2) \bar{f}(-z/2) | q \rangle e^{-iP \cdot z} d^4 z \quad (5.3)$$

and

$$A_{II}^{(+)} = (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle k^1 k^{2(+)} | \bar{f}(-z/2) f(z/2) | q \rangle e^{-iP \cdot z} d^4 z. \quad (5.4)$$

From Appendix 2, the invariant scalars $A_{II\lambda_j}^{(\pm)}$ of the matrix element $A_{II}^{(\pm)}$ are seen to be related with those of $A_I^{(\pm)}$ as follows:

$$A_{II\lambda_j}^{(\pm)}(\nu, (P \cdot k), (k \cdot \kappa), k^2, \kappa^2) = \pm A_{I\lambda_j}^{(\pm)}(-\nu, -(P \cdot k), (k \cdot \kappa), k^2, \kappa^2), \quad (5.5)$$

where signs \pm are to be taken as same as in (4.12). Therefore, we have only to investigate the matrix element $A_I^{(+)}$, if the range of the variable ν is unrestricted in $(0, \infty)$ as in (4.13).

We next expand the matrix element $\langle k^1 k^{2(+)} | f(z/2) \cdot \bar{f}(-z/2) | q \rangle$ in $A_I^{(+)}$, (5.3), in the complete set of intermediate states n which are taken as the eigenstates of the total energy and momentum,

$$A_I^{(+)} = (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} (2\pi)^4 \sum_n \delta(p+q-n) \langle k^1 k^{2(+)} | f(0) | n \rangle \langle n | \bar{f}(0) | q \rangle, \quad (5.6)$$

where $\langle k^1 k^{2(+)} | f(0) | n \rangle \langle n | \bar{f}(0) | q \rangle$ may be taken as

$$\frac{1}{2} \{ \langle k^1 k^{2(\text{out})} | f(0) | n^{(\text{in})} \rangle \langle n^{(\text{in})} | \bar{f}(0) | q \rangle + \langle k^1 k^{2(\text{in})} | f(0) | n^{(\text{out})} \rangle \langle n^{(\text{out})} | \bar{f}(0) | q \rangle \} \quad (5.7)$$

in order to ensure formally the reality of the invariant scalars $A_{I\lambda}^{(\pm)}$ in each order of n . The matrix elements of the single Heisenberg operators in the above then may be expressed by the scattering matrix elements, when all the intermediate particles interact with the nuclear matter. It may happen, however, that the intermediate states n include the one pion of the state k^1 or k^2 , the two pions of k^1 and q or k^2 and q and the three pions of k^1 , k^2 and q , respectively. In these cases, there will be the additional "delta type"²⁶⁾ contributions to the matrix elements $\langle k^1 k^{2(+)} | f(0) | n \rangle$ and/or $\langle n | \bar{f}(0) | q \rangle$ from the process that those pions move between initial and final states without the interaction with the nuclear matter. After the separation of these contributions from the $\langle k^1 k^{2(+)} | f(0) | n \rangle \langle n | \bar{f}(0) | q \rangle$, $A_I^{(+)}$ may be easily written as the sum of four terms,

$$A_I^{(+)} = (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} (2\pi)^4 \times \left\{ \sum_n \langle k^1 k^{2(+)} | f(0) | n \rangle \langle n | \bar{f}(0) | q \rangle \delta(p+q-n) \right.$$

$$\begin{aligned}
 & + \sum_n [\langle k^2 | f(0) | n \rangle \langle n, k^1 | \bar{f}(0) | q \rangle \delta(p' + k^2 - n) + k^1, k^2 \text{ exchange term}] \\
 & + \sum_n [\langle k^1 | f(0) | n, q \rangle \langle n, k^2 | \bar{f}(0) | 0 \rangle \delta(p - k^2 - n) + k^1, k^2 \text{ exchange term}] \\
 & + \sum_n \langle 0 | f(0) | n, q \rangle \langle n, k^1, k^2 | \bar{f}(0) | 0 \rangle \delta(p' - q - n) \}. \quad (5.8)
 \end{aligned}$$

Here, the particles in the intermediate states have to be understood as interacting always with the matter. We hereafter denote these terms as a, b, c and d in order from the top in (5.8). We illustrate these graphically in Fig. 1, where the states of nucleon, pion and intermediate particles are represented by the fine, dotted and thick lines, respectively. $A_{II}^{(+)}$ may then be expressed by exchanging the lines of the nucleon p and p' in Fig. 1.

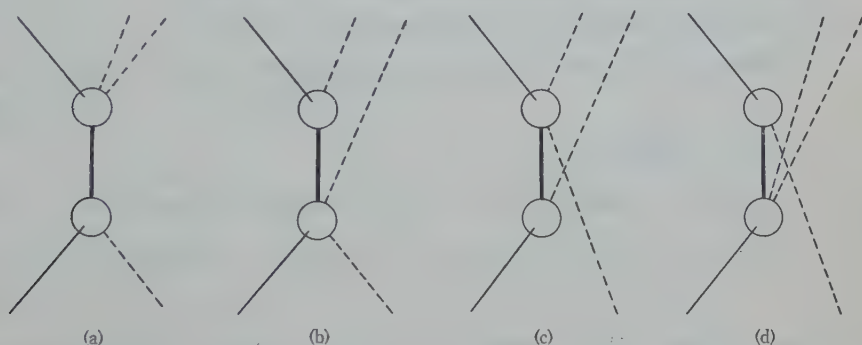


Fig. 1

Now, first we consider the contribution a ,

$$a \equiv (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} (2\pi)^4 \sum_n \delta(p + q - n) \langle k^1 k^{2(+)} | f(0) | n \rangle \langle n | \bar{f}(0) | q \rangle. \quad (5.9)$$

This will be expressed as the sum of the contributions a' from the states of the propermasses $m_n ((p+q)^2 + m_n^2 = 0)$,

$$\begin{aligned}
 a' = & (-\pi) \theta(p_0 + q_0) \delta((p+q)^2 + m_n^2) \sum_n (8q_0 k_0^1 k_0^2)^{1/2} (2\pi)^4 \langle k^1 k^{2(+)} | f(0) | n \rangle \langle n | \bar{f}(0) | q \rangle \\
 & \times \langle k^1 k^{2(+)} | f(0) | n \rangle \langle n | \bar{f}(0) | q \rangle, \quad (5.10)
 \end{aligned}$$

where n is the substate of the propermass m_n and momentum $n = p + q$. may be written in the invariant form as

$$a' = \theta(p_0 + q_0) \delta((p+q)^2 + m_n^2) O_\lambda Q_j a_{\lambda j}(m_n^2, (P \cdot k), (k \cdot \kappa), k^2, \kappa^2), \quad (5.11)$$

where $\theta(p_0 + q_0) \delta((p+q)^2 + m_n^2)$ is also to be understood as the function of ν , $(P \cdot k)$, $(k \cdot \kappa)$, k^2 , κ^2 and m_n^2 . In the similar way, the contribution to b, c and d from the states of the propermass m_n may be written as follows:

$$\begin{aligned}
b' &= \theta(p_0' + k_0^2) \delta((p' + k^2)^2 + m_n^2) O_\lambda Q_j b_{\lambda j}(m_n^2, (P \cdot k), (k \cdot \kappa), k^2, \kappa^2) + \text{ex.}, \\
c' &= \theta(p_0 - k_0^2) \delta((p - k^2)^2 + m_n^2) O_\lambda Q_j c_{\lambda j}(m_n^2, (P \cdot k), (k \cdot \kappa), k^2, \kappa^2) + \text{ex.}, \quad (5 \cdot 12) \\
d' &= \theta(p_0' - q_0) \delta((p' - q)^2 + m_n^2) O_\lambda Q_j d_{\lambda j}(m_n^2, (P \cdot k), (k \cdot \kappa), k^2, \kappa^2),
\end{aligned}$$

where ex. expresses the term given by exchanging the pion states k^1 and k^2 in the original one.

The following symmetric properties, however, will be proved,

$$d_{\lambda j}(m_n^2, (P \cdot k), (k \cdot \kappa), k^2, \kappa^2) = \mp a_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2), \quad (5 \cdot 13)$$

$$c_{\lambda j}(m_n^2, (P \cdot k), (k \cdot \kappa), k^2, \kappa^2) = \mp b_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2), \quad (5 \cdot 14)$$

where the *minus* sign is to be taken for $\lambda=1, 2, 3$ and $j=4$ and for $\lambda=4$ and $j=1, 2, 3$, otherwise the *plus* sign is to be taken. The proof is as follows. First, consider the original expression of d , the last term of (5.8),

$$d = (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} (2\pi)^4 \sum_n \delta(p' - q - n) \langle 0 | f(0) | n, q \rangle \langle n, k^1, k^2 | \bar{f}(0) | 0 \rangle. \quad (5 \cdot 15)$$

This may be easily rewritten as

$$\begin{aligned}
d &= (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} (2\pi)^4 \sum_n \delta(p' - q - n) \langle -q | f(0) | n \rangle \langle n | \bar{f}(0) | -k^1, -k^2 \rangle \\
&= \{\gamma_4 [(-1/2) (8q_0 k_0^1 k_0^2)^{1/2} (2\pi)^4 \sum_n \delta(p' - q - n) \langle -k^1, -k^2 | f(0) | n \rangle \langle n | f(0) | \\
&\quad -q \rangle] \gamma_4\}^\dagger \quad (5 \cdot 16)
\end{aligned}$$

where the pion states $-k^1, -k^2$ and $-q$ are to be taken as those obtained by changing the signs of the 4-momenta k^1, k^2 and q in the original expression (5.15).

The expression in the bracket [] of (5.16), however, is just the one obtained from the matrix element a , the first term in (5.8), by the transformation of 4-momenta as $p' \rightarrow p, p \rightarrow p', q \rightarrow -q, k^1 \rightarrow -k^1$ and $k^2 \rightarrow -k^2$. Using the above result, the contribution to d from the states of the propermass m_n may be written as

$$d' = [\gamma_4 \theta(p_0' - q_0) \delta((p' - q)^2 + m_n^2) O_\lambda' Q_j a_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2) \gamma_4]^\dagger, \quad (5 \cdot 17)$$

where O_λ' are those obtained from O_λ by the transformation of the 4-momenta as $k^1 \rightarrow -k^1, k^2 \rightarrow -k^2$ and $q \rightarrow -q$. As $[\gamma_4 O_\lambda' Q_j \gamma_4]^\dagger = \mp O_\lambda Q_j$ is easily verified, (5.13) is proved taking into account the reality of $a_{\lambda j}$. Similarly, (5.14) may be proved.

We here write down the contributions to $A_{\text{II}}^{(\pm)}$, (5.4), from the states of the propermass m_n , by the use of (5.5), corresponding to a', b', c' , and d' , as follows:

$$\begin{aligned}
a_c' &= \pm \theta(-p_0' + q_0) \delta((p' - q)^2 + m_n^2) O_\lambda Q_j a_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2), \\
b_c' &= \pm \theta(-p_0 + k_0^2) \delta((p - k^2)^2 + m_n^2) O_\lambda Q_j b_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2) + \text{ex.}, \\
c_c' &= \pm \theta(-p_0' - k_0^2) \delta((p' + k^2)^2 + m_n^2) O_\lambda Q_j c_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2) + \text{ex.}, \\
d_c' &= \pm \theta(-p_0 - q_0) \delta((p + q)^2 + m_n^2) O_\lambda Q_j d_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2).
\end{aligned} \quad (5 \cdot 18)$$

Substituting (5.13) and (5.14) in (5.12) and (5.18), and combining a' , a'_e , b' and b'_e with d'_e , d' , c'_e and c' respectively, we finally obtain the contributions to the absorptive part $A^{(+)}$ from the states of the propermass m_n , $A^{(++)}$, in the closed form,

$$\begin{aligned} A^{(++)} = & \varepsilon(p_0 + q_0) \delta((p+q)^2 + m_n^2) O_\lambda Q_j a_{\lambda j}(m_n^2, (P \cdot k), (\kappa \cdot k), k^2, \kappa^2) \\ & \pm \varepsilon(-p'_0 + q_0) \delta((p'-q)^2 + m_n^2) O_\lambda Q_j a_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2) \\ & + [\varepsilon(p'_0 + k_0^2) \delta((p'+k)^2 + m_n^2) O_\lambda Q_j b_{\lambda j}(m_n^2, (P \cdot k), (\kappa \cdot k), k^2, \kappa^2) + \text{ex.}] \\ & \pm [\varepsilon(-p_0 + k_0^2) \delta((p-k)^2 + m_n^2) O_\lambda Q_j b_{\lambda j}(m_n^2, -(P \cdot k), (\kappa \cdot k), k^2, \kappa^2) + \text{ex.}] \end{aligned} \quad (5.19)$$

where signs \pm are, of course, to be taken according to the rule mentioned below (4.12).

The values of the sign functions ε 's in (5.19) may be evaluated in the natural frame introduced in § 3. They generally depend upon the values of m_n^2 , $(P \cdot k)$, k^2 , and κ^2 , and are fairly complicated.

Now, we consider the one nucleon state in n . It will give the δ -type contributions to the absorptive part $A^{(+)}$ with respect to ν . It will be clear that these contributions are expressed as the product of the elastic scattering matrix element and the pion-nucleon vertex with the renormalized coupling constant. The one nucleon state, however, cannot be realized physically, so it will be necessary to continue the scattering angle and energy in the unphysical domains. Next contributing intermediate states are the one nucleon and one pion. These contributions will be expressed by using the elastic pion-nucleon scattering matrix and the one additional pion production matrix. The dispersion relations, now, will turn out to the linear inhomogeneous equation to the one additional pion production matrix with the kernels of the elastic scattering matrix. Here also the continuation of the variables to the unphysical domain will naturally be necessary.

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Appendix I

According to Lüders,²³⁾ the time-reversal operation is defined as follows.

Two time-reversed states $T\psi_1$ and $T\psi_2$ of the original states ψ_1 and ψ_2 satisfy the relation,

$$\langle T\psi_1 | T\psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle^* \quad (\text{A} \cdot 1)$$

Hence, any complex number α is transformed into its conjugate complex α^* ,

$$T\alpha T^{-1} = \alpha^* \quad (\text{A} \cdot 2)$$

Nucleon field operators $\phi_s(\mathbf{x})$ and $\bar{\phi}_s(\mathbf{x})$ and pion field operators $\phi_s^i(\mathbf{x})$ and $\pi_s^i(\mathbf{x})$ in the Schroedinger representation are transformed as

$$\begin{aligned} T\phi_s(\mathbf{x})T^{-1} &= u\phi_s(\mathbf{x}), & T\bar{\phi}_s(\mathbf{x})T^{-1} &= \bar{\phi}_s(\mathbf{x})u^{-1}, \\ T\phi_s^i(\mathbf{x})T^{-1} &= \begin{cases} +\phi_s^i(\mathbf{x}) & \text{for } i=2, \\ -\phi_s^i(\mathbf{x}) & \text{for } i=1 \text{ and } 3, \end{cases} & T\pi_s^i(\mathbf{x})T^{-1} &= \begin{cases} -\pi_s^i(\mathbf{x}) & \text{for } i=2, \\ +\pi_s^i(\mathbf{x}) & \text{for } i=1 \text{ and } 3, \end{cases} \end{aligned} \quad (\text{A} \cdot 3)$$

where unitary matrix u is defined as

$$u^{-1}\gamma_\mu^*u = \gamma_\mu. \quad (\text{A} \cdot 4)$$

Using (A.2) and (A.3), the pion state of momentum \mathbf{k} and isospin index i is transformed as

$$T|\mathbf{k}^{(i)}\rangle = \begin{cases} +|-\mathbf{k}^{(i)}\rangle & \text{for } i=2, \\ -|-\mathbf{k}^{(i)}\rangle & \text{for } i=1 \text{ and } 3. \end{cases} \quad (\text{A} \cdot 5)$$

Now, we define the interaction operator as

$$O_I(\mathbf{x}, t) = e^{iH_0 t} O_s(\mathbf{x}) e^{-iH_0 t}. \quad (\text{A} \cdot 6)$$

Then, it follows that

$$\begin{aligned} T\phi_I(\mathbf{x}, t)T^{-1} &= u\phi_I(\mathbf{x}, -t), & T\bar{\phi}_I(\mathbf{x}, t)T^{-1} &= \bar{\phi}_I(\mathbf{x}, -t)u^{-1} \\ \text{and} & & T\phi_I^i(\mathbf{x}, t)T^{-1} &= \pm\phi_I^i(\mathbf{x}, -t). \end{aligned} \quad (\text{A} \cdot 7)$$

Next we define the Heisenberg operator so as to coincide with the interaction operator when t approaches $-\infty$,

$$O(\mathbf{x}, t) = [U(t, -\infty)]^{-1} O_I(\mathbf{x}, t) U(t, -\infty), \quad (\text{A} \cdot 8)$$

where the transformation function $U(t, t')$ is written as

$$U(t, t') = e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'}. \quad (\text{A} \cdot 9)$$

Hence, it follows that

$$TU(t, t')T^{-1} = U(t, -t'), \quad (\text{A} \cdot 10)$$

especially,

$$TST^{-1} = TU(\infty, -\infty)T^{-1} = U(-\infty, \infty) = S^\dagger \quad (\text{A} \cdot 11)$$

We thus obtain the transformation rule for the Heisenberg operators ϕ , $\bar{\phi}$ and ϕ ,

$$T\phi(\mathbf{x}, t)T^{-1} = u(S\phi(\mathbf{x}, -t)S^\dagger), \quad T\bar{\phi}(\mathbf{x}, t)T^{-1} = (S\bar{\phi}(\mathbf{x}, -t)S^\dagger)u^{-1}$$

and

$$T\phi^i(\mathbf{x}, t)T^{-1} = \pm(S\phi^i(\mathbf{x}, -t)S^\dagger). \quad (\text{A} \cdot 12)$$

The "out" state $|a^{(\text{out})}\rangle = S^\dagger|a^{(\text{in})}\rangle$ is transformed as

$$T|a^{(\text{out})}\rangle = S|Ta^{(\text{in})}\rangle. \quad (\text{A} \cdot 13)$$

Here, we shall verify the reality of $A_{\lambda_j}^{(\pm)}$ and $D_{\lambda_j}^{(\pm)}$ in (4.8) and (4.7). Using (A.1), (A.12) and (A.13) the conjugate complex of $A^{(\pm)}$ is written as

$$A^{(\pm)*} = (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle Tk^1, Tk^{2(\pm)} | S^\dagger S [uf(z/2, -z_0/2), \bar{f}(-z/2, +z_0/2) u^{-1}]_+ S^\dagger | Tq \rangle e^{+iP \cdot z} d^4 z$$

Using (A.5), this is rewritten as

$$A^{(\pm)*} = \pm u \{ (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle k_T^1 k_T^{2(\pm)} | [f(z/2), \bar{f}(-z/2)]_+ | q_T \rangle e^{-iP_T z} d^4 z \} u^{-1}, \quad (\text{A.15})$$

where $k_T^1 = (-\mathbf{k}^1, k_0^1)$, $k_T^2 = (-\mathbf{k}^2, k_0^2)$, $q_T = (-\mathbf{q}, q_0)$ and $P_T = (-\mathbf{P}, P_0)$, and signs \pm arise from (A.5). The expression in the bracket $\{ \}$ is just the one obtained from the original $A^{(\pm)}$ by the transformation: $k^1 \rightarrow k_T^1$, $k^2 \rightarrow k_T^2$, $q \rightarrow q_T$ and $P \rightarrow P_T$. Rewriting the both sides of (A.15) in the invariant form, according to (4.8), and comparing the both sides in this form, we easily see the reality of $A_{\lambda_j}^{(\pm)}$. Similarly, the reality of $D_{\lambda_j}^{(\pm)}$ may be verified.

Appendix 2

The charge conjugation is defined, as is well-known, as

$$\begin{aligned} \bar{f}'(x) &= c^{-1} f(x), \\ f'(x) &= c \bar{f}(x), \\ \phi'_i(x) &= \begin{cases} +\phi_i(x) & \text{for } i=1, 3, \\ -\phi_i(x) & \text{for } i=2, \end{cases} \end{aligned} \quad (\text{A.16})$$

where unitary matrix c satisfies the relations

$$c^{-1} \gamma_\mu c = -\gamma_\mu^t, \quad c^t = -c \quad (\text{A.17})$$

where γ_μ^t , c^t are the transposed matrices of γ_μ and c respectively. Now, we decompose the absorptive part A into A_I and A_{II} as (5.4) and (5.5). A_{II} is rewritten from the charge conjugation invariance as follows:

$$(A_{II})_{\alpha\beta} = (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle ck^1 ck^2 | (c^{-1} f(-z/2))_\beta, (\bar{f}(z/2) c^t)_\alpha | cq \rangle e^{-iP \cdot z} d^4 z, \quad (\text{A.18})$$

where ck^1 , ck^2 and cq are the charge conjugate of k^1 , k^2 and q respectively. In the matrix form, it turns out to

$$A_{II}^t = \pm c^{-1} \{ (-1/2) (8q_0 k_0^1 k_0^2)^{1/2} \int \langle k^1 k^2 | f(z/2) \bar{f}(-z/2) | q \rangle e^{iP \cdot z} d^4 z \} c^t \quad (\text{A.19})$$

where signs \pm arise from (A.16). The expression in the above bracket $\{ \}$ is just the one obtained from the original A_I , (5.3), by the transformation $P \rightarrow -P$

or $p \rightarrow -p'$ and $p' \rightarrow -p$. Writing the both sides of (A·18) in the invariant form, using (4·4) or (4·8), and comparing the both sides in this form, we easily arrive at (5·5). (4·12) may be obtained in consequence of (5·5).

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The Dipole Giant Resonance in the High Energy Proton Scattering

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The characteristic peaks in the energy spectra of high energy (p, p') scattering accompanied by residual excitation of 20 Mev, experimentally observed by Tyren and Maris, are analyzed in terms of the Coulomb excitation. It is shown that in the Born approximation the cross section for this process can be directly connected with the absorption cross section of electric dipole γ -rays, without referring to any special nuclear model. Using the known photonuclear cross sections, (p, p') energy spectra are calculated and are compared with experiment. Shapes of the energy spectra are reasonably well reproduced by the calculation. Also the absolute magnitude of the calculated cross sections are in agreement with the observed ones as far as the order of magnitude is concerned. On closer examination, however, it seems that the former may be smaller than the latter by a factor of 2 to 4. Several discussions are given.

§ 1. Introduction

In a series of papers¹⁾, Tyren and Maris have recently reported the experiments on the inelastic scattering of high energy protons, in which the energy spectra of protons inelastically scattered in the forward direction have been measured for a number of nuclei, extending from deuteron up to Zn⁶⁵.

One of the most conspicuous features of the observed energy spectra is the presence of a large and broad resonance-like peak at the proton energy which corresponds to the excitation energy of about 20 Mev of the residual nucleus. The peak has been actually observed for almost all the nuclei investigated, except for the lightest ones. Furthermore, the appearance of the peak is confined to the forward scattering direction of which the scattering angles are smaller than about 20°. Examples of such energy spectra are shown in Fig. 1.

From the properties of the peak mentioned above it is tempting to assume that it has some interrelation with the well-known giant resonance observed in the photonuclear reaction. In fact, Tyren and Maris²⁾ assumed that the incident proton exerts the Coulomb force to the bulk of proton charge of the target nucleus and gives rise to the dipole oscillation of the latter which is responsible for the well-known giant resonance of electric dipole γ -ray absorption. Assuming the Goldhaber-Teller model for the latter process and the uniform distribution of the proton charge over the spherical nucleus, Tyren and Maris have calculated the angular distribution of inelastically scattered protons in the case of C¹² target nucleus, which shows very sharply forward peaked angular distribution, having the maximum at 0° with

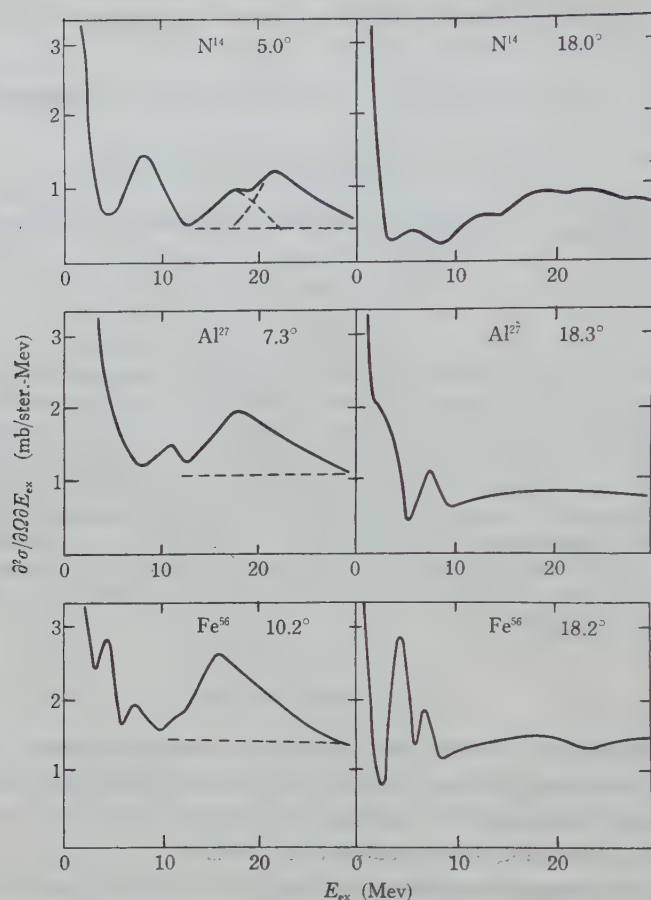


Fig. 1 Typical examples of energy spectra of inelastically scattered protons in the (p, p') reaction at 180 Mev incident energy observed in the forward scattering angles after Tyren and Maris. The differential cross sections for N, Al and Fe are plotted as functions of excitation energy of the residual nucleus. The scattering angles are shown in each section of the Figure.

a half width of about 3° for the choice of the nuclear radius, $R = 1.2 \times 10^{-13} A^{1/3}$ cm. The absolute magnitude of the cross section has been found to be about 1.5 mb/ster. · Mev at 3° which is of the order of magnitude of the experimentally observed ones. After examining several possible mechanisms for the reaction, they have reached the conclusion that the Coulomb excitation described above is the most probable, main process for the "giant resonance peak" around 20 MeV of residual excitation in the (p, p') reaction under consideration.

In view of the above situations it seems worth while to investigate the high energy Coulomb excitation process further in detail. In particular, it is desirable to obtain the cross section of the process without referring to a specific nuclear

model such as the one adopted by Tyren and Maris, so that the result can be free from ambiguity which may arise from the assumed nuclear model.

In the present paper the cross section for high energy Coulomb excitation is obtained by means of the Born approximation in such a way that no special assumption with regard to the nuclear states concerned is made in deriving the cross section formula. In fact, it is shown to be directly proportional to the cross section for absorption of electric dipole photons by the target nucleus, so that it is possible to evaluate the cross section for the high energy Coulomb excitation using available photonuclear cross section data.

An attempt will be made at the direct comparison of the experimentally observed (p, p') energy spectra of Tyren and Maris with the calculated ones for high energy Coulomb excitation using available photonuclear cross section data. The cross section for the photon absorption is estimated using $\sigma(\gamma, p)$ and $\sigma(\gamma, n)$, i.e., the cross sections for the emission of particles after the absorption of the photon.

Through such comparison it is found that for most of target nuclei investigated the shapes of the energy spectra are reasonably well reproduced by the calculation. The existence of the "giant resonance peaks" in the (p, p') energy spectra are accounted for as a reflection of the well-known giant resonance peaks in the photon absorption cross sections, as is expected. The calculated angular distribution of scattered protons is sharply peaked forward, having the maximum at 0° , which is in qualitative agreement with experiment. However, the former seems to be a little sharper than the latter, which fact has already been noted by Tyren and Maris.

The absolute magnitude of the calculated cross sections are all in agreement with the experimental ones as far as the order of magnitude is concerned, with the single exception of Ne which has the (p, p') cross section about ten times as large as the calculated ones. A closer examination, however, reveals that the calculated cross sections are always a factor of 2 to 4 smaller than the observed ones. On account of the crudeness in the present calculations and of the experimental error included in the absolute magnitude of the relevant cross sections, it is not quite clear whether the discrepancy mentioned above should really be taken seriously or not. It might, however, be possible that the discrepancy has some significance, since it is not restricted to any particular nuclei but is quite common to almost all the nuclei investigated. If so, the discrepancy would have to be attributed to the failure either of the Coulomb excitation hypothesis or of the Born approximation in the present calculation. The Born approximation may not be sufficiently valid since, for example, the incident as well as the scattered protons are under the influence of average nuclear potential of the target nucleus. It turns out, however, that the main contribution to the Coulomb excitation cross section comes from those protons which pass outside the nucleus. It is, therefore, expected that the effect of the distorting nuclear potential will not be very large, since it effects the cross section only indirectly. Crude estimation of such effect will be indicated in a later paragraph.

In § 2, the cross section formula is derived for the high energy Coulomb

excitation process, by means of the Born approximation.

In § 3, the cross sections for the Coulomb excitation are calculated for various target nuclei using available photonuclear cross section data.

In § 4, several discussions and the concluding remarks are given.

§ 2. The derivation of the cross section for (p, p') reaction

Since the energy of the incident proton is high, we shall, as a first step, assume that the Born approximation is valid for the calculation of the cross section. Discussion on the validity of the Born approximation shall be carried out in the last section.

Let the wave functions of the incident proton and the target nucleus be $e^{i\mathbf{k}_i \cdot \mathbf{r}}$ and $\Psi_0(\hat{\xi})$, respectively, where \mathbf{k}_i and \mathbf{r} are the wave number and the coordinate vectors of the incident proton and $\hat{\xi}$ represents the aggregate of the coordinates of the nucleons in the target nucleus.

The Born approximation to the cross section for excitation of the nucleus to an excited state $\Psi_n(\hat{\xi})$ by the Coulomb force of the proton which is scattered with the final wave number vector \mathbf{k}_n is given by

$$\left(\frac{\partial \sigma}{\partial \Omega} \right)_{on} = \frac{m^2 k_n}{4\pi^2 \hbar^4 k_i} \left| \iint e^{-i\mathbf{k}_n \cdot \mathbf{r}} \Psi_n^* \sum_{\alpha=1}^Z \frac{e^2}{|\mathbf{r} - \mathbf{r}_\alpha|} e^{i\mathbf{k}_i \cdot \mathbf{r}} \Psi_0 d\mathbf{r} d\hat{\xi} \right|^2, \quad (1)$$

where m is the mass of the incident proton and \mathbf{r}_α the coordinate of the α -th proton in the nucleus. Making use of the relation

$$\int \frac{1}{|\mathbf{r} - \mathbf{r}_\alpha|} e^{i\mathbf{q} \cdot \mathbf{r}} d\mathbf{r} = \frac{4\pi}{q^2} e^{i\mathbf{q} \cdot \mathbf{r}_\alpha}, \quad (2)$$

(1) may be written as

$$\left(\frac{\partial \sigma}{\partial \Omega} \right)_{on} = \frac{e^4 m^2 k_n}{4\pi^2 \hbar^4 k_i} \frac{(4\pi)^2}{q^4} \left| \int \Psi_n^* \sum_{\alpha=1}^Z e^{i\mathbf{q} \cdot \mathbf{r}_\alpha} \Psi_0 d\hat{\xi} \right|^2, \quad (3)$$

where $\mathbf{q} = \mathbf{k}_i - \mathbf{k}_n$.

For the energy and the scattering angle of the proton under consideration, q is of the order of $0.1 \times 10^{13} \text{ cm}^{-1}$ and the range of r which yields appreciable contribution to the integral is of the order of nuclear radius R which is a few times of 10^{-13} cm . Therefore, we have $qR < 1$, for the present case, which permits the expansion of $e^{i\mathbf{q} \cdot \mathbf{r}}$ into power series of $(\mathbf{q} \cdot \mathbf{r})$. Retaining only the second term of this expansion, it follows,

$$\left(\frac{\partial \sigma}{\partial \Omega} \right)_{on} = \frac{e^4 m^2 k_n}{4\pi^2 \hbar^4 k_i} \frac{(4\pi)^2}{q^2} \left| \int \Psi_n^* \sum_{\alpha=1}^Z z_\alpha \Psi_0 d\hat{\xi} \right|^2, \quad (4)$$

where z_α is the component of \mathbf{r}_α in the direction of \mathbf{q} .

It is now seen from (4) that the matrix element responsible for the process is just the one that causes the electric dipole transition of the nucleus. The

higher order terms in the expansion of $e^{i\mathbf{q}\cdot\mathbf{r}}$ would yield the matrix elements responsible for the γ -transition of higher multipolarity, which will safely be neglected, especially for Ψ_n which is situated near the giant resonance maximum.

Now, the transition probability for the excitation of the nucleus from 0 to n through the electric dipole absorption of γ -ray is given by⁴⁾

$$\sigma_{ph}(0 \rightarrow n) = 4\pi^2 \frac{e^2}{\hbar c} E_{ex} \left| \int \Psi_n^* \sum_{\alpha=1}^Z z_{\alpha} \Psi_0 d\xi \right|^2, \quad (5)$$

where E_{ex} is the energy of the state Ψ_n above the ground state. Substituting from (5) the electric dipole transition matrix element into (4) and, furthermore, summing (4) over the possible excited states n with the excitation energies between E_{ex} and $E_{ex} + dE_{ex}$, we have energy spectrum of the emitted protons given by

$$\frac{\partial^2 \sigma}{\partial E_{ex} \partial \Omega} = \frac{1}{2\pi^2} \frac{e^2}{\hbar c} \frac{mc^2}{E} \frac{1}{E_{ex}} \sqrt{1 - \frac{E_{ex}}{E}} \frac{\sigma_{ph}(E_{ex})}{2 - (E_{ex}/E) - 2\sqrt{1 - (E_{ex}/E)} \cos \theta}, \quad (6)$$

where E is the energy of the incident proton and $\sigma_{ph}(E_{ex})$ the cross section for the excitation of nucleus by energy E_{ex} through absorption of electric dipole γ -ray. The energy of the emitted proton is given by

$$E_f = E - E_{ex}.$$

If $E \gg E_{ex}$ and $\theta \ll 1$, (6) may be simplified approximately:

$$\frac{\partial^2 \sigma}{\partial E_{ex} \partial \Omega} = \frac{1}{2\pi^2} \cdot \frac{e^2}{\hbar c} \cdot \frac{mc^2}{E} \cdot \frac{1}{E_{ex}} \cdot \frac{1}{\theta^2} \cdot \sigma_{ph}(E_{ex}). \quad (6a)$$

(6) and (6a) give the cross section $\partial^2 \sigma / \partial E_{ex} \partial \Omega$ in terms of the photonuclear cross section σ_{ph} . Thus, we can calculate the cross section of the process under consideration from the known experimental data of σ_{ph} without referring to any special model of nucleus. It should be noted that (6) and (6a) contain no adjustable parameter so that they would provide us with the means of making a rather critical test of the Coulomb excitation hypothesis, so far as the Born approximation is valid.

$\partial^2 \sigma / \partial E_{ex} \partial \Omega$ given by (6) and (6a) have all qualitative features observed by experiments as mentioned in the preceding paragraph. The peak at $E_{ex} = 20$ Mev is due to the giant resonance of $\sigma_{ph}(E_{ex})$ which is common to all nuclei and, moreover, is larger in magnitude for heavier nuclei. The angular distribution is concentrated very sharply in the forward direction because of the factor $1/\theta^2$. Thus, it is clear that the broad peak in the high energy (p, p') scattering under consideration is qualitatively explained by the assumption that it is due to the Coulomb excitation of the dipole giant resonance state of the nucleus.

§ 3. Numerical calculation and comparison with experiment

In order to evaluate $(\partial^2 \sigma / \partial E_{ex} \partial \Omega)$ by (6) or (6a), a knowledge of σ_{ph} is

required. Unfortunately, however, there is practically no direct experimental data of the γ -absorption cross section but we have only individual reaction cross sections such as (γ, n) , (γ, p) , and so on. In terms of these cross sections σ_{ph} may be expressed

$$\sigma_{ph} = \sigma(\gamma, p) + \sigma(\gamma, n) + \sigma(\gamma, np) + \dots$$

Now, there are considerable accumulation of data of $\sigma(\gamma, n)$ ⁵⁾ while we have a few $\sigma(\gamma, p)$.⁶⁾ Very few data exist for $\sigma(\gamma, np)$, $\sigma(\gamma, 2p)$, $\sigma(\gamma, 2n)$, etc. Unfortunately, $\sigma(\gamma, p)$ is considerably larger than $\sigma(\gamma, n)$ for most of the nuclei in this region of mass numbers except for the heaviest ones investigated.⁷⁾ Therefore, in the case of $\sigma(\gamma, p)$ being not measured it is rather difficult to evaluate the absolute magnitude of $\partial^2\sigma/\partial E_e \partial \Omega$ from (6a). We shall discuss this point later again.

On the other hand, the shape of the energy spectra, i.e., relative value of the cross section may be obtained from $\sigma(\gamma, n)$ alone, since the excitation function of $\sigma_{ph}(E_e)$ would be approximately proportional to that of $\sigma(\gamma, n)$ except for E_e near the threshold of (γ, n) or E_e higher than the thresholds of (γ, np) , $(\gamma, 2n)$, $(\gamma, 2p)$, etc. The favorable situation is expected most likely to hold for E_e corresponding to the giant resonance peak.

Thus, as a first step, we shall evaluate the relative energy spectra of inelastically scattered protons, using the known $\sigma(\gamma, n)$ data. Typical examples of the results of such calculations are shown in Fig. 2 in comparison with the energy spectra observed by Tyren and Maris. Rather flat back-ground in proton energy spectra is subtracted as is shown in Fig. 1. This subtraction procedure is rather arbitrary, but for most of nuclei the back-ground cross section is so flat (in many cases, almost constant) that the subtraction procedure will cause no serious ambiguity in the final results. C^{12} and N^{14} are exceptions, which will be discussed further in detail in the next paragraph.

Agreement between the calculated cross sections and the experimental ones is reasonable, allowing for the crude assumption underlying the calculation as well as the experimental uncertainty of the relevant cross sections, especially that of photonuclear cross sections. Generally speaking, agreement seems to be slightly better for heavier nuclei than for lighter ones.

As is seen in Fig. 2, the observed width of the peak is larger than the calculated one. Inclusion of such processes as (γ, p) , (γ, np) , $(\gamma, 2p)$, $(\gamma, 2n)\dots$ in σ_{ph} will considerably improve this situation. However, the origin of the discrepancy is not clear at present.

Next, we shall go on to the discussion of the absolute magnitude of the cross sections, particularly at the top of the giant resonance peaks.

As has already been mentioned, the experimental data of photonuclear cross sections other than $\sigma(\gamma, n)$ are very scanty. The excitation functions of the (γ, p) reaction are measured only for a few nuclei for which σ_{ph} may be obtained by

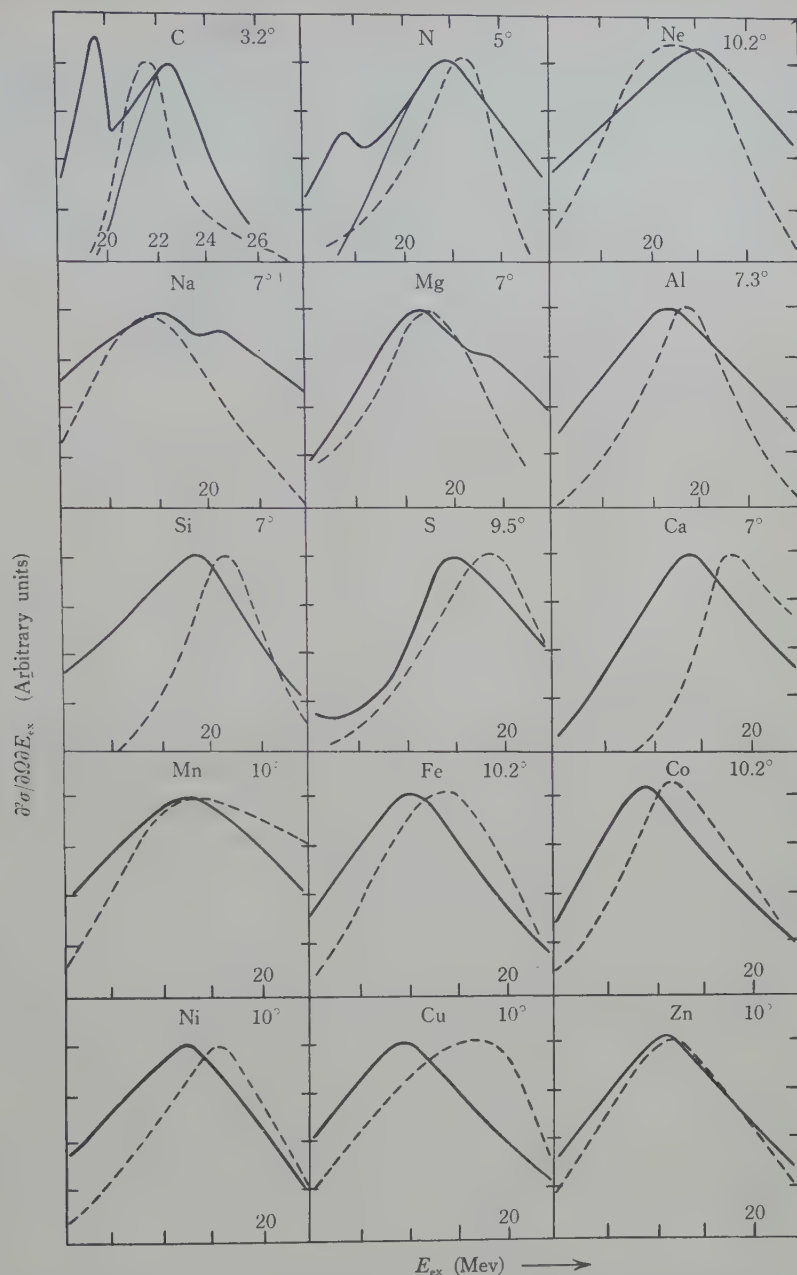


Fig. 2 Comparison of the observed and calculated shapes of energy spectra of (p, p') reaction for various target nuclei. The solid curves represent the observed proton energy spectra at the scattering angle indicated in each section of the Figure. The broken curves represent the results of the calculation by means of (6a) in the text and the observed (γ, n) cross section data. The flat back-ground is arbitrarily subtracted from the original energy spectra in order to derive solid curves. For C and N, thick solid curves represent the original energy spectra and thin solid curves represent tentative division of the thick curves into contributions of two different peaks.

Table 1 Comparison of the calculated and observed absolute values of the cross sections at the top of the giant resonances. The first column represents the target nuclei, the second column represents the angle of observation. In the third column the first column represents the calculated value of the cross section using various methods indicated by a, b and c in the last column. The second of the third column is the observed cross section at the top of the resonance peak. For the implication of the symbols, a, b and c, see the text.

Nucleus	θ°	$\left(\frac{\partial^2 \sigma}{\partial \Omega \partial E}\right)_{\max} \frac{\text{mb}}{\text{ster.} \cdot \text{Mev}}$		
		cal.	exp.	
C	3.2	1.2	1.0	a
O	5.5	0.23	1.0	c
Ne	10.2	0.08	0.8	c
Mg	7.0	0.34	0.9	b
Al	7.3	0.22	0.8	a
Si	7.0	0.59	0.9	b
S	9.5	0.24	1.0	b
Ca	5.0	1.5	2.1	b
Fe	10.0	0.27	1.1	c
Ni	10.0	0.36	0.7	a

summing $\sigma(\gamma, n)$ and $\sigma(\gamma, p)$. The calculated cross sections together with the observed peak height for some of these nuclei are listed in Table 1 marked as a. It is seen that the calculated cross sections are of the same order of magnitude as the observed ones, which fact supports the conclusion drawn by Tyren and Maris.

However, it is at the same time observed that the calculated cross sections are about a factor of 2 to 3 smaller than the experimental ones, although the uncertainty in the absolute magnitude of the photonuclear cross sections used in the calculations is rather large. In order to see whether this tendency is peculiar to those nuclei investigated above or common to all nuclei the comparison should be extended also to other nuclei for which there are no experimental data of (γ, p) excitation functions.

Since, in this case, it is impossible to calculate σ_{ph} directly from the experimental data, we have to resort to more indirect method. Now, we shall assume that the ratio of $\sigma(\gamma, n)$ to $\sigma(\gamma, p)$ at the top of the giant resonance is approximately equal to that of the corresponding integrated cross sections. This assumption will be valid only in the case in which $\sigma(\gamma, n)$ is proportional to $\sigma(\gamma, p)$. However, it will be at least approximately valid for those nuclei for which the width of the giant resonance peak of $\sigma(\gamma, n)$ is not greatly different from that of $\sigma(\gamma, p)$.

Fortunately, the experimental data⁸⁾ for both integrated (γ, n) and (γ, p) cross sections exist for nuclei of Mg^{24} , Si^{28} , S^{32} and Ca^{40} , and may be used for calculation of the (p, p') cross section under the assumption mentioned above. The results are listed in Table 2, marked as b.

For the cross sections in Table 1, marked as c, the following method of calculation has been adopted. First we shall assume that the ratio of $\sigma(\gamma, p)$ to $\sigma(\gamma, n)$ is correctly given by the evaporation theory. Then, the branching ratio of (γ, n) to (γ, p) , i.e., F_p/F_n , is given by

$$\frac{F_p}{F_n} = \frac{\int_0^{E_{ex}-S_p} \epsilon \sigma_{cp}(\epsilon) w_p(E_{ex}-S_p-\epsilon) d\epsilon}{\int_0^{E_{ex}-S_n} \epsilon \sigma_{cn}(\epsilon) w_n(E_{ex}-S_n-\epsilon) d\epsilon} \quad (7)$$

where S_n and S_p , W_n and W_p and σ_{cn} and σ_{cp} are the separation energies, the level densities of residual nuclei and the compound nucleus formation cross sections for neutron and proton emissions, respectively. For even-even nuclei we may assume

$$W_p(E) = W_n(E).$$

$W(E)$ is assumed to be of the form

$$W(E) = C \exp \sqrt{aE},$$

where C and a are constants. σ_{cp} and σ_{cn} are given by continuum theory.^{9),10)}

Then, it is possible to adjust the quantity a for nuclei Mg^{24} , Si^{28} , S^{32} and Ca^{40} so that the calculation yields correct experimental values of F_p/F_n . Along these procedure we find

$$a \simeq 0.10A \text{ (Mev)}^{-1}, \quad (8)$$

where A is the mass number of the nucleus. The above value of a is about factor 4 smaller than the one determined from particle reaction data. This tendency has already been pointed out by Igo and Wegner.¹¹⁾

Assuming, then, the formula (8) for all the nuclei investigated we have calculated F_p/F_n using (7). The examples of the calculated results are shown in Table 2. It should be noted that F_p/F_n calculated in this way shows good agreement with the experimental ones for nuclei C^{12} and Ni^{58} for which direct measurements of both (γ, n) and (γ, p) excitation functions exist. Making use of the branching ratio F_p/F_n mentioned above, we estimate the (γ, p) cross sections by means of the formula

$$\sigma(\gamma, p) = \sigma(\gamma, n) \frac{F_p}{F_n},$$

which leads to

$$\sigma_{ph} = \sigma(\gamma, n) + \sigma(\gamma, p) = \left(1 + \frac{F_p}{F_n}\right) \sigma(\gamma, n). \quad (9)$$

The cross sections for (p, p') , at the top of the giant resonance, evaluated using (9), are tabulated in Table 1, marked as c.

Table 2 The branching ratio of the (γ, p) and (γ, n) reactions. The first column represents the target nuclei, the second column represents the branching ratio of (γ, p) to (γ, n) process calculated by the evaporation theory at the top of the giant resonance. For assumptions adopted in the calculation, see text. The last column represents the experimental branching ratio at the top of the giant resonance obtained either by direct measurement or by estimation from the integrated cross section. For detailed explanation, see text.

Nucleus	$\left(\frac{F_p}{F_n}\right)_{\text{cal}}$	$\left(\frac{F_p}{F_n}\right)_{\text{exp}}$
C	2.4	2.6
O	2.0	—
Ne	2.8	—
Mg	6.4	2.8
Si	4.2	3.9
S	4.3	4.6
Ca	6.1	6.8
Fe	0.1	—
Ni	0.7	1.1

As seen in Table 1, a, b and c, the comparison of the absolute magnitudes of both calculated and observed cross sections shows that the discrepancy by a factor of 2 to 4 is common to all the nuclei investigated, with a single exception of Ne for which the observed cross section becomes ten times as large as the calculated one.

§ 4. Discussion

The analysis in the preceding paragraph has shown that the high energy Coulomb excitation process may account for the general qualitative behavior of the giant resonance peak in the energy spectrum of (p, p') reaction. It has been found at the same time that the absolute magnitudes of the calculated cross sections seem to be about 2 to 4 times smaller than the observed ones. On account of the various crude assumptions made in the calculation and of the experimental error in the relevant data, it might be dangerous to overemphasize the significance of such discrepancy. Since, however, the discrepancy of nearly equal order of magnitudes appears for all the nuclei investigated, it seems possible that it is not fortuitous and is inherent to the present calculation.

Now, the Born approximation in the present calculation might be subject to doubt, since the effect of average nuclear potential on the incident and outgoing protons would not in general be negligible even at 185 Mev under consideration.

The effect of such distorting potential will be large if the protons which pass through the interior of the nucleus have large contribution to the whole Coulomb excitation process. To estimate the relative importance of such process we calculate $(\partial\sigma/\partial\Omega)_{0 \rightarrow n}^{\text{int}}$ by (1) restricting the integral over \mathbf{r} from R to ∞ instead of 0 to ∞ , and

see the relative change of the resulting cross section. It is easy to see that

$$(\partial\sigma/\partial\Omega)_{on}^{out}/(\partial\sigma/\partial\Omega)_{on}=|j_0(qR)|^2\simeq 1-\frac{1}{3}(qR)^{2*}.$$

Assuming, for example, $qR=0.5$, we have 0.92 for this ratio. In other words, the protons passing through the interior of the nucleus contribute only 8% of the total Coulomb excitation cross section. Therefore, it is expected that the effect of the distorting potential will not be very large.

The conclusion reached above has been confirmed also by the conventional perturbation treatment of the distorting potential. Taking the first order effect of distorting potential into account, the modification of the Born scattering amplitude due to the distorting potential has been estimated to be of the order of 10% of the Born amplitude for the case of C^{12} target nucleus and 0° scattering angle. The modification seems to be slightly larger for larger scattering angles. For the sake of analytic simplicity the complex distorting potential has been assumed to be of Yukawa shape. The potential parameters have been adjusted so that the forward scattering due to this potential is approximately the same as that due to the usual square well optical potential appropriate to this energy region. It seems likely, therefore, that the effect of the distorting potential would not drastically change the qualitative features of the calculated cross section. A more accurate calculation of the effect of the distorting potential is desirable for wider range of scattering angles for various target nuclei. Then, a more definite conclusion will be obtained as to the absolute magnitude of the cross section for the Coulomb excitation together with the origin of the discrepancy mentioned previously. Furthermore, the distorted Born calculation might account for the observed proton angular distribution satisfactorily which, as mentioned in the preceding section, is less sharply peaked forward than expected from the present Born approximation.**

On the other hand, the possibility of the collective dipole excitation of the target nucleus may be important,¹²⁾ which is due to the τ -spin dependent part of the nuclear force exerted by the incident proton. The angular distribution of emitted protons in such process is dipped at 0° and spreads out to larger angles than in the case of Coulomb excitation, on account of the short range of the nuclear force, and then such process cannot be the main process in this reaction because the observed proton angular distribution is sharply peaked forward. However, the inclusion of this process in the present analysis might improve the agreement between calculation and experiment, for the observed angular distribution is less sharply peaked than expected from the present calculation.

Returning, now, to the discussion of the analysis of the experimental data using

* Our thanks are due to Drs. M. Soga and K. Nakamura for suggesting the use of this equation.

** Also, the angular distribution would be slightly changed, if $\exp[i\mathbf{q}\cdot\mathbf{r}]$ is expanded as

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l 4\pi i^l j_l(qr) Y_{lm}^*(\Omega_q) \cdot Y_{lm}(\Omega_r).$$

(6) and (6a), we should like to add a few supplementary remarks.

For some of the nuclei two giant resonance peaks are seen in the (p, p') energy spectrum instead of single one present in σ_{ph} . Typical examples are C^{12} and N^{14} as are seen in Fig. 2. The presence of such two peaks cannot be accounted for by the simple Coulomb excitation mechanism. It is clear in Fig. 2 that resonance peak of σ_{ph} is situated approximately at the excitation energy of one of these peaks. Therefore, it would be tempting to assume that only one of these two peaks lying at the approximate position of the giant resonance peak of σ_{ph} is due to the Coulomb excitation mechanism under consideration. If we tentatively assume the above interpretation to be correct we shall be able to separate each of the observed cross section curves into two parts, as shown by thin curves in Fig. 2 for C and N. The shapes of the curves for different peaks are rather arbitrarily assumed; for C the observed shape of the peak at the lower excitation energy is smoothly extrapolated towards the higher energy region, and for N the shape of the lower peak is assumed to be approximately symmetrical for the energies above and below the top of the peak. As is clearly seen in Fig. 2, the comparison with the calculation of the resulting curves belonging to the higher excitation peaks is greatly improved for both C and N. The possible mechanism responsible for the peaks at lower excitation energies is not yet clarified at present. In order to see what process might possibly be responsible for these peaks further study would be necessary.

Rather a similar situation exists also for the case of Mg and Na, for which the giant resonance (p, p') energy spectra seem to consist of more than one peak. In these cases, however, the unambiguous separation of the resonance curve into individual parts is more difficult than in cases of C and N. It seems, therefore, difficult to see whether the agreement between the calculated curve and the experimental one is improved by the similar procedure as in cases of C and N.

Now, we shall make a little remark about the possibility of observing the giant resonance peak of Coulomb excitation under different condition from the one discussed so far. It is easy to see that the cross section for the Coulomb excitation is proportional to MZ^2 , where M and Z are the mass and the charge of the incident particle. If we use an α -particle as projectile, the cross section will become 16 times larger, in comparison with that of proton. In this case, however, the effect of nuclear distortion and, in particular, absorption of α -particles in the target nucleus will be much stronger than in the present case, and the giant resonance peak might be marked by such disturbances.

On the other hand, the experiment using the electrons as projectile will be free from such back-ground, although the cross section will be 10^{-3} times that for protons.

Since direct measurement of σ_{ph} is usually very difficult in the γ -ray experiment, the σ_{ph} calculated from (6) using experimental energy spectrum of (p, p') reaction might serve as a tool for estimating σ_{ph} . Furthermore, integration of (6) over E_{ec} yields

$$\frac{\partial \sigma}{\partial \Omega} = \frac{1}{2\pi^2} \frac{e^2}{\hbar c} \frac{mc^2}{E} \frac{1}{\theta^2} \int \frac{\sigma_{ph}(E_{ex})}{E_{ex}} dE_{ex}.$$

Thus, the measurement of integrated cross section $\partial\sigma/\partial\Omega$ of this process will give useful information on the harmonic mean energy, for the absorption of the electric dipole γ -rays. Then, using the dipole sum rule of Levinger and Bethe,⁴⁾ we have

$$\frac{\partial \sigma}{\partial \Omega} = 2 \cdot \left(\frac{e^2}{\hbar c} \right)^2 \frac{mc^2}{E} \left(\frac{NZ}{A} \right)^2 \frac{1}{\theta^2} \cdot \left\langle \left\{ \frac{1}{Z} \sum_i z_i - \frac{1}{N} \sum_j z_j \right\}^2 \right\rangle_{00}$$

where N , Z and A are neutron, proton and mass number of the target nucleus, respectively, z_i and z_j are the z -components of the coordinate vectors of proton and neutron respectively, and $\langle O \rangle_{00}$ is the expectation value of operator O in the ground state.

Thus, we see that the Coulomb dipole excitation process may provide a useful tool for investigating the photonuclear reaction in the neighbourhood of the dipole giant resonance.

Before this method may be put into practice, however, further study, both experimental and theoretical, of "giant resonance" peak in the inelastic scattering spectra will be highly desirable.

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On the Inversion of the Specific-Heat Function

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In this paper we present a method for the inversion of specific heat data to find the vibrational frequency spectrum of a solid. Our results are given in terms of a power series in ω , the frequency. In this series the coefficients depend on $C_v(T)$ and its derivatives at $T=0$. The series thus derived is valid only in an asymptotic sense, but represents an improvement over inversions by Montroll and Kroll which require a knowledge of functions formed from the data, and the evaluation of double integrals over these functions. A method for forming the successive values of $C_v^{(n)}(0)$ from experimental data is also presented.

One of the most frequently measured physical characteristics of a solid is the specific heat at constant volume $C_v(T)$, where T is the temperature. The behavior of this function has quite accurately been predicted by various characterizations of a lattice of atoms which have harmonic interactions. If it is assumed that the specific heat can be attributed solely to harmonic effects, then it can be written as an integral over the vibrational frequency spectrum¹⁾

$$C_v(T) = k \int_0^{\infty} \frac{(\beta\omega)^2}{\sinh^2 \beta\omega} g(\omega) d\omega \quad (1)$$

where k is Boltzmann's constant, $\beta = \hbar/(2kT)$, and $g(\omega)d\omega$ is the number of frequencies in the interval $(\omega, \omega+d\omega)$ for $d\omega$ small. Although the upper limit of this integral is written as ∞ it is well known that there exists an upper limit ω_L beyond which $g(\omega)=0$. One of the problems suggested by the representation of Eq. (1) is the inversion of the integral to find $g(\omega)$ as a function of $C_v(T)$. So far there have been three more or less exact inversions of Eq. (1) by Montroll²⁾, Stanley and Grayson-Smith³⁾, and Kroll⁴⁾. The inversions of Montroll and Kroll are given in the form of double integrals over functions defined in terms of the experimentally-observed specific heat. These formulations, because of the complicated mathematical representations, will probably not be of use in the foreseeable future. On the other hand, the work of Stanley and Grayson-Smith, which assumes a definite form for the frequency spectrum, does not contain any difficulties from the mathematical point of view, and does lead to a reasonably simple representation of the frequency spectrum in terms of specific heat data. It is the purpose of this paper to derive another representation with the same mathematical simplicity

as the result of Stanley and Grayson-Smith. While it is probable that the method herein presented will not be of immediate use because the present limits of experimental precision are not narrow enough, it is hoped that our theory may be of future value.

A three-dimensional periodic lattice, according to the results of van Hove⁵⁾, will generally have a continuous frequency spectrum with a discontinuous first derivative. While special forms of the dispersion relation might lead to a singularity in the frequency spectrum, we will assume that this is not the case, and that the frequency spectrum is in fact continuous. Then, by the theorem of Weierstrass⁶⁾, the frequency spectrum can be arbitrarily closely approximated by a polynomial, i.e., we have

$$g(\omega) \sim \sum_{n=0}^N a_n \omega^{2n} \quad (2)$$

where we have taken advantage of the known properties of the frequency spectrum to expand $g(\omega)$ in even powers of ω . We then have

$$\frac{C_v(T)}{k} \sim \sum_{n=0}^N a_n \int_0^{\omega_L} \frac{\beta^2 \omega^{2n+2}}{\sinh^2 \beta \omega} d\omega. \quad (3)$$

We will solve for the coefficients a_n in terms of $C_v(T)$. We therefore consider the integral

$$F_n(\beta) = \int_0^{\omega_L} \frac{\beta^2 \omega^{2n+2}}{\sinh^2 \beta \omega} d\omega = \beta^{-(2n+1)} \int_0^{\beta \omega_L} \frac{x^{2n+2}}{\sinh^2 x} dx. \quad (4)$$

This can be integrated by parts to yield

$$\beta^{2n+1} F_n(\beta) = -(\beta \omega_L)^{2n+2} \operatorname{ctnh} \beta \omega_L + (2n+2) \int_0^{\beta \omega_L} y^{2n+1} \operatorname{ctnh} y dy. \quad (5)$$

We now expand the hyperbolic cotangent as follows

$$\operatorname{ctnh} y = 1 + 2 \sum_{n=1}^{\infty} e^{-2ny} \quad (6)$$

and substitute into Eq. (5). This gives the result

$$\begin{aligned} \beta^{2n+1} F_n(\beta) = & -2(\beta \omega_L)^{2n+2} \sum_{n=1}^{\infty} e^{-2n\beta \omega_L} \\ & + (4n+4) \int_0^{\beta \omega_L} y^{2n+1} (e^{-2y} + e^{-4y} + \dots) dy. \end{aligned} \quad (7)$$

The integrals appearing in this expression can be decomposed to yield

$$\int_0^{\beta\omega_L} = \int_0^\infty - \int_{\beta\omega_L}^\infty. \quad (8)$$

The contribution from the interval $(0, \infty)$ can be written

$$(4n+4) \int_0^\infty y^{2n+1} \sum_{k=1}^\infty e^{-2ky} dy = \frac{(2n+2)!}{2^{2n+1}} \left(1 + \frac{1}{2^{2n+2}} + \frac{1}{3^{2n+2}} + \dots \right) = \pi^{2n+2} B_{n+1} \quad (9)$$

where B_n is the n 'th Bernoulli number, as defined in the Smithsonian Tables⁷⁾.

We now concentrate our attention on the case of low temperatures, i.e., $\beta \sim \infty$. Then the contribution from the integral over the interval $(\beta\omega_L, \infty)$ is just

$$\int_{\beta\omega_L}^\infty y^{2n+1} \frac{e^{-2y}}{1-e^{-2y}} dy < 2 \int_{\beta\omega_L}^\infty y^{2n+1} e^{-2y} dy \quad (10)$$

where the inequality holds for β sufficiently large. Also, for β sufficiently large the majorizing integral behaves at worst like $(\beta\omega_L)^{2n+1} e^{-2\beta\omega_L}$ which, for a fixed n , is negligible. Hence, in three dimensions we have the asymptotic expansion at low temperatures

$$\frac{C_v(T)}{k} \sim \sum_{n=0}^N a_n \pi^{2n+2} B_{n+1} \left(\frac{2kT}{\hbar} \right)^{2n+1}. \quad (11)$$

Thus, at sufficiently low temperatures $C_v(T)/k$ can be represented asymptotically by a power series in T . This implies that a_n is given by

$$a_n = \frac{1}{k} \frac{1}{\pi^{2n+1} B_{n+1}} \left(\frac{\hbar}{2k} \right)^{2n+1} \frac{C_v^{(2n+1)}(0)}{(2n+1)!} \quad (12)$$

or

$$g(\omega) \sim \frac{1}{k} \sum_{n=0}^N \frac{1}{\pi^{2n+1} B_{n+1}} \left(\frac{\hbar}{2k} \right)^{2n+1} \frac{C_v^{(2n+1)}(0)}{(2n+1)!} \omega^{2n} \quad (13)$$

where $C_v^{(n)}(0)$ denotes the n 'th derivative of $C_v(T)$ evaluated at $T=0$. It must be emphasized that this relation is asymptotic and hence probably will only give a good representation of the frequency spectrum at low frequencies. This is the final form of the representation of $g(\omega)$ in terms of $C_v(T)$.

Numerical representations of the first three coefficients a_n are

$$\begin{aligned} a_1 &= 6.505 \times 10^{-14} C^{(1)}(0) \text{ sec}^3 \\ a_2 &= 3.432 \times 10^{-35} C^{(3)}(0) \text{ sec}^5 \\ a_3 &= 5.313 \times 10^{-57} C^{(5)}(0) \text{ sec}^7. \end{aligned} \quad (14)$$

If measurements of $C_v(T)$ are available at a set of points $0 < T_1 < T_2 < T_3 \dots$ where the T 's are small then approximate values for $C_v^{(2n+1)}(0)$ are easily found in terms of them. We consider a representation of $C_v^{(2n+1)}(0)$ in the form

$$C_v^{(2n+1)}(0) = A_1 C_v(T_1) + A_2 C_v(T_2) + \cdots + A_{2n+1} C_v(T_{2n+1}) \quad (15)$$

and seek to determine the A 's. We expand each function $C_v(T)$ around $T=0$ retaining $2n+1$ terms,

$$\begin{aligned} C_v^{(2n+1)}(0) = & A_1 \left[C_v^{(1)}(0) T_1 + C_v^{(2)}(0) \frac{T_1^2}{2!} + \cdots + C_v^{(2n+1)}(0) \frac{T_1^{2n+1}}{(2n+1)!} \right] \\ & + A_2 \left[C_v^{(1)}(0) T_2 + C_v^{(2)}(0) \frac{T_2^2}{2!} + \cdots + C_v^{(2n+1)}(0) \frac{T_2^{2n+1}}{(2n+1)!} \right] \\ & + \cdots \end{aligned} \quad (16)$$

where we have assumed $C_v(0)=0$. In order for this equation to be correct the A 's must satisfy

$$\begin{aligned} T_1 A_1 + T_2 A_2 + \cdots + T_{2n+1} A_{2n+1} &= 0 \\ T_1^2 A_1 + T_2^2 A_2 + \cdots + T_{2n+1}^2 A_{2n+1} &= 0 \\ \vdots & \\ T_1^{2n} A_1 + T_2^{2n} A_2 + \cdots + T_{2n+1}^{2n} A_{2n+1} &= 0 \\ T_1^{2n+1} A_1 + T_2^{2n+1} A_2 + \cdots + T_{2n+1}^{2n+1} A_{2n+1} &= (2n+1)! \end{aligned} \quad (17)$$

The solution for A_j can be written as

$$A_j = (2n+1)! D_j / \Delta \quad (18)$$

where Δ is the determinant

$$\Delta = \begin{vmatrix} T_1 & T_2 & \cdots & T_{2n+1} \\ T_1^2 & T_2^2 & \cdots & T_{2n+1}^2 \\ T_1^3 & T_2^3 & \cdots & T_{2n+1}^3 \\ \vdots & \vdots & \ddots & \vdots \\ T_1^{2n+1} & T_2^{2n+1} & \cdots & T_{2n+1}^{2n+1} \end{vmatrix} = T_1 T_2 \cdots T_{2n+1} \prod_{\substack{k, l \\ k > l}} (T_k - T_l) \quad (19)$$

by the properties of Vandermonde determinants. Similarly D_j is given by

$$D_j = \begin{vmatrix} T_1 & \cdots & 0 & \cdots & T_{j+1} \\ T_1^2 & \cdots & 0 & \cdots & T_{j+1}^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ T_1^{2n+1} & \cdots & 1 & \cdots & T_{j+1}^{2n+1} \end{vmatrix} = \{ T_1 T_2 \cdots T_{2n+1} \prod_{k > l} (T_k - T_l) \}' \quad (20)$$

where the prime indicates that terms containing T_j are to be omitted. Hence we find

$$A_j = \frac{(2n+1)!}{T_j} \frac{1}{\prod_{k > j} (T_k - T_j) \prod_{j > k} (T_j - T_k)} \quad (21)$$

This relation, Eqs. (13), (15) and (18) formally solve the problem of relating low temperature measurements of the specific heat to the low frequency behavior of the frequency spectrum.

The specialization of this method to fitting data to two-dimensional lattice models is straightforward. In this case it is known that in the vicinity of $\omega=0$ the frequency spectrum is of the form

$$g(\omega) = \sum_{n=0}^{\infty} a_n \omega^{2n+1}. \quad (22)$$

Then the integrals we must consider, and their form at low temperatures are given by

$$\begin{aligned} F_n(\beta) &= \int_0^{\omega_D} \frac{\beta^2 \omega^{2n+1}}{\sinh^2 \beta \omega} d\omega \sim \frac{1}{2^{2n} \beta^{2n}} (2n+1)! \left(1 + \frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}} + \dots \right) \\ &= \frac{1}{2^{2n} \beta^{2n}} (2n+1)! \zeta(2n+1) \end{aligned} \quad (23)$$

where the first few values of the zeta function which appear on the right side of this equation are given in the Smithsonian Tables. Thus, if we denote

$$\frac{1}{2^{2n} \beta^{2n}} (2n+1)! \zeta(2n+1) \quad (24)$$

by C_n we find that $g(\omega)$ is approximated by

$$g(\omega) \sim \frac{1}{k} \sum_{n=0}^N \frac{C_n^{(2n)}(0)}{(2n)! C_n} \omega^{2n+1}. \quad (25)$$

The methods which are described in this note suffice to relate the low temperature behavior of the specific heat to the low frequency end of the frequency spectrum. It is probable that there is no good way of finding the behavior of $g(\omega)$ over the entire range. One could, in theory at least, find the moments of the dynamical matrix from an expansion of the specific heat in powers⁸⁾ of T^{-1} . These moments could then be used in an application of Montroll's moment-trace method⁹⁾. However, for a three dimensional lattice one needs a prohibitively large number of moments for a detailed description of the frequency spectrum, hence the impracticability of the method.

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Evolution of Massive Stars. II[†]

—Helium-Burning Stage—

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To investigate the evolution of massive stars in the helium-burning stage, four sample models ($M=15.6M_{\odot}$) consisting of the following four regions are constructed: (1) hydrogen-rich envelope, (2) radiative zone with varying hydrogen content, (3) radiative helium region, (4) convective helium core. The results show that these models lie in the Hertzsprung gap of the h and χ Persei cluster and do not explain the existence of the red supergiants branch of this cluster. These models are also compared with those in which the rate of helium burning is slower by a factor of 10. The comparison shows that the properties of models having the double energy-source are very sensitive to the rates of both the hydrogen- and helium-burning.

§ 1. Introduction

The photometric observations of the double cluster h and χ Persei show the high-temperature stars which extend upward and to the right from the main-sequence in the H - R diagram.¹⁾ This band is then interrupted by a wide Hertzsprung gap before it again appears as another sequence of low-temperature supergiants. It is quite tempting to interpret these observed features in terms of the theory of stellar structure and of its evolution. So far efforts have been concentrated on the construction of models for massive stars in their initial and early evolutionary phases.²⁾ General features of the stellar structure which came out of these studies are the models having three regions, *i.e.*, a) an outer radiative envelope where the initial chemical composition is maintained, b) a convective core where the amount of hydrogen is gradually decreased as a result of hydrogen-burning and efficient mixing, and c) an inhomogeneous intermediate zone where the molecular weight μ changes continuously between the values in regions a) and b).

In the previous paper^{3)***} of this series, the models of $15.6 M_{\odot}$ have been

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*** Hereafter referred to as Paper I.

constructed and its evolutionary sequence in the early phase has been discussed in detail. A problem which might be raised at this point is what will happen next in the stellar structure beyond the last stage of evolutionary sequence treated in Paper I. When hydrogen in the core is depleted so as not to be able to keep the energy production, the gravitational contraction of the core may begin and cause a heating in the core, while the hydrogen-burning will shift to the shell surrounding the core. Consequently, the helium-burning will commence in the very core and the convection will occur because the energy generation of this process depends on a high power of the temperature.⁴⁾⁻⁶⁾ The purpose of this paper is to discuss the stellar models of the same mass as in Paper I for such stage of evolution.

Our models consist of the following four regions from surface to center: a) Outer radiative envelope, b) inhomogeneous intermediate radiative zone, c) radiative helium zone, and d) convective helium core.

§ 2. Definitions and assumptions

Subscripts: The subscripts used in this paper are the following: e for the outer radiative envelope, 1 for the interface between the inhomogeneous radiative zone and the radiative helium zone, 2 for the interface between the radiative helium zone and the convective core, and c for the center of star.

Radiation pressure: The effect of radiation pressure is fully taken into account.

Mixing: We have assumed no mixing in the envelope or between the envelope and the helium regions. Inside the convective helium core complete turbulent mixing keeps the composition homogeneous throughout.

Mass ejection and outer convective zone: The possible mass ejection from the star, and the existence of the outer convection zone near the surface are neglected. Since the estimated time is relatively short ($\sim 10^5$ years) to arrive at Model I from the beginning of gravitational contraction, we are partly justified in neglecting the effect of mass ejection. As to the outer convection zone, most of our models have higher effective temperatures than 6600°K , hence this convection zone is not supposed to play an important role as in the low-temperature stars.

Opacity: The absorption coefficient is due to electron scattering only, i.e.,

$$\kappa = 0.19(1 + X). \quad (1)$$

For the models of 15.6 solar masses, this approximation is reasonably accurate except for the low-temperature stars in red-supergiant region. We have checked the validity of this approximation for our models after the computation (see Section 5).

Energy generation: In our models energy generation takes place by the 3α -process in the core and by the CN-cycle in the shell just outside the helium zone. The formula for the former process is given by

$$\varepsilon_{8\alpha} = 1.3 \times 10^{-2} \rho^2 Y^3 (T/1.5 \times 10^8)^{26.8}, \quad (2)$$

where ρ is density. The numerical values in this expression are taken from Salpeter's work.⁶⁾ The rate for the CN cycle is

$$\varepsilon_{\text{CN}} = 3.14 \times 10^7 \rho X X_N (T/4 \times 10^7)^{14.2}, \quad (3)$$

where a new result of the off-resonant measurements for $\text{N}^{14}(p, \gamma)$ is used.

Chemical composition: The outer envelope retains the composition

$$X_e = 0.90, \quad Y_e = 0.08, \quad Z_e = 0.02, \quad X_N = \frac{1}{3} Z_e, \quad (4)$$

which are assumed in the initial model in Paper I. The variation of the hydrogen content in the intermediate zone for the subsequent models in Paper I can be reasonably well approximated by

$$X = 3.25q - 0.448 \quad \text{for } q_1 \leq q \leq 0.415, \quad (5)$$

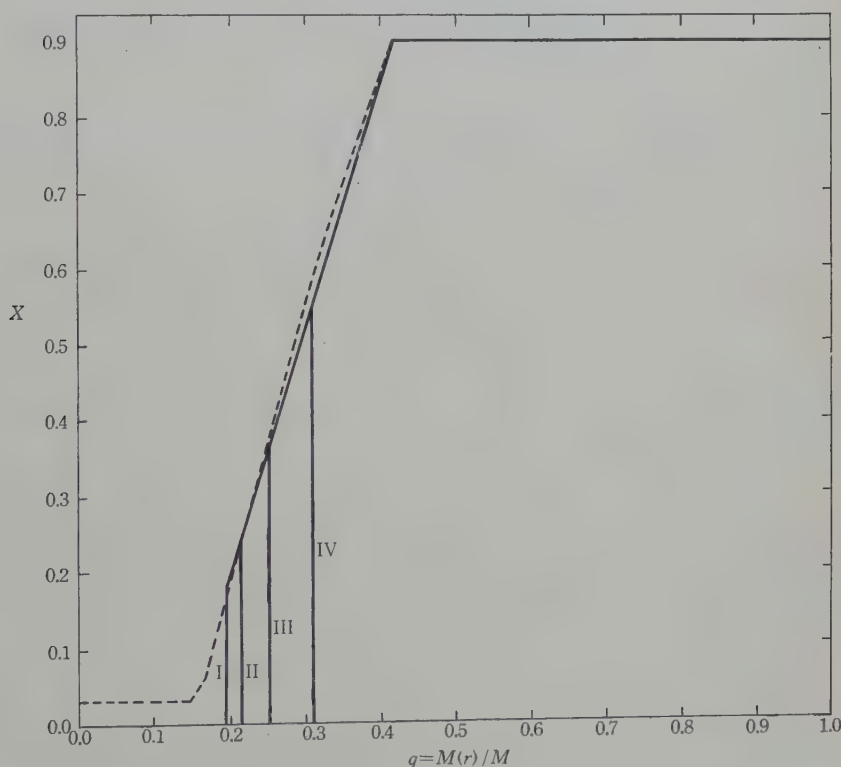


Fig. 1. The solid line shows the variation of hydrogen content from the center (left) to the surface, while the dashed line represents the q - X relation given by Sakashita et al. for the earlier evolutionary phases. The numbers at the interface between the envelope and the helium region denote the model numbers.

where q is the mass fraction $M(r)/M$ and q_1 denotes the value at the interface 1. We have constructed the models for four different values of q_1 ; here the model with a larger value of q_1 corresponds to an advanced stage of evolution. The value of X inside the interface 1 is zero as a result of the hydrogen-burning in the preceding stage (see Fig. 1).

The composition indicated above may not represent the chemical composition of Population I stars such as those belonging to the h and χ Persei cluster. Recently, Cameron⁸⁾ revised a table of the cosmical chemical composition compiled by Suess and Urey.⁹⁾ It gives

$$X=0.61, Y=0.37, Z=0.02, X_N=0.001, \quad (6)$$

for which the mean molecular weight in the envelope becomes

$$\mu_e=0.664, \quad (7a)$$

instead of

$$\mu_e=0.535, \quad (7b)$$

as is determined from the values shown in Eq. (4). The models having the larger μ_e (7a) will also be discussed in Section 5.

Composition parameters: The chemical composition enters the basic equations through the mean molecular weight, the opacity, and the energy generation. Accordingly, we define the following two quantities:

$$l = \frac{\mu}{\mu_e} = \frac{2X_e + 0.75Y_e + 0.5Z_e}{2X + 0.75Y + 0.5Z}, \quad (8)$$

$$j = \frac{\mu}{\mu_e} \frac{1+X}{1+X_e}.$$

These values are functions of q through Eq. (5) in the inhomogeneous region and change abruptly at the interface 1.

§ 3. Basic equations

Outer and Inhomogeneous Radiative Zone.

In terms of the dimensionless variables,¹⁰⁾

$$P = p \frac{GM^2}{4\pi R^4}, \quad T = t \frac{\mu_e M}{k} \frac{GM}{R}, \quad M(r) = qM, \quad r = xR, \quad (9)$$

the basic equations from the conditions of hydrostatic and thermal equilibrium are expressed as

$$\frac{dp}{dx} = -\beta \frac{pq}{x^2 t} l, \quad \frac{dq}{dt} = \beta \frac{x^3 p}{t} l, \quad \frac{dt}{dx} = -C\beta \frac{p}{t^4 x^2} j, \quad (10)$$

with

$$\beta = 1 - At^4/p, \quad (11)$$

where

$$A \equiv \frac{4\pi a}{3} \left(\frac{\mu_e H}{k} \right)^4 G^3 M^3, \quad (12)$$

$$C \equiv \frac{3}{4ac} \frac{0.19(1+X_e)}{(4\pi)^2} \left(\frac{k}{\mu_e H G} \right)^4 \frac{L}{M^3}, \quad (13)$$

and

$$L = L_{\text{shell}} + L_{\text{core}}. \quad (14)$$

Then homology variable are

$$U = \frac{x^3 p}{qt} l\beta, \quad V = \frac{q}{xt} l\beta, \quad n+1 = \frac{q}{C} \frac{t^4}{p}. \quad (15)$$

Convective Helium Core.

The equation of convective equilibrium in terms of the differential increments for P and ρ can be expressed in the form⁽¹¹⁾

$$\frac{dP}{P} = \frac{32-24\beta-3\beta^2}{3(8-7\beta)} \frac{d\rho}{\rho}. \quad (16)$$

From the above equation and the equation of state, using Henrich's notation,⁽¹²⁾ we have the basic relations:

$$P = P_c \frac{\Pi}{\Pi_c}, \quad \rho = \rho_c \frac{\xi}{\xi_c}, \quad T = T_c \frac{\theta}{\theta_c}, \quad (17)$$

where

$$\begin{aligned} \Pi &= e^{32y/3} y^{5/3} (1+y), \\ \xi &= e^{8y} y, \\ \theta &= e^{8y/3} y^{2/3}. \end{aligned} \quad (18)$$

In Eq. (17) Π_c , ξ_c , and θ_c are the values when $y=y_c$, add y is defined by

$$y = (1-\beta)/\beta. \quad (19)$$

Equation of the hydrostatic equilibrium is

$$\frac{1}{\eta^3} \frac{d}{d\eta} \left(\eta^3 \frac{dZ}{d\eta} \right) = -\zeta, \quad (20)$$

where

$$Z = e^{8y/3} y^{2/3} \left(1 + \frac{8}{5} y \right), \quad (21)$$

and η is defined by

$$r = \sqrt{\frac{5k}{8\pi G \mu_e H} \frac{T_c}{\rho_c}} e^{8y_c/3} y_c^{1/6} \eta. \quad (22)$$

Homology variables in the core are expressed by

$$U = -\frac{\eta\zeta}{dZ/d\eta}, \quad V = -\frac{5}{2} \frac{\eta\zeta}{\Pi} \frac{dZ}{d\eta}, \quad n+1 = \frac{\theta}{\Pi} \frac{d\Pi}{d\theta}. \quad (23)$$

Radiative Helium Zone.

Equations for the hydrostatic and thermal equilibrium conditions are transformed by Eqs. (17) and by the equation of state into

$$\frac{d\varphi}{d\eta} = \eta^2 \frac{\Pi\beta}{\theta}, \quad \frac{d\Pi}{d\eta} = -\frac{5}{2} \frac{\varphi\Pi}{\eta^2\theta} \beta, \quad \frac{d\theta}{d\eta} = -C^* \frac{\Pi\beta}{\theta^5\eta}, \quad (24)$$

$$\text{and} \quad 1 - \beta = \theta^4 / \Pi. \quad (25)$$

The quantity φ introduced here is proportional to M_r , i.e.,

$$M(r) = A^* \varphi, \quad (26)$$

and is related to Henrich's notation by

$$\varphi = -\eta^2 \frac{dZ}{d\eta}. \quad (27)$$

In Eqs. (24) and (26), constants are given by

$$C^* = \frac{3}{4ac} \frac{0.19\mu_c H}{k} \frac{L_{\text{core}}}{4\pi} \left(\frac{8\pi G \mu_c H}{5k} \right)^{1/2} \left(\frac{a}{3} \right)^{3/2} \left(\frac{\mu_c H}{k} \right)^{1/2} \quad (28)$$

and

$$A^* = 4\pi \left(\frac{1}{4\pi G} \right)^{3/2} \left(\frac{3}{a} \right)^{1/2} \left(\frac{k}{H} \right)^2 \left(\frac{5}{2} \right)^{3/2} \frac{1}{\mu_c^2}. \quad (29)$$

Finally, homology variables are

$$U = \eta^3 \Pi\beta/\theta\varphi, \quad V = 5\varphi\beta/2\eta^2\theta, \quad n+1 = 5\varphi\theta^4/2C^*\Pi. \quad (30)$$

§ 4. Construction of models

The envelope solutions form a family of one free parameter C , when the mass and μ_e of the models are fixed, i.e., $M = 15.6 M_\odot$ and $\mu_e = 0.535$. We have utilized four solutions specified by $\log C = -2.70, -2.75, -2.80$ and -2.85 ; their numerical integrations have been performed up to the point q_1 . We have constructed four models with $q_1 = 0.194, 0.221, 0.252$ and 0.31 , and these will later be designated as Models I to IV.

The solutions of the basic equations in the convective core have a free parameter β_c or y_c . Let us, for the moment, regard η_2 as another disposable constant. Then, with β_c and η_2 specified, the solutions in the helium region can be obtained from Eqs. (17) and (24). Here the value of C^* in Eq. (28) is determined by the continuity condition of $(\theta/\Pi)(d\Pi/d\theta)$ at $\eta = \eta_2$,

$$\dot{C}^* = \left[\frac{\varphi(1+4y)}{1+8y+(32/5)y^2} \right]_{\eta=\eta_2} \quad (31)$$

We have carried out the computations of core solutions for $y_c=0.1$, 0.2 and 0.25, where Henrich's tables have been utilized for $y_c=0.1$ and 0.25. For each β_c , several values of η_2 are taken as the starting points of the integration in the radiative region. These integrations should be stopped at $\eta=\eta_1$ where η_1 is fixed by the requirement that φ should take the assigned value φ_1 which is related to q_1 by

$$q_1 = \frac{4.420}{(M/M_\odot)\mu_c^2} \varphi_1. \quad (32)$$

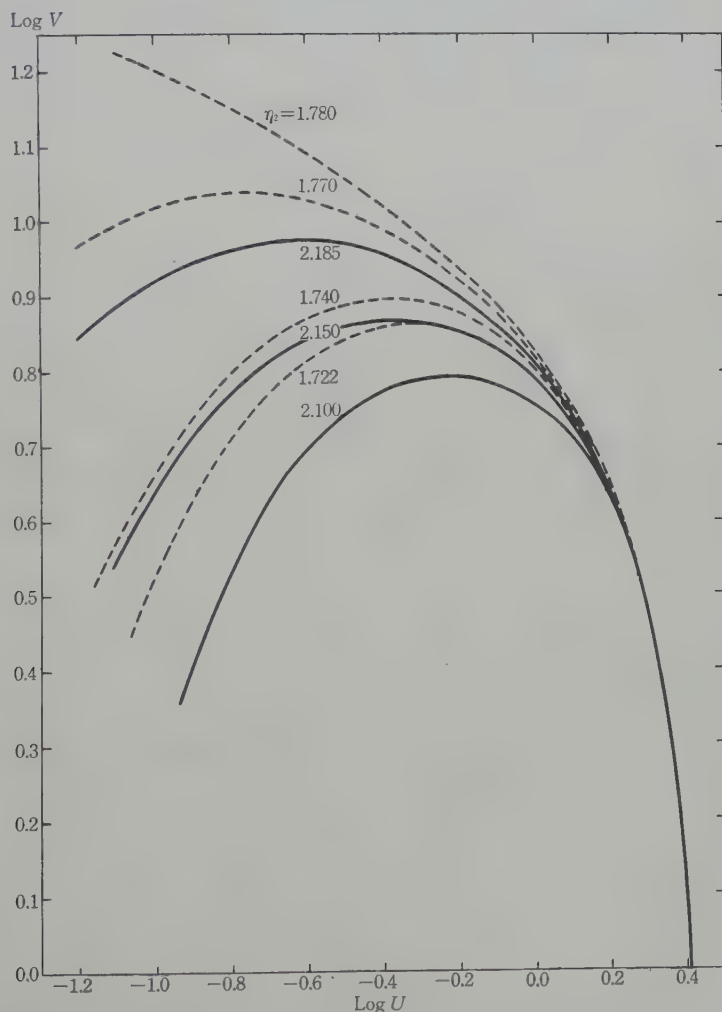


Fig. 2. Family of solutions for helium region in the U - V plane. Solutions are drawn for $y_c=0.10$ (solid lines) and 0.20 (dashed lines) with various values of η_2 .

The U - V curves in the helium region are shown in Fig. 2.

The condition of fitting between the solutions of envelope and of core are

$$U_{1e}/l_1\mu_e = U_{1c}/\mu_c, \quad V_{1e}/l_1\mu_e = V_{1c}/\mu_c. \quad (33)$$

With these conditions, β_c and γ_2 are fixed by interpolation for a given envelope parameter C , hence the mathematical characteristics are completely determined.

Next we turn to the evaluation of physical parameters involved in our models. First we define the effective polytropic index n_c in the core solution by

$$n_c = 3(8 - 7\beta_c)/(8 - 3\beta_c - 3\beta_c^2). \quad (34)$$

Then the contribution to the total luminosity by the energy-generation in the core can be given by⁴⁾

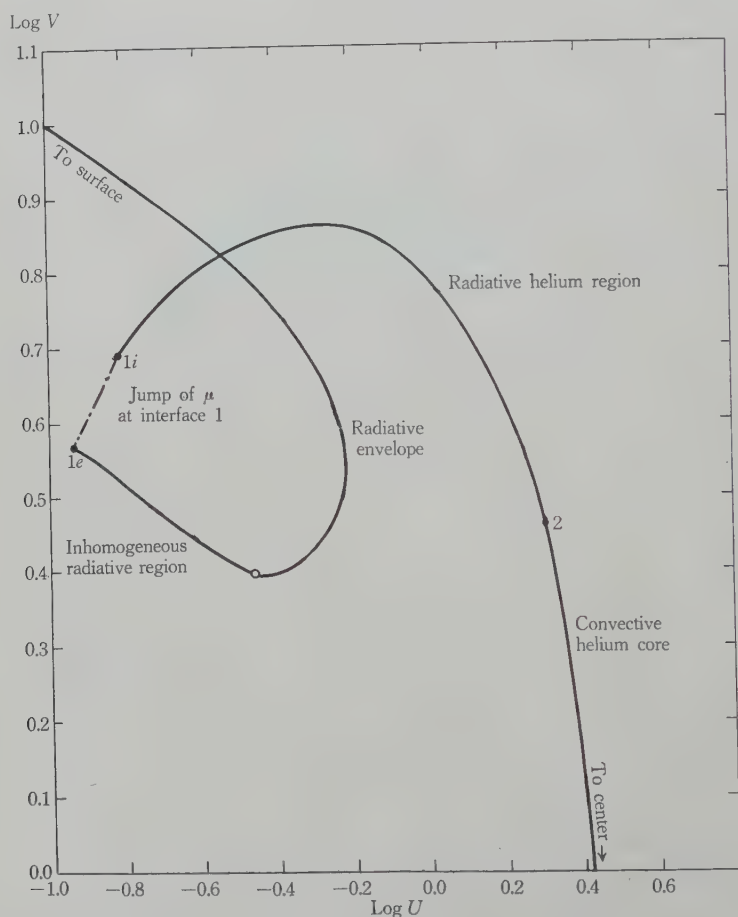


Fig. 3. Representation of Model I in the U - V plane. The open circle shows the point where the hydrogen content begins to decrease.

$$L_{\text{core}} = \epsilon_{3\alpha}^0 Y^3 \left(\frac{3n_c + 3}{3n_c + s} \right)^{3/2} \left(\frac{k}{2G\mu_c H} \right)^{3/2} \rho_c^{3/2} T_c^{3/2} \left(\frac{T_c}{T_0} \right)^s, \quad (35)$$

where $\epsilon_{3\alpha}^0 = 1.3 \times 10^{-2}$, $s = 25.8$, and $T_0 = 1.5 \times 10^8$ are adopted. From Eq. (35) and the equation of state at the center, we can determine T_c and ρ_c , hence T_1 and ρ_1 by Eq. (17). Consequently, the contribution to the luminosity from the shell L_{CN} can be computed by¹³⁾

$$L_{\text{CN}} = \epsilon_{\text{CN}}^0 X X_N \rho_{1e}^2 \left(\frac{T_1}{T_0} \right)^s \frac{4\pi r_1^3}{V_{1e} [2 + (s-2)/(n+1)_{1e}] - 3}, \quad (36)$$

where $\epsilon_{\text{CN}}^0 = 3.14 \times 10^7$, $s = 14.2$ for $T_0 = 4 \times 10^7$ are used. However, this L_{CN} is not necessarily equal to $L - L_{\text{core}}$ for an arbitrarily given C , and actually it supplies the last condition required to fix the value of C . This completes our scheme for the determination of the physical parameters of models. Fig. 3 illustrates the solution for Model I on the U - V plane.

§ 5. Results and discussions

In Table 1, we summarize the mathematical and physical characteristics of the models which are constructed by the procedures described in the preceding section. The Hertzsprung-Russell diagram in Fig. 4 shows the evolutionary tracks for the star in terms of the luminosity and the effective temperature. The results of our computations show the following features: 1) The temperature at the hydrogen-burning shell, T_1 , is approximately constant, *i.e.*, $\log T_1 = 7.64$. 2) The central temperature T_c increases in order of the model numbers. This fact corresponds to the increases of $L_{\text{core}}/L_{\text{shell}}$ from 0.15 for Model I to 0.79 for Model IV. 3) Accompanied with increase of the whole mass of helium region, the mass of the inner convective core increases as well. 4) The reduced radius $x_1 = r_1/R$, which indicates the location of the hydrogen-burning shell, increases as the evolution proceeds; this gives rise to the increase of the effective temperature. 5) The time necessary to evolve from one model to the next is nearly proportional to the variation in $\log T_c$ between these two successive models. The point plotted on the $\log T_c$ - $\log L/L_\odot$ diagram moves toward the main-sequence with the roughly constant velocity $d \log T_c / dt \sim 0.03 \times 10^{-5} \text{ year}^{-1}$. 6) The time required to reach Model I since the beginning of the gravitational contraction is estimated as about 1.5×10^5 years.

In checking the validity of the assumption that the opacity is contributed only from electron scattering, we take the following procedure. Assuming that the stellar radius is $100 R_\odot$, we solve the equations for the envelope adopting Kramers' opacity, instead of the electron scattering, with the starting values $T = T_c$ and $\rho = \mu HGM/kT\kappa R^2$ at $r = R$. At the point where Kramer's opacity becomes equal to the opacity by electron scattering, we switch the opacity from the former to the latter, thus continuing our numerical integrations until q reaches q_1 as before. The comparison of this new solution with that previously obtained only by electron

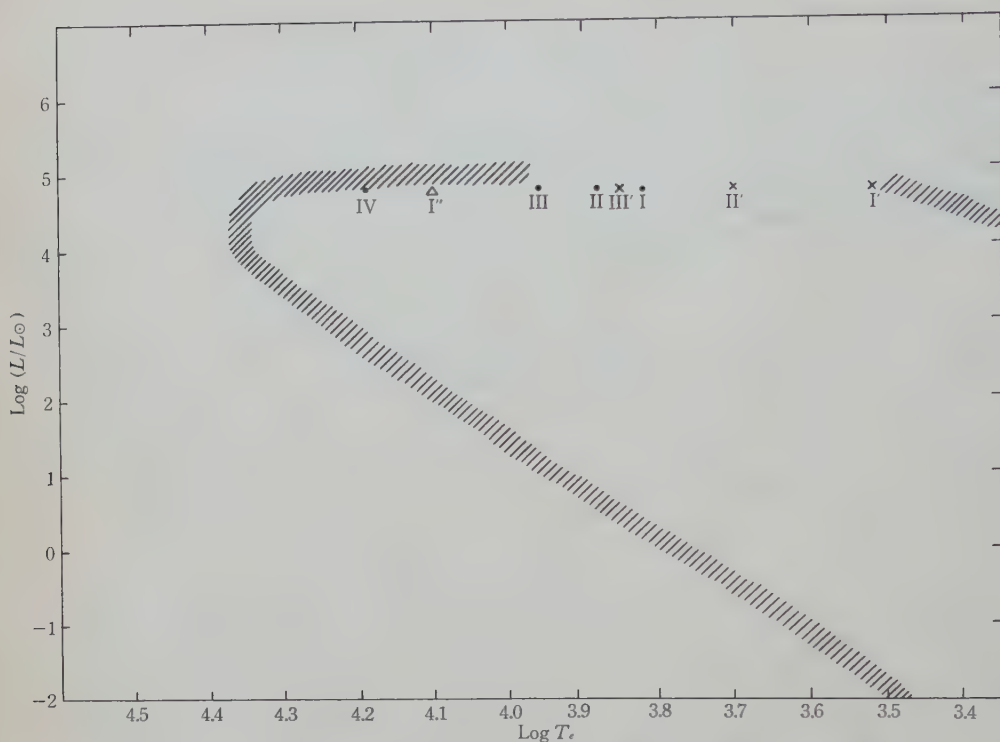


Fig. 4. The evolutionary track in the Hertzsprung-Russell diagram for the star with $M=15.6 M_{\odot}$ at the helium-burning stage. The hatched band shows the observed locations for the double cluster h and χ Persei.

scattering shows no substantial differences in our final results. For instance, the deviation of $\log T_e$ is -0.03 , while $\log L$ has the same value for Model I.

We did not take account of the effect of the conversion of He^4 to C^{12} in our computations. A rough estimate indicates that the degree of depletion of the helium content in the core amounts to 0.1, 0.4, and 1.5 for Models II, III, and IV, respectively. Since the decrease of helium content in the core implies the increase of the ratio $L_{\text{shell}}/L_{\text{core}}$, it is supposed that the real models corresponding to the stages of Models III and IV must have lower effective temperature than those given in Table 1 and Fig. 4. After helium in the core diminishes, the star will have a configuration consisting of a core made of heavy-elements surrounded by the helium-burning shell as well as the hydrogen-burning shell in the outer region. Such a sequence of models will be studied in one of the forthcoming papers of this series.

Since Salpeter's values for the 3α process⁶⁾ can be in error by a factor of about 10, it may be of interest to construct the models by using the values of one tenth of his reaction rate. In this case, Models I, II, and III in Table 1 should be replaced by Models I', II', and III' in Table 2. The following feature can be

Table 1. Mathematical and physical characteristics of inhomogeneous models with double energy-source ($M=15.6 M_{\odot}$, $\mu_e=0.533$).

Model	I	II	III	IV
$\log C$	-2.814	-2.804	-2.791	-2.782
$\log L/L_{\odot}$	4.74	4.75	4.77	4.78
L_{shell}/L	0.87	0.81	0.72	0.56
$\log x_1$	-2.77	-2.64	-2.45	-1.97
$\log q_1$	-0.71	-0.66	-0.60	-0.51
$\log p_1$	8.13	7.72	7.05	5.31
$\log t_1$	1.61	1.50	1.33	0.89
β_1	0.69	0.69	0.70	0.71
$\log l_1$	0.28	0.24	0.19	0.11
X_1	0.18	0.27	0.37	0.56
$\log U_{1e}$	-0.94	-0.96	-0.98	-1.01
$\log V_{1e}$	0.57	0.57	0.56	0.53
$(n+1)_{1e}$	4.0	4.0	3.9	3.9
β_e	0.89	0.87	0.85	0.81
$\log II_e$	-0.97	-0.72	-0.43	+0.05
$\log \theta_e$	-0.49	-0.41	-0.32	-0.17
$\log II_1$	-3.77	-3.45	-3.15	-2.62
$\log \theta_1$	-1.07	-1.00	-0.92	-0.79
$\log T_e$	8.22	8.23	8.24	8.25
$\log \rho_e$	3.20	3.15	3.09	3.01
$\log T_1$	7.64	7.64	7.64	7.64
$\log \rho_{1e}$	0.76	0.74	0.69	0.62
$\log \eta_2$	0.32	0.29	0.26	0.22
$\log \varphi_2$	-0.29	-0.21	-0.12	0.01
q_2	0.081	0.098	0.120	0.162
$\log II_2$	-1.56	-1.34	-1.12	-0.75
$\log \theta_2$	-0.70	-0.62	-0.54	-0.42
β_2	0.94	0.93	0.91	0.89
$\log U_2$	0.31	0.30	0.28	0.26
$\log V_2$	0.47	0.49	0.52	0.55
$\log R/R_{\odot}$	2.25	2.15	1.98	1.54
$\log T_e$	3.82	3.88	3.96	4.19
life (10^5 yr)	0	2.2	5.0	12.5

Table 2. Mathematical and physical characteristics of models with reduced reaction rate for helium-burning and with larger $\mu_e=0.664$.

Model	I'	II'	III'	I''
$\log C$	-2.80	-2.79	-2.78	-2.84
$\log L/L_{\odot}$	4.76	4.76	4.77	4.60
L_{shell}/L	0.87	0.81	0.74	0.81
$\log x_1$	-3.60	-3.04	-2.70	-2.29
$\log q_1$	-0.71	-0.66	-0.60	-0.71
$\log p_1$	11.39	9.15	7.97	6.41
$\log t_1$	2.40	1.86	1.56	1.15
β_1	0.68	0.69	0.71	0.71
$\log l_1$	0.28	0.24	0.19	0.28
X_1	0.18	0.27	0.37	0.18
$\log U_{1e}$	-1.01	-1.03	-1.06	-0.77
$\log V_{1e}$	0.58	0.57	0.57	0.54
$(n+1)_{1e}$	3.9	3.8	3.8	3.9
β_e	0.89	0.87	0.85	0.90
$\log II_e$	-0.97	-0.72	-0.43	-0.99
$\log \theta_e$	-0.49	-0.41	-0.32	-0.49
$\log II_1$	-3.87	-3.55	-3.25	-3.57
$\log \theta_1$	-1.09	-1.02	-0.95	-1.01
$\log T_e$	8.25	8.26	8.27	8.22
$\log \rho_e$	3.30	3.24	3.19	3.20
$\log T_1$	7.65	7.65	7.64	7.70
$\log \rho_{1e}$	0.78	0.76	0.70	1.01
$\log \eta_2$	0.32	0.29	0.26	0.31
$\log \varphi_2$	-0.29	-0.21	-0.12	-0.30
q_2	0.081	0.098	0.120	0.080
$\log II_2$	-1.56	-1.34	-1.12	-1.56
$\log \theta_2$	-0.70	-0.62	-0.54	-0.70
β_2	0.94	0.93	0.91	0.93
$\log U_2$	0.31	0.30	0.28	0.32
$\log V_2$	0.47	0.49	0.52	0.43
$\log R/R_{\odot}$	3.05	2.50	2.21	1.64
$\log T_e$	3.52	3.70	3.85	4.09

pointed out by comparison of these two sequences of models with and without primes. The reduced radius of the hydrogen-burning shell, x_1 , now becomes smaller by factors 0.15, 0.40, and 0.56 for Models I', II', and III' respectively than Models I, II, and III. This gives rise to the increase of radius R by factors 6.3, 2.2, and

1.7 for these models and consequently to the decrease of effective temperature. It must be emphasized that the characteristics of models having the double energy-source as considered here are very sensitive to the values of reaction rates both in the core and in the shell, and particularly to their ratio. As is discussed by one of us (M. N.),¹⁴⁾ such a strong tendency will generally be revealed as an inherent property of the model with double energy-source.

Finally, we consider the case in which the larger value of mean molecular weight $\mu_e=0.664$ is adopted according to the argument given in Section 2. Then the mass of our models is reduced to $M=10.1M_\odot$ by Eq. (2) when the electron scattering is the only source of opacity. Although the shape of X - q curves shown in Fig. 1 changes, the composition parameters given in Eq. (8) are invariant as functions of q . Then we can carry out the fitting in the same way without new numerical integrations for the hydrogen regions. Results are presented in Table 2 as Model I'', which has the same value of q_1 corresponding to Model I. In this case, we note that x_1 becomes larger than that of Model I, while the ratio of $L_{\text{shell}}/L_{\text{core}}$ decreases due mainly to the smaller values of X and X_N . Accordingly, the model has a higher effective temperature and approaches the main sequence more compared with Model I.

Since the present study does not treat the models in the stages of gravitational contraction, we cannot draw any definite conclusion about the contracting stars in their final stage, although their effective temperature is certainly lower than that of Model I. Whatever low temperature it has in this stage, it is difficult to explain the existence of the red supergiants branch of η and χ Persei by these contracting stars, since the life is too short for this stage as indicated above. We conclude, therefore, that our models do not clarify the physical meaning of the Hertzsprung gap, and this is the reason why we need a further investigation on the models having double-shell source for energy generation as discussed above.

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Note added in proof: Recently, Eccles and Bodansky (*Phys. Rev.* **113** (1959), 608.) attempted to determine the probability that the 7.65 MeV state of C^{12} decays to its lower states and found that it is less than 0.1%. This limit is an order of magnitude smaller than the best previous experimental limits from which Salpeter estimated the rate of helium-burning. Therefore, it is reasonable to assume a value lower than Salpeter's one for this rate, as considered in § 5.

A Lower Limit on the π^0 Lifetime

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The assumption is made that the $\gamma + P \rightarrow \gamma + P$ scattering amplitude can be analytically continued to the pole arising from exchange of a π^0 between P and γ . Experiment limits the possible size of the residue at the pole, thus providing a lower limit $\tau \geq 6 \times 10^{-19}$ sec. on the π^0 lifetime.

§ 1. Introduction

Present experimental knowledge¹⁾ limits the π^0 lifetime τ to less than 10^{-15} sec. On the other hand, application of the uncertainty principle $\tau \Delta E \gtrsim \hbar$ to the experimental π^0 mass uncertainty $\Delta E \sim 0.3 \text{ Mev}^2)$ yields the lower limit $\tau \gtrsim 2 \times 10^{-21}$ sec. The purpose of this paper is to present an argument for $\tau \gtrsim 6 \times 10^{-19}$ sec. Perturbation theory³⁾ and a dispersion calculation⁴⁾ suggest that $\tau \sim 6 \times 10^{-17}$ sec., which remains well inside the suggested limits for τ .

Several alternative proposals to limit τ have been made. J. M. Cassels⁵⁾ has undertaken to raise the lower limit on τ by examining the photon energy spectrum in $(\pi^- P \text{ atom}) \rightarrow \pi^0 + N \rightarrow 2\gamma + N$, where the π^0 separates from the field of N rather slowly. Also H. Primakoff⁶⁾ has pointed out that the cross section for photopion production in complex nuclei contains, in addition to the usual $A^{2/3}$ -dependent terms, a Z^2 term which represents π^0 production in the nuclear Coulomb field and is related to $\pi^0 \rightarrow 2\gamma$. These proposals are less speculative than the present paper. Cassel's method probably cannot do better than $\tau \gtrsim 10^{-20}$ sec., but the method proposed by Primakoff may be the most sensitive available. The decay $\pi \rightarrow e + \nu + \gamma$ provides another speculative approach, if one assumes conservation of the vector current in weak decay. Under this assumption the vector part of $\pi \rightarrow e + \nu + \gamma$ is proportional to the rate of $\pi^0 \rightarrow 2\gamma^6)$. From the calculations of Vaks and Joffe⁶⁾, one easily finds that the rates w for $\pi \rightarrow e + \nu$ and w_v for vector $\pi \rightarrow e + \nu + \gamma$ decay are in the ratio

$$\frac{w(\pi \rightarrow e + \nu)}{w_v(\pi \rightarrow e + \nu + \gamma)} = \frac{\tau}{10^{-19} \text{ sec.}} \quad (1)$$

(in the vector part photons are emitted from virtual nuclear particles, not the final electron, so the vector part has no infrared divergence). Thus experiments on $\pi-e$

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decay give a limit $\tau \gtrsim 10^{-19}$ sec. at present.* The decay $\pi \rightarrow \mu + \nu + \gamma$ is much less sensitive to $\tau^{(6),(7)}$ and provides no new information on τ .

§ 2. Derivation of the limit

Our argument is based on two assumptions made by several physicists⁽⁸⁾ recently: (1) in principle scattering amplitudes can be analytically continued, as a function of momentum transfer, to poles which appear at unphysical scattering angles (2) in practice it is possible to distinguish sufficiently large pole terms from all other terms by an examination of scattering over a range of physical momentum transfers. We shall apply these assumptions, without proof, to $\gamma + P \rightarrow \gamma + P$. Low⁽⁹⁾ has pointed out that in addition to the usual diagrams for $\gamma + P \rightarrow \gamma + P$, which lead to the Thomson limit at low energies, there is one diagram (Figure 1) which possesses a pole at momentum transfer $q^2 = m_\pi^2$ (center of mass scattering angle $\cos \theta = 1 + m_\pi^2/2k^2$ where k is photon energy). The residue at the pole is proportional to the amplitude for the physical process $\pi^0 \rightarrow 2\gamma$.⁽⁹⁾ From these facts and our first assumption it follows that the matrix element \mathcal{M} has the form

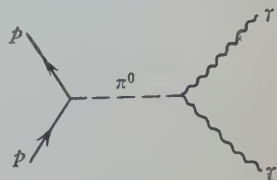


Fig. 1 The diagram for $\gamma + P \rightarrow \gamma + P$ which contains a pole at $q^2 = m_\pi^2$.

$$\mathcal{M} = \frac{P}{\sqrt{\tau}(q^2 - m_\pi^2)} + Q \quad (2)$$

where P and Q are analytic at $q^2 = m_\pi^2$, P arises from Figure 1, and Q arises from all other diagrams. In equation (2) we have ignored, as negligible for our purposes, the correction to the physical π^0 propagator $i/(2\pi)^4(q^2 - m_\pi^2)$ arising from π^0 decay. Then the differential cross section $d\sigma/d\Omega$ has the form

$$\frac{d\sigma}{d\Omega} = \frac{A}{\tau(q^2 - m_\pi^2)^2} + \frac{B}{\sqrt{\tau}(q^2 - m_\pi^2)} + C \quad (3)$$

where A , B , and C are analytic at $q^2 = m_\pi^2$, A represents the square of Figure 1, and B is an interference term.

Before proceeding further it is comforting to see that as $k \rightarrow 0$, and the Thomson limit becomes precisely correct⁽¹⁰⁾ independently of proton structure, the pole at $\cos \theta = 1 + m_\pi^2/2k^2$ goes away to infinity. We also note that small angle Compton scattering is not complicated by Coulomb scattering, and that diffraction scattering is negligible below the pion production threshold.

Now the precise value of A at the pole (which is easily computed from Figure 1 in renormalized Born approximation), and the Thomson limit for C are

* The results of Fazzini, Fidecaro, Merrison, Paul and Tollestrup, Phys. Rev. Lett. 1 (1958), 247 and Impeduglia, Plano, Prodell, Samios, Schwartz, and Steinberger, Phys. Rev. Lett. 1 (1958), 249, provide some evidence that π -e decays occur with a discrete electron energy.

$$A = \frac{-g^2 m_\pi^3 \hbar}{4c^2} \left(\frac{\hbar}{M_p c} \right)^2 \quad (4)$$

$$C = \frac{\alpha^2}{2} \left(\frac{\hbar}{M_p c} \right)^2 (1 + \cos^2 \theta) \quad (5)$$

where $\alpha = 1/137$ and $g^2 \sim 15$.* If τ is correctly given by perturbation theory as $\tau \sim 6 \times 10^{-17}$ sec., then the A and B terms (with q^2 dependence estimated by Born approximation) are each less than 2% of the total $d\sigma/d\Omega$ for all physical angles and energies below resonance ($k \sim 275$ Mev). A and B are small mainly because the pseudoscalar π^0 coupling picks out P wave mesons which occur within the proton Compton wavelength with a factor $\sim q^2/4M_p^2$. As we decrease τ from its perturbation value the A/τ term dominates the $B/\sqrt{\tau}$ term, and we shall not consider B any further.

Let us then consider, for a fixed center of mass energy, the extrapolated residue of the experimental quantity $(q^2 - m_\pi^2)^2 d\sigma/d\Omega$ at the pole. Since present experimental accuracy is not sufficient to distinguish the residue from zero, the theoretical residue A/τ must be less than the experimental and extrapolational uncertainties, and this gives us the lower limit on τ . For optimum results, the photon energy should be large enough to place the pole rather near the physical region, but small enough to minimize other mesonic corrections. The best choice may be $k \sim 10$ Mev. Existing data^(11,12) at $k \sim 50$ to 95 Mev and $\theta \sim 50^\circ$ to 150° appears to agree with theory within 50%, without introducing the A term. With this degree of accuracy, we arrive at the limit $\tau \gtrsim 6 \times 10^{-19}$ sec. This very imprecise limit could be improved by measurements of $d\sigma/d\Omega$ over a range of angles at a well-defined, fixed k , but accuracy of order 0.5% would be required to raise the limit to the perturbation theory result, $\tau \sim 6 \times 10^{-17}$ sec.

It may clarify matters to repeat that we have used the pole: (1) to justify the separation of Figure 1 from all other diagrams (2) to estimate Figure 1 in renormalized Born approximation, which is exactly correct at the pole.

Nambu⁽¹³⁾ has suggested a neutral vector meson ρ^0 , which might produce another pole similar to the one in Figure 1. But a spin one particle cannot decay into

* In ordinary dispersion relations g^2 is determined from the residue of a pole at angle $\theta=0$ and an unphysical energy. In the present method g^2 occurs in the residue of a pole at a physical energy and unphysical angle. But in both methods, the *covariants* upon which g^2 depends are the same: m_π^2 and M_p^2 . So both methods should yield the same $g^2 \sim 15$ (except for electromagnetic corrections which depend on the charge) if the assumptions of the theory are correct. The results of Taylor et al., Phys. Rev. **113** (1959), 689 are consistent with this expectation, and thus we feel it is reasonable to consider g^2 as known for our purposes.

It is interesting to note that whereas a unique coupling constant applies to the "3 particle vertex" at the pole in our problem, the "4 particle vertex" at the pole in $\pi + \mathcal{N} \rightarrow \pi + \pi + \mathcal{N}$ (C. Goebel, Phys. Rev. Lett. **1** (1958), 337) has two degrees of freedom, which can be taken as the momentum transfer and total energy in $\pi + \pi \rightarrow \pi + \pi$. Thus the residue at the pole in $\pi + \mathcal{N} \rightarrow \pi + \pi + \mathcal{N}$ could in principle give the $\pi - \pi$ scattering cross section at all energies and angles.

two photons; therefore the residue of the pole at $q^2=m_\rho^2$ vanishes independently of the ρ^0 coupling strength and one cannot obtain any information on the ρ^0 from our line of reasoning.

A neutral, spin zero, strangeness zero, strongly interacting X boson has also been proposed by various authors.^{14)15)16)*} If the X boson is pseudoscalar the analysis of this paper is applicable with A and the perturbation theory value of $1/\tau$ each multiplied by $(g_x/g)^2(m_x/m_\pi)^3$, and the position of the pole as a function of $(\cos\theta-1)^2$ further removed from the physical region by a factor $(m_x/m_\pi)^4$. Thus we might say that

$$(m_x/m_\pi)^2(g_x/g)^4 \lesssim 100.$$

In the case of scalar X bosons¹⁶⁾ a more restrictive result is obtained because A is enhanced by the S to P wave ratio $4M_p^2/q^2$ (about 200 at $q^2=m_\pi^2$).

In perturbation theory^{3),**} the residue at the pole $q^2=m_x^2$ is a factor

$$\frac{4}{9} \left(\frac{g_x}{g} \right)^4 \left(\frac{2M}{m_\pi} \right)^2 \left(\frac{m_x}{m_\pi} \right)^4 \simeq 90 \left(\frac{g_x}{g} \right)^4 \left(\frac{m_x}{m_\pi} \right)^4$$

larger than the residue at $q^2=m^2$. In this case we might say that

$$\left(\frac{g_x}{g} \right)^4 \lesssim 1.$$

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Note on the Spin-Orbit Coupling and Tensor Forces[†]

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The spin-orbit coupling in heavy nuclei is investigated on the basis of strong tensor interaction between nucleons, characteristic of the pion theory of nuclear force. It is shown that the process which includes the exchange of the incident nucleon with those of the target nucleus deformed by itself is important in the low energy region, and that this process provides the spin-orbit coupling with a correct sign and reasonable magnitude.

§ 1 Introduction

The nuclear shell model proposed by Mayer and Jensen¹⁾ succeeded in describing considerable parts of the static nature of nuclei. In this model, the existence of the spin-orbit coupling is essential in explaining the presence of the magic numbers. Further, experiments on the scattering of a nucleon from nuclei confirm the existence of the spin-orbit coupling between the incident nucleon and the target nucleus. The sign of this spin-orbit coupling is the same with and the magnitude of it is comparable to that of the shell model.

Thus, it is an almost well established fact that there exists a term which can be expressed, at least approximately, as a spin-orbit potential in the interactions between a nucleon and nucleus. But it still has been open to question how this coupling term can be interpreted on the basis of a more fundamental theory.

The effects which have been considered to give rise to the spin-orbit term are²⁾: 1) the relativistic effect (Thomas term), 2) the spin-orbit force between two nucleons derived from meson theory (recoil effect), 3) the effect coming from the many-body force, and 4) the tensor force between nucleons. Of these, contributions to the spin-orbit term coming from effects 1)–3) are small and amount to about one tenth of the experimental value. Some attempts^{3)–6)} have been made to describe the main part of the spin-orbit coupling in terms of tensor part of nucleon-nucleon interaction. But the results in some of them are unsatisfactory with respect to its magnitude and others give the sign for this coupling opposite to the experiment. Feingold⁴⁾ investigated the spin-orbit splitting of the energy levels of light nuclei,

[†] A part of this paper was read at the meeting on nuclear force held at Research Institute for Fundamental Physics, on March, 1959¹³⁾.

considering the effect of tensor potential. The origin of the spin-orbit coupling, however, does not seem to be thoroughly investigated in his work.

Now, the feature of the nuclear force acting between free nucleons has been clarified by the recent works of Japanese group⁷⁾. According to them, the tensor force is strong and plays an important role. This feature originates from the one-pion-exchange potential with utterly stronger tensor part than the central one. In the triplet even state, especially, the nuclear force is represented almost completely by the one-pion-exchange potential when the distance r between two nucleons is larger than $0.7 \times \kappa^{-1}$, κ being the reciprocal Compton wave length of pion. As to the inner region $r < 0.4 \times \kappa^{-1}$ it is almost well established that the interaction between nucleons can be represented by a strong repulsive potential called the "hard core".

Recently, Nagata, Sawada, Sasakawa and Tamagaki⁸⁾ showed that these characteristic features of nuclear force, especially the strongness of tensor force, can explain the doublet splitting of the p -wave phase shifts of nucleon scattered from He^4 . Further, Terasawa⁹⁾ showed the doublet splitting of energy levels of He^5 , N^{15} and O^{17} can be explained by the similar nature of nuclear force.

In these works it is to be noted that the so-called "target exchange term"¹⁰⁾ with the deformed target is more important in the low energy region than the induced polarization term and gives rise to the spin-orbit coupling which has the same sign as that derived experimentally. This term comes from the exchange of the incident nucleon—the spin-orbit coupling of which with the nucleus is to be investigated—with a nucleon in the nucleus which has already been deformed by the tensor type interactions among the nucleons in it. On the other hand, the induced polarization term caused by the interaction between the incident nucleon and those in the target nucleus, has been calculated by many authors and gives rise to the spin-orbit coupling which may have in the low energy region the sign opposite to the experimental one provided the high momentum part of the potential is not suitably cut off.*

Since the spin-orbit coupling term is proportional to the derivative of nuclear density, the nucleons near the nuclear surface may largely contribute to this coupling. Thus it can be easily conjectured that the target exchange term, in which these nucleons play a dominant role, is effective in the course of derivation of spin-orbit coupling. Now these nucleons may not be much influenced by the Pauli principle. Therefore, it would be a reasonable approximation to take the nucleon-nucleon interaction in free state as the two-body interaction in the nucleus in the course of calculation.

In this note we shall treat the problem of the spin-orbit coupling generalizing Nagata et al.'s treatment and clarify the origin of this coupling. It will be shown that the strongness of tensor part of the nuclear force in the region $r > 0.7 \times \kappa^{-1}$ is an important fact in explaining the sign and magnitude of the spin-

* See, for instance, reference 6).

orbit coupling and, moreover, the cut-off in the higher momentum part in the tensor potential due to the hard core is also a favourable effect to the magnitude of spin-orbit coupling.

In Section 2, the spin-orbit potential will be derived from the Schrödinger equation of many-body system. Antisymmetrization of the total system will be taken into account. In Section 3 processes which contribute to the problem will be described, and in Section 4 the results and the comparison with experiment will be given.

§ 2. Construction of the spin-orbit potential

First we shall derive the one-particle Schrödinger equation from the many-body equation. The Schrödinger equation of $A+1$ nucleons is

$$H\Psi(x_0, x_1, \dots, x_A) = E\Psi(x_0, x_1, \dots, x_A), \quad (2.1)$$

$$H = \sum_{i=0}^A T_i + \sum_{i < j} V_{ij},$$

where x_i denote the coordinates (including charge and spin) of the i -th particle, T_i is its kinetic energy and V_{ij} is the interaction between i -th and j -th nucleon. Now we shall conveniently call the system composed of the particles $1, \dots, A$ 'target', and denote the Hamiltonian of it as H_T ;

$$H_T = \sum_{i=1}^A T_i + \frac{1}{2} \sum_{i,j=1}^A V_{ij}.$$

The wave function Ψ can be developed by the set of eigenfunctions ϕ_α of H_T which constitutes a complete orthonormal set under suitable boundary conditions:

$$\Psi = (A+1)^{-1/2} \sum_{\alpha} \sum_{i=0}^A \delta(P_{oi}) P_{oi} \phi_\alpha(0) \phi_\alpha(1, \dots, A), \quad (2.2)$$

$$H_T \phi_\alpha = E_c^\alpha \phi_\alpha,$$

$$\delta(P_{oi}) = \begin{cases} 1, & \text{for } i=0, \\ -1 & \text{for } i \neq 0, \end{cases}$$

where P_{oi} is the operator which interchanges 0 and i . ϕ_α is antisymmetrized with respect to x_i and x_j ($i, j=1, \dots, A$). Then the equation for $\phi_\alpha(0)$ can be obtained as follows:

$$(T_0 - \epsilon_\alpha) \phi_\alpha(0) = \sum_{\beta} (S_{\alpha\beta}(0) \phi_\beta(0) + \int L_{\alpha\beta}(0, 1) \phi_\beta(1) dx_1) \quad (2.3)$$

where

$$S_{\alpha\beta}(0) = - \int \phi_\alpha^*(1, \dots, A) \left(\sum_{i=1}^A V_{oi} \right) \phi_\beta(1, \dots, A) dx_1 \dots dx_A, \quad (2.4)$$

$$L_{\alpha\beta}(0, 1) = A \left\{ \int \phi_\alpha^*(1, \dots, A) (T_0 - \epsilon_\alpha + \sum V_{oi}) \phi_\beta(0, 2, \dots, A) dx_2 \dots dx_A \right\},$$

$$\epsilon_\alpha = E - E_c^\alpha.$$

Eq. (2.3) may be solved by iteration:

$$\begin{aligned} (T_0 - \epsilon_\alpha) \varphi_\alpha(0) = & \left(S_{\alpha\alpha}(0) - \sum' S_{\alpha\beta} \frac{1}{a_\beta(0)} S_{\beta\alpha} + \dots \right) \varphi_\alpha(0) \\ & + \int \left\{ L_{\alpha\alpha}(0, 1) - \sum' S_{\alpha\beta} \frac{1}{a_\beta} L_{\beta\alpha}(0, 1) - \sum' L_{\alpha\beta}(0, 1) \frac{1}{a_\beta(1)} S_{\beta\alpha}(1) + \dots \right\} \varphi_\alpha(1) dx_1 \\ & + \dots \end{aligned} \quad (2.5)$$

where the operator $1/a_\beta(0)$ denotes the integral operator $1/(\epsilon_\beta - T_0)$, the energy denominator being modified appropriately according to the boundary conditions imposed on φ . \sum' means the sum over all possible β except α . This equation describes the motion of the 0-th nucleon when added to the target which is in the state α . The last line of Eq. (2.5) means the term which ultimately contributes to other channels.

The first line of the right-hand side of Eq. (2.5) may be written as $-v_\alpha \varphi_\alpha$ where v_α satisfies the following integral equation:

$$\begin{aligned} v_\alpha &= -S_{\alpha\alpha} - \sum' S_{\alpha\beta} \frac{1}{a_\beta} v_{\beta\alpha}, \\ v_{\beta\alpha} &= -S_{\beta\alpha} - \sum' S_{\beta\tau} \frac{1}{a_\tau} v_{\tau\alpha}. \end{aligned}$$

If we did not antisymmetrize the whole system, only this term would appear and would usually be interpreted as one-particle potential. This potential contains the effect of polarization of the target nucleus in the intermediate state induced by the incident nucleon which is called "the induced polarization".

The second line of Eq. (2.5), on the other hand, comes from the antisymmetrization of the total system, and gives rise, in general, to a non-local potential working on φ . Strictly speaking, however, we can extract a term which can be interpreted as a local potential working on φ . Let the 2nd line of Eq. (2.5) be

$$\int K_\alpha(0, 1) \varphi_\alpha(1) dx_1, \quad (2.6)$$

and let $\varphi_\alpha(1)$ be

$$\varphi_\alpha(1) = \zeta_\alpha(\tau_1) \chi_\alpha(\sigma_1) \phi_\alpha(\mathbf{r}_1),$$

where ζ , χ and ϕ are the charge, spin and space wave function respectively. Then if we define $V_\alpha^{(e)}$ to be

$$\int e^{-i\mathbf{k}_0' \cdot \mathbf{r}_0} K_\alpha(0, 1) e^{i\mathbf{k}_0 \cdot \mathbf{r}_1} d\mathbf{r}_0 d\mathbf{r}_1 = \langle \mathbf{k}_0' | V_\alpha^{(e)} | \mathbf{k}_0 \rangle, \quad (2.7)$$

we obtain, denoting the Fourier transform of $\phi_\alpha(\mathbf{r}_1)$ as $\phi_\alpha(\mathbf{k})$,

$$\int e^{-i\mathbf{k}_0' \cdot \mathbf{r}_0} K_\alpha(0, 1) \phi_\alpha(\mathbf{r}_1) d\mathbf{r}_1 d\mathbf{r}_0 = \int \langle \mathbf{k}_0' | V_\alpha^{(e)} | \mathbf{k}_0 \rangle \phi_\alpha(\mathbf{k}_0) d\mathbf{k}_0, \quad (2.8)$$

which is a usual form of a potential acting on particle 0 in the momentum representation.

As to the spin part, we can rewrite the terms including $\chi_\alpha(\sigma_1)$ into those operating on $\chi_\alpha(\sigma_0)$ (See Appendix I). The same holds as to the charge state function. Thus we see Eq. (2.6) is equivalent to a potential operating on $\varphi_\alpha(0)$.

This equivalent potential $V_\alpha^{(e)}$ contains the exchange term of the incident nucleon with one of the nucleons in the target nucleus which is represented by the wave function φ_α . In the state α , the nucleons in the target nucleus are correlated with each other through the interactions between nucleons. As will be discussed in the next section, the "target exchange term with a deformed target" is derived from this potential.

Now from these potentials, v_α and $V_\alpha^{(e)}$, we can bring out terms linear in σ_0 . These terms can be gathered together and put in a form

$$\int d\mathbf{r} e^{-i(k_0' - k_0)r} a \frac{1}{r} \frac{\partial \rho}{\partial r} (\mathbf{l} \cdot \boldsymbol{\sigma}_0) \quad (2.9)$$

under some approximations as shown in Appendix II. There are several processes contributing to a , which will be discussed and calculated in the next section.

§ 3. Concrete procedure of the calculation

3-1. Representation of the contributing process

Here, the target is considered as a closed shell, having a little deviation from spherical symmetry even in its ground state. Regarding the tensor part of the inter-nucleon force as perturbing interaction, we can obtain the ground state wave function of target φ_0 from the spherical symmetric state $\varphi_0^{(0)}$ with the perturbation theoretical calculation. In order to derive the spin-orbit potential by means of the tensor part of the nuclear force, the perturbing interaction must be contained more than twice in the matrix element, because the matrix element of the tensor interaction vanishes, when averaged over the unperturbed target state. It is only of the lowest order—the second order—that is calculated here. Not only the interaction between the incident nucleon and a nucleon within the nucleus, but also the one between nucleons within the target must be taken into account. Let us present several probable processes, which may contribute to the derivation of the spin-orbit coupling.

These processes are classified with the number of particles taking

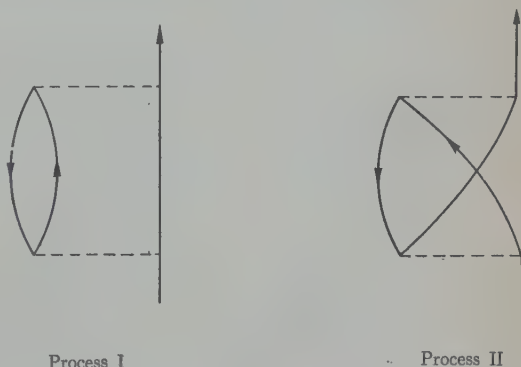


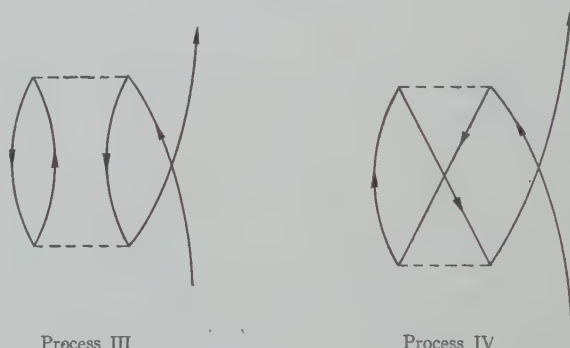
Fig. 1. Processes concerning with two-particle correlation.

part in. Firstly, for two-particle correlation, two processes are concerned. These processes are visualized by the Bethe-Goldstone diagram. (Fig. 1)

In process I, the intermediate state is formed by the interaction of the incident nucleon with the nucleon in the target—in this stage, the polarization of target may be induced in general because of the tensor interaction—and the final state may be set up with next interaction. Thus, this process is called the “induced target polarization.” Process II is almost the same as I, except for the interchange of the incident nucleon with a nucleon in the target in the intermediate stage, and this is a kind of “target exchange.”⁽¹⁰⁾ We shall call this process “induced target polarization”, too. We, therefore, use this nomenclature for the processes in which two nucleons are concerned, i.e., processes I and II.

Next we consider the case in which three particles, containing the incident particle, are concerned. In this case two processes may be considered (Fig. 2).

In both processes, two particles within the target jump by the mutual interaction of the tensor type—in this stage, the target polarizes by itself independently of the incident particle—and this intermediate state goes back to the ground state by the interaction of the incident particle with either of the nucleons which have jumped on. Since this intermediate state is included in the original state Φ_0 —not the unperturbed state $\Phi_0^{(0)}$ —in Eq. (2.4), processes above



Process III

Process IV

Fig. 2. Processes concerning with three-particle correlation.

mentioned contain the exchange process of the incident nucleon with that in the deformed part of the target nucleus. We call these processes “target exchange processes” with the deformed target. The difference between III and IV lies in the fact that the exchange of two particles takes place or not in the course of the transition to the final state. Terms coming from these processes are included in the equivalent one-particle potential $V_0^{(e)}$.

Other than above mentioned processes, there may be several 2nd order diagrams which evidently do not contribute to the spin-orbit coupling.

We put the tensor potential as

$$V_{ij} = S_{ij} \Gamma(\tau_i, \tau_j) v(r_{ij}),$$

where S_{ij} is the tensor operator: $S_{ij} = 3(\sigma_i \mathbf{r})(\sigma_j \mathbf{r})/r^2 - (\sigma_i \sigma_j)$, $\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$, and Γ is an operator depending on τ_i and τ_j . Consider the case where the target nucleus is in its ground state (i.e., put $\alpha=0$ in the equations of the preceding section). Taking the plane wave approximation for $\Phi_0^{(0)}$ above mentioned, we can write a in Eq. (2.9) as follows,

$$a = \sum_{i=1}^{IV} a_i, \quad a_i = C \alpha_i L_i, \quad (3.1)$$

where

$$C = \frac{9}{4} \frac{M}{\hbar^2} \frac{(4\pi)^2}{(2\pi)^3} \frac{3}{4\pi k_f^3} \frac{1}{k_0^2}. \quad (3.2)$$

In the above expression, k_f and k_0 are the local Fermi momentum and the momentum of the incident nucleon. α_i 's are the constants depending on the exchange character of V_{ij} . If we take it as $\frac{1}{3}(\tau_i \tau_j)$, we get $\alpha_I = \alpha_{III} = \frac{4}{3}$, $\alpha_{II} = \alpha_{IV} = \frac{2}{3}$. This exchange character is the same as that of one-pion-exchange potential. It would be natural to take it for our purpose. The subscripts attached to L refer to each of processes I, II, III and IV.

L_i 's are given as follows,

$$L_I = \iint dk_j dq \cdot \frac{\mathbf{k}_0(\mathbf{q}-\mathbf{k})}{k^2 - q^2} \cdot \left[\frac{w(|\mathbf{q}-\mathbf{k}|)}{|\mathbf{q}-\mathbf{k}|} \right]^2, \quad (3.3a)$$

$$\begin{aligned} L_{II} = \iint dk_j dq \cdot \frac{w(|\mathbf{q}-\mathbf{k}|)}{|\mathbf{q}-\mathbf{k}|^2} \cdot \left\{ \left[-\mathbf{k}_0 \mathbf{k} + \frac{(\mathbf{k}_0 \mathbf{k}) q^2 - (\mathbf{k}_0 \mathbf{q})(\mathbf{k} \mathbf{q})}{k^2 - q^2} \right] \frac{w(|\mathbf{q}+\mathbf{k}|)}{|\mathbf{q}+\mathbf{k}|^2} \right. \\ \left. + [(\mathbf{k}_0 \mathbf{q})(\mathbf{k} \mathbf{q}) - (\mathbf{k}_0 \mathbf{k}) q^2 + k^2(\mathbf{k}_0 \mathbf{q}) - (\mathbf{k}_0 \mathbf{k})(\mathbf{k} \mathbf{q})] \right. \\ \left. \times \frac{1}{|\mathbf{q}+\mathbf{k}|} \cdot \frac{\partial}{\partial |\mathbf{q}+\mathbf{k}|} \cdot \frac{w(|\mathbf{q}+\mathbf{k}|)}{|\mathbf{q}+\mathbf{k}|^2} \right\}, \end{aligned} \quad (3.3b)$$

$$L_{III} = \iint dk_i dq_j \cdot \frac{\mathbf{k}_0 \cdot (\mathbf{k}_0 - \mathbf{k}_i)}{(\mathbf{q}_j - \mathbf{k}_i)(\mathbf{k}_0 - \mathbf{k}_i)} \cdot \left[\frac{w(|\mathbf{k}_0 - \mathbf{k}_i|)}{|\mathbf{k}_0 - \mathbf{k}_i|} \right]^2, \quad (3.3c)$$

$$\begin{aligned} L_{IV} = \iint dk_i dk_j \cdot \frac{w(|\mathbf{k}_0 - \mathbf{k}_j|)}{|\mathbf{k}_0 - \mathbf{k}_j|^2} \cdot \left\{ \left[2\mathbf{k}_0 \cdot (\mathbf{k}_0 - \mathbf{k}_j) + \frac{(\mathbf{k}_0 \cdot (\mathbf{k}_0 - \mathbf{k}_i))(\mathbf{k}_0 - \mathbf{k}_j)^2}{(\mathbf{k}_0 - \mathbf{k}_i)(\mathbf{k}_0 - \mathbf{k}_j)} \right] \right. \\ \left. \times \frac{w(|\mathbf{k}_0 - \mathbf{k}_i|)}{|\mathbf{k}_0 - \mathbf{k}_i|^2} \right. \\ \left. + \left[(\mathbf{k}_0 \cdot (\mathbf{k}_0 - \mathbf{k}_j))(\mathbf{k}_0 - \mathbf{k}_i)^2 - (\mathbf{k}_0 \cdot (\mathbf{k}_0 - \mathbf{k}_i))((\mathbf{k}_0 - \mathbf{k}_i) \cdot (\mathbf{k}_0 - \mathbf{k}_j)) \right] \right. \\ \left. \times \frac{1}{|\mathbf{k}_0 - \mathbf{k}_i|} \cdot \frac{\partial}{\partial |\mathbf{k}_0 - \mathbf{k}_i|} \cdot \frac{w(|\mathbf{k}_0 - \mathbf{k}_i|)}{|\mathbf{k}_0 - \mathbf{k}_i|^2} \right\}, \end{aligned} \quad (3.3d)$$

where

$$w(k) \equiv \int v(r) j_2(kr) r^2 dr. \quad (3.4)$$

$j_2(r)$ is the spherical Bessel function of the second order. \mathbf{k}_0 is the momentum vector of the incident nucleon and \mathbf{k}_i and \mathbf{k}_j are those of nucleons in the target. \mathbf{k} and \mathbf{q} are defined as

$$\begin{aligned} 2\mathbf{k} &= \mathbf{k}_0 - \mathbf{k}_j, \\ 2\mathbf{q} &= \mathbf{q}_0 - \mathbf{q}_j, \end{aligned} \quad (3.5)$$

where q_0 and q_j are momenta of the nucleon 0 and j in the intermediate stage respectively. In the above equations the regions of integrations must be limited because of the Pauli principle. They are given as

$$k_i, k_j < k_f, \quad (3.6a)$$

$$q_0, q_j > k_f, \quad (3.6b)$$

and so

$$|\mathbf{q} \pm \frac{1}{2} \mathbf{K}| > k_f, \quad (3.6c)$$

where

$$\mathbf{K} = \mathbf{k}_0 + \mathbf{k}_j.$$

Detailed calculation of L_i will be given in Appendix III with process III as an example. Calculations of processes I and II have been given by Jancovici (see reference 6) in a special case where $k_0 = k_f$. L_I and L_{III} are reduced to a single integral, and can be written in rather simple forms as follows.

$$L_I = (2\pi)^2 k_f^4 \int_0^\infty \phi_I(x) [w(x)]^2 dx, \quad (3.7)$$

$$L_{III} = (2\pi)^2 k_f^4 \int_{(k_0 - k_f)/k_f}^{(k_0 + k_f)/k_f} \phi_{III}(x) [w(x)]^2 dx, \quad (3.8)$$

where

$$x = \frac{|\mathbf{q} - \mathbf{k}|}{k_f} \quad \text{in Eq. (3.7),} \quad (3.9)$$

and

$$x = \frac{|\mathbf{k}_0 - \mathbf{k}_i|}{k_f} \quad \text{in Eq. (3.8).}$$

The explicit form of $\phi_I(x)$ will be given in Appendix IV. Here we write the expression of it in the case $k_0 = k_f$

$$\left. \begin{aligned} \phi_I(x) &= \frac{1}{8} \left(1 - \frac{x^2}{4} \right)^2 \ln \frac{1+x/2}{1-x/2} + \frac{x}{24} (2+x)^2 (x-4) \ln 2 \\ &\quad + \frac{1}{24} x \left(-\frac{x^3}{4} - \frac{3}{4} x^2 + 6x + 13 \right) \quad \text{if } x \leq 2, \\ \phi_I(x) &= \frac{x}{24} (x-2)^2 (x+4) \ln \left(\frac{x}{2} - 1 \right) \\ &\quad + \frac{x}{24} (x+2)^2 (x-4) \ln \left(\frac{x}{2} + 1 \right) + \frac{x^2}{3} \left(3 - \frac{x^2}{4} \right) \ln \frac{x}{2} \\ &\quad + \frac{x^2}{6} + 1 \quad \text{if } x > 2. \end{aligned} \right\} \quad (3.10)$$

And

$$\begin{aligned}
 \phi_{III}(x) = & -\frac{1}{8}(\lambda^2-1)^2 \ln[(\lambda+1)x] + \frac{x}{48}(2\lambda-x)^2(4\lambda+x) \ln\left[\left(\lambda-\frac{x}{2}\right)x\right] \\
 & -\frac{1}{24}[3(x-1)^4-8(x-1)^3x+6\{(x-1)^2-\lambda^2\}(x^2-1)+(8x-3\lambda)\lambda^3] \\
 & \quad \times \ln[(\lambda+1-x)x] \\
 & +\frac{1}{8}\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)^2-1\right\}^2 \ln\left[\left(\frac{\lambda^2-1+x^2}{2x}+1\right)x\right] \\
 & -\frac{1}{24}\left[3\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)^4-(x-1)^4\right\}-8\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)^3-(x-1)^3\right\}x\right. \\
 & \quad \left.+6(x^2-1)\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)^2-(x-1)^2\right\}\right] \ln\left[\frac{\lambda^2-1-x(x-2)}{2}\right] \\
 & -\frac{1}{48}(\lambda^2-1)^2\left[\ln\frac{\lambda^2-1}{2}\right]\cdot\left[\frac{2(\lambda^2-1)}{x^2}+3\right] \\
 & -\frac{1}{8}\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)^3-\lambda^3\right\}x+\frac{5}{48}\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)^2-\lambda^2\right\}x(x-2) \\
 & +\frac{1}{24}\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)-\lambda\right\}x(x-5) \quad \text{for } x \leq 2, \\
 \phi_{III}(x) = & \frac{1}{8}\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)^2-1\right\}^2 \ln\frac{\lambda^2-1+x^2+2x}{\lambda^2-1+x^2-2x} \\
 & -\frac{1}{4}\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)^3-\lambda^3\right\}-\frac{1}{4}\left\{\left(\frac{\lambda^2-1+x^2}{2x}\right)-\lambda\right\} \\
 & -\frac{1}{8}(\lambda^2-1)^2 \ln\frac{\lambda+1}{\lambda-1} \quad \text{for } x \geq 2, \quad (3.11)
 \end{aligned}$$

where

$$\lambda = k_0/k_f.$$

L_{II} and L_{IV} cannot, in general, be reduced into single integrals, and cannot be written in such simple forms as L_I and L_{III} .

§ 4. Detailed discussions for results of the calculation

In order to estimate the value of a , we assume some appropriate forms for $v(r)$. That is,

$$v_M(r) = V_0 \left(1 + \frac{3}{\kappa r} + \frac{3}{\kappa^2 r^2} \right) \cdot \frac{e^{-\kappa r}}{\kappa r} \quad (4.1a)$$

or

$$v_M^{(c)}(r) = V_0(1 + \kappa r) \frac{e^{-\kappa r}}{\kappa r}. \quad (4.1b)$$

Then, $w_M(k)$ and $w_M^{(c)}(k)$ are obtained from Eq. (3.4) as follows,

$$w_M(k) = V_0 \frac{k^2}{\kappa^3(k^2 + \kappa^2)}, \quad (4.2a)$$

and

$$w_M^{(c)}(k) = V_0 \frac{2k^2}{\kappa(k^2 + \kappa^2)^2} \quad (4.2b)$$

where, κ is the reciprocal Compton wave length of pion. $w(k)$ is shown graphically in Fig. 3. Here we take $V_0 = 0.08 \mu c^2 = 13 \text{ Mev}$ in accordance with the one-pion-exchange potential suggested by meson theory in region $r \gtrsim 1/\kappa$. μ is the mass of pion. $v_M^{(c)}$ is the potential which is essentially the same as v_M in the region $r \gtrsim 0.7 \kappa^{-1}$ and does not have the singularity appearing in v_M in the region of small r . $\phi_I(x)$ and $\phi_{III}(x)$ are plotted in Fig. 4.

a 's corresponding to processes I and III are calculated with use of these potentials. In Table 1 we show the values of a 's when k_0 is nearly equal to k_f (We took $k_f = 1.09 \times 10^{13} \text{ cm}^{-1}$, i.e., $k_f = 1.5 \kappa$.)

The numerical value of a depends on the shape and magnitude of $v(r)$. Especially, process I is influenced even in its sign, because $\phi_I(x)$ of I has rather

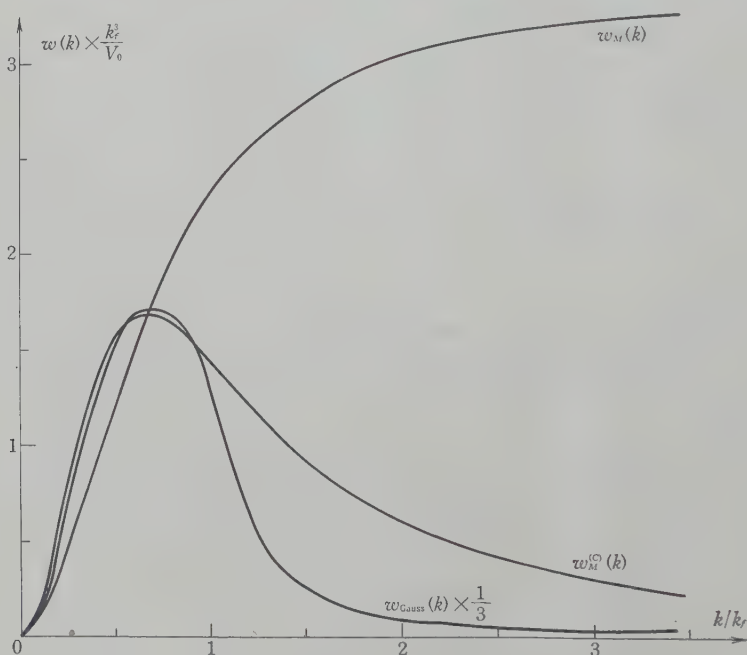
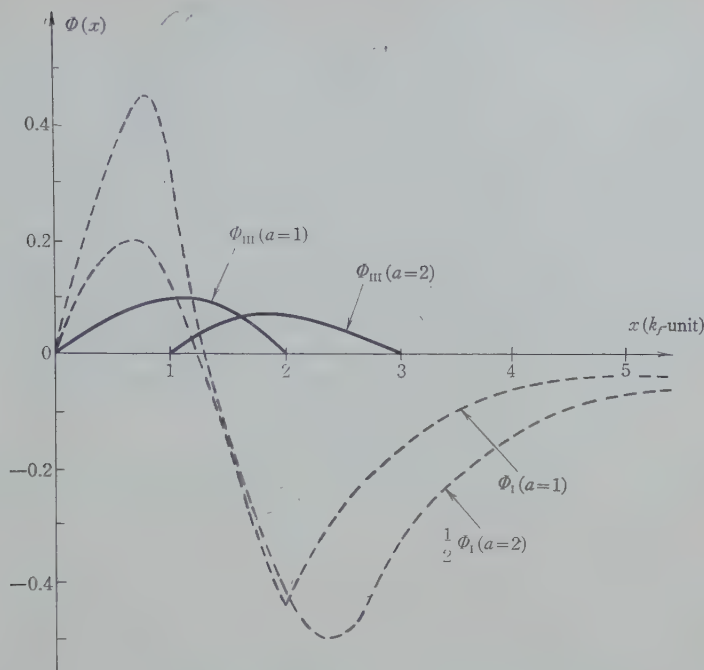


Fig. 3. Bessel transforms of v 's.


 Fig. 4. $\phi_I(x)$ and $\phi_{III}(x)$ for $k_0 = k_f$ and $k_0 = 2k_f$.

sharp dip at $k = 2k_f$ and the overlapping integral with $[w(x)]^2$ violently varies depending on the cutting-off of the high momentum part of $v(r)$. In the case of III, however, the overlapping does not vary so remarkably, due to the nature of positive definiteness of $\phi_{III}(x)$.

As can be seen from the graph of $\phi_{III}(x)$, the parts of $w(x)$, which contribute to a_{III} , correspond to the region $k < 2k_f$, that is, $r > 1/(3\kappa)$ in configuration space. And, as to process I, the large negative contribution comes from the region $k \gtrsim 1.5 k_f$, i.e., $r \lesssim 1/(2\kappa)$, if we use $v_M(r)$ literally in this case. But the potentials used in our calculation should not be extended to such an inner region. High momentum part of the potential may be cut off for some reason—the hard core or many particle correlations due to the Pauli principle or something like this. Thus $v_M^{(c)}$ may be a reasonable potential in such a sense. So the magnitude of a_I derived by Jancovici is reinvestigated here. In evaluating a_I in Table I for the case of v_M , we cut off $w_M(k)$ for $k > 2k_f$. In the region $1/\kappa > r > 1/(2\kappa)$, there are some ambiguities concerning the shape and magnitude of $v(r)$, but we suppose that those in the region $r > 1/\kappa$ would not be essentially altered from those adopted here even if we take into account the many-particle correlation with the Pauli-principle. Thus we can conclude that our processes play important roles concerning the spin-orbit coupling and give the correct sign and the reasonable order of magnitude.

As already mentioned, a_{II} and a_{IV} can neither be written in simple forms nor be easily evaluated if we use the general shape of potential. Here we take, for

the sake of easiness of integration, the following potential,

$$v(r) = V_0' k_f^2 r^2 e^{-\alpha r^2}. \quad (4.3)$$

The parameters in Eq. (4.3) are determined by the condition that it reproduces the low momentum part of the one-pion-exchange potential, namely

$$V_0' = 0.3 \text{ Mev}, \quad \alpha = 0.1 k_f^2. \quad (4.4)$$

The numerical results in this case are written in Table 1. Although we cannot unambiguously see how they depend on the shape of $v(r)$, we may plausibly conclude from the values shown in Table 1 that both a_{II} and a_{IV} are, in general, small compared with a_I and a_{III} in the low energy region.

Table 1. Values of a_i (Mev $\cdot (10^{-13} \text{ cm})^5$) at $\lambda=1$

Potential	a_I	a_{II}	a_{III}	a_{IV}	Experimental value of $a = \sum a_i$
v_M	-46	-4	53	3.2	several tens Mev $\cdot (10^{-13} \text{ cm})^5$
$v_M^{(c)}$	7	1	13	1	

Experimentally, the magnitude of a is not so definite. Assuming the nuclear wave function as uniform, we obtain the doublet splitting of energy level by spin-orbit coupling Eq. (2.9), as follows,

$$\langle \Delta E \rangle \simeq a \cdot (9/8\pi) \kappa^{-5} (2l+1) A^{-2/3}. \quad (4.5)$$

From the above equation, we can get the empirical values of a with use of $\langle \Delta E \rangle$ measured by the experiment for various nuclei. The values of a determined by the above procedure have large fluctuations from nucleus to nucleus since the actual nuclei hardly have the doublet splitting of pure single particle level structure. The order of magnitude of a thus obtained is about $40 \sim 90 \text{ Mev} \cdot (10^{-13} \text{ cm})^5$.¹⁰⁾

In high energy nucleon scattering from nuclei, the spin-orbit coupling term of the one-particle potential plays also important roles, especially in reproducing the polarization phenomena.¹¹⁾ In this case the magnitude of a has the same order as above,—that is, one adopted in the shell model. We should, however, pay attention to the difference of the mechanism of spin-orbit coupling in low energy and high energy. Process III which is dominant in the low energy region gives only small contribution in the high energy region. For one thing, the expression of a_{III} explicitly contains the factor k_0^{-2} , and for another, ϕ_{III} itself becomes small for large k_0 as indicated in Fig. 4. In fact, for $k_0 = 2k_f$, a_{III} becomes $27 \text{ Mev} \cdot (10^{-13} \text{ cm})^5$ for v_M and $1 \text{ Mev} \cdot (10^{-13} \text{ cm})^5$ for $v_M^{(c)}$ respectively, which shows that this process is less important in the high energy region than in the low energy region.

On the other hand, a_I , which comes from the induced polarization term, does not become so small even when the factor k_0^{-2} is taken into account. This is also seen from Fig. 4.

Thus, in the high energy case, process I mainly contributes to the spin-orbit coupling and gives the correct sign and reasonable order of a , as justified by the argument in ref. 12). Moreover, as Riesenfeld and Watson¹²⁾ showed, if we appropriately take the nucleon-nucleon interaction into account we can reproduce the spin-orbit coupling in the very high energy region by the induced polarization term.

Summarizing the results, we may conclude that, at least in the low energy region, the origin of spin-orbit coupling may be understood qualitatively on the basis of strong tensor interaction between nucleons and that the effect of exchange of the incident nucleon with those of the target nucleus deformed by itself plays an important role in explaining this coupling. Even if we took into account higher order effects our conclusion would not require any drastic alteration.

Acknowledgement

The authors would like to express their thanks to Dr. R. Tamagaki for valuable discussions.

Appendix I. Spin dependence of processes III and IV.

The spin dependent factor of processes III and IV can be written in the form

$$M = \frac{1}{16} Sp. \left\{ [\chi_{\alpha_i}^*(1) \chi_{\alpha_j}^*(2) + \chi_{\alpha_j}^*(1) \chi_{\alpha_i}^*(2)] \left(\frac{3(\sigma_1 \mathbf{k})(\sigma_2 \mathbf{k})}{k^2} - \sigma_1 \sigma_2 \right) \right. \\ \times [\chi_{\beta_i}(1) \chi_{\beta_j}(2) + \chi_{\beta_j}(1) \chi_{\beta_i}(2)] \left. \right\} \\ \times \left\{ [\chi_{\beta_i}^*(1) \chi_{\beta_j}^*(2) + \chi_{\beta_j}^*(1) \chi_{\beta_i}^*(2)] \left(\frac{3(\sigma_0 \mathbf{k}')(\sigma_2 \mathbf{k}')}{k'^2} - \sigma_0 \sigma_2 \right) \right. \\ \times [\chi_{\alpha_i}(0) \chi_{\alpha_j}(2) + \chi_{\alpha_j}(0) \chi_{\alpha_i}(2)] \chi_{\alpha}(1) \left. \right\},$$

since the tensor operator vanishes when operated on the spin singlet state. $Sp.$ means the sum over all possible spin states of α_i , α_j , β_i and β_j . In these expressions, the final term can be written as

$$(\chi_{\alpha_i}(0) \chi_{\alpha_j}(2) + \chi_{\alpha_j}(0) \chi_{\alpha_i}(2)) \chi_{\alpha}(1) = \frac{1}{2} (1 + \sigma_0 \sigma_1) \left\{ (\chi_{\alpha_i}(1) \chi_{\alpha_j}(2) + \chi_{\alpha_j}(1) \chi_{\alpha_i}(2)) \right. \\ \left. \times \chi_{\alpha}(0) \right\}.$$

Performing the spin operator calculation and taking the trace, we finally obtain

$$M = 9 \left\{ 1 + \frac{(\mathbf{k} \mathbf{k}')^2}{k^2 k'^2} + i \frac{(\mathbf{k} \mathbf{k}')}{k^2 k'^2} \sigma_0 [\mathbf{k}' \times \mathbf{k}] \right\} \chi_{\alpha}(0).$$

Thus we find the term linear in σ_0 as

$$9 i \frac{(\mathbf{k} \cdot \mathbf{k}')}{k^2 k'^2} \sigma_0 [\mathbf{k}' \times \mathbf{k}],$$

where j_2 is the spherical Bessel function of order 2. After performing the integration explicitly, we have

$$\langle k_0' | V_{\alpha}^{(e)} | k_0 \rangle = \text{const} \cdot 9i \sum_{\alpha_i} \int \phi_{\alpha_i}(k_i) \delta(k_0' - k_0 - k_i + k_i') \\ \times \frac{\sigma_0 [k' \times k] (k \cdot k')}{k^2 k'^2} \cdot \frac{1}{k_i^2 + k_j^2 - k_0^2 - k_\beta^2} \phi_{\alpha_i}(k_i') \cdot w(k) w(k'),$$

where

$$\left. \begin{aligned} k &= k_0 - k_j \\ k' &= k_i' - k_i - k_0 + k_j \\ k_\beta &= k_i + k_j - k_0 \end{aligned} \right\} \text{ for III, or } \left. \begin{aligned} k &= k_0 - k_j \\ k' &= k_i' - k_0 \end{aligned} \right\} \text{ for IV,}$$

and $w(k)$ defined in Eq. (3.4).

Now we use the procedure used by Jancovici; namely, we first perform the state sum. This cannot be done easily without restriction. Here we shall assume that the state sum can be obtained for $\phi_{\alpha_i}^* \phi_{\alpha_i}$ without appreciably changing the value of other factors. This assumption may be valid when the nucleon is incident with high energy. When the energy of the incident nucleon is low, this assumption may not be a good one. But, since the states contributing to this sum may be restricted to those which are near the Fermi surface, this approximation may not appreciably alter the situation in our case.

Thus $\sum_{\alpha_i} \phi_{\alpha_i}^*(k) \phi_{\alpha_i}(k')$ may be approximated as

$$\sum_{\alpha_i} \phi_{\alpha_i}^*(k) \phi_{\alpha_i}(k') = \frac{1}{(2\pi)^3} \int e^{i(k-k')r'} \rho(r') \frac{3}{16\pi} k_f^{-3} dr',$$

where ρ is the density of nucleus and k_f is the Fermi momentum. Then

$$\langle k_0' | V_{\alpha}^{(e)} | k_0 \rangle = \text{const} \cdot \frac{3}{16\pi k_f^3} \int dr' \cdot \rho(r') \delta(k_0' - k_0 - k_i + k_i') e^{i(k_i' - k_i)r'} \\ \times 16\pi^2 \cdot \frac{\sigma_0 [k' \times k] (k \cdot k')}{k^2 k'^2} \frac{1}{k_i^2 + k_j^2 - k_0^2 - k_\beta^2} w(k) w(k').$$

Putting $g = k_i' - k_i$ and taking the term linear in g , we can rewrite the above expression as

$$\langle k_0' | V_{\alpha}^{(e)} | k_0 \rangle = \text{const} \cdot \int dr' \rho(r') \delta(k_0' - k_0 + g) e^{ig \cdot r'} \cdot \sigma_0 [g \times k_0] \cdot L \\ = \int dr \cdot e^{-i(k_0' - k_0)r} \cdot a \frac{1}{r} \frac{\partial \rho}{\partial r} \cdot (\sigma_0 \cdot [r \times k_0]) \\ = \int dr \cdot e^{-i(k_0' - k_0)r} \cdot a \frac{1}{r} \frac{\partial \rho}{\partial r} \cdot (l \cdot \sigma_0),$$

where

$$a=i \cdot CL.$$

Thus we obtain Eq. (2·9).

Appendix III. Calculation of L for process III.

In the plane wave approximation the term we have to calculate is

$$L_{\text{III}} \sim \iint dk_i dk_j \cdot \frac{\mathbf{k}_0 \cdot (\mathbf{k}_0 - \mathbf{k}_i)}{k_i^2 + k_j^2 - k_0^2 - k_p^2} \cdot \left[\frac{w(|\mathbf{k}_0 - \mathbf{k}_i|)}{|\mathbf{k}_0 - \mathbf{k}_i|} \right]^2,$$

where $\mathbf{k}_p = \mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_0$. The domain of integration is restricted by the condition

$$k_i, k_j < k_f, \quad |\mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_0| > k_f.$$

Now the energy denominator is $-2(\mathbf{k}_0 - \mathbf{k}_i)(\mathbf{k}_0 - \mathbf{k}_j)$. Taking $\mathbf{k}_0 - \mathbf{k}_i$ and \mathbf{k}_j as the integral variables, we have an integral

$$\int \frac{(\mathbf{k}_0 \cdot \mathbf{q})}{\mathbf{q}(\mathbf{k}_0 - \mathbf{p})} \cdot \frac{w^2(q)}{q^2} d\mathbf{q} d\mathbf{p},$$

the domain of integration being

$$p < k_f, \quad |\mathbf{p} - \mathbf{q}| > k_f.$$

This integral is evaluated straightforwardly and becomes to Eq. (3·8).

Appendix IV. Explicit form of $\Phi_1(x)$

$$\begin{aligned} \Phi_1(x) = & +\frac{1}{8}(\lambda^2-1)^2 \ln(|\lambda+1|) - \frac{1}{8}(\lambda^2\xi^2-1)^2 \ln(|\lambda\xi+1|) \\ & -\frac{1}{48}x(2\lambda\xi+x)^2(4\lambda\xi-x) \ln\left(\left|\lambda\xi+\frac{x}{2}\right|\right) \\ & +\frac{1}{48}x(2\lambda+x)^2(4\lambda-x) \ln\left(\left|\lambda+\frac{x}{2}\right|\right) \\ & +\frac{1}{24}(\lambda+x+1)^2[x^2-2(\lambda+1)x-3(\lambda-1)^2] \ln(\lambda+x+1) \\ & -\frac{1}{24}(\lambda\xi+x+1)^2[x^2-2(\lambda\xi+1)x-3(\lambda\xi-1)^2] \ln(|x+\lambda\xi+1|) \\ & -\frac{1}{8}\lambda^3x(\xi^2-1) - \frac{5\lambda^2}{48}x(x+2)(\xi^2-1) - \frac{\lambda x}{24}(x+5)(\xi-1), \end{aligned}$$

for $x \leq 2$,

$$\begin{aligned} \Phi_1(x) = & \frac{1}{24} \left[-(\lambda+x+1)^2\{3\lambda^2-(6-2x)\lambda+(3-x)(1+x)\} \ln(x+\lambda+1) \right. \\ & +(\lambda+x-1)^2\{3\lambda^2+(6+2x)\lambda+(3+x)(1-x)\} \ln(x+\lambda-1) \\ & \left. +(\lambda\xi+x+1)^2\{3\lambda^2\xi^2-(6-2x)\lambda\xi+(3-x)(1+x)\} \ln(x+\lambda\xi+1) \right] \end{aligned}$$

$$\begin{aligned}
 & -(\lambda\xi+x-1)^3\{3\lambda^2\xi^2+(6+2x)\lambda\xi+(3+x)(1-x)\}\ln(x+\lambda\xi-1)\Big] \\
 & +\frac{a}{2}(1-\xi)\Big[\frac{1}{2}+\frac{1}{2}\lambda^2+\frac{5}{6}\lambda x+\frac{1}{6}x^2+\lambda\xi\Big(-\frac{5}{6}x+\frac{1}{2}\lambda\Big) \\
 & +\frac{1}{2}\lambda^2\xi^2\Big], \qquad \qquad \qquad \text{for } x>2
 \end{aligned}$$

where

$$\begin{aligned}
 \xi &= \frac{1-\lambda^2-x^2}{2\lambda x} & \text{for } |x-\lambda| < 1 \\
 \xi &= -1 & \text{for } |x-\lambda| \geq 1.
 \end{aligned}$$

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Two-Nucleon Potential with the "One-Pion-Exchange Tail"

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It is shown that p - p single and double scattering up to 150 Mev can be reproduced by a potential with the "one-pion-exchange tail" in its outer part, without assuming any seriously energy-dependent interactions and spin-orbit coupling potentials. The intermediate part of the potential has all features that are characteristic of the two-pion-exchange potential. Qualitative reasoning of this potential is given together with the results of calculation.

§ 1. Introduction and summary

The pion theory of nuclear forces has been verified by analysis of all low energy two-nucleon phenomena below about 20 Mev¹⁾, taking properly into account the reliability of the theory in accordance with the Taketani theory²⁾. Outer parts of the forces have been shown to be given by the static pion-theoretical potential correctly ($x \gtrsim 1.5$)^{§§} or almost correctly ($1.5 \gtrsim x \gtrsim 0.7$). The effective pion-nucleon coupling constant has been determined as $g_e^2/4\pi\hbar c = 0.080 \pm 0.010$.

Comparing with phenomenological potentials usually used, the pion-theoretical static potential has the following characteristics:

(i) Outside $x \gtrsim 1$, the central potential is weak, while the tensor potential is very strong. The latter is almost correctly given by the outer part of the one-pion-exchange potential.

(ii) Inside $x \lesssim 1.5$, the two-pion-exchange central potential becomes strong. There-

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§§ x is the inter-nucleon distance in the unit of the pion Compton wave length $\hbar/\mu c \equiv 1/\kappa$. Notations are the same as those used in Ref. 1.

fore the shape of the potential is not simple. It has two ranges and is not always monotonic.

(iii) The spin-orbit coupling potential has the range $1/2\kappa$ and is negligibly small around the pion range.* Its depth is of the order of $(\mu/M) \cdot (g_e^2/4\pi)^2$, μ and M being the pion and nucleon mass respectively.

As the kinetic energy of the two-nucleon system increases, it would become inadequate to describe the nuclear forces in terms of the static potential. Some interactions that are possibly energy-dependent would have to be added to the "potential", or the potential itself has to be gradually modified. The features of such modifying interactions are of interest, since they are related to the properties of the pion cloud and also possibly to the nature of the nucleon core. But at present, we scarcely know either field-theoretically or phenomenologically whether the modifying interactions are weak or strong, or how we should reasonably choose the energy-dependent parameters of the interactions.

We could expect, however, that when the energy is not very high (say, below a few hundred Mev), the characteristics of the pion-theoretical static potential (i) ~ (iii) mentioned above are still important. Actually our previous results of a kind of phase shift analysis at 90 and 150 Mev gave a set of phase shifts that is consistent with a strong tensor force (positive in the odd state and negative in the even state as the one-pion-exchange potential).³⁾

Since among the above characteristics, (i) ~ (iii), the most important one (i) is well reproduced by the one-pion-exchange potential, we will analyse the proton-proton scattering at 95 and 150 Mev using potentials with "the one-pion-exchange tail". Namely:

(i) We assume the one-pion-exchange potential with $g_e^2/4\pi\hbar c = 0.08$ in the outer region $x \gtrsim 1 \sim 1.5$.

(ii) We take arbitrary (possibly energy-dependent) potentials in the intermediate region $x \sim 1$.

(iii) We also investigate effects of inner interactions assuming them phenomenologically for $x \leq 0.7$. In most of the low energy phenomena, the phenomenological inner interactions do not play any important roles. But in the high energy region in which we are concerned, the effects are no longer negligible.

Thus we use the potential

$$\begin{aligned}
 V(x) &= V^{(1)}(x) \text{ (the one-pion-exchange potential) for } x \geq x^{(1)}, \\
 &= V_{oo} + S_{12} V_{ot} && \text{for } x^{(1)} \geq x \geq 0.7, \\
 &= V'_{oo} + S_{12} V'_{ot} && \text{for } 0.7 \geq x \geq x_0, \\
 &= \infty && \text{for } x_0 \geq x.
 \end{aligned} \tag{1}$$

In the triplet odd state

* See Table 4.

$$V^{(1)}(x) = (1/3) (g_e^2/4\pi\hbar c) \mu c^2 \left[1 + S_{12} \left\{ 1 + (3/x) + (3/x^2) \right\} \right] e^{-x}/x,$$

and in the singlet even state

$$V^{(1)}(x) = - (g_e^2/4\pi\hbar c) \mu c^2 e^{-x}/x,$$

with $g_e^2/4\pi\hbar c = 0.08$. V_{oc} , V'_{oc} , V_{ot} and V'_{ot} are constants. In the triplet odd state $x^{(1)} = 1.0$ or 0.7 , since the two-pion-exchange potential is rather weak in the region $x \gtrsim 1.0$, especially in the tensor potential, as can be seen in Fig. 7. In the singlet even state $x^{(1)} = 1.5$ or 1.0 , since the two-pion-exchange potential is appreciable even outside $x \sim 1.0$, as can be seen in Fig. 10. Of course the tensor potential $S_{12}V_{ot}$ and $S_{12}V'_{ot}$ is absent in this state.

In § 2, the triplet odd potential is investigated by analysing experimental data at 95 Mev. In order to bring out essential features of nuclear forces, we try to fully clarify relations between phase shifts and potential features and between the data and the phase shifts, so that our paper may not be mere accumulations of cumbersome tables and figures. Such discussions are made in § 2-2 and, making use of the results, qualitative reasoning for the proposed potentials is given in § 2-3. In § 3, the triplet odd potential is discussed independently at 150 Mev. In § 4 discussions in § 2 and § 3 are summarized.

It is found that all the data up to 150 Mev can be reproduced with either of two proposed potentials Eq. (2-I, 13-I) and Eq. (2-II, 13-II) with the one-pion-exchange tail. These potentials retain all the features that are characteristic of the two-pion-exchange potential: a moderately strong tensor potential (i.e., the one-pion-exchange tensor potential is suppressed by the two-pion-exchange one) and a deep, attractive central potential around the pion range. We feel it rather surprising that no positive evidence is found on the energy-dependence of nuclear forces.

In § 5 the singlet even potential is investigated, showing no positive evidence on the energy-dependence. In § 6 some comments are made. Especially, hasty introduction of *strong* spin-orbit coupling forces is criticized since it has no sound basis not only from the pion-theoretical view-point but from semi-phenomenological ones.

§ 2. Triplet odd potential, discussions at 95 Mev

2-1. Proposal of the potential

Our potentials which reproduce all 90 Mev single and double scattering data are as follows:

$$\begin{aligned} \text{(I)} \quad x^{(1)} &= 1.0, & x_0 &= 0.32 \\ V_{oc} &= -20 \text{ Mev}, & V'_{oc} &= -100 \text{ Mev}, & (2\text{-I}) \\ V_{ot} &= V_T^{(1)}(x=1), & V'_{ot} &= 0. \end{aligned}$$

$$\begin{aligned}
 \text{(II)} \quad & x^{(1)}=1.0, & x_0=0.34, \\
 & V_{oo}=-20 \text{ Mev}, & V'_{oo}=-100 \text{ Mev},
 \end{aligned} \tag{2-II}$$

$$V_{oT}=V'_{oT}=V_T^{(1)}(x=1).$$

$$\begin{aligned}
 \text{(III)} \quad & x^{(1)}=1.0, & x_0=0.30, \\
 & V_{oo}=V'_{oo}=-20 \text{ Mev},
 \end{aligned} \tag{2-III}$$

$$V_{oT}=V'_{oT}=V_T^{(1)}(x=1).$$

$V_T^{(1)}(x=1)$ denotes the value of the one-pion-exchange tensor potential at the pion range, namely, V_{oT} (and/or V'_{oT}) = $V_T^{(1)}(x=1)$ means the straight cut-off at $x=1$. $V_T^{(1)}(x=1) \sim 10$ Mev.

Phase shifts and mixing parameters due to these potentials are summarized in

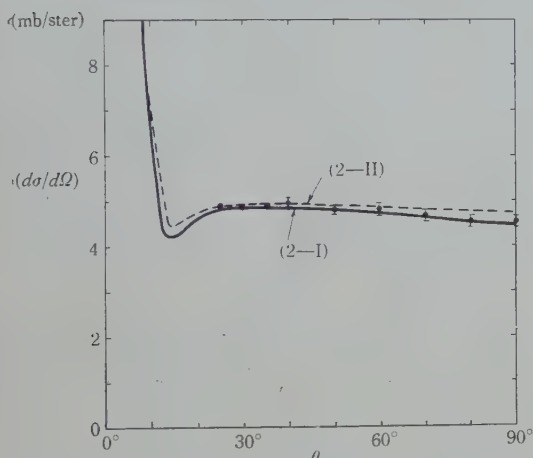


Fig. 1. p - p angular distribution at 95 Mev. Theoretical curves are due to potentials (2-I) and (2-II).

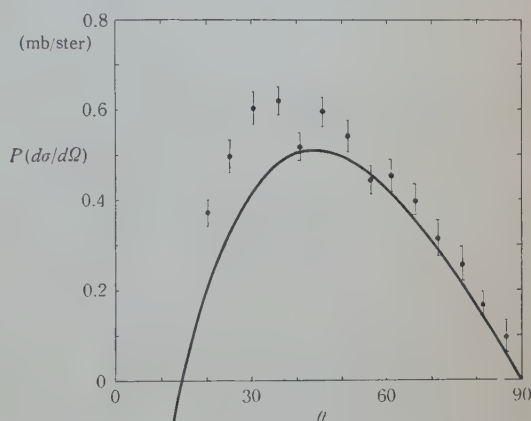


Fig. 2. p - p polarization cross section at 95 Mev. Theoretical curve is due to potential (2-I).

Table 1. Phase shifts at 95 Mev

potential	phase shifts ${}^3\delta_{J^P}$ ($\rho=\alpha,\beta,\gamma$) (Rad.)						(phenomenologically determined)
	${}^3\delta_0^T$	${}^3\delta_1^B$	${}^3\delta_2^A$	${}^3\delta_2^T$	ϵ_2	$f(\epsilon_2)$	${}^1\delta_0$
Eq. (2-I)	0.560	-0.170	0.129	-0.012	-0.438	0.483	0.27
Eq. (2-II)	0.572	-0.181	0.120	-0.014	-0.469	0.428	0.27
Eq. (2-III)	0.53	-0.20	0.10	-0.02	-0.52	0.34	0.10
Eq. (10)	0.61	-0.21	0.10	-0.02	-0.55	0.25	
Eq. (11)	0.48	-0.09	0.16	-0.006	-0.30	0.72	0.30

Other phase shifts

$${}^1\delta_2=0.09, \quad {}^3\delta_3^B=-0.028, \quad {}^3\delta_4^A=0.012, \quad {}^3\delta_4^T=-0.006, \quad \epsilon_4=-0.708$$

$${}^1\delta_4=0.006, \quad {}^3\delta_5^B=-0.005, \quad {}^3\delta_5^A=0.001, \quad {}^3\delta_6^T=0.000, \quad \epsilon_6=-0.744$$

* See Eq. (7) and its explanation.

Table 1. Comparison with data is made in Figs. 1 and 2. Phase shifts with the higher angular momenta shown in Table 1 are taken consistently with the one-pion-exchange tail. They are scarcely dependent on the inner interactions. Singlet *S*-wave phase shifts ${}^1\delta_0$ are chosen phenomenologically and discussed in § 5.

Preliminary discussions necessary for qualitative reasoning why the above potentials (2-I, II, III) can reproduce the data are made in the next subsection § 2-2. On the basis of these discussions the following points are argued in § 2-3:

- (i) A strong positive "intermediate" tensor potential makes the (total) cross section σ too large. In the present paper, an "intermediate" potential means the part of the potential around the pion range.
- (ii) For cases in which the intermediate tensor potential is weak or negative:
 - (a) If the intermediate central potential is very strongly attractive, a large forward peak appears in the angular distribution $d\sigma(\theta)/d\Omega$.
 - (b) If the intermediate central potential is weak or repulsive, the polarization cross section $P(\theta)d\sigma(\theta)/d\Omega$ becomes vanishingly small or even negative.
- (iii) Therefore, the intermediate tensor potential should be moderately strong. If its strength is described by the depth of a constant potential, it is almost equal to $V_T^{(1)}(x=1)$.
- (iv) Under the condition (iii), an attractive intermediate central potential is necessary. Otherwise, $Pd\sigma/d\Omega$ appears too small. The order of magnitude of the strength are of several ten Mev as given in Eqs. (2-I, II, III).

2-2. Preliminaries necessary to qualitative reasoning of the potential

(a) Signs and order of the *P*-phase shifts

Matrix elements of the tensor operator are given in Table 2. Since the one-pion-exchange tensor potential is much stronger than the other forces, it is expected that

$${}^3\delta_0^\gamma \gg 0, {}^3\delta_1^\beta < 0 \text{ and } {}^3\delta_2^\alpha > 0, \tag{3}$$

where ${}^3\delta_\rho^\gamma$ is the Blatt-Biedenharn triplet phase shift with the total angular momentum $J\hbar$ and $\rho=\alpha, \beta$ and γ .

Table 2. Matrix elements of the non-central operators

State	S_{12}	$L.S$
3P_0	-4	-2
3P_1	2	-1
3P_2	-(2/5)	+1
3P_2	-(8/5)	-4

Non-diagonal element of S_{12} between 3P_2 - 3F_2 is $(6/5)\sqrt{6}$.

(b) *Correlations among ${}^3\delta_2^a$, ${}^3\delta_2^r$ and ϵ_2 in the presence of a positive tensor potential*

Assume that only a positive tensor potential $V_T(x)$ is present in the triplet odd state. Then the radial parts of the wave function with $J=2$, $u(x)$ for P -wave and $w(x)$ for F -wave, obey a pair of coupled differential equations:

$$\begin{aligned} \left\{ \frac{d^2}{dx^2} + k^2 - \frac{2}{x^2} + \frac{2}{5} U_T(x) \right\} u(x) &= \left\{ \frac{6}{5} \sqrt{6} U_T(x) \frac{w(x)}{u(x)} \right\} u(x), \\ \left\{ \frac{d^2}{dx^2} + k^2 - \frac{12}{x^2} + \frac{8}{5} U_T(x) \right\} w(x) &= \left\{ \frac{6}{5} \sqrt{6} U_T(x) \frac{u(x)}{w(x)} \right\} w(x), \end{aligned} \quad (4)$$

where $U_T(x) = (M/\hbar^2 \kappa^2) V_T(x) \geq 0$.

The right-hand sides of Eqs. (4) are due to the non-diagonal terms of the tensor potential. They are rewritten for convenience' sake as if they were due to central potentials. We call them non-diagonal potentials. In most cases, these non-diagonal potentials are decisively important as can be seen below, not only because their coefficients $(6/5)\sqrt{6}$ are large, but also because either the ratio $|u(x)/w(x)|$ or its inverse becomes very large. The sign of the non-diagonal potentials depends on the relative sign of $u(x)$ to $w(x)$ inside the force range. Outside the force range, an eigenwave whose phase shift is denoted as δ behaves as

$$\frac{w(x)}{u(x)} \xrightarrow{x \rightarrow \infty} \frac{\sin \epsilon \cdot \sin(kx - (3\pi/2) + \delta)}{\cos \epsilon \cdot \sin(kx - (\pi/2) + \delta)} = -\tan \epsilon.$$

where ϵ is the mixing parameter. Therefore we generally have:

	signs of u and w for $x \lesssim 1$	non-diagonal potentials
$\pi/2 > \epsilon > 0$	same	repulsive
$0 > \epsilon > -\pi/2$	different	attractive

(see Fig. 3 and note that the relative sign of $u(x)$ to $w(x)$ inside the force range

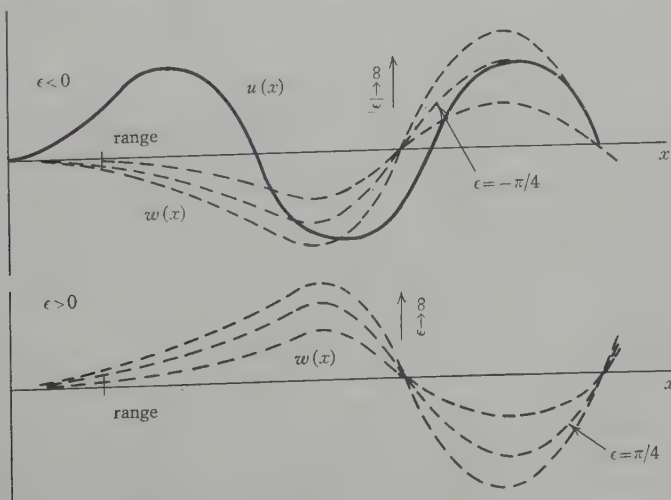


Fig. 3. Schematic shapes of $u(x)$ and $w(x)$ for the mixing parameter ϵ_2

is opposite to that in the asymptotic behaviour.) On the contrary, the diagonal terms of the tensor potential in the left-hand side of Eqs. (4) are always attractive.

Assume that we have an eigenwave solution of which $\delta > 0$. For this eigenwave solution, the total (diagonal plus non-diagonal) potential for $w(x)$ must be by far stronger than that for $u(x)$ so that $w(x)$ and $u(x)$ may be shifted by the same amount δ , since the centrifugal potential for $w(x)$ is 6 times stronger than that for $u(x)$. But the diagonal (attractive) potential for $w(x)$ is not stronger enough than that for $u(x)$. So, in general, only when the non-diagonal potentials contribute more attractively to $w(x)$ than to $u(x)$, an eigenwave solution with $\delta > 0$ is obtained. Thus we usually have

$$\text{for } (u/w)_{av} < -1, \epsilon < 0 \text{ and } \delta > 0, \quad (5)$$

where $(u/w)_{av}$ means the averaged effect inside the force range.

There is one more independent eigenwave solution of Eqs. (4) whose phase shift and mixing parameter are denoted respectively by δ' and ϵ' . After the well-known Wronskian condition,

$$\tan \epsilon = -\cot \epsilon', \quad (5')$$

$$\text{or } \pi/2 > \epsilon' = (\pi/2) + \epsilon > 0.$$

In this case, $u(x)$ and $w(x)$ have the same sign inside the force range and the non-diagonal potentials become repulsive. These non-diagonal potentials must contribute more repulsively to $w(x)$ than to $u(x)$ for the eigenwave solution*. These non-diagonal repulsive potentials compete with the attractive diagonal potentials. In most cases in our actual calculations, the former surpasses the latter, so a negative δ' is obtained. At any rate, the absolute value of δ' can not be large. Thus we usually have

$$\epsilon' > 0, (u/w)_{av} > 1 \text{ and } \delta' \lesssim 0. \quad (5'')$$

Summarizing the above results, (5), (5') and (5''), we say that in the presence of a positive tensor potential, the usual eigenwave solution is such that**

$$\delta \equiv {}^3\delta_2^\alpha > 0, \delta' \equiv {}^3\delta_2^\tau \lesssim 0, 0 > \epsilon \equiv \epsilon_2' > -\pi/2. \quad (6)$$

Next we will examine correlations between these parameters. Assume that ϵ_2 increases somewhat by some reason or other (for example, effects due to non-tensor forces, variation of the tensor force, etc.). Then the non-diagonal attractive potential for $w(x)$ of the α -wave becomes stronger since $(u/w)_{av}$ increases.

* Owing to the centrifugal potentials, the u -wave amplitude is usually larger than the w -wave amplitude inside the force range.

** We identify δ and δ' as the α - and τ -wave phase shifts respectively. However, even if we identify differently, i. e., $\delta \equiv {}^3\delta_2^\tau$, $\delta' \equiv {}^3\delta_2^\alpha$ and $\epsilon' \equiv \epsilon_2$, all our results are the same, since expressions for $d\sigma/d\Omega$ and P are symmetric with respect to the interchanges $\delta \leftrightarrow \delta'$ and $\epsilon \leftrightarrow \epsilon'$. Physical identification should be made so that ${}^3\delta_2^\alpha \rightarrow P$ -wave phase shift as $V_T \rightarrow 0$.

(See Fig. 3.) This is the necessary condition for ${}^3\delta_2^\alpha$ to increase and actually ${}^3\delta_2^\alpha$ increases more or less. At the same time, ϵ' increases according to Eq. (5'), then the non-diagonal repulsive contribution to $w(x)$ of the γ -wave decreases and so ${}^3\delta_2^\gamma$ increases. Thus, ${}^3\delta_2^\alpha$, ${}^3\delta_2^\gamma$ and ϵ_2 increase and decrease together. One example showing these tendencies is given in Fig. 4, where the depth of the intermediate central potential is chosen as the variable parameter. Although the above discussions are rather crude, correlations among ${}^3\delta_2^\alpha$, ${}^3\delta_2^\gamma$ and ϵ_2 summarized in Eq. (6) and shown in Fig. 4 are found in all of our actual calculations using the potential (1) with the one-pion-exchange tail. The values themselves of these parameters are dependent on the details of the potential and cannot be easily estimated.

Therefore the problem to be investigated in the present paper is the following :

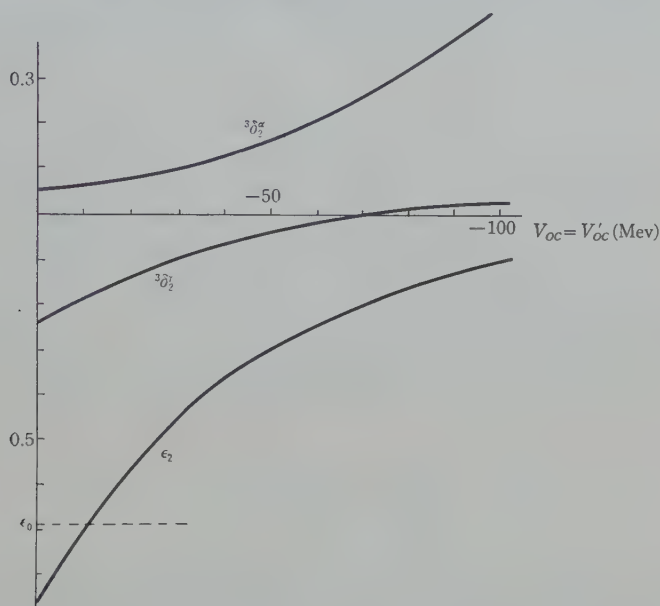


Fig. 4. One example showing the typical correlations among ${}^3\delta_2^\alpha$, ${}^3\delta_2^\gamma$ and ϵ_2 . At 90 Mev, with $x^{(1)}=1.0$, $x_0=0.3$ and $V_{0T}=V'_{0T}=0$, ${}^3\delta_2^\alpha$, ${}^3\delta_2^\gamma$ and ϵ_2 are plotted against $V_{0C}=V'_{0C}$. (Some discussions about this potential is made below Eq. (12).)

Using the one-pion-exchange tail, giving the above correlations among ${}^3\delta_2^\alpha$, ${}^3\delta_2^\gamma$ and ϵ_2 (Eq. (6) and Fig. 4), and assuming suitable intermediate potentials determining the actual values of the parameters, can we reproduce all experimental data or not?

(c) $P(\theta)d\sigma(\theta)/d\Omega$, σ and $d\sigma/d\Omega$ in the presence of a positive tensor potential

In the preceding discussions we see that a positive tensor potential gives rise to relations (3) and (6) and the correlations as shown in Fig. 4. Making use of these results, main terms of $P(\theta)d\sigma(\theta)/d\Omega$, σ and $d\sigma(\theta)/d\Omega$ are given and their features are discussed below.

Polarization cross section $P(\theta)d\sigma(\theta)/d\Omega$

$$P(\theta)d\sigma(\theta)/d\Omega \sim (6/k^2\kappa^2)f(\epsilon_2)\{[0\gamma|2\alpha]+[2\gamma|0\gamma] \\ + (3/2)[1\beta|2\alpha] + (3/2)[2\gamma|1\beta] - (5/2)[2\alpha|2\gamma]\}^*, \quad (7)$$

where $[J\rho|J'\rho'] \equiv \sin^3\delta_J^{\rho} \sin^3\delta_{J'}^{\rho'} \sin^3(\delta_J^{\rho} - \delta_{J'}^{\rho'})$,

and $f(\epsilon_2) = (5/2\sqrt{6})\sin 2(\epsilon_2 - \epsilon_0)$, $\epsilon_0 = -(1/2)\tan^{-1} 2\sqrt{6}$.

All five terms in the parentheses contribute additively after Eqs. (3) and (6). So $Pd\sigma/d\Omega$ depends very sensitively on ϵ_2 through the sinusoidal function $f(\epsilon_2)$, and, in consequence, on the details of the forces as discussed in (b). Especially, since $f(\epsilon_2)$ has its zero point at $\epsilon_2 = \epsilon_0 = -0.685$, it is impossible to have the correct sign of $Pd\sigma/d\Omega$ if $\epsilon_2 < \epsilon_0$.** Actually many sets of parameters of the potential (1) were excluded in our calculation because they could not give $\epsilon_2 > \epsilon_0$.

Total cross section σ

$$\sigma \sim (2\pi/k^2\kappa) \left[(\sin^2\delta_0)^2 + 5(\sin^2\delta_2)^2 \right. \\ \left. + (\sin^2\delta_0^{\gamma})^2 + 3(\sin^2\delta_1^{\beta})^2 + 5(\sin^2\delta_2^{\alpha})^2 \right]. \quad (8)$$

Angular distribution $d\sigma(\theta)/d\Omega$

$$d\sigma(\theta)/d\Omega \sim (1/k^2\kappa^2)[B_0P_0(\cos\theta) + B_2P_2(\cos\theta)], \quad (9)$$

where

$$B_0 = (k^2\kappa^2/2\pi)\sigma,$$

$$B_2 = 10({}^1\delta_0|{}^1\delta_2) + \sum_{J\rho J'\rho'} b_{J\rho J'\rho'}(\epsilon_2)(J\rho|J'\rho')$$

and $(J\rho|J'\rho') \equiv \sin^3\delta_J^{\rho} \sin^3\delta_{J'}^{\rho'} \cos^3(\delta_J^{\rho} - \delta_{J'}^{\rho'})$.

$b_{J\rho J'\rho'}(\epsilon_2)$ are non-negative functions of ϵ_2 and each of main terms contributes to B_2 as below:

sign and magnitude of $b_{J\rho,J\rho'}(J\rho J'\rho')$ at					
J	J'	sign of $(J\rho J'\rho')$	$\epsilon_2 \sim \epsilon_0$		$\epsilon_2 = 0$
1β	1β	+	+	\rightarrow	+
0γ	2α	+	0	\nearrow	+
0γ	2γ	-	-	\nearrow	-
1β	2α	-	-	\nearrow	-
1β	2γ	+	0	\nearrow	+
<div style="display: flex; justify-content: space-between; align-items: center;"> (a constant) each term increases </div> <div style="display: flex; justify-content: space-between; align-items: center;"> as ϵ_2 increases. </div>					

* For the complete formula, see Ref. 3c).

** For more detailed discussions, see Refs. 3a) and 3c).

The arrows mean that the latter four terms increase as ϵ_2 increases.

In order to reproduce the isotropic angular distribution, it is desired that B_2 almost vanishes. To cancel the large positive contributions to B_2 from $(1\beta|1\beta)$ and $(^1\delta_0|^1\delta_2)$, which are independent of ϵ_2 , it is necessary that $\epsilon_2 < 0$, as can be seen from the table above*. Otherwise, B_2 is largely positive and a strong forward peak arises. It is interesting to note that the condition imposed by $Pd\sigma/d\Omega$

$$\epsilon_2 > \epsilon_0 = -0.685,$$

and that imposed by $d\sigma/d\Omega$

$$\epsilon_2 < 0$$

are consistent with a positive tensor potential as discussed and shown in Eq. (6). However, since effects due to several terms cancel one another in B_2 , $d\sigma/d\Omega$ is not so sensitive to the details of the forces as $Pd\sigma/d\Omega$.

2-3. Qualitative reasoning of the potential

Referring to the preliminary discussions given above, we will show the qualitative reasoning of the proposed potentials.

(i) When the intermediate tensor potential around the pion range is stronger than Eqs. (2-I, II, III), it makes one or both of $^3\delta_0^\pi$ and $^3\delta_1^\pi$ larger, irrespective of the central potential. (See Table 2.) Therefore σ exceeds its experimental value, even if there were no force in the singlet even state (cf. Eq. (8)). One example is the case in which the positive one-pion-exchange tensor potential is present for $x \geq 0.7$:

$$\begin{aligned} x^{(1)} &= 0.7, & x_0 &= 0.30, \\ V'_{o\sigma} &= -100 \text{ Mev}, & V'_{o\pi} &= 0. \end{aligned} \quad (10)$$

This potential gives σ that is larger by about 40% than the data. Note the large values of $^3\delta_0^\pi$ and $^3\delta_1^\pi$ of this potential shown in Table 1.

(ii) When the intermediate tensor potential is weaker than Eqs. (2-I, II, III), the following two cases must be considered separately.

(a) When the intermediate central potential is strongly attractive, this central potential together with the weak tensor potential makes $^3\delta_1^\pi \sim 0$ and $^3\delta_2^\pi \sim 0$. (See, e.g., Fig. 4.) Then all negative terms in B_2 , $(0\bar{7}|2\bar{7})$ and $(1\beta|2\alpha)$, vanish (see the table below Eq. (9)) and a forward peak appears. One example is

$$\begin{aligned} x^{(1)} &= 1.0, & x_0 &= 0.3, \\ V_{o\sigma} &= V'_{o\sigma} = -50 \text{ Mev}, \\ V_{o\pi} &= V'_{o\pi} = 0. \end{aligned} \quad (11)$$

* For more detailed discussions, see Ref. 3a).

This potential gives ${}^3\delta_1^\pi = -0.09$, ${}^3\delta_2^\pi = -0.006$ and a consequent forward peak in $d\sigma/d\Omega$. (See Table 1.)

(b) When the intermediate central potential is weak or repulsive, the argument developed in § 2-2 (b) shows that ${}^3\delta_2^\pi$ decreases and at the same time ϵ_2 decreases to ϵ_0 (the zero point of $f(\epsilon_2)$). So $Pd\sigma/d\Omega$ becomes vanishingly small. For example, for

$$\begin{aligned} x^{(1)} &= 1.0, & x_0 &= 0.3, \\ V_{oT} &= V'_{oT} = 0, \end{aligned} \quad (12)$$

we can estimate from Fig. 4 that $\epsilon_2 = \epsilon_0$ for $V_{oC} = V'_{oC} \sim -10$ Mev.

(iii) Therefore, we can conclude that the intermediate tensor potential should be moderately strong. If this strength is described by the depth of a constant potential, our actual calculation shows that it must be almost equal to $V_T^{(1)}(x=1)$, the straight cut-off at the pion range (Eq. (2-II)) or a little less (Eq. (2-I)).

(iv) Under the condition (iii), the intermediate central potential must be attractive. If it is weak or repulsive, ϵ_2 approaches to ϵ_0 as discussed in § 2-2 (b), and $Pd\sigma/d\Omega$ becomes vanishingly small. Our actual calculation shows that the strength of the attractive force is of several ten Mev as given in Eqs. (2-I, II, III).

(v) Thus, essential properties of the potentials (2-I, II, III) are worked out and the dependence of the cross sections on the shape of the potentials is clarified.

§ 3. Triplet odd potential, discussions at 150 Mev

Analyses at 150 Mev are made independently of that at 90 Mev. Potentials obtained are as follows:

$$\begin{aligned} \text{(I)} \quad x^{(1)} &= 1.0, & x_0 &= 0.28, \\ V_{oC} &= -20 \text{ Mev}, & V'_{oC} &= -100 \text{ Mev}, \\ V_{oT} &= V_T^{(1)}(x=1), & V'_{oT} &= 0. \end{aligned} \quad (13\text{-I})$$

$$\begin{aligned} \text{(II)} \quad x^{(1)} &= 1.0, & x_0 &= 0.30, \\ V_{oC} &= -20 \text{ Mev}, & V'_{oC} &= -100 \text{ Mev}, \\ V_{oT} &= V'_{oT} = V_T^{(1)}(x=1). \end{aligned} \quad (13\text{-II})$$

Table 3. Phase shifts at 150 Mev

potential	${}^3\delta_0^\pi$	${}^3\delta_1^\pi$	${}^3\delta_2^\pi$	${}^3\delta_3^\pi$	ϵ_2	$f(\epsilon_2)$	${}^1\delta_0$
Eq. (13-I)	0.627	-0.139	0.213	-0.003	-0.349	0.635	-0.01
Eq. (13-II)	0.663	-0.169	0.203	-0.006	-0.379	0.556	0.03
Eq. (2-III)			0.07	-0.10	-1.02	-0.35	

$${}^1\delta_2 = 0.15, \quad {}^3\delta_3^\pi = -0.046, \quad {}^3\delta_4^\pi = 0.023, \quad {}^3\delta_4 = -0.005, \quad \epsilon_4 = -0.690$$

$${}^1\delta_4 = 0.013, \quad {}^3\delta_5^\pi = -0.010, \quad {}^3\delta_6^\pi = 0.005, \quad {}^3\delta_6 = -0.003, \quad \epsilon_6 = -0.741$$

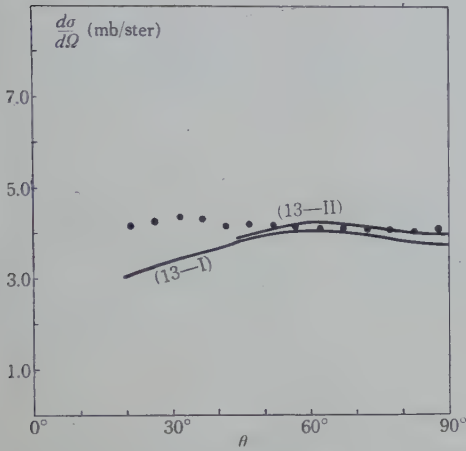


Fig. 5. p - p angular distribution at 150 Mev. Theoretical curves are due to potentials (13-I,II).

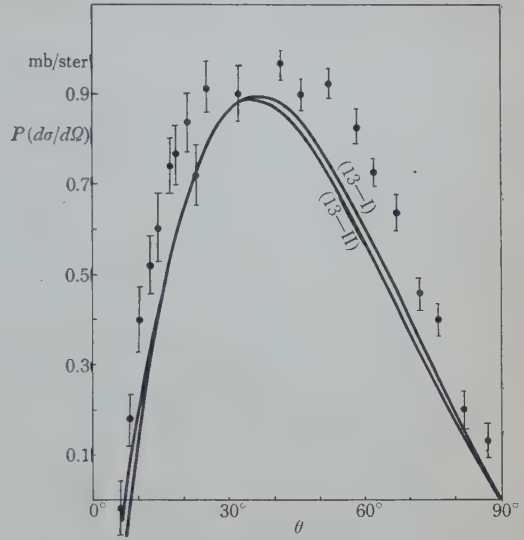


Fig. 6. p - p polarization cross section at 150 Mev. Theoretical curves are due to potentials (13-I,II).

Their phase shifts are shown in Table 3. Comparison with data is made in Figs. 5 and 6. The triple scattering experiment is discussed in § 6.

It is interesting that the above potentials are quite the same with those at 95 Mev. Discussions about the shape of the potential can be developed in parallel with those at 95 Mev. One important point at 150 Mev is that constant central potentials inside the pion range are excluded. For example, attractive central potential (2-III) predicts a negative $Pd\sigma/d\Omega$ as $\epsilon_2 = -1.02 < \epsilon_0$. (See Table 3.) It means that, in the state with $J=2$, the attractiveness of a constant intermediate central potential is too weak to increase ϵ_2 , and to reproduce $Pd\sigma/d\Omega$. The central potential therefore, must, be deeper inside.

§ 4. Triplet odd potential, summary and comments

Using the one-pion-exchange potential outside the pion range, we get quite the same intermediate potentials by independent analyses at 90 and 150 Mev. These potentials are characterized by the following features, reasons for which have been discussed in detail in § 2.

(i) The intermediate tensor potential is positive and moderately strong, and can be simulated by the straight cut-off of the one-pion-exchange potential at the pion range.

(ii) The intermediate central potential is attractive and is deeper inside.

These two features of the potential around the pion range are in good agreement with the static two-pion-exchange potentials in that the latter predict (i) a

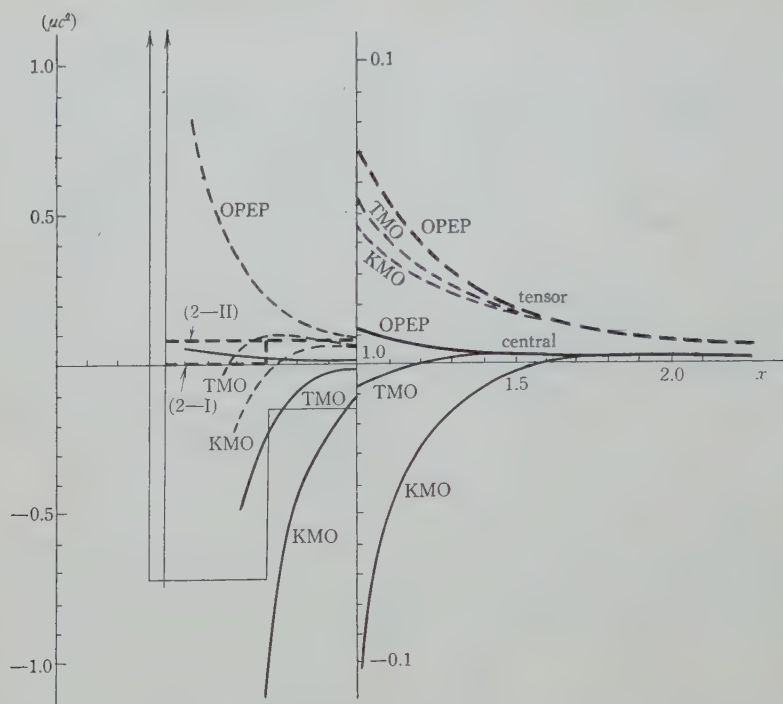


Fig. 7. Potentials in the triplet odd state. For the two-pion-exchange potential, see Refs. 1) and 5). weak attractive tensor potential and (ii) a strong attractive central potential, as shown in Fig. 7.

The presence of the attractive intermediate central force was also found in the analyses of 3P -wave phase shifts in p - p scattering at low energies and 18 Mev.⁵⁾ It is rather remarkable that $Pd\sigma/d\Omega$ is very sensitive to this intermediate central force, though no polarization appears by a purely central force.

(iii) We have some indications that a repulsive interaction like a hard core with the radius $x_0 \sim 0.3$ is present. Otherwise, the intermediate tensor and central potentials, the presence of the latter being confirmed also in the low energy data,⁵⁾ make $^3\delta_0^r$ too large to reproduce σ ; ($^3\delta_0^r \sim 0.7$).

Comparing Eqs. (2-I, II) with Eqs. (13-I II), it is seen that the radius of the hard cord x_0 decreases by about 0.05 as the energy increases from 90 to 150 Mev. We do not know whether this energy dependence is only apparent or real. However, since this energy dependence is not serious at any rate, we could substitute it by one or more of a soft core, a more realistic shape of the intermediate potentials, and a spin-orbit coupling force of very short range. We may also change the coupling constant of the one-pion-exchange tail within the extent 0.08 ± 0.01 . Considering these various factors we should say that the figures of our phase shifts in Tables, especially those at 150 Mev, are not the final ones. We rather feel it surprising that 3P -wave phase shifts can be successfully reproduced

by the simple potentials with the pion theoretical characteristics even up to 150 Mev.

§ 5. Singlet even potential

In the course of our analysis, the S -wave phase shift ${}^1\delta_0$ was chosen phenomenologically. The energy-dependence of ${}^1\delta_0$ is shown in Fig. 8.

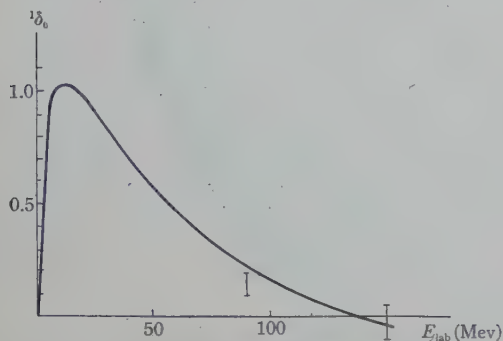


Fig. 8. Energy-dependence of ${}^1\delta_0$. In the low energy region, it is determined by the effective range and the scattering length. The theoretical curve is due to the potential Eq. (14-I) with the parameter of Eq. (15).

By analyzing the low energy scattering parameters :

the scattering length

$${}^1a = -15 \sim -20 \times 10^{-13} \text{ cm},$$

the effective range

$${}^1r_e = 2.5 \sim 2.85 \times 10^{-13} \text{ cm},$$

we have shown in our previous work¹⁾ that, since the singlet even potential has weak one-pion-exchange tail, it must have (i) a strong attraction around and inside the pion range, which is consistent with the prediction by the two-pion-exchange potential and (ii) a strong repulsion

inside. These facts can be shown in the following examples with the one-pion-exchange tail, Eq. (1) :

$$(I) \quad x^{(1)} = 1.5, \quad V_{oo} = -20 \text{ Mev}, \quad (14-I)$$

$$x_0, V'_{oo} \text{ variable},$$

$$(II) \quad x^{(1)} = 1.0, \quad V_{oo} = -20 \text{ Mev}, \quad (14-II)$$

$$x_0, V'_{oo} \text{ variable}.$$

Fig. 9 shows that the potential (14-I) reproduces 1a and 1r_e correctly, while for (14-II), when 1a is correctly given, 1r_e becomes too small ($< 2.3 \times 10^{-13} \text{ cm}$) due to too weak an attraction around the pion range.*

In Fig. 9, it is to be noted that, even if $x^{(1)}$ and V_{oo} are kept fixed, there are a great number of inner potentials that reproduce ${}^1\delta_0$ at 90 and 150 Mev within the experimental errors with different V'_{oo} and x_0 . Therefore ${}^1\delta_0$ at high energies cannot be a crucial test for the shape of the singlet even potential. A crucial test comes from the low energy parameters 1a and 1r_e , which require the strong attraction around and inside the pion range. Therefore, even if any energy-dependent interactions existed, it would not be possible to find its evidence in the behaviour of ${}^1\delta_0$.

From Fig. 9, we can estimate

* For more detailed discussions, see Ref. 1).

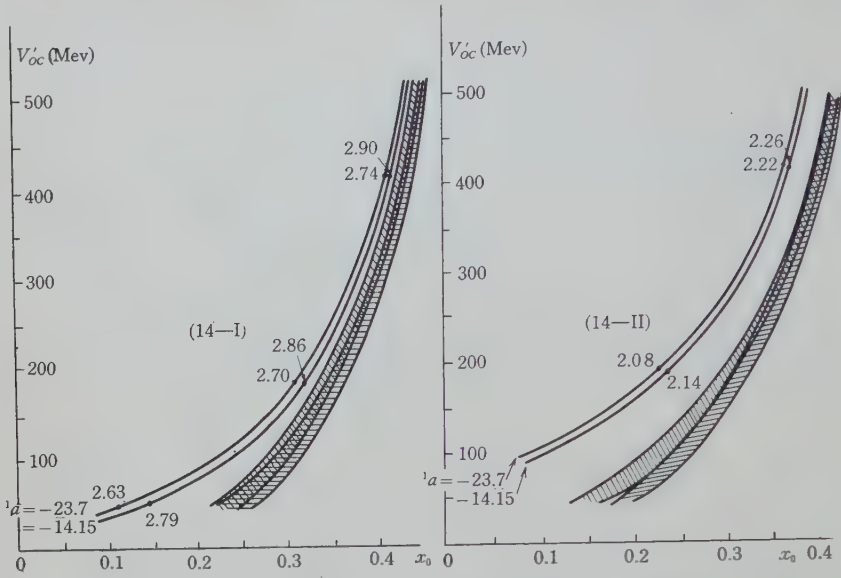


Fig. 9. V'_{oc} in Eq. (14-I,II) versus x_0 that reproduce the following quantities within the experimental errors:
 --- 1d_0 at 90 Mev,
 --- 1d_0 at 150 Mev,
 --- 1a (1r_e is shown parametrically.)

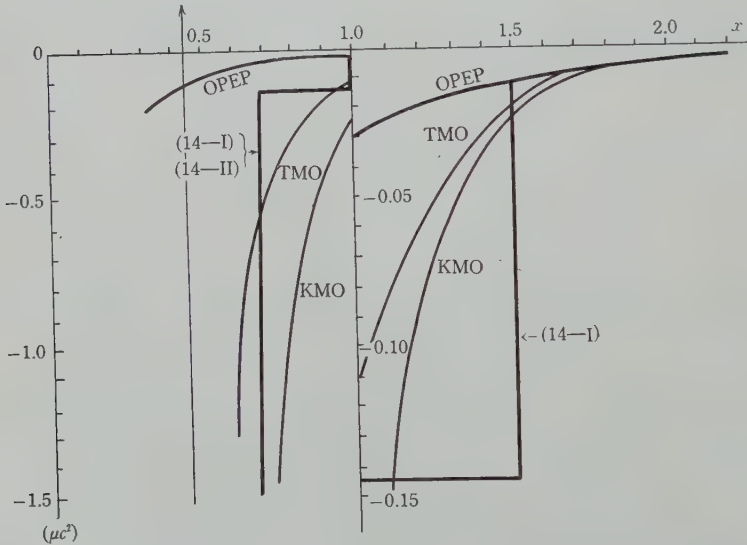


Fig. 10. Potentials in the singlet even state. For the two-pion-exchange potential, see Refs. 1) and 5).

$$x_0 \sim 0.45 \text{ and } V'_{oc} \sim -5 \times 10^2 \text{ Mev} \quad (15)$$

for the potential (14-I). These figures have no quantitative meaning. They may change largely if the assumed potential around the pion range (e.g. $x^{(1)}$ and V_{oc} in

Eq. (1) changes.

Among various two-pion-exchange potentials proposed so far, our potential (14-I) with the parameters of Eq. (15) is substituted well by the KMO potential⁴⁾ as seen in Fig. 10. This potential gives the strongest attraction around the pion range of various two-pion-exchange potentials.

§ 6. Concluding remarks

6-1 Conclusions

It was shown in the present paper that p - p scattering up to 150 Mev can be reproduced by a potential without assuming any serious energy dependent interactions. This potential has all the features that are characteristic of the static pion-theoretical potential:

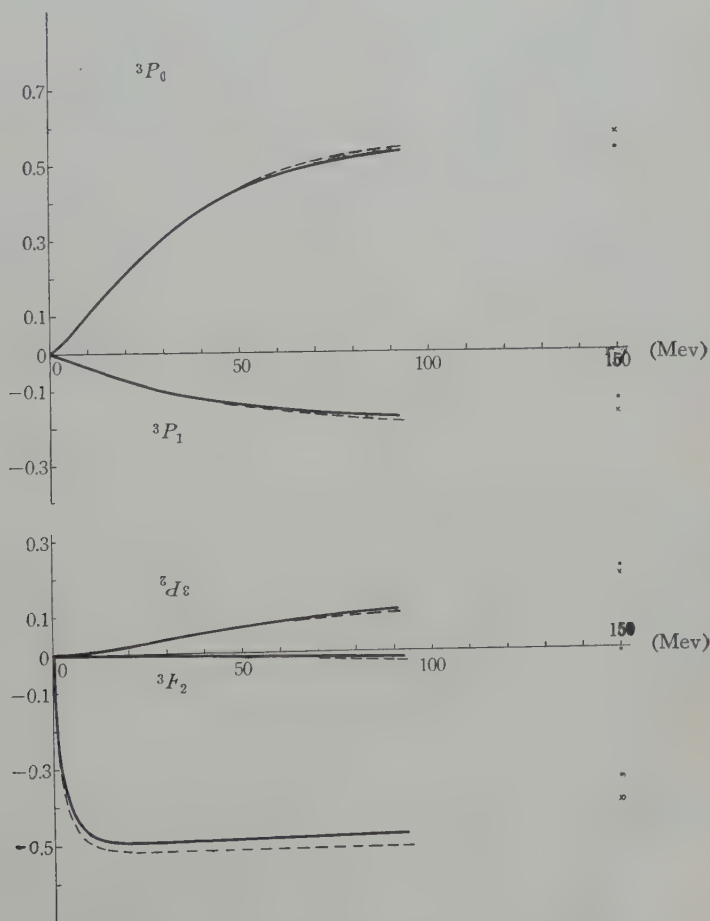


Fig. 11. Energy-dependence of the phase shifts and the mixing parameter in the triplet odd state

- (i) The one-pion-exchange tail
- (ii) Around the pion range, in the triplet odd state, a moderately strong positive tensor potential and a deep, attractive central potential becoming deeper inside; and, in the singlet even state, a strong attractive potential.

Furthermore, it is indicated that

- (iii) At very small two-nucleon distances, there would be a strong repulsion like a hard core with the radius $x_0 \sim 0.3$.

It is rather surprising that any necessity for serious modifications to the static pion-theoretical potential has not been found.

Qualitative reasoning for the potential was given. It should be noted that crucial tests are $Pd\sigma/d\Omega$ for the potential around the pion range in the triplet odd state and 1r_e for the singlet even potential. Energy dependence of the phase shifts and the mixing parameter in the triplet odd-state due to our potential are given in Fig. 11. Their values are consistent with existing experimental data.

Depolarization $D(\theta)$ at 150 Mev is calculated and shown in Fig. 12, though a detailed analysis is put off because of the too large uncertainty in the experimental data. The shape of the theoretical curve in Fig. 12 seems to be universal, if $\epsilon_2 < 0$ and ${}^3\partial_0^r$ is large due to the positive tensor potential. If it is decisively found that $D(\theta)$ is positive, it may mean something not so simple for the pion theory of nuclear forces, and then various kinds of new factors should be taken into account, for example, a strong spin-orbit coupling in the inner part, large recoil corrections, effects of strong coupling, effects ϕ of K -mesons, possibility of isoscalar mesons and others.

n - p scattering will be analyzed in a forthcoming paper.

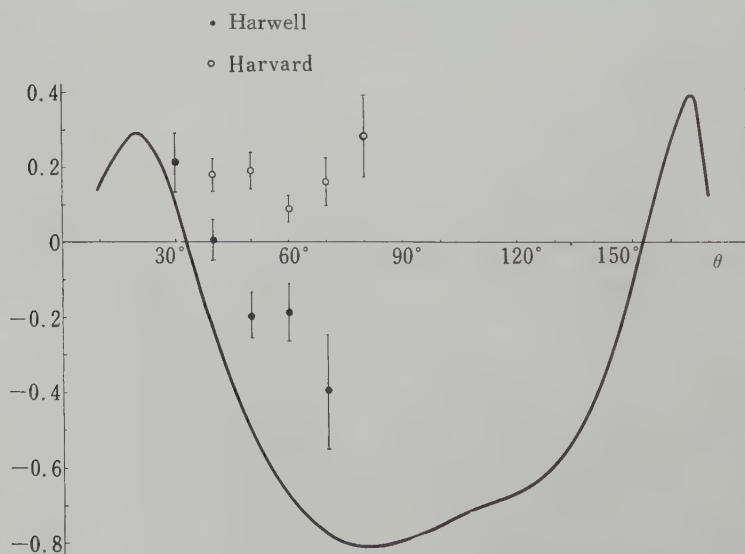


Fig. 12. Depolarization $D(\theta)$ at 150 Mev due to the potential(13-II)

Among various two-pion-exchange potentials proposed so far, our potential is much the same with the KMO potential. This potential is supposed to include the effects of the pion cloud correctly, though somewhat phenomenologically in the static limit. This point and the problems of recoil corrections should be investigated more in detail.

6-2 Remarks on spin-orbit coupling potentials

Signell and Marshak⁶ⁱ⁾, and Gammel and Thaler⁷⁾ introduced *large* spin-orbit coupling potentials to explain the p - p scattering data. However, we have found no evidence in favour of them. They have also been criticized in detail in our previous papers.^{3a)3c)} The points are briefly repeated here as follows:

(i) The spin-orbit coupling potentials are by an order of magnitude larger than that predicted by the pion theory, as shown in Table 4. The latter has the depth of the order $(\mu/M)(g_\sigma^2/4\pi\hbar c)^2$ and the range $(1/2\kappa)$.

Table 4. Magnitude of the spin-orbit coupling potential in the triplet odd state (aside the negative sign, in unit of $0.01\mu c^2$)

	$x=1$	$x=2$
pion-theoretical one*	0.77	0.0058
Gammel-Thaler ⁷⁾	5.4	0.019
Signell-Marshak ^{6a)}	5.7	0.30
Signell-Zinn-Marshak ^{6b)}	6.1	0.17

$$* V_{LS} = -(g_\sigma^2/4\pi\hbar c)^2 (2\mu^2/M) \{3 + 2(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})\} \\ \times (1 + 2/x + 2/x^2 + 1/x^3) \exp(-2x)/x^3,$$

calculated by the perturbation method by S. Sato⁸⁾.

(ii) Signell and Marshak's spin-orbit coupling potential was introduced merely to cancel unwanted effects for ${}^3\partial_0^\tau$ and ${}^3\partial_1^\alpha$ due to a deep attractive well (-1.5 BeV) of Gartenhaus' potential. If the nuclear forces near the origin were treated phenomenologically and a hard (or soft) core with $x_0 \sim 0.3$ were introduced, such a large spin-orbit coupling potential should not be required. It should be pointed out that, though the shorter range $(1/2\kappa)$ and the smaller depth are assumed in their new proposal⁶ⁱⁱ⁾ than in their old one,⁶ⁱⁱ⁾ the new one is even stronger for $x < 1$ than the old one, since their spin-orbit coupling potentials have a derivative form.

(iii) Gammel and Thaler's spin-orbit coupling potential was introduced from a purely phenomenological point of view. Their potential has not the one-pion-exchange tail, and so has no sound basis of the pion theory.

6-3. Analyses at the higher energies

We do not expect that nuclear forces can be described by a simple potential

quite above 150 Mev. However, in view of the success of our approach, we would like to point out that the "one-pion-exchange tail" should not be neglected. This tail would be important, though not necessarily the most. Recoil corrections to the one-pion-exchange tail also have to be included. Phase shift analyses at the higher energies should be made, taking the one-pion-exchange tail properly into account.

Recently, Chew proposed a method to determine the pion-nucleon coupling constant by extrapolating nucleon-nucleon angular distribution.⁹⁾ If $g_{\pi}^2/4\pi\hbar c$ comes out with the numerical value of about 0.08 above 150 Mev by his method, which we expect, it also suggests the presence and the importance of the one-pion-exchange tail at the higher energies.

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Spin-Orbit Coupling and Tensor Forces

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The possible explanation of the spin-orbit coupling, in heavy nuclei, by second-order effects of the tensor forces is investigated. The various second order terms are described by 4 graphs, in the case of one particle outside closed shells. The main contribution comes from an exclusion effect: two particles of the closed shells cannot, by mutual excitation, jump into the orbit which is already occupied by the outside particle. This effect could account, at least partially, for the observed spin-orbit splitting. The case of a hole is also investigated, and shows the same kind of agreement. A comparison with other works is discussed.

§ 1. Introduction

It is very likely that the properties of complex nuclei could in principle be computed from the two-body forces between nucleons. The spin-orbit coupling in complex nuclei, which is so important in the shell model, should be explained in terms of non-central two-body forces.

The existence of mutual spin-orbit forces in the two-body interaction is still under discussion. Such forces, in addition to the usual central and tensor ones, are sufficient to explain the two-body scattering and polarization data,^{1,2)} but there was raised some doubt whether these spin-orbit forces are actually necessary, and it was recently claimed that central and tensor forces, with hard cores, could suffice.³⁾

A related problem arises in the consideration of complex nuclei. Two-body spin-orbit forces could account for the shell-model spin-orbit splitting; a first-order perturbation calculation is perhaps too crude, but a more refined calculation,⁴⁾ using the reaction matrix, could pretend to quantitative results. On the other hand, it was attempted to use only tensor forces; since tensor forces give no splitting in first-order perturbation theory, second-order effects had to be computed. For light nuclei, it was found that tensor forces could account for an important fraction of the observed splittings.^{5,6)} But, in the case of heavy nuclei, a calculation in a previous paper,⁷⁾ using the reaction matrix as an intermediate step, predicted a much too small effect of the tensor forces.

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The purpose of the present paper is to emphasize the general importance of some exclusion effects which might escape a calculation based on the use of an approximate reaction matrix, and to apply these considerations to the effects of tensor forces in the spin-orbit splitting of heavy nuclei.* It will be found that these exclusion effects might indeed be important and account, at least partially, for the observed splittings. The general formulation of the problem, for one particle outside the closed shells, is reviewed in § 2. § 3 describes an approximate procedure to reduce the problem to a calculation of the Thomas-Fermi kind. This calculation is actually carried out in § 4 and the results are compared with the experimental data. The case of one hole in the closed shells is studied in § 5. The conclusions are stressed in § 6.

§ 2. Energies of nuclei

The general prescription for computing the first levels of a nucleus made of closed shells plus some particles or holes has been given by Bloch and Horowitz.⁸⁾ We here consider the special case of one particle outside the closed shells (the case of one hole will be very similarly treated in § 5). The total hamiltonian is

$$H = H_0 + V, \quad (1)$$

where H_0 is some zeroth-order hamiltonian and V is the residual interaction. V causes a total energy shift of the nucleus

$$\Delta E = \Delta E^e + \Delta E^p. \quad (2)$$

ΔE^e is the contribution of the closed shells and does not depend on the presence of outside particles. ΔE^p is the contribution of outside particles. This separation is actually just a matter of convenient definition, and some terms of ΔE^p may alternatively be thought of as involving particles in the closed shells. ΔE^p is implicitly given by a Brillouin-Wigner perturbation series as follows: A vacuum state $|0\rangle$ is defined as the unperturbed closed shell nucleus. Other eigenstates of H_0 are conveniently described by using annihilation and creation operators η and η^\dagger ; the unperturbed nucleus with one particle outside the closed shells is therefore in the state $\eta^\dagger|0\rangle$ with an energy E_0 . ΔE^p is then conveniently described as a sum of graphs⁹⁾

$$\Delta E^p = \langle 0 | \eta_p \Phi \eta_p^\dagger | 0 \rangle = \left\langle 0 | \eta_p V \sum_{n=0}^{\infty} \left(\frac{Q_0}{E_0 + \Delta E^p - H_0} V \right)^n \eta_p^\dagger | 0 \right\rangle_L, \quad (3)$$

where L means that only linked graphs are included and where Q_0 is the projector

* In the case of light nuclei, these exclusion effects, although they had already been implicitly taken into account in Ref. 5), were explicitly shown to be important, in Ref. 6) for the bound state problem, and in Ref. 6 bis) for the He^4 -neutron scattering problem. The spin-orbit coupling at low energy could then be explained by strong tensor forces.

outside the subspace of the unperturbed states with an energy E_0 , i. e., the intermediate states represented by one particle line.*

(3) is exact. A second-order approximation, in which ΔE^P is neglected in the energy denominator, is

$$\Delta E^P = \langle 0 | \eta_P \left[V + V \frac{Q_0}{E_0 - H_0} V \right] \eta_P^\dagger | 0 \rangle. \quad (4)$$

This expression is nothing but the usual second order perturbation result. The main advantage of having used the language of graphs is to clearly exhibit all possible exchange effects. If V consists of two-body interactions, represented by a horizontal dotted line, the second-order terms in (4) are represented by the four graphs of Fig. 1, which we shall now discuss in some detail.

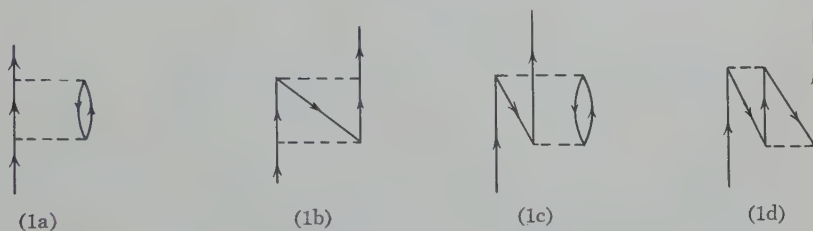


Fig. 1. The second-order graphs for one particle.

Graphs (1a) and (1b) are the direct and exchange second-order terms of the reaction matrix for interactions between the outside particle and one of the particles in the closed shells. These graphs appear in a naive extension of the Brueckner method to the degenerate nuclei. They were considered by many authors for various kinds of problems. If we are interested in the spin-orbit splitting which appears from (4) when V contains tensor forces, graphs (1a) and (1b) correspond to these terms which were considered in I, where it was shown that their contribution to the spin-orbit splitting is small.

Graphs (1c) and (1d) describe exclusion effects involving two core particles. By mutual interaction, two particles in the closed shells can get excited to intermediate states; but the state $\eta_P^\dagger | 0 \rangle$ already occupied by the outside particle is not available. This effect can be visualized before the elimination of disconnected graphs by saying that the graphs of Fig. 2 are forbidden by the Pauli principle. It is easy to see that graphs (1c) and (1d) provide the necessary subtraction. This modification of the reaction matrix for a pair of particles in the closed shells, because of the presence of an outside particle, would escape calculations based on an approximate reaction matrix computed beforehand for two particles imbedded in infinite nuclear matter, an approximate procedure which was often used.¹⁰⁾ Because

* The states $\eta_P^\dagger | 0 \rangle$ which differ by spin or orbital angular momentum orientation are degenerate, and this proper linear combination must be chosen which diagonalizes \emptyset .



Fig. 2. The excluded graphs.

of such approximations, graphs (1c) and (1b) did not appear in I. The main purpose of the present paper is to investigate these graphs and to show their importance.

We shall therefore proceed to compute the contribution to (4) of graph (1c), which is

$$\langle P | \mathcal{V}^{(c)} | P \rangle \equiv \Delta E^{P(c)} = \sum_{m,n,s} \frac{\langle mn | v | Ps \rangle \langle Ps | v | mn \rangle}{E_P + E_s - E_m - E_n}, \quad (5)$$

where P, m, n, s are single particle states; E_P, E_s, E_m, E_n are the corresponding single particle energies; and v is the two-body interaction. The summation is on unoccupied states s and on occupied closed shell states m and n .

§ 3. Reduction of the problem to a Fermi-gas calculation

If the interactions v are central and tensor forces, only the term of (4) which contains twice the tensor force will contribute to the spin-orbit splitting. The tensor force between nucleons i and j is

$$v = A[1 - \chi + \chi(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)]u(r)S_{ij}(\mathbf{r}), \quad (6)$$

where

$$S_{ij}(\mathbf{r}) = 3(\boldsymbol{\sigma}_i \cdot \mathbf{r})(\boldsymbol{\sigma}_j \cdot \mathbf{r})/r^2 - (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j). \quad (7)$$

We shall now make some approximation of the same kind as in I.* If we took all intermediate states as plane waves, the spin-orbit potential would vanish, because there is no spin-orbit potential in a constant density medium. Since, however, we are considering large nuclei, we shall assume the minimum departure from plane waves, which gives a non-vanishing result: intermediate particle states such as $|s\rangle$ will be chosen as plane waves; and similarly for the hole states $|m\rangle$ (the normalization of these plane waves will be chosen to be unity in the unit volume). In order to obtain a finite result, we must take one of the hole states different from a plane wave, and it can be seen that this hole state must be $|n\rangle$. The momentum representatives $\langle \mathbf{k}_n | n \rangle$

* The numerical factors in the definitions of the present paper and the notations may differ from those used in I.

and $\langle \mathbf{k} | P \rangle$ will, however, be strongly peaked, so that E_n and E_p may be replaced by $k_n^2/2M$ and $k^2/2M$. With all these simplifications, (5) may be rewritten in a momentum representation

$$\begin{aligned} \langle \mathbf{k}' | \psi^{(c)} | \mathbf{k} \rangle &= \sum_{\text{occupied}} \frac{2M}{(2\pi)^{12}} \int d^3 \mathbf{k}_n' \int d^3 \mathbf{k}_n \int_{k_m < k_0} d^3 \mathbf{k}_m \int_{k_s > k_0} d^3 \mathbf{k}_s \\ &\langle n | \mathbf{k}_n \rangle \langle \mathbf{k}_n' | n \rangle \frac{\langle \mathbf{k}_m \mathbf{k}_n | v | \mathbf{k} \mathbf{k}_s \rangle \langle \mathbf{k}' \mathbf{k}_s | v | \mathbf{k}_m \mathbf{k}_n' \rangle}{k^2 + k_s^2 - k_m^2 - k_n^2}. \end{aligned} \quad (8)$$

In other words, at each point the closed shell core is pictured as a local Fermi gas. The summations on occupied or unoccupied states are performed on momenta smaller or greater than the local Fermi momentum k_0 , which is a function of the position \mathbf{r} :

$$\rho(r) = (2/3\pi^2) k_0^3. \quad (9)$$

The mixed density function is approximated by one of this local Fermi gas:

$$\sum_{\text{occupied}} \langle \mathbf{r} | n \rangle \langle n | \mathbf{r}' \rangle = \rho(r) (3/4\pi k_0^3) \int_{\kappa < k_0} d^3 \kappa \exp[-i\kappa \cdot (\mathbf{r}' - \mathbf{r})]; \quad (10)$$

transforming (10) to the momentum representation we obtain

$$\sum_{\text{occupied}} \langle \mathbf{k}_n' | n \rangle \langle n | \mathbf{k}_n \rangle = (2\pi)^3 \int d^3 \mathbf{r} \exp[i(\mathbf{k}_n - \mathbf{k}_n') \cdot \mathbf{r}] \rho(r) (3/4\pi k_0^3) \mathcal{E}(k_0 - k_n), \quad (11)$$

where \mathcal{E} is the step function.

We shall only keep in the v matrix element product of (8) that term which is linear in the spin σ of the \mathbf{k} or \mathbf{k}' states, and scalar in the isotopic spin, since that term is the only one which contributes to the spin-orbit splitting:

$$\begin{aligned} \langle \mathbf{k}_m \mathbf{k}_n | v | \mathbf{k} \mathbf{k}_s \rangle \langle \mathbf{k}' \mathbf{k}_s | v | \mathbf{k}_m \mathbf{k}_n' \rangle &= 9iA^2 (2\pi)^6 (1 - 2\chi + 4\chi^2) \delta(\mathbf{k}_m + \mathbf{k}_n - \mathbf{k} - \mathbf{k}_s) \\ &\delta(\mathbf{k}' + \mathbf{k}_s - \mathbf{k}_m - \mathbf{k}_n') \sigma \cdot [(\mathbf{k}_n' - \mathbf{k}_s) \times (\mathbf{k}_n - \mathbf{k}_s)] \frac{(\mathbf{k}_n' - \mathbf{k}_s) \cdot (\mathbf{k}_n - \mathbf{k}_s)}{|\mathbf{k}_n' - \mathbf{k}_s|^2 |\mathbf{k}_n - \mathbf{k}_s|^2} \\ &\times w(|\mathbf{k}_n' - \mathbf{k}_s|) w(|\mathbf{k}_n - \mathbf{k}_s|) + \dots, \end{aligned} \quad (12)$$

where

$$w(k) = 4\pi \int_0^\infty j_2(kr) u(r) r^2 dr \quad (13)$$

(j_2 is the spherical Bessel function¹¹⁾).

In a large nucleus, the outside particle undergoes only small angle scatterings when propagating through nuclear matter. Therefore $\mathbf{k}' - \mathbf{k}$ and, because of the δ functions in (12),

$$\mathbf{g} = \mathbf{k}_n' - \mathbf{k}_n \quad (14)$$

are small quantities; we shall expand (12) into a series of \mathbf{g} , keeping only the first non-vanishing term which is

$$\langle \mathbf{k}_m \mathbf{k}_n | v | \mathbf{k} \mathbf{k}_s \rangle \langle \mathbf{k}' \mathbf{k}_s | v | \mathbf{k}_m \mathbf{k}_n' \rangle = 9iA^2 (2\pi)^6 (1 - 2\chi + 4\chi^2) \delta(\mathbf{k}' - \mathbf{k} - \mathbf{g}) \\ \delta(\mathbf{k}_m + \mathbf{k}_n - \mathbf{k} - \mathbf{k}_s) \dot{\boldsymbol{\sigma}} \cdot [\mathbf{g} \times (\mathbf{k} - \mathbf{k}_m)] \frac{[\mathcal{W}(|\mathbf{k} - \mathbf{k}_m|)]^2}{|\mathbf{k} - \mathbf{k}_m|^2}. \quad (15)$$

Using now (11), (14) and (15) in (8), we obtain

$$\langle \mathbf{k}' | \mathcal{V}^{(c)} | \mathbf{k} \rangle = \int d^3 \mathbf{r} e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} \boldsymbol{\sigma} \cdot (\mathbf{r} \times \mathbf{k}) \frac{1}{r} \frac{d}{dr} a^{(c)} \rho \quad (16)$$

where

$$a^{(c)} = (1 - 2\chi + 4\chi^2) \frac{27MA^2}{16\pi^4 k_0^3 k^2} \int_{k_n < k_0} d^3 \mathbf{k}_n \int_{k_m < k_0} d^3 \mathbf{k}_m \int_{k_s > k_0} d^3 \mathbf{k}_s \delta(\mathbf{k} + \mathbf{k}_s - \mathbf{k}_m - \mathbf{k}_n) \\ \frac{\mathbf{k} \cdot (\mathbf{k}_m - \mathbf{k})}{k^2 + k_s^2 - k_m^2 - k_n^2} \frac{[\mathcal{W}(|\mathbf{k}_m - \mathbf{k}|)]^2}{|\mathbf{k}_m - \mathbf{k}|^2}. \quad (17)$$

From (16) and (17) it is seen that graph (1c) contributes an effective spin-orbit coupling with a radial dependence $(1/r)(d/dr)a^{(c)}\rho$. Strictly speaking, $a^{(c)}$ is a function of k and of r (through k_0). However, in the case where $a^{(c)}$ does not depend too strongly on r , (16) is the familiar $a^{(c)}(1/r)(d\rho/dr)(\boldsymbol{\sigma} \cdot \mathbf{l})$ spin-orbit coupling. Here we are interested in the spin-orbit potential acting on a bound particle with an energy just above the last closed shell; the momentum k of this particle will be replaced by the Fermi momentum k_0 as an average value.

In the case where the even and odd tensor forces are given by different w functions, the even and odd contributions add together without interference.

§ 4. Practical calculation of the spin-orbit potential magnitude and comparison with experiment

The magnitude of the term (1c) of the spin-orbit potential can be computed without further approximations, by the same techniques as in I. We choose as integration variables $\mathbf{K} = \mathbf{k}_m - \mathbf{k}$, \mathbf{k}_s , \mathbf{k}_n , and obtain

$$a^{(c)} = (1 - 2\chi + 4\chi^2) \frac{27MA^2}{32\pi^4 k_0^3} \int_{|\mathbf{K} - \mathbf{k}| > k_0} d^3 \mathbf{K} \frac{\mathbf{k} \cdot \mathbf{K}}{k^2} \frac{\mathcal{W}^2(K)}{K^2} \int_{\substack{|\mathbf{k}_s + \mathbf{K}| > k_0 \\ k_s < k_0}} d^3 \mathbf{k}_s \frac{1}{(\mathbf{k} - \mathbf{k}_s) \cdot \mathbf{K} - K^2}. \quad (18)$$

We successively perform the integrals on \mathbf{k}_s and on the angles of \mathbf{K} ; it is necessary to subdivide \mathbf{k} and \mathbf{K} spaces into regions. The result of these integrations is a "universal" function $2\pi^2 k_0 C(K, k)$ which is given in the Appendix for the case $k = k_0$, appropriate to the last bound nucleon. The last integration on K must be done numerically:

$$a^{(c)} = (1 - 2\chi + 4\chi^2) \frac{27M\mu^2}{16\pi^2 k_0^2 k} \int_{k=k_0}^{k+k_0} dK \cdot KC(K, k) w^2(K). \quad (19)$$

(19) was actually computed (with $k=k_0$) for the Gammel-Thaler¹⁾ and the second-order meson theory³⁾ tensor potentials, the Bessel transforms (13) of which can be calculated analytically. No hard core was taken into account; it may be noted, however, that any Fourier component with a momentum higher than $2k_0$ would not affect (19). k_0 is related by (9) to the local density and should vary throughout the nuclear surface; $a^{(c)}$ was actually computed for several values of k_0 , but appears to be not very sensitive to k_0 (the value of k_0 which corresponds to the average density of nuclear matter is about $1.27 \times 10^{13} \text{ cm}^{-3}$). The results are tabulated in Table I. We see that $a^{(c)}$ depends somewhat on the tensor force, but is always of the same order of magnitude.

The experimental strength of the spin-orbit coupling in heavy nuclei is a bit confusing. The doublet splittings in Pb^{207} and Pb^{209} can be explained¹²⁾ by a spin orbit force $a(1/r)(d\rho/dr)(\sigma \cdot l)$ with $a \sim 57 \text{ Mev}$ (10^{-13} cm)⁵. But values of a four times greater were suggested to be necessary to account for the sequence of the single-particle levels in an average potential¹³⁾. The former evidence, from doublet splittings, is perhaps to be preferred, since it is more direct. It must also be remembered that the relation between a and the doublet splitting is extremely sensitive to detailed assumptions about the nuclear surface and the radial wave functions. For all these reasons, we cannot hope to check the agreement beyond mere orders of magnitude.

In a second-order calculation, the value of a would be the sum of the contributions from the four graphs of Fig. 1. It was shown in I that graphs (1a) and (1b) give a negligible contribution.* From Table I, it appears that the contribution

Table I. The value of the spin-orbit constant contribution $a^{(c)}$.

Tensor force	k_0 10^{13} cm^{-3}	Mev $a^{(c)}$ (10^{-13} cm) ⁵
Gammel-Thaler	1.27	30
	1.05	37
	0.87	41
2nd-order meson theory	1.27	50
	0.9	51

* The contributions from graphs (1a) and (1b) are actually somewhat dependent on the detailed shape of the tensor potential. The contribution from graph (1a), however, is an integral over a function with a change in sign. The quantity Φ which in Ref. 7) is similar to the function $C(K, k)$ in formula (19) of the present paper, changes its sign as K increases, and $a^{(a)}$ may become vanishingly small by a slight adjustment of the tensor force. On the contrary, $C(K, k)$ in $a^{(c)}$ has a fixed sign and $a^{(c)}$ is much less sensitive to the details of the tensor force. The contribution from (1b) involves an exchange and is usually smaller than one from (1a).

of graph (1c) is *not* negligible and could be of the same order of magnitude (with the proper sign) as the "experimental" value 57. However, we have not computed the contribution of graph (1d), because it would be an extremely tedious calculation.* Our main point is that there is at least one graph, (1c), the contribution of which can explain an appreciable part of the experimental spin-orbit splitting.

§ 5. The case of a hole in the closed shells

In the case of a hole in the closed shells, the second order graphs are represented on Fig. 3. These graphs are obtained from the corresponding ones in the

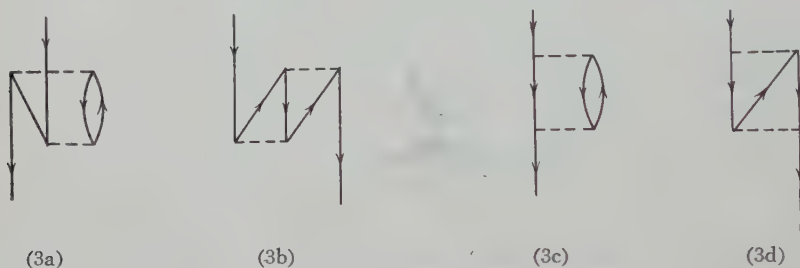


Fig. 3. The second-order graphs for hole.

particle case (Fig. 1) by replacing the incoming and outgoing particle lines by hole lines. For the bound states which are considered here, the particle is just above and the hole is just under the Fermi surface; in the limiting case of a large nucleus, the spacing between single particle levels goes to zero, and the particle and hole states become identical. It is therefore easy to see that the hole graphs of Fig. 3 give contributions which are just the opposite to those ones from the corresponding particle graphs of Fig. 1. The spin-orbit potential for one hole would therefore be $-a(1/r)(d\rho/dr)(\sigma \cdot \mathbf{l})$, with the same value of a as for one particle. The important contributions for a hole would now come from graphs (3c) and perhaps (3d), which are of the "reaction matrix" type.

The agreement with experiment is of the same kind as in the particle case. The experimental analysis which was quoted in § 4 involved also hole doublet splittings in Pb.²⁰⁷

§ 6. Conclusion

We can summarize the above discussion in the following way: The evaluation of the second order effect of the tensor forces, for the case of one particle outside the closed shells, first involves terms of the "reaction matrix" type, which describe collisions between the outside particle and one of the particles in the closed shells;

* This calculation could be very similar to the calculations of a_1 (graph (1b)) in I, and would be possible, in principle, for the case of a tensor force $r^2 \exp(-r^2/r_0^2)$.

these terms provide no appreciable spin-orbit effect. But there are also terms describing additional exclusion effects, which might be forgotten in a too naive extension of the Brueckner method to non-degenerate systems; those terms can explain, at least in part, the experimental spin-orbit splitting.

Those important exclusion effects may be described as the interdiction, for two nucleons of the closed shells, to jump by mutual excitation into an outer orbit which would already be occupied by another nucleon. This effect was here considered in the case of heavy nuclei. Similar calculations were carried out in the case of light nuclei. In that latter case, the important effective three-body vector forces which were derived by Feingold⁵⁾ from the tensor forces, or the exclusion effect of Terasawa⁶⁾ are of the same kind as the exclusion effect which was just described here. The smallness of the "reaction matrix" type graphs was also shown in light nuclei,^{5,6)} and even at all orders in heavy nuclei.¹⁰⁾

In the case of a hole in the closed shells, however, the important effect is of the "hole reaction matrix" type: the relevant intermediate states are those ones in which the hole is excited to some state of lower energy.

Turning again to the particle case, let us finally note that these exclusion effects (1c) and (1d) which appear to be important in the case of a bound outside particle would become negligible in the scattering problem for a particle of high energy, because the wave function of the closed shells would have little Fourier components in the state of the incident particle. Therefore, the exclusion mechanism, which provides a spin-orbit coupling for bound states, cannot account for the polarization of the high energy scattered nucleons. On the other hand, at high energy, the impulse approximation should be valid, and the nucleon-nucleus scattering problem is closely related to the two-body problem; the important effects therefore are of the "reaction", or rather "scattering matrix" type (Fig. 1a and 1b).^{*} If the low energy spin-orbit coupling is actually the result of exclusion type effects, the low and high energy spin-orbit couplings would result from quite different mechanisms, a result which is perhaps a little surprising and even unsatisfactory.

The author is indebted to the Institute of Nuclear Study for its kind hospitality, to many of his colleagues in this Institute and in the Research Institute for Fundamental Physics in Kyoto for stimulating discussion (the importance of exchange effects was emphasized by S. Takagi at a seminar held in March 1959 at the R. I. F. P), and to Miss Adachi for her help in numerical calculations.

^{*} At this two-body stage, there is still to be found the proper nucleon-nucleon potential. The problem of determining if this potential includes ^{1,2)} or not ³⁾ an elementary spin-orbit force does not seem to be uniquely set up at the present time.

Appendix

$C(K, k)$ is defined by

$$2\pi^2 k_0 C(K, k) = \int_{|\mathbf{K}-\mathbf{k}| < k_0} d\Omega_K \frac{\mathbf{k} \cdot \mathbf{K}}{kK} \int_{\substack{|\mathbf{K}_s + \mathbf{K}| < k_0 \\ k_s > k_0}} \frac{d^3 \mathbf{k}_s}{(\mathbf{k} - \mathbf{k}_s) \cdot \mathbf{K} - K^2}, \quad (20)$$

where $\int d\Omega_K$ denotes integration on \mathbf{K} angles. We consider only the case $k = k_0$. C is then found to be

$$\begin{aligned} C(K, k) = & (1/48) (K/k_0)^3 - (1/16) (K/k_0)^2 - (1/2) (K/k_0) + (13/12) \\ & + [(1/64) (K/k_0)^3 - (1/8) (K/k_0) + (1/4) (k_0/K)] \log \left| \frac{2k_0 + K}{2k_0 - K} \right| \\ & + [-(1/12) (K/k_0)^3 + (K/k_0) - (4/3)] \log 2. \end{aligned} \quad (21)$$

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Photodisintegration of the Alpha-Particle

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July 25, 1959

The study of the photodisintegration of the alpha-particle is very important in three respects. First, it gives information about the "rigidity" of the alpha-particle. This information would provide a significant clue to the knowledge in the nuclear structure, in the alpha-decay or in the mechanism of nuclear reactions induced by the alpha-particle and of the inverse processes of them. Secondly, this reaction plays an important role in thermo-nuclear reactions. Lastly, this reaction is very interesting in view of the consistency problem of the light nuclei. This point requires some discussion.

Many authors have made their efforts to interpret consistently many experimental data, the phase shifts of the scatterings, the binding energies and so on, of light nuclei. Recently, Bransden et al.¹⁾ studied the photodisintegration of the alpha-particle. They found that using the wave function consistent with the binding energy of He^4 , one gets the maximum (γ, p) cross section

at the energy which is too high (20~30 Mev above the threshold). On the other hand, Rustigi and Levinger²⁾ showed that using the same wave function one gets too small a value of the bremsstrahlung-weighted cross section. Thus, the consistency problem still remains and one could not find the wave function which is consistent both with the correct binding energy of He^4 and with the (γ, p) cross-section.

It was suggested by the above authors¹⁾²⁾ that to get the correct cross sections we may use the wave function, which gives the root mean square radius R found by the electron scattering.³⁾ Indeed, the wave function used in the above calculations gives a lower value of R .

In the previous papers,⁴⁾⁵⁾ we found that the wave function determined to meet the requirement of reproducing the right mean square radius of He^4 is actually consistent with the binding energy of it, if we take the interaction given by meson theory and at the same time introduce the effect of the hard core correlation. This wave function can give the wide splitting of p -phases of He^5 and can reproduce the elastic cross section of the high energy scattering of proton by He^4 . Then we may hope our wave function will give a good answer to the photodisintegration.

However, it is made clear that if we calculate the (γ, p) cross section using our wave function, neglecting the final state interaction, the maximum cross section appears still at a too high energy (18~19 Mev above the threshold). (It is clear that the D -state in He^4 and the hard core correlation is unimportant in the electric dipole matrix elements, so we consider only the S -part, neglecting the hard core correlation. The wave function is assumed to be the Gaussian type both for H^3 and He^4 , as before :

$$\left. \begin{aligned} \psi_{\text{H}^3} &= \exp \left\{ - (1/2) \mu \sum_{i>j} r_{ij}^2 \right\}, \\ \psi_{\text{He}^4} &= \exp \left\{ - (1/2) \alpha \sum_{i>j} r_{ij}^2 \right\}, \\ \alpha &= 0.14 \times 10^{26} \text{cm}^{-2}. \end{aligned} \right\} \quad (1)$$

In this case the position of the maximum (γ, p) cross section is independent of the assumed value of μ , as Flowers and Mandl⁶⁾ showed.)

It is well known that the elastic scattering of proton by H^3 and $T(p, n)\text{He}^3$ shows the resonance at an energy about 3 Mev.^{7,8)} This corresponds to the existence of the maximum in the (γ, p) cross section. Then, it is worth-while to examine whether the outgoing proton shows the large amplitude near the corresponding energy. If so, the (γ, p) cross section shows the maximum at the energy.

In our calculation, it is found that the (γ, p) cross section is reproduced well, if we assume that the outgoing particle is subject to the square well potential in the final state. This potential is adjusted so as to produce the p -wave resonance of the outgoing particle at the maximum cross section. The

adopted values of parameters are $V_0 = 50$ Mev, $R = 2.0 \times 10^{-13} \text{cm}$ and $\mu = 0.15 \times 10^{26} \text{cm}^{-2}$. The (γ, p) cross section is insensitive to μ . The form of the wave functions and the value of α are the same as (1). The Coulomb effect is neglected. The result is shown in Fig. 1.

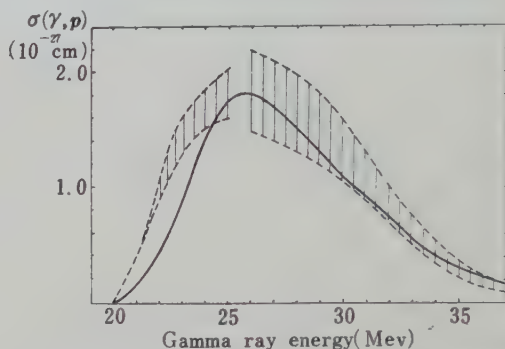


Fig. 1 $\text{He}^4(\gamma, p)\text{H}^3$ cross section; The shaded region shows the experimental value.⁷⁾ The solid line is the calculated value.

It is concluded that (1) the alpha particle has an excited state (1P_1 -state) at the energy of about 26 Mev above the ground state and (2) it seems to be hopeful to solve the consistency problem in view of our previous successes^{4,5)} and the present result. More detailed account will appear shortly.

The author wishes to thank Professor M. Kobayasi for his keen interest in the present work.

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Pairing Energy for Deformed Nuclei

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An attempt is made to estimate the pairing energy for strongly deformed nuclei by dividing the energy into two parts, one of which is the interaction energy between the nucleons forming the last pair, while the other part arises from the deformation of the nucleus. Assuming an attractive inter-nucleon force with a short range, the interaction energy in the last pair is first estimated on the Nilsson model, and is then estimated in the case where the deformation is extremely strong. The other part is estimated on the unified model. The calculation is carried out for the last neutron pairs of the five nuclei Gd^{158} , Dy^{164} , Yb^{174} , Hf^{178} and Hf^{180} in the rare earth region. The results obtained by using the Nilsson model as well as the unified model are in fairly good agreement with the experimental data.

§ 1. Introduction

The pairing energies for several deformed heavy nuclei have been observed mass-spectroscopically with sufficient accuracy by Johnson and Bhanot,¹⁾ and have been estimated to be close to 1.5 Mev for each element. Another observation for the pairing energies of such nuclei made by Johnson and Nier²⁾ gave also results of the same order of magnitude. Though it has been generally recognized³⁾ that the pairing energy for deformed heavy nuclei has the value of this order, no satisfactory explanation of such a pairing energy has been presented. The purpose of this paper is to give an explanation of the above experimental values.

The pairing energy E_p for a deformed nucleus X_N of neutron (or proton) number N , N being an even number, can be expressed by

$$E_p = 2E_{N-1}(\delta_1) - E_N(\delta_0) - E_{N-2}(\delta_2), \quad (1.1)$$

where δ are the deformation parameters defined by Nilsson,⁴⁾ and $E_{N-i}(\delta_i)$ are energies of the nuclei X_{N-i} at their equilibrium deformations δ_i . If the energy of $(N-1)$ st neutron (or proton) in the nucleus X_{N-1} is $E_1(\delta_1)$ when the deformation of this nucleus is δ_1 , the expression (1.1) becomes

$$E_p = [2E_{N-2}(\delta_1) - E_{N-2}(\delta_0) - E_{N-2}(\delta_2) + 2\{E_1(\delta_1) - E_1(\delta_0)\}] + E_{12}(\delta_0). \quad (1.2)$$

Here E_{12} is the interaction energy in the last neutron (or proton) pair of the nucleus X_N , and it is called the nucleonic part in this paper. The first term in Eq. (1.2), namely, the term enclosed by square brackets is a contribution from

the deformation energies, which arise from differences in deformations of adjacent nuclei in the rare earth region.

Elliott⁵⁾ and Moszkowski⁶⁾ showed that the nucleonic correlation generates collective effects. Therefore, the interaction between the nucleons forming the last pair of the nucleus X_N mentioned above may have an effect on the equilibrium deformation δ_0 . So the first term in Eq. (1.2) is never independent of the interaction between the nucleons of the last pair. However, in the region of large deformation such as the equilibrium deformations of rare earth nuclei to be considered here, the effect of this interaction upon such a term is small, as shown in the following paragraphs.

From Eq. (1.2), for nuclei belonging to the spherical (equilibrium) shape region, $E_{12}(\delta_0)$ (where $\delta_0=0$) is equal to E_p itself. In fact, the pairing energies for light and intermediate weight nuclei have been successfully explained^{7), 8)} on the shell model as the interaction energy in the last pair. Therefore, for a deformed heavy nucleus, the interaction energy in the last pair also might have a considerable magnitude. Bohr, Mottelson and Pines⁹⁾ suggested recently that for such a problem an analogy with the correlation between electrons on the Fermi surface in a super-conducting metal may be used. However, the correlation we shall adopt in this paper is a type which is usually used for the nucleons, no use being made of such an analogy.

When the nucleus begins to deform from the spherical shape, the interaction energy in the pair varies rapidly owing to the removal of the Ω -degeneracy.¹⁰⁾ If the deformation increases, the mixing of many different j -states begins. Such an interaction energy might come to differ enough from that of the spherical nucleus. Therefore, it will be of interest to see the magnitude of the interaction energy when the nucleus is strongly deformed, assuming the correlation by which the pairing energy for the spherical nucleus is successfully explained.

For the explanation of the character of the deformed odd nuclei in the ground state and the low excited states, the Nilsson model⁴⁾ is extremely successful.¹¹⁾ By using, therefore, the Nilsson wave function⁴⁾ for the nucleon, the interaction energy in the pair (i.e. the nucleonic part of pairing energy) will be estimated in Part A of § 2.

In order to see the magnitude of the interaction energy at stronger deformations than the ones in Nilsson's calculation,⁴⁾ we shall consider (as zero approximation) a pure anisotropic harmonic oscillator potential for the nuclear potential and treat $\mathbf{l} \cdot \mathbf{s}$ - and l^2 -terms, which are included in the Nilsson Hamiltonian,⁴⁾ as perturbation. In Part B of § 2, the calculation in such a case will be carried out, and then some interpretation of the results obtained in Part A will be given in the light of the finding in this part.

In the calculation in § 2, we shall assume the nucleonic interaction to be an attractive force with short range because of the following reason: The size of the nucleus considered here is sufficiently larger than the range of nuclear force.

In fact, this approximation for the nuclear force has made a good success for the level scheme of Pb^{206} ,¹²⁾ the mass of which is close to those of the nuclei considered here.

In § 3 an estimation will be made of the part of pairing energy arising from deformation (i.e. the first term in Eq. (1.2)) on the unified model.¹³⁾ The result will be compared with the nucleonic part and the observed values. In § 4, the results obtained in §§ 2 and 3 will be discussed.

§ 2. Nucleonic part of the pairing energy

A. Estimation on the Nilsson model

The nucleonic levels in a non-spherical potential with an axial symmetry are characterized only by the quantity \mathcal{Q} which is the component of angular momentum of the nucleon along the symmetry axis and they are doubly degenerate according to the values $\pm\mathcal{Q}$. The pair is then made of two identical nucleons, with $+\mathcal{Q}$ and $-\mathcal{Q}$, occupying the same level.

The energy of the pair, suffixes 1 and 2 each standing for one particle of the pair, are given as follows:

$$E_1 + E_2 = T_1 + T_2 + V_{12} + \sum_{i=1,2} \sum_{k \substack{(\pm 1, 2)}} V_{ik} = H_1 + H_2 + V_{12}, \quad (2.1)$$

where T represents the kinetic energy of the particle, and V_{ik} represents the interaction potential between the i th particle and the k th particle. H_1 or H_2 in Eq. (2.1) is the Hamiltonian of one nucleon of the pair in a case where the other is removed. Because of the success of the Nilsson model⁴⁾ for an explanation of the levels and behaviour of the last nucleon of odd- A nucleus, we assume here the Nilsson wave function to be the eigen solution for H_1 or H_2 .

Considering the Pauli principle, the interaction energy E_{12} in the pair is given by

$$E_{12} = I_{12} - K_{12},$$

$$I_{12} = \int \phi_1^*(x) \phi_2^*(x') V_{12}(x, x') \phi_1(x) \phi_2(x') dx dx', \quad (2.2)$$

K_{12} is an exchange integral,

where ϕ is the wave function of the particle, and we adopt here the Nilsson wave function⁴⁾; x or x' stands for the coordinates of this particle, including the spin coordinate. For the reason that the nucleonic state is characterized only by \mathcal{Q} , the wave function ϕ can be expressed as follows:

$$\phi(x) = \sum_j a_{j\Omega} \phi(j\mathcal{Q}), \quad (2.3)$$

where \sum denotes a summation over the whole of possible values of j in the shell with a definite total quantum number \bar{N} , $a_{j\Omega}$ are constants which give the mixing rates, and all the functions ϕ have the same parity.

Substituting (2.3) into (2.2), and changing the representation to that of the resultant j and m by means of

$$\phi(j_1\Omega)\phi(j_2-\Omega) = \sum_{j'm} (j_1j_2\Omega-\Omega/j_1j_2jm)\phi(j_1j_2, jm), \quad (2.4)$$

the interaction energy E_{12} becomes

$$\begin{aligned} E_{12} = & \sum_{j_1j_2j_1'j_2'} a_{j_1\Omega}^* a_{j_2-\Omega}^* a_{j_1'\Omega} a_{j_2'-\Omega} \sum_{j'mj'm'} (j_1j_2\Omega-\Omega/j_1j_2jm) \\ & \times (j_1'j_2'\Omega-\Omega/j_1'j_2'j'm') [\langle j_1j_2jm | V_{12} | j_1'j_2'j'm' \rangle \\ & - (-1)^{j_1'+j_2-j'} \langle j_1j_2jm | V_{12} | j_2'j_1'j'm' \rangle], \end{aligned} \quad (2.5)$$

where $(j_1j_2\Omega-\Omega/j_1j_2jm)$ are the Clebsch-Gordan coefficients.

For the nucleonic interaction we assume, following de-Shalit,¹⁴⁾ a short range attractive potential with an arbitrary mixture of ordinary and spin-exchange forces as follows:

$$V_{12} = - \sum_{n=0,1} \sum_{\kappa=0}^n f^{(n)} \sigma_{\kappa}^{(n)}(1) \sigma_{\kappa}^{(n)}(2) \delta(\mathbf{r}_1 - \mathbf{r}_2) \equiv - \sum_{n=0,1} V_{12}^{(n)}, \quad (2.6)$$

$$\sigma_0^{(0)} = 1, \quad \sigma_1^{(1)} = -\frac{1}{\sqrt{2}}(\sigma_x + i\sigma_y), \quad \sigma_0^{(1)} = \sigma_z, \quad \sigma_{-1}^{(1)} = \frac{1}{\sqrt{2}}(\sigma_x - i\sigma_y),$$

where $f^{(n)}$ represents the force strength.

Substituting (2.6) into (2.5), and making use of the Racah technique,¹⁵⁾ the matrix elements in (2.5) can be easily estimated in a quite analogous way to the calculation of the interaction energy between two particles with a definite j -value (see, for example, reference 14)). Thus, remembering that all the orbital angular momenta being related to one Nilsson level have the same parity, the matrix element is estimated as follows:

$$\begin{aligned} \langle j_1j_2jm | V_{12}^{(n)} | j_1'j_2'j'm' \rangle = & \delta_{jj'} \delta_{mm'} \frac{[(2j_1+1)(2j_2+1)(2j_1'+1)(2j_2'+1)]^{1/2}}{2(2j+1)} \\ & \times F_0^{(n)} \left[\left(j_1j_2 \frac{1}{2} \frac{1}{2} / j_1j_2j1 \right) \left(j_1'j_2' \frac{1}{2} \frac{1}{2} / j_1'j_2'j'1 \right) \right. \\ & \left. + \varepsilon_j^{(n)} (-1)^{j_2'-j_2} \left(j_1j_2 \frac{1}{2} \frac{-1}{2} / j_1j_2j0 \right) \left(j_1'j_2' \frac{1}{2} \frac{-1}{2} / j_1'j_2'j'0 \right) \right], \end{aligned} \quad (2.7)$$

$$F_0^{(n)} = \frac{f^{(n)}}{4\pi} \int R(n_1l_1, r) R(n_2l_2, r) R(n_1'l_1', r) R(n_2'l_2', r) r^2 dr, \quad \begin{aligned} \varepsilon_j^{(0)} &= 1 \text{ for all } j, \\ \varepsilon_j^{(1)} &= 2(-1)^{1+j} - 1, \end{aligned}$$

where n_l and l_l , etc., are the radial quantum number and the orbital angular momentum quantum number, respectively, corresponding to j_l , etc.; $R(nl, r)$ is the radial wave function of a nucleon lying in the state characterized by the quantum numbers n and l . By using (2.7), (2.5) becomes

$$E_{12} = \sum_{j_1j_2j_1'j_2'} a_{j_1\Omega}^* a_{j_2-\Omega}^* a_{j_1'\Omega} a_{j_2'-\Omega} \sum_j \frac{[(2j_1+1)(2j_2+1)(2j_1'+1)(2j_2'+1)]^{1/2}}{2(2j+1)}$$

$$\begin{aligned} & \times (j_1 j_2 \Omega - \Omega / j_1 j_2 j 0) (j_1' j_2' \Omega - \Omega / j_1' j_2' j 0) \left(j_1 j_2 \frac{1}{2} \frac{-1}{2} / j_1 j_2 j 0 \right) \\ & \times \left(j_1' j_2' \frac{1}{2} \frac{-1}{2} / j_1' j_2' j 0 \right) (-1)^{j_2' - j_2} [1 + (-1)^j] \sum_{n=0,1} \varepsilon_0^{(n)} F_0^{(n)}. \quad (2.8) \end{aligned}$$

The expression (2.8) shows that the potential of type (2.6) between two identical particles yields an interaction only if they are in the singlet state, which is familiar result of the Pauli principle.

We apply (2.8) to the last neutron pairs of the five nuclei, Gd¹⁵⁸, Dy¹⁶⁴, Yb¹⁷⁴, Hf¹⁷⁸ and Hf¹⁸⁰, in the rare earth region, all of which have a deformation close to $\delta=0.3$. The configurations of the levels occupied by these pairs can be found from the Nilsson model⁴⁾ as seen in Table I. They agree almost completely with those provided from the spins and parity of the odd-A nuclei that result by taking out one neutron from the above nuclei.

Transferring the representation from that of Nilsson⁴⁾ to the $Nj\Omega$ -representation, the mixing rate $a_{j\Omega}$ can be estimated from the Nilsson wave function. The

Table I. Mixing rates of the wave functions at the three different deformations. The numbers in the first column denote the level number assigned by Nilsson. These levels are occupied by the last neutron pair of the elements Gd¹⁵⁸, Dy¹⁶⁴, Yb¹⁷⁴, Hf¹⁷⁸ and Hf¹⁸⁰, and are given below from top to bottom.

Level	Configuration				$a_{j\Omega}$		
	\bar{N}	Ω	Parity	j	$\eta=2$	4	6
(52)	5	3/2	-	11/2	-0.094	-0.240	-0.335
				9/2	-0.772	-0.569	-0.505
				7/2	0.604	0.742	0.727
				5/2	-0.124	-0.050	0.002
				3/2	0.127	0.258	0.324
(44)	5	5/2	-	11/2	-0.127	-0.193	-0.246
				9/2	0.793	0.878	0.888
				7/2	0.571	0.365	0.277
				5/2	0.172	0.240	0.273
(50)	5	5/2	-	11/2	-0.091	-0.188	-0.250
				9/2	-0.593	-0.426	-0.376
				7/2	0.800	0.883	0.887
				5/2	0.001	0.062	0.101
(41)	5	7/2	-	11/2	-0.082	-0.134	-0.174
				9/2	0.927	0.959	0.963
				7/2	0.367	0.251	0.206
(49)	6	9/2	+	13/2	0.997	0.990	0.982
				11/2	0.043	0.082	0.114
				9/2	0.072	0.119	0.151

calculated $a_{j\Omega}$ for each level at three different deformations $\gamma=2, 4, 6$ are listed in Table I, where γ is defined by the following relationship,

$$\gamma = \frac{\delta}{0.05} \left[1 - \frac{4}{3} \delta^2 - \frac{16}{27} \delta^3 \right]^{-1/6}. \quad (2.9)$$

The quantity $F_0^{(n)}$ is calculated with the use of the harmonic oscillator radial wave function $R(n l, r)$, and the results are listed in Table II.

Table II Values of $F_0^{(n)}$ in a unit of $f^{(n)}\alpha^3/2\pi\sqrt{2\pi}$.

States: ($n_1 l_1$) ($n_2 l_2$) ($n_1' l_1'$) ($n_2' l_2'$)	$F_0^{(n)}$
(1h) ⁴	0.124
(1h) ³ (2f)	-0.042
(1h) ³ (3p)	0.0077
(1h) ² (2f) ²	0.067
(1h) ² (2f) (3p)	-0.053
(1h) ² (3p) ²	0.057
(1h) (2f) ³	-0.0078
(1h) (2f) ² (3p)	0.0167
(1h) (2f) (3p) ²	-0.0098
(1h) (3p) ³	0.0089
(2f) ⁴	0.136
(2f) ³ (3p)	-0.057
(2f) ² (3p) ²	0.084
(2f) (3p) ³	-0.0117
(3p) ⁴	0.228
(1i) ⁴	0.106
(1i) ³ (2g)	-0.032
(1i) ² (2g) ²	0.056

Since the potential (2.6) yields the interaction only if the pair occupies a singlet state, E_{12} depends rather on the force strength of singlet state $f_s (= f^{(0)} - 3f^{(1)})$ than on each of $f^{(0)}$ and $f^{(1)}$. The numerical value of f_s may be estimated from the observed value¹⁾ for the pairing energy of the last neutron pair of Pb²⁰⁸, because this nucleus is expected to have a spherical shape and then the [pairing energy may depend only on the interaction between the nucleons forming the last pair. The shell model is applicable to this nucleus and then E_{12} has the following form,⁸⁾ when the interaction potential has the type (2.6) and the pair is on a definite j -level

$$E_{12} = \frac{2j+1}{8\pi} f_s \int R^4(nl, r) r^2 dr. \quad (2.10)$$

The last neutron pair of Pb²⁰⁸ is in the state characterized by $j=1/2$, $n=3$ and $l=1$. So, by using the harmonic oscillator radial wave function as $R(n l, r)$, E_{12} of (2.10) becomes

$$E_{12} = 0.228 \frac{f_s \alpha^3}{2\pi\sqrt{2\pi}}, \quad (2.11)$$

where the quantity α stands for $(M\omega_0/\hbar)^{1/2}$ with the nucleon mass M and the characteristic frequency ω_0 defined by Nilsson.⁴⁾ By taking the mean value of r^2 for the pair to be equal to 3/5 of square of the radius of the nucleus considered here, we get the expression $\alpha^3 = (N+3/2)^{3/2} c/A$ with a constant c . By taking E_{12} of (2.11) to be equal to the observed value 0.63 Mev of pairing energy for Pb²⁰⁸, we have

$$f_s c = 546 \text{ Mev}. \quad (2.12)$$

This value for f_{sc} is larger by about $3/2^*$ than the average value obtained from the analysis⁸⁾ of the pairing energies for light nuclei, but this is rather reasonable because such analysis for light nuclei has shown that the value for f_{sc} increases gradually with A (see reference 8)).

By using the value (2.12) for the force strength, and making use of the values listed in Tables I and II for $a_{j\Omega}$ and $F_0^{(n)}$, respectively, E_{12} for each of the nuclei which are considered here is estimated from Eq. (2.8), and the values calculated at $\gamma=2, 4, 6$ are listed in Table III and are graphically shown in Fig. 1. As seen from Table III and Fig. 1, for all the elements, the interaction energy E_{12} at $\gamma=6$, which is close to the deformation expected for these elements, is close to 0.5 or 0.6 Mev.

Table III. Pairing energies of the last neutron pairs of the nuclei in the rare earth region. E_{obs} denotes the observed value, E_{12} and E_{def} denote the nucleonic part and the part arising from deformation respectively.

Nucleus	Ω and Parity of the level	E_{obs} (Mev)	E_{12} (Mev)			E_{def} (Mev)	$E_{\text{obs}} - E_{12}$ at $\gamma=6$ (Mev)
			$\gamma=2$	4	6		
Gd ¹⁵⁸	3/2-	1.57	0.436	0.446	0.475	1.49	1.10
Dy ¹⁶⁴	5/2-	1.37	0.361	0.475	0.491	0.83	0.88
Yb ¹⁷⁴	5/2-	1.13	0.431	0.496	0.517	1.40	0.61
Hf ¹⁷⁸	7/2-	1.29	0.500	0.537	0.546	1.47	0.74
Hf ¹⁸⁰	9/2+	1.15	0.600	0.588	0.577	0.79	0.57

B. Estimation in the limit of very strong deformations

In a case with a very strong deformation, we can use as zero approximation a pure harmonic oscillator wave function for the nucleon, and can treat $l \cdot s$ - and l^2 -terms, which are included in the Nilsson Hamiltonian,⁴⁾ as perturbations. The unperturbed state is characterized by the quantum numbers n_z , n_p , Λ and Σ , where n_z is the number of oscillator quanta along the symmetry axis, $n_p = \bar{N} - n_z$, Λ and Σ are the components of orbital and spin angular momenta along the symmetry axis, respectively. The quantity Λ has the same parity with n_p and is restricted by a condition $|\Lambda| \leq n_p$.

The first-order energy E of the level characterized by the above quantum numbers at the deformation δ is given by (see Appendix B of the Nilsson thesis⁴⁾)

$$E = E_0 + \bar{C}\Lambda\Sigma + \bar{D}(\Lambda^2 + 2n_p n_z + 2n_z + n_p),$$

$$E_0 = \hbar\omega(\delta) \left(\bar{N} + \frac{3}{2} + \varepsilon \frac{n_p - 2n_z}{3} \right), \quad (2.13)$$

* The value (2.12) for f_{sc} corresponds to $-V_s/8\pi r_0^3 = 21.5$ Mev with the notations in reference 8).

$$\omega(\delta) = \bar{\omega}_0 \left[1 + \frac{1}{9} \delta^2 + O(\delta^3) \right], \quad \dot{\epsilon} = \delta + \frac{1}{6} \delta^2 + O(\delta^3),$$

where \bar{C} and \bar{D} are the strengths of $\mathbf{l} \cdot \mathbf{s}$ - and l^2 -couplings, respectively. For the quantities \bar{C} and \bar{D} we assume also the relationships $C = -0.1 \hbar \bar{\omega}_0$ and $\bar{D} = 0.225 \bar{C}$ as on the Nilsson model.⁴⁾

From Fig. 4a and the rule mentioned in Chapter II of the Nilsson thesis,⁴⁾ we can easily assign the quantum numbers for the unperturbed levels corresponding to the levels considered in Part A, as seen in Table IV. Each level is doubly degenerate with (Λ, Σ) and $(-\Lambda, -\Sigma)$, and so we consider the two nucleons on the same level as a pair.

By using again the form (2.6) for the interaction potential, the zero-order interaction energy E_{12}^0 of the pair which is in a singlet state is given by

$$E_{12}^0 = \frac{f_s \alpha^3}{2\pi} \int R_{n_p \Lambda}^2 R_{n_p - \Lambda}^2 r dr \int Z_{n_z}^4 dz, \quad (2.14)$$

where $R_{n_p \Lambda}$ and Z_{n_z} are the cylindrical harmonic wave function and the one-dimensional harmonic wave function, respectively, the explicit expressions being given by Tamura.¹⁶⁾

By using again the value (2.12) for the force strength, E_{12}^0 for the pair on each level being considered here is estimated as seen in the third column of Table IV and is graphically shown in the column (b) in Fig. 1.

We now consider only the main contribution of $\mathbf{l} \cdot \mathbf{s}$ - and l^2 -terms to E_{12}^0 . Apart from diagonal elements, the non-vanishing matrix elements of $\mathbf{l} \cdot \mathbf{s}$ -term occur only between the states with Σ different by one unit. The first order correction to E_{12}^0 owing to the $\mathbf{l} \cdot \mathbf{s}$ -term may come from substituting the first-order correction into any one of the wave functions in the integral with the same form as (2.2). However, this correction to E_{12}^0 vanishes, since V_{12} yields to an interaction only for the pair in the singlet state. Thus the largest correction to E_{12}^0 coming from the $\mathbf{l} \cdot \mathbf{s}$ -coupling is that of the second order.

The off-diagonal and non-vanishing matrix elements of the l^2 -term occur only between the states with n_p or n_z different by two and with the equal $\bar{\Lambda}$ and Λ .

Table IV. Zero-order interaction energy E_{12}^0 in the limit of very strong deformation, and the correction ΔE_{12} for E_{12}^0 owing to the $\mathbf{l} \cdot \mathbf{s}$ -coupling. The signs \pm in the second column correspond to $\Sigma = \pm \hbar/2$.

Level	n_z	Configuration			E_{12}^0 (Mev)	ΔE_{12} at $\delta=0.5$ (Mev)
		n_p	Λ	Σ		
(52)	2	3	1	+	0.728	-0.117
(44)	2	3	3	-	0.702	-0.105
(50)	1	4	2	+	0.620	-0.064
(41)	1	4	4	-	0.662	-0.085
(49)	2	4	4	+	0.693	-0.086

The first-order correction to E_{12}^0 owing to the l^2 -term is obtained in a quite similar way to the above procedure for the $\mathbf{l} \cdot \mathbf{s}$ -term. In this case the first-order correction does not vanish, but it is very small due to the smallness of the overlapping of the wave functions. So the second-order correction is rather large.

Therefore, the second-order correction to E_{12}^0 , which comes from the $\mathbf{l} \cdot \mathbf{s}$ -term, is the largest one, since the strength of $\mathbf{l} \cdot \mathbf{s}$ -coupling is sufficiently larger than that of l^2 -coupling. For the off-diagonal and non-vanishing matrix elements of the quantity $\mathbf{l} \cdot \mathbf{s}$, we have

$$\begin{aligned} \langle n_z+1, n_p-1, A+1, -|\mathbf{l} \cdot \mathbf{s}|n_z, n_p, A, + \rangle &= -\frac{\hbar^2}{2} \sqrt{(n_p-A)(n_z+1)} \equiv M_-, \\ \langle n_z-1, n_p+1, A+1, -|\mathbf{l} \cdot \mathbf{s}|n_z, n_p, A, + \rangle &= -\frac{\hbar^2}{2} \sqrt{(n_p+A+2)n_z} \equiv M_+, \end{aligned} \quad (2.15)$$

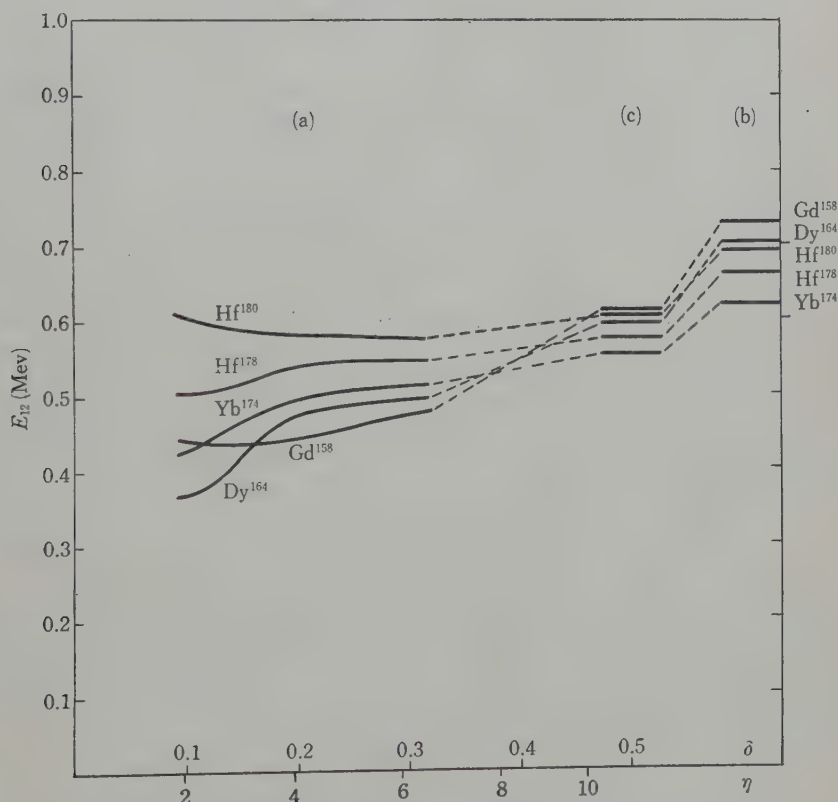


Fig. 1. Nucleonic part E_{12} of the pairing energy for the five nuclei Gd^{158} , Dy^{164} , Yb^{174} , Hf^{178} and Hf^{180} in the rare earth region against the deformation δ . Column (a) shows the values estimated on the Nilsson model. Column (b) shows the values in the case of the strongly deformed pure oscillator potential, and column (c) shows the values at $\delta=0.5$ obtained by regarding the $\mathbf{l} \cdot \mathbf{s}$ -coupling as a perturbation.

where the signs \pm in the basic vector expressions stand for $\Sigma = \pm \hbar/2$. Making use of the matrix elements (2.15), and substituting the perturbed wave functions to the second order for the functions R and Z in the integral (2.14), we have as the largest correction for E_{12}^0 , when the non-perturbed state has $\Sigma = \hbar/2$ and a positive A ,

$$\begin{aligned} \Delta E_{12} = & \frac{C\alpha^2}{\pi} \sum_{\pm} \left[\frac{CM_{\pm}}{\varepsilon \hbar \omega(\delta)} \right]^2 \left[\int R_{n_p, \pm 1, \Lambda+1} R_{n_p, \pm 1, \Lambda-1} R_{n_p, \Lambda} R_{n_p, -\Lambda} r dr \right. \\ & \times \left. \int Z_{n_z, \pm 1}^2 Z_{n_p}^2 dz - \int R_{n_p, \Lambda}^2 R_{n_p, -\Lambda}^2 r dr \int Z_{n_z}^4 dz \right], \end{aligned} \quad (2.16)$$

where \sum_{\pm} denotes a summation over the double sign \pm under the conditions $A \sim n_p$ and $n_p \leq N$. In a case where the non-perturbed state has $\Sigma = -\hbar/2$ and a positive A , we must make the following changes in M_{\pm} and the wave functions in Eq. (2.16):

$$A \rightarrow A-1, \quad n_p \rightarrow n_p \pm 1, \quad \text{and} \quad n_z \rightarrow n_z \mp 1,$$

where the double sign corresponds to that in (2.16) with the same order.

Making use the value (2.12) for the force strength and the preceding values for the strength of $\mathbf{l} \cdot \mathbf{s}$ -coupling, the correction ΔE_{12} at $\delta=0.5$ are estimated as seen in the 4th column of Table IV, and $E_{12}^0 + \Delta E_{12}$ are shown in the column (c) in Fig. 1. As seen from Fig. 1, the perturbation for the $\mathbf{l} \cdot \mathbf{s}$ -coupling makes E_{12} close to the exact value.

§ 3. Part of pairing energy arising from deformation

Since the nuclei in the rare earth region are expected to have a prolate shape, we shall restrict ourselves to the discussion on this shape. The energy E_{N-2} of the nucleus X_{N-2} with a deformation δ which is close to the equilibrium deformation δ_2 can be expressed as follows:

$$E_{N-2}(\delta) = E_{N-2}(\delta_2) + \frac{p}{2}(\delta - \delta_2)^2, \quad (3.1)$$

where p represents the rigidity of the nucleus under consideration. In the rare earth region the difference between the two equilibrium deformations δ_1 and δ_2 of the nuclei X_{N-1} and X_{N-2} , respectively, is small, so the energy $E_1(\delta_1)$ of the last nucleon of nucleus X_{N-1} can be expressed in the following form to the first order of $\delta_1 - \delta_2$, making use of A as the expansion coefficient,

$$E_1(\delta_1) = E_1(\delta_2) + A(\delta_1 - \delta_2). \quad (3.2)$$

Thus, from (3.1) and (3.2), the energy $E_{N-1}(\delta_1)$ of the nucleus X_{N-1} with the equilibrium deformation δ_1 is

$$\begin{aligned} E_{N-1}(\delta_1) &= E_{N-2}(\delta_1) + E_1(\delta_1) \\ &= E_{N-2}(\delta_2) + E_1(\delta_2) - (A^2/2p), \quad \delta_1 = \delta_2 - (A/p). \end{aligned} \quad (3.3)$$

Since the particles of the last neutron (or proton) pair of the nucleus X_N occupy the same level with the last single neutron (or proton) of the nucleus X_{N-1} , the energy $E_N(\delta_0)$ of the nucleus X_N with the equilibrium deformation δ_0 is

$$E_N(\delta_0) = E_{N-2}(\delta_0) + 2E_1(\delta_0) - E_{12}(\delta_0). \quad (3.4)$$

As seen in Fig. 1, the interaction energy E_{12} in the last pair varies little against δ in the region of such deformation as the equilibrium deformations of the rare earth nuclei. So, by regarding $E_{12}(\delta_0)$ as a constant, we can estimate, with fairly good approximation, δ_0 as $\delta_0 \approx \delta_2 - 2J/p$. Thus, from Eq. (3.1) and the expression obtained by replacing δ_1 in Eq. (3.2) by δ_0 , we have

$$E_N(\delta_0) = E_{N-2}(\delta_2) + 2E_1(\delta_2) - \frac{4J^2}{2p} - E_{12}(\delta_0). \quad (3.5)$$

From Eqs. (3.1), (3.3) and (3.5), we have the following expression for the part of pairing energy arising from deformation, which we denote by E_{def} ,

$$E_{def} = 2E_{N-2}(\delta_1) - E_{N-2}(\delta_0) - E_{N-2}(\delta_2) + 2\{E_1(\delta_1) - E_1(\delta_0)\} = J^2/p. \quad (3.6)$$

For each element considered in § 2, the value of J can be found from the tangent (at $\delta = \delta_2$) of the curve which shows the Nilsson energy level as a function of δ . The results found upon J are shown in Table V.

Table V. Gradient J of the five Nilsson levels against δ at $\delta = \delta_2$.

Level	δ_2^*	J in $\hbar\omega_0$ unit
(52)	0.39	1.53
(44)	0.34	1.15
(50)	0.27	1.52
(41)	0.27	1.54
(49)	0.26	1.14

* The values are computed from the data of Alder, Bohr, Huus, Mottelson and Winther (reference 17).

For the quantity p we have a relationship $p \approx 1.10C^*$ with the effective surface tension C ,⁽¹³⁾ following the unified model. The quantity C might depend on the detail of the nucleonic level structure, but the theoretical estimation for C is very difficult for a strongly deformed nucleus in which the mixing of many j -states occurs. In such nuclei the nucleonic level structure is far from that of the shell model, and the distribution of nucleonic states may approach rather to the statistical one. Thus we expect that the value of C is obtained with a fairly good

* In the paper⁽¹³⁾ by A. Bohr, the relationship $p \approx C$ is given, making use of $\beta (\approx 1.05 \delta)$ as the deformation parameter. The relationship in this paper is obtained by transforming the deformation parameter from β to δ .

approximation from the semi-empirical mass formula. Following the mass formula presented by Green,¹⁸⁾ which fits well with the experimental data in the rare earth region, we get

$$C = 5.72 A^{2/3} - 0.114 Z^2 A^{-1/3} \text{ Mev.} \quad (3.7)$$

By using (3.7) and the values listed in Table V for A , for each element mentioned in the preceding paragraph, E_{def} is calculated from (3.6), and the results are shown in the 7th column of Table III. Though the calculated E_{def} for each element (with the exception of Dy¹⁶⁴) is large in comparison with $E_{obs} - E_{12}$ listed in the 8th column of the same table, the agreement is fairly good considering that we have used C unrelated with the detailed structure of the nucleus.

§ 4. Conclusion

Assuming that the correlation of nucleons in a deformed nucleus is the same with that in a spherical nucleus with a similar mass number, the interaction energy in the last neutron pair at $\delta=0.3$ for all of the considered nuclei is close to 0.5 or 0.6 Mev, and depends slightly on the mass number and on the character of the level occupied by the pair.

These nuclei are expected to have a deformation close to $\delta=0.3$, and E_{12} varies little in the neighbourhood of $\delta=0.3$, so the values of E_{12} at the exact deformation of these nuclei are expected to be really close to the value at $\delta=0.3$. Thus, for any nuclei deformed strongly with the similar deformation we may generally expect E_{12} to be of the same order with the above value.

With the exception of Hf¹⁸⁰, E_{12} at the small deformation $\delta=0.1$ is smaller than that at $\delta=0.3$ (see Fig. 1 or Table III), for the reason that mixing rates of the main states for each level are rather small at $\delta=0.1$ as seen in Table I. However, if the deformation decreases more, the level will tend to a level characterized by a pure j -value and E_{12} of each element will increase with decreasing δ as seen in the case of Hf¹⁸⁰. In the limit of small deformation, the coupling of angular momenta of the nucleons forming the pair becomes strong, and E_{12} will tend to the value to be obtained by using the shell model. In the last case, if the particles forming the pair occupy a definite j -state, we may get $E_{12} \simeq 0.63 \times (2j+1)/2$ Mev, which is considerably larger than any values at $\delta = 0.1 \sim 0.3$ obtained in this paper.

If the nucleus has a pure harmonic oscillator potential deformed strongly without $\mathbf{l} \cdot \mathbf{s}$ - and \mathbf{l}^2 -terms, the interaction energy in the pair is close to 0.7 Mev. By using the perturbation method we can see that the $\mathbf{l} \cdot \mathbf{s}$ -coupling makes E_{12} reduce to the exact value.

We can see from the comparison between E_{def} and $E_{obs} - E_{12}$ listed in Table III, and from Table V, that the value (3.7) for C is good for the nucleus with stronger deformation ($\delta \gtrsim 0.3$), while it is too small for the nucleus with a smaller deformation ($\delta \lesssim 0.3$), which is a tendency we would expect. We can see also from the com-

parison between E_{12} , E_{def} and E_{obs} that for the rare earth nuclei with $\delta \simeq 0.3$ the part of pairing energy arising from deformation is 1~2 times as large as the nucleonic part.

Johnson and Bhanot¹⁾ found that the neutron pairing energy is large for the nuclei on both sides (close to $N=90$ and 116 with the neutron number N) of the region characterized by strongly deformed nuclei. For such nuclei, our results will be considerably modified. The modification is that: (1) The part arising from deformation is small, since C for such nuclei is large. (2) A shape oscillational mode becomes remarkable to some extent, introducing a finite probability that the nucleus has the extremely small deformation. Therefore, the average of the nucleonic part increases, because the magnitude of E_{12} at such a small deformation is considerably large. Thus it is expected that the pairing energy for the nuclei on both sides mentioned above increases finally owing to the modification (2), in spite of decreasing on account of (1).

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On the Possible Role of $\pi^{0'}$ -Meson in Decay Process

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The existence of the heavy neutral pseudoscalar meson with isotopic spin zero ($\pi^{0'}$) is assumed and its effects on the K-meson decay are discussed. In particular the K_e^+ experimental data are re-examined by assuming the K_{e3}^+ ($K^+ \rightarrow \pi^0 + e^+ + \nu$) and $K_{e3'}$ ($K^+ \rightarrow \pi^{0'} + e^+ + \nu$) modes. Analysis shows that the V-A Fermi interaction can give the energy spectrum of the electron consistent with the present experiments.

§ 1. Introduction

It has been suggested in our previous note that the V-A Fermi interaction is not favorable for explaining the present experimental electron energy spectrum of the K_e^+ decay process.^{1), 2), 3)} Although the experimental data so far accumulated are still scanty and to draw a definite conclusion will be too premature, the fact that the V-A theory is rather unprobable and that S-T combination seems to be required in the K_{e3}^+ -decay process is embarrassing, since except in the case of K_{e3}^+ there is no apparently negative evidence for the universal weak V-A Fermi theory.

In our previous analysis we have assumed only the K_{e3}^+ mode ($K^+ \rightarrow \pi^0 + e^+ + \nu$) and compared the theoretical energy spectrum of electron in K_{e3}^+ with the experimental data of K_e^+ . However, if other K-meson decay process having an electron as a decay product exists beside the mode K_{e3}^+ , the previous analysis will no longer be appropriate, and the suggestion that the V-A Fermi interaction is unlikely in the K_{e3}^+ process will have to be altered.

In this paper we shall reanalyse the K_e^+ spectrum by assuming the decay mode $K^+ \rightarrow \pi^{0'} + e^+ + \nu$. Here $\pi^{0'}$ is the heavy neutral meson with spin zero. The existence of such a neutral meson can be anticipated by Sakata's composite theory⁴⁾ for the elementary particles, in which the proton, neutron and lambda particles are taken as the fundamental particles. As will be seen in the following, the existence of $\pi^{0'}$ with $m_{\pi^{0'}} \sim 700 m_e$ does improve the situation of K_{e3}^+ and gives a solution consistent with K_e^+ data by V-A Fermi interaction and also introduces no difficulties for the interpretation of other decay processes. The decay of $\pi^{0'}$ which is interesting in the respect of observability is also discussed and it is suggested that its dominant decay mode will be two photon decay process.

§ 2. Effects of $\pi^{0'}$ on the K-meson decay processes

In this section we discuss the possible effect of the $\pi^{0'}$ on the K-meson decay

process. Ikeda, Ogawa, and Ohnuki have suggested that the $\pi^{0'}$ meson which is anticipated in Sakata's composite particle model would be a pseudoscalar particle with isotopic spin zero and even charge parity, assuming a symmetry among p , n and Λ .^{5)*} Due to the lack of theoretical information about the mass of $\pi^{0'}$, the value of $m_{\pi^{0'}}$ is rather arbitrary. We assume that it is much heavier than ordinary π^0 , since we are interested in such $\pi^{0'}$ which can affect the K_e^+ spectrum.

As to the mass of $\pi^{0'}$, if we take

$$m_{K^+} - m_{\pi^+} < m_{\pi^{0'}} < 3m_{\pi^+}, \quad (1)$$

the following mode will be forbidden energetically:

$$\begin{aligned} K^\pm &\rightarrow \pi^\pm + \pi^{0'}; \quad (K_{2\pi^{0'}}^\pm), \\ &\rightarrow \pi^\pm + \pi^0 + \pi^{0'}, \\ &\text{etc.,} \end{aligned} \quad (2)$$

and

$$\begin{aligned} \Lambda &\rightarrow n + \pi^{0'}, \\ \Sigma^+ &\rightarrow p + \pi^{0'}, \\ \Xi^0 &\rightarrow \Lambda + \pi^{0'}, \end{aligned} \quad (3)$$

while the allowed modes will be

$$\begin{aligned} K^\pm &\rightarrow \pi^{0'} + e^\pm + \nu; \quad (K_{e\nu}^\pm) \\ &\rightarrow \pi^{0'} + \mu^\pm + \nu; \quad (K_{\mu\nu}^\pm). \end{aligned} \quad (4)$$

If we assume a smaller mass for the $\pi^{0'}$ than (1), some of the processes of (2) and (3) become possible. However, since our main interests are in the $\pi^{0'}$ which can influence largely the electron spectrum of the K -meson decay, we discuss only the case of $\pi^{0'}$ with mass $\sim m_k - m_\pi$.

(i) K_e^+ -mode

First we discuss the case of K_{e3}^+ . Since in the $K_{\mu 3}^+$ the energy released is small, it will give relatively small contribution. The available maximum energy of the electron and muon in K_{L3}^+ are given in Table I.

Table I Available maximum kinetic energy of electron and muon in K_{L3}^+

Mass of $\pi^{0'}$ (Mev/c ²)	330	360	Ordinary K_{L3}^+
K_{e3}^+	136 (Mev)	115	228
$K_{\mu 3}^+$	42.4	21.4	134

* There appear two neutral mesons with isotopic spin 0 in the composite particle model, $\Pi^{0'} = \frac{1}{\sqrt{6}}(\bar{p}p + \bar{n}n - 2\bar{\Lambda}\Lambda)$ and $\Pi^{0''} = \frac{1}{\sqrt{3}}(\bar{p}p + \bar{n}n + \bar{\Lambda}\Lambda)$. Our $\pi^{0'}$ here considered could be either $\Pi^{0'}$ or $\Pi^{0''}$ if it is pseudoscalar. Another possible spin-parity assignments for the $\pi^{0'}$ meson which could give a consistent interpretation of K_{e3}^+ spectrum is 1^+ . In this case the most rapid decay mode of $\pi^{0'}$ would be $\pi^{0'} \rightarrow \pi^0 + \gamma$.

The present experimental distribution of the electron energy in K_e^+ shows the characteristic features: (a) the bimodal distribution and (b) the preponderance of the low energy electron. The experimental data⁶⁾ are given in Fig. 1 together with the theoretical curve of the combined spectrum of K_{e3}^+ and $K_{e3'}^+$.

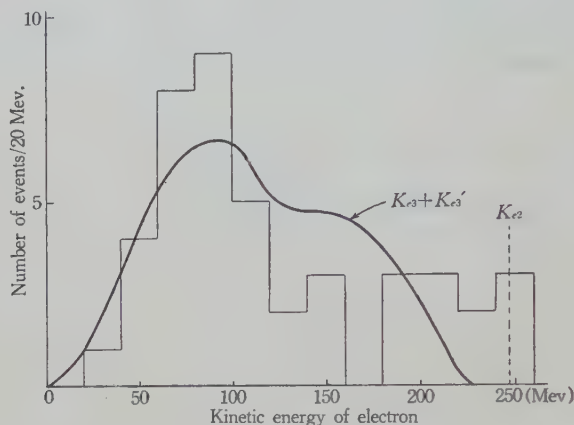


Fig. 1. Theoretical and experimental energy distribution of the electron in K_e^+ decay process.

The combined spectrum of K_{e3}^+ and $K_{e3'}^+$ will generally show the bimodal feature and produce the above experimental features. We have applied the χ^2 -test to the theoretical spectrum of V-A Fermi interaction when the effective coupling constants are approximated by the momentum independent constant.

Table II χ^2 -probability for the combined spectrum of K_{e3}^+ and $K_{e3'}^+$

F'^V/F^V^*	0	1	2	3	4
Mass of $\pi^{0'}$					
330 Mev/c ²	0.0004 (0)**	0.02 (0.05)	0.15 (0.20)	0.38 (0.45)	0.30 (0.80)
360 Mev/c ²		0.008 (0.02)	0.04 (0.08)	0.15 (0.18)	0.23 (0.33)

* F^V and F'^V are the effective coupling constants of the vector interaction in K_{e3}^+ and $K_{e3'}^+$ respectively.

** The figures in the brackets are the branching ratio $K_{e3'}/K_{e3}^+$.

As is seen in Table II, the combined spectrum of K_{e3}^+ and $K_{e3'}^+$ with $\pi^{0'}$ of mass 330 Mev/c² and the mixing ratio $F'^V/F^V=3$ gives a good fit of χ^2 -probability ~ 0.4 . In this case the high energy/low energy ratio of electron is 0.84 (the experimental data ~ 0.65). This is a fine improvement on the V-A theory if we recall the fact that the V-A Fermi interaction with K_{e3}^+ mode alone gives such poor correspondence with the experiment as the χ^2 -probability 0.0004, taking the

momentum independent effective coupling constant. (If we permit the very strong energy dependency for the effective coupling constant the fitness of the V-A theory with K_{e3}^+ mode alone can be improved only as far as 0.07 at most.)*

Here we note some differences between (a) the spectrum of the V-A theory with K_{e3}^+ and $K_{e3'}^+$ modes and (b) the spectrum of the S-T theory with K_{e2}^+ mode. The main difference is in the location of the second peak.

In (a) the second peak will be at <180 Mev, and in (b) it will be at >180 Mev. If the clear second peak appears in experiments and its position can be determined, it can be answered which of (a) or (b) is right.

The K_{e3}^0 spectrum will be much different from the K_e^+ spectrum, if the $K_{e3'}^+$ exists, since in the former case the decay mode corresponding to the $K_{e3'}^+$ will not appear.

(ii) Other modes

Next we shall discuss the effects of $\pi^{0'}$ on other decay modes of the K-meson. The $K_{\mu3'}$ mode is possible if $m_{\pi^{0'}} < m_K - m_\mu$ and contributes to the energy spectrum of the muon at low energy. The curve of $K_{\mu3}^+ + K_{\mu3'}^+$ for the $\pi^{0'}$ of mass $330 \text{ Mev}/c^2$ and the effective coupling constants $F'^V/F^V=3$ is given in Fig. 2. It fits well to the experimental data⁷⁾.

If $m_{\pi^{0'}} < m_K - m_{\pi^+}$, the process $K^+ \rightarrow \pi^+ + \pi^{0'}$ is energetically possible. This may be the contaminating process to the τ' if $m_{\pi^{0'}} > 270 \text{ Mev}/c^2$, and the π^+ energy spectrum has a peak at the corresponding energy of π^+ in $K^+ \rightarrow \pi^+ + \pi^{0'}$ (for example, the kinetic energy of π^+ is $16 \text{ Mev}/c^2$ for $330 \text{ Mev}/c^2$ $\pi^{0'}$) if the $K_{2\pi'}$ process is not greatly suppressed. The branching ratio of the $K_{2\pi'}$ to $K_{2\pi}^+$ is (for the direct coupling without the derivatives)

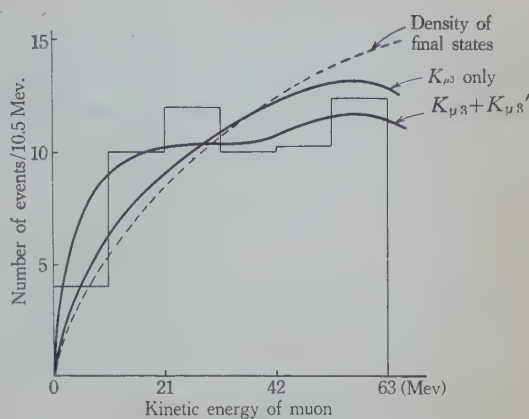


Fig. 2 Theoretical and experimental energy distribution of the muon in $K_{\mu3}^+$ decay process.

* One may have an impression that it is necessary to take a large F'/F in order to obtain good fit with the K_{e3} experiment, and this might conflict with the symmetry consideration of Ikeda et al. and also with the information on the nuclear force at low energies. However, this will not be the case. Within the scope of the global symmetry of p , n and Λ (this symmetry will be violated by the electromagnetic interaction and Λ -nucleon mass difference in reality) $(F'/F)^2$ may be taken to be $8/3$ if we assume $\pi^{0'} = \Pi^{0'}$. This gives the χ^2 -probability of about 0.02 for the $\pi^{0'}$ of mass $360 \text{ Mev}/c^2$ which would be taken as a satisfactory improvement for the V-A theory if we consider the fact that F 's could have energy dependence. In this case $\pi^{0'}$ contribution to the nuclear force will probably not produce any serious problem. We thank Dr. W. Watari for his kind information of this point.

$$K_{2\pi'}^+/K_{2\pi}^+ = g'^2 p'/g^2 p, \quad (5)$$

where $g(g')$ is the effective coupling constant of $K_{2\pi}^+(K_{2\pi'}^+)$ and $p(p')$ is the $\pi^+(\pi')$ momentum. The present experimental data on the τ' -energy spectrum show that the $K_{2\pi'}$ mode is greatly suppressed compared with the $K_{2\pi}$ mode or the mass of $\pi^{0'}$ $m_{\pi^{0'}} > m_K - m_{\pi}$.

§ 3. Some remarks on $\pi^{0'}$ -meson

Now we discuss the decay of the $\pi^{0'}$, which is closely connected with the observability of the $\pi^{0'}$. Since the strangeness of $\pi^{0'}$ meson is zero, it can decay into pions+photons very rapidly. However, the rapid decay of the following modes will not occur.

$$\pi^{0'} \rightarrow 2\pi, \quad (6a)$$

$$\rightarrow 3\pi, \quad (6b)$$

$$\rightarrow \pi^0 + \gamma, \quad (6c)$$

$$\rightarrow 2\pi^0 + \gamma. \quad (6d)$$

(6a) is forbidden by the pseudoscalar nature of $\pi^{0'}$, (6b) is energetically forbidden if $m_{\pi^{0'}} < 3m_{\pi}$,* (6c) is strictly forbidden, since it cannot conserve angular momentum, and (6d) is excluded by the charge conjugation invariance requirements. The allowed modes will be

$$\pi^{0'} \rightarrow \pi^+ + \pi^- + \gamma, \quad (7a)$$

$$\rightarrow 2\gamma. \quad (7b)$$

It is very interesting to study the branching ratio of (7a) and (7b). When the decay mode (7a) is overwhelming to (7b), the K -decay event seen as if the K -meson decays into $\pi^+ + \pi^- + e$ would be observed, since the $\pi^{0'}$ emitted in the $K_{e\pi'}$ process will decay into $\pi^+ + \pi^- + \gamma$ very rapidly. If this is the case, the analysis of the preceding section will no longer be held, since the K_e^+ data consists of the ones $K^+ \rightarrow e^+ + \pi^0 + \pi^0$. Such an event seen as if $K^+ \rightarrow e^+ + \pi^+ + \pi^-$ has not been reported so far, though it might be confused with the three charged pions decay of K -meson. We shall estimate the decay probability of (7a) and (7b) by taking the following matrix elements that satisfy the gauge invariance requirements,

$$\mathfrak{M}(\pi^{0'} \rightarrow \pi^+ + \pi^- + \gamma) = (2\pi)^4 \delta^4(P - p - q - k) e G^2 f M^{-3} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} p_{\rho} q_{\sigma} \quad (8a)$$

$$\mathfrak{M}(\pi^{0'} \rightarrow 2\gamma) = (2\pi)^4 \delta^4(P - k - k') e^2 f' M^{-1} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}' \quad (8b)$$

where e and G are the coupling constants of the electromagnetic interaction and the pion-nucleon interaction respectively. P_{μ} , p_{μ} and q_{μ} are the energy-momentum vector

* The rapid decay of the pseudoscalar $\pi^{0'}$ meson into three pions is forbidden by the isotopic and charge conjugation invariances.

of $\pi^{0'}$, π^+ and π^- and $F_{\mu\nu} = k_\mu e_\nu - k_\nu e_\mu$ where k_μ is the photon momentum and e_μ is the polarization vector. M is the quantity having the dimension of mass and is inserted to make f and f' dimensionless constants. The calculation gives the decay probability for these processes,

$$w(\pi^{0'} \rightarrow \pi^+ + \pi^- + \gamma) = \frac{e^2}{4\pi} \left(\frac{G^2}{4\pi} \right)^2 \frac{16}{3} f^2 \left(\frac{m_{\pi^{0'}}}{M} \right)^6 \int_{m_\pi}^{m_{\pi^{0'}/2}} dp_0 p^3 \left(\frac{m_{\pi^{0'}}}{2} - p_0 \right)^3 \times m_{\pi^{0'}}^{-2} (m_{\pi^{0'}}^2 + m_\pi^{-2} - 2m_{\pi^{0'}} p_0)^{-2}, \quad (9a)$$

$$w(\pi^{0'} \rightarrow 2\gamma) = \left(\frac{e^2}{4\pi} \right)^2 f'^2 4\pi m_{\pi^{0'}} \left(\frac{m_{\pi^{0'}}}{M} \right)^2. \quad (9b)$$

From (9a) and (9b) we obtain the following value for the branching ratio of these decay modes for the $\pi^{0'}$ with mass 360 Mev/c²,

$$\frac{w(\pi^{0'} \rightarrow \pi^+ + \pi^- + \gamma)}{w(\pi^{0'} \rightarrow 2\gamma)} \sim 4.2 \times 10^{-4},$$

if we assume $M = \text{nucleon mass}$, $G^2/4\pi = 15$ and $f = f'$. This is a rather remarkable result, since $\pi^{0'} \rightarrow \pi^+ + \pi^- + \gamma$ is the process of order e , while $\pi^{0'} \rightarrow 2\gamma$ is of order e^2 . In spite of the crudeness of the above estimation we may accept

$$\frac{w(\pi^{0'} \rightarrow \pi^+ + \pi^- + \gamma)}{w(\pi^{0'} \rightarrow 2\gamma)} \ll 1,$$

and the process $\pi^{0'} \rightarrow 2\gamma$ will be the dominant decay mode of $\pi^{0'}$ decay. This is consistent with the experimental observation. To find the evidence of $\pi^{0'}$ in K_{e3}^+ , it is necessary to carry out the experiment by using the bubble chamber.

We have discussed in this paper the behavior of the $\pi^{0'}$ in the weak decay process. The introduction of the pseudoscalar neutral $\pi^{0'}$ with mass $700 m_e$ improves the situation of K_e^+ decay process and brings no difficulties so long as the weak decay process is concerned. Another important aspect of the $\pi^{0'}$ is its effect in the strong interaction. If the existence of $\pi^{0'}$ is real, the evidence may be obtained more easily in the high energy phenomena. In this respect we only note that the heavy $\pi^{0'}$ discussed in this paper will not affect the low energy pion physics and the existence of resonance states of isotopic spin 1/2 in the pion-nucleon scattering around 1 Bev is interesting in connection with the iso-singlet $\pi^{0(8)}$.

Finally it should be stressed that the $\pi^{0'}$ introduced here is a logical consequence of the composite model for the strange particle which gives us the natural understanding of the existing mass level of baryon and meson system and gives a clue to the law of their mutual transformation.

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Properties of the Pion-Pion Interaction Derived from the Analysis of Pion-Nucleon Scatterings

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The properties of the pion-pion interaction are discussed by the use of dispersion relation techniques. New dispersion relations for the pion-nucleon scattering are obtained, by applying the reduction formula to the initial pion and final nucleon operators. In each of these dispersion relations the absorptive part is divided into two parts, one of which is written as an integral of the pion-pion scattering-amplitude multiplied by a certain matrix element similar to the pion-nucleon scattering-amplitude, while the other is written in terms of the phase shifts for the pion-nucleon scattering. Therefore, if the pion-pion scattering-amplitude is expressed in terms of unknown parameters, such as scattering lengths and effective ranges, their values can be determined by the use of these dispersion relations. Thus one can obtain some knowledge about the sign and the strength of the pion-pion interaction. In this way we are led to the conclusion that, in the isotopic spin state $I=0$ of the pion-pion system, the pion-pion interaction is attractive and its strength is such that the scattering length for the pion-pion scattering is of the same order of magnitude as that of the pion Compton wave length, and that, in the isotopic spin state $I=1$ of the pion-pion system, the interaction is also attractive and is of the strength at least comparable to that of the pion-nucleon interaction in the $(3, 3)$ state.

§ 1. Introduction and summary

From the existence of the strong pion-nucleon interaction, the existence of strong pion-pion interactions is theoretically expected. Such an interaction, if it exists, will play an important role in the various pion phenomena such as the pion-nucleon scattering, the electromagnetic structure of the nucleon, the multiple production of pions from a nucleon, and so on. Unfortunately we have neither experimental nor theoretical methods to attack this interaction, hence little has been known about this until now. Since a nucleon has pion cloud around it, the pion-nucleon scattering must include the effect due to the pion-pion interaction. If this effect can be separated, the analysis of the pion-nucleon scattering will give some information concerning the pion-pion interaction. Indeed, this separation is possible, if one uses new dispersion relations for the pion-nucleon scattering, which will be explained in this section.

As is well known, the amplitude for the pion-nucleon scattering can be written in the form

$$T(p_2, k_2 i_2; p_1, k_1 i_1) \\ = \bar{u}(p_2) \left\{ -\delta_{i_2 i_1} A^{(+)} - (1/2) [\tau_{i_2}, \tau_{i_1}] A^{(-)} \right.$$

$$+ (i/2)\gamma(k_1+k_2)[\partial_{i_2 i_1} B^{(+)} + (1/2)[\tau_{i_2}, \tau_{i_1}] B^{(-)}] \Big\} u(p_1), \quad (1.1)$$

where p_1 and k_1 are four-momenta of the initial nucleon and the pion respectively, p_2 and k_2 are those of final ones, and i_1 and i_2 are charge indices of pion before and after the scattering. $A^{(\pm)}$ and $B^{(\pm)}$ are functions of two independent scalars composed of the four-momenta p_1 , p_2 , k_1 and k_2 . Usually the independent variables are taken as:

$$\nu = -(p_1 + p_2)(k_1 + k_2)/4M, \quad \kappa^2 = (k_1 - k_2)^2/4. \quad (1.2)$$

Then it is proved that $A^{(\pm)}$ and $B^{(\pm)}$, as functions of ν , are analytic in the upper half of the complex ν -plane. The ordinary dispersion relations¹⁾ are obtained from this fact.

In order to get alternative dispersion relations, we write the scattering amplitude as

$$\begin{aligned} T(p_2, k_2 i_2; p_1, k_1 i_1) = & -i(2p_{10}k_{20}/M)^{1/2} \int d^4x \exp[-i(p_2 + k_1)x/2] \\ & \times \bar{u}(p_2) \langle k_2 i_2 | \theta(x) [\gamma(x/2), O_{i_1}(-x/2)] \\ & - \delta(x_0) [\gamma(x/2), \dot{\phi}_{i_1}(-x/2)] | p_1 \rangle, \end{aligned} \quad (1.3)$$

with the abbreviations:

$$\gamma = (\gamma \partial / \partial x + M) \psi, \quad O_i = (\mu^2 - \square) \phi_i,$$

where ψ and ϕ_i are the Heisenberg field operators for nucleon and pion respectively. Using (1.3), and taking the Lorentz frame in which $\mathbf{p}_1 + \mathbf{k}_2 = 0$, we can prove in the usual way that, if $A^{(\pm)}$ and $B^{(\pm)}$ are written as functions of the two variables,

$$\omega = -(p_2 + k_1)(p_1 + k_2)/4M \quad \text{and} \quad \sigma^2 = -(p_1 + k_2)^2/4, \quad (1.4)$$

they are analytic in the upper-half of the complex ω -plane for fixed value of σ^2 . The proof is not rigorous, as in the usual treatment, and, moreover, it can be carried out only for a restricted region of σ^2 . However, we have found that the first few terms in the perturbation expansion of the scattering amplitude actually satisfy this analyticity property for any fixed σ^2 . Therefore we will assume this in the following.

Denoting the total kinetic energy and the scattering angle in the center-of-mass system by w and θ respectively, we have

$$(w + M + \mu)^2/2 = M\omega + \sigma^2 = M\nu + \kappa^2 + (M^2 + \mu^2)/2, \quad (1.5)$$

$$\begin{aligned} 1 - \cos \theta = & \frac{2(M\omega + \sigma^2)(M\omega - \sigma^2)}{[M\omega + \sigma^2 - (M + \mu)^2/2][M\omega + \sigma^2 - (M - \mu)^2/2]} \\ = & \frac{4\kappa^2[M\nu + \kappa^2 + (M^2 + \mu^2)/2]}{(M\nu + \kappa^2 - M\mu)(M\nu + \kappa^2 + M\mu)}. \end{aligned} \quad (1.6)$$

The physical region is determined by the conditions $w \geq 0$ and $-1 \leq \cos \theta \leq 1$.

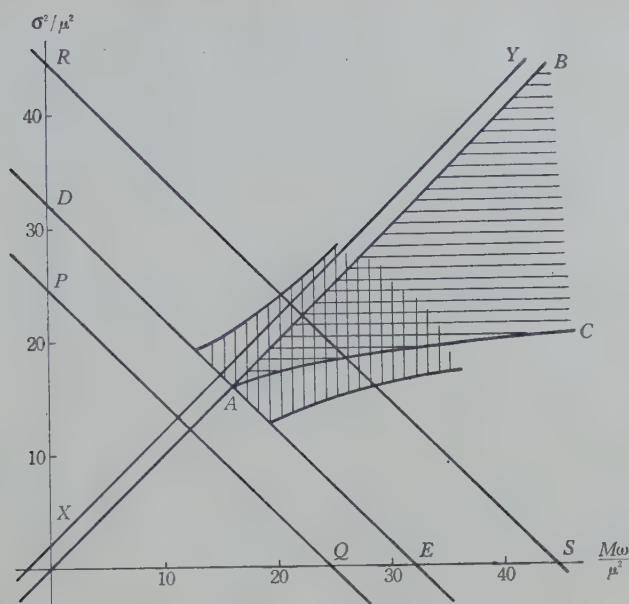


Fig. 1 Physical and unphysical regions in the ω - σ^2 plane.

Various lines are determined as follows:

$$AB: \cos \theta = 1$$

$$AC: \cos \theta = -1$$

$$DE: M\omega + \sigma^2 = (M + \mu)^2/2, \text{ i. e., } w = 0$$

$$PQ: M\omega + \sigma^2 = M^2/2$$

$$RS: w = w_r, \text{ where } w_r \text{ is the resonance energy.}$$

$$XY: \sigma^2 - M\omega = 2\mu^2, \text{ i. e., } \kappa^2 = -\mu^2$$

This is represented in Fig. 1 by the area shaded with horizontal lines.

The absorptive part of the scattering amplitude is obtained from (1.3) by replacing $\theta(x)$ by $-i/2$ and by dropping the term with $\delta(x_0)$. Then, the absorptive part is the sum of the following two terms:

$$F_+ = -(1/2) (2p_{10} k_{20}/M)^{1/2} \int d^4 x \exp[-i(p_2 + k_1)x/2] \times \bar{u}(p_2) \langle k_2 i_2 | \eta(x/2) O_{i_1}(-x/2) | p_1 \rangle, \quad (1.7)$$

$$F_- = (1/2) (2p_{10} k_{20}/M)^{1/2} \int d^4 x \exp[-i(p_2 + k_1)x/2] \times \bar{u}(p_2) \langle k_2 i_2 | O_{i_1}(-x/2) \eta(x/2) | p_1 \rangle. \quad (1.8)$$

First, we consider F_+ . Introducing the complete set of eigenstates $|n\rangle$ of the total energy-momentum operator, we may rewrite F_+ as

$$F_+ = -(1/2) (2\pi)^4 (2p_{10} k_{20}/M)^{1/2} \sum_n \delta^4(P_n - p_1 - k_1) \times \bar{u}(p_2) \langle k_2 i_2 | \eta(0) | n \rangle \langle n | O_{i_1}(0) | p_1 \rangle, \quad (1.9)$$

where P_n is the total energy-momentum of the state $|n\rangle$. Only the intermediate states with nucleon number one give nonvanishing contributions to F_+ , of which the total mass is $[-(p_1+k_1)^2]^{1/2}$. Among the states with nucleon number one, only the single-nucleon state has the discrete mass M , and all the others have masses M_n continuously varying in the region $M_n \geq M + \mu$. We denote the contribution of the single-nucleon intermediate state by $F_+^{(p)}$ and the rest by $F_+^{(e)}$, writing

$$F_+ = F_+^{(p)} + F_+^{(e)}, \quad (1.10)$$

then $F_+^{(p)}$ is different from zero at $-(p_1+k_1)^2 = M^2$, and $F_+^{(e)}$ does not vanish, only when $-(p_1+k_1)^2 \geq (M+\mu)^2$. $F_+^{(p)}$ is easily calculated, and is given as

$$F_+^{(p)} = (\pi g^2/2M) \delta(\omega + \sigma^2/M - M/2) \\ \times \bar{u}(p_2) (i/2) \gamma(k_1+k_2) \left\{ \delta_{i_2 i_1} + (1/2) [\tau_{i_2}, \tau_{i_1}] \right\} u(p_1), \quad (1.11)$$

where g is the renormalized and rationalized pseudoscalar coupling constant. This is different from zero only on the line PQ in Fig. 1. The region in which $F_+^{(e)}$ is different from zero is determined by the condition

$$\omega \geq [(M+\mu)^2 - 2\sigma^2]^{1/2}/2M,$$

because $-(p_1+k_1)^2 = 2(M\omega + \sigma^2)$. This region is to the right side of the line DE in Fig. 1. Especially when point (ω, σ^2) lies in the physical region, $F_+^{(e)}$ can be written as

$$F_+^{(e)} = \bar{u}(p_2) \left\{ -\delta_{i_2 i_1} \text{Im} A^{(+)}(\omega, \sigma^2) - (1/2) [\tau_{i_2}, \tau_{i_1}] \text{Im} A^{(-)}(\omega, \sigma^2) \right. \\ \left. + (i/2) \gamma(k_1+k_2) [\delta_{i_2 i_1} \text{Im} B^{(+)}(\omega, \sigma^2) + (1/2) [\tau_{i_2}, \tau_{i_1}] \text{Im} B^{(-)}(\omega, \sigma^2)] \right\} u(p_1). \quad (1.12)$$

Moreover the imaginary parts can be expanded in the partial waves. Lehmann²⁾ showed that this partial wave expansion is valid in an area larger than the physical region. This region is shown in Fig. 1 by the area shaded with vertical lines. More strictly speaking, $F_+^{(e)}$ has the expression obtained by omitting the single-nucleon intermediate state from (1.9). By virtue of this expression one can show, following Lehmann, that (1.12) holds in the above-mentioned region provided that in the right-hand side $\text{Im} A^{(\pm)}$ and $\text{Im} B^{(\pm)}$ are being expanded in the partial waves. Since the outside of this region is found to be not important in our dispersion relations, we will use (1.12) for all values of ω and σ^2 .

Next we write F_- in the form similar to (1.9). Since operator η lowers the nucleon number by unity, the intermediate states must have nucleon number 0, and the total mass of these states is $[-(k_1-k_2)^2]^{1/2}$. Moreover, among the intermediate states consisting of pions, only the intermediate states with even number of pions contributes to F_- . This follows from the invariance under charge conjugation and a rotation around the 2 axis in the isotopic spin space. Therefore, the

least massive state is a two-pion state, and consequently F_- differs from zero only when $-(k_1 - k_2)^2 \geq 4\mu^2$. F_- can be written analogously to (1.1) in the following form:

$$F_- = \bar{u}(p_2) \left\{ -\delta_{i_2 i_1} u^{(+)}(\omega, \sigma^2) - (1/2) [\tau_{i_2}, \tau_{i_1}] u^{(-)}(\omega, \sigma^2) \right. \\ \left. + (i/2) \gamma(k_1 + k_2) [\delta_{i_2 i_1} v^{(+)}(\omega, \sigma^2) + (1/2) [\tau_{i_2}, \tau_{i_1}] v^{(-)}(\omega, \sigma^2)] \right\} u(p_1). \quad (1.13)$$

Since $-(k_1 - k_2)^2 = 2(\sigma^2 - M\omega)$, $u^{(\pm)}$ and $v^{(\pm)}$ are different from zero only when $\omega \leq (\sigma^2 - 2\mu^2)/M$. In Fig. 1, this region is to the left side of the line XY .

From the above consideration, we have the following dispersion relations³⁾:

$$\text{Re } A^{(\pm)}(\omega, \sigma^2) = (P/\pi) \int_{[(M+\mu)^2 - 2\sigma^2]/2M}^{\infty} d\omega' \text{Im } A^{(\pm)}(\omega', \sigma^2) / (\omega' - \omega) \\ + (1/\pi) \int_{-\infty}^{(\sigma^2 - 2\mu^2)/M} d\omega' u^{(\pm)}(\omega', \sigma^2) / (\omega' - \omega), \quad (1.14)$$

$$\text{Re } B^{(\pm)}(\omega, \sigma^2) = -g^2 [1/(2M\omega + 2\sigma^2 - M^2) \pm 1/(4\sigma^2 - M^2 - 2\mu^2)] \\ + (P/\pi) \int_{[(M+\mu)^2 - 2\sigma^2]/2M}^{\infty} d\omega' \text{Im } B^{(\pm)}(\omega', \sigma^2) / (\omega' - \omega) \\ + (1/\pi) \int_{-\infty}^{(\sigma^2 - 2\mu^2)/M} d\omega' v^{(\pm)}(\omega', \sigma^2) / (\omega' - \omega). \quad (1.15)$$

Here, no subtraction has been made. The first term between brackets in the right-hand side of (1.15) comes from $F_-^{(p)}$. The second term does not depend on ω . We have added this term so that (1.15) holds in the lowest order perturbation theory. But it may be doubtful that the coefficient of this term is really the square of a renormalized coupling constant, g^2 . To avoid this ambiguity, we will carry out one subtraction in the above dispersion relations.

Let us consider the last terms in (1.14) and (1.15). Owing to the denominator of the integrand, $u^{(\pm)}$ and $v^{(\pm)}$ at small values of $|\omega|$ are important in these terms, and this becomes surer if the subtraction is carried out. Now, for small values of $|\omega|$, the most important contribution to F_- comes from the two-pion intermediate states. Indeed, for $2\mu^2 - \sigma^2 \leq -M\omega < 8\mu^2 - \sigma^2$, only the two-pion intermediate states contribute to F_- . (For $-M\omega \geq 8\mu^2 - \sigma^2$, contribution of the four-pion intermediate state appears.) Therefore, we approximate F_- by the contribution of two pion intermediate states. Then we have

$$F_-(p_2, k_2 i_2; p_1, k_1 i_1) \\ = (1/4) (2\pi)^{-2} (2p_{10} k_{20}/M)^{1/2} \sum_{i_1' i_2'} \int d\mathbf{k}_1' d\mathbf{k}_2' \delta^4(k_2 - k_1 - k_1' - k_2') \\ \times \langle k_2 i_2 | O_{i_1}(0) | k_1' i_1', k_2' i_2' \rangle \bar{u}(p_2) \langle k_1' i_1', k_2' i_2' | \eta(0) | p_1 \rangle. \quad (1.16)$$

Here, the integral over intermediate states should be understood to be the average of the integral over "in" states and the one over "out" states. The matrix element of $O_{i_1}(0)$ in (1.16) is the scattering amplitude for the pion-pion scattering, and is written by the requirement of Lorentz invariance in the following form:

$$\begin{aligned} & \langle k_2 i_2 | O_{i_1}(0) | k_1' i_1', k_2' i_2' \text{ in} \rangle \\ &= 32\pi (8k_{20} k_{10}' k_{20}')^{-1/2} \left\{ L_0(-\kappa'^2, \xi) (1/3) \delta_{i_1 i_2} \delta_{i_1' i_2'} \right. \\ &+ L_1(-\kappa'^2, \xi) (1/2) (\partial_{i_1 i_1'} \partial_{i_2 i_2'} - \partial_{i_1 i_2'} \partial_{i_2 i_1'}) \\ &+ L_2(-\kappa'^2, \xi) [(1/2) (\partial_{i_1 i_1'} \partial_{i_2 i_2'} + \partial_{i_1 i_2'} \partial_{i_2 i_1'}) \\ &\left. - (1/3) \delta_{i_1 i_2} \delta_{i_1' i_2'}] \right\}, \end{aligned} \quad (1.17)$$

where

$$\kappa'^2 = (k_1' + k_2')^2/4, \quad \xi = (k_1' + k_2' - 2k_2)(k_1' - k_2')/4. \quad (1.18)$$

Here, L_l is a Lorentz invariant scattering amplitude in the state where the total isotopic spin is I . This is related to the pion-pion phase shifts $\delta_{I,l}$ as

$$L_l(\omega_c^2, k_c^2 \cos \theta) = (\omega_c/k_c) \sum_l (2l+1) \exp(i\delta_{I,l}) \sin \delta_{I,l} P_l(\cos \theta), \quad (1.19)$$

where ω_c and k_c are energy and magnitude of momentum of either pion and θ is the scattering angle in the center-of-mass system. The sum over l should be understood as the sum over even l for even I and the sum over odd l for odd I . This follows from the invariance of the matrix element (1.17) under the exchange of $k_1' i_1'$ and $k_2' i_2'$.

It is easily found that L_2 does not contribute to F_- . This follows from the fact that the system of a nucleon and an antinucleon cannot have a total isotopic spin larger than one.

Next we assume that all the terms with $l > 1$ can be neglected in (1.19). The assumption may be reasonable from the following reason. The range of the pion-pion interaction is considered to be $(2\mu)^{-1}$ at most. Then the condition under which the d -wave plays an important role in the scattering is that $k_c \gtrsim 4\mu$, which is also written as $M\omega \lesssim \sigma^2 - 34\mu^2$. From this, we see that for small values of $|\omega|$, the d -wave can be neglected more safely than the contribution of four-pion intermediate states which has already been neglected. For s - and p -waves, we write

$$L_0(\omega_c^2) = (\omega_c/k_c) \exp(i\delta_0) \sin \delta_0; \quad (1.20)$$

$$L_1(\omega_c^2) = \mu^2 (\omega_c/k_c^3) \exp(i\delta_1) \sin \delta_1, \quad (1.21)$$

where we have written δ_0 and δ_1 in place of $\delta_{0,0}$ and $\delta_{1,1}$ respectively. Then L_0 and L_1 become

$$L_0(-\kappa'^2, \xi) = l_0(-\kappa'^2), \quad L_1(-\kappa'^2, \xi) = 3\mu^{-2} l_1(-\kappa'^2) \xi. \quad (1.22)$$

At low energies l_0 and l_1 can be characterized by a few parameters such as

the scattering lengths and the effective ranges. Therefore, if the matrix element of $\gamma(0)$ in (1.16) has been calculated, and if F_- and, consequently, $u^{(\pm)}$ and $v^{(\pm)}$ can be expressed in terms of these parameters, then we can estimate these parameters numerically by means of our dispersion relations (1.14) and (1.15), since all the terms in (1.14) and (1.15) other than those containing $u^{(\pm)}$ and $v^{(\pm)}$ can be computed by using pion-nucleon data. Numerical calculations will be given in § 3. The results obtained in this way lead us to the following conclusions: the pion-pion interaction is attractive for the both states $I=0$ and $I=1$. For the state $I=0$, the scattering length of the pion-pion scattering is of the same order as that of the pion Compton wave length. For the state $I=1$, strength of the interaction is at least comparable to that of the pion-nucleon interaction in the $(3, 3)$ state.

The matrix element of $\gamma(0)$ in (1.16) can be written formally, as the pion-nucleon scattering amplitude, but it must be noticed that the matrix element lies in the unphysical region. As has been done in the problem of the electromagnetic structure of nucleon,⁴⁾ the scattering amplitude in the unphysical region can be related to that of the physical region by virtue of the ordinary dispersion relations¹⁾ and the expansion in the partial waves. In § 2, we first calculate the matrix element in question in this method. In this calculation, however, as has already been pointed out,⁴⁾ we have to use the partial-wave expansion in the region where its convergence has not been proved. To avoid such an unjustified procedure, we next make in § 2 a calculation in the static theory. We shall see in § 3 that both calculations yield qualitatively the same results.

§ 2. Calculations of $u^{(\pm)}$ and $v^{(\pm)}$

By the requirement of invariance, the matrix element of $\gamma(0)$ in (1.16) can be written in the form

$$\begin{aligned} & \bar{u}(p_2) \langle k'_1 i'_1, k'_2 i'_2 \text{ out} | \gamma(0) | p_1 \rangle \\ &= -(1/2) (p_{10} k'_{10} k'_{20} / M)^{-1/2} \bar{u}(p_2) \left\{ -\delta_{i_2' i_1'} A^{(+)}(\nu', \kappa'^2) \right. \\ & \quad - (1/2) [\tau_{i_2'}, \tau_{i_1'}] A^{(-)}(\nu', \kappa'^2) \\ & \quad + (i/2) \gamma(k_2' - k_1') [\partial_{i_2' i_1'} B^{(+)}(\nu', \kappa'^2) \\ & \quad \left. + (1/2) [\tau_{i_2'}, \tau_{i_1'}] B^{(-)}(\nu', \kappa'^2) \right\} u(p_1), \end{aligned} \quad (2.1)$$

where

$$\nu' = -(p_1 + p_2) \cdot (k_2' - k_1') / 4M$$

and κ'^2 is given by (1.18). If the "out" state in the left-hand side is changed into "in" state, $A^{(\pm)}$ and $B^{(\pm)}$ in the right-hand side are changed into their complex conjugates. This follows from the invariance under charge conjugation. For the same reason, if the "in" state in the left-hand side of (1.17) is changed into the "out" state, L_I 's in the right-hand side are changed into their complex conjugates.

Taking account of these facts, we substitute (2.1) and (1.17) with (1.22) into (1.16). Transforming the three-dimensional integrals into four-dimensional ones by adding the delta-function $\delta(k_i'^2 + \mu^2)$ to the integrand, and carrying out the integration over $k_1' + k_2'$, we have

$$\begin{aligned}
 F_-(p_2, k_2 i_2; p_1, k_1 i_1) &= -(1/\pi) \int d^4 Q \theta(k_{20} - k_{10} + 2Q_0) \theta(k_{20} - k_{10} - 2Q_0) \\
 &\times \delta(Q^2 + \kappa^2 + \mu^2) \delta[-(k_2 - k_1)Q] \\
 &\times \bar{u}(p_2) \left\{ -\delta_{i_2 i_1} \text{Re}(l_0^*(-\kappa^2) A^{(+)}(\nu', \kappa^2)) \right. \\
 &+ (1/2)[\tau_{i_2}, \tau_{i_1}] \text{Re}(l_1^*(-\kappa^2) A^{(-)}(\nu', \kappa^2)) (3(k_1 + k_2)Q/2\mu^2) \\
 &+ i\gamma Q [\delta_{i_2 i_1} \text{Re}(l_0^*(-\kappa^2) B^{(+)}(\nu', \kappa^2)) \\
 &\left. + (1/2)[\tau_{i_2}, \tau_{i_1}] \text{Re}(l_1^*(-\kappa^2) B^{(-)}(\nu', \kappa^2)) (3(k_1 + k_2)Q/2\mu^2) \right\} u(p_1), \quad (2.2)
 \end{aligned}$$

where

$$\kappa^2 = (k_1 - k_2)^2/4, \quad \nu' = -(p_1 + p_2)Q/2M.$$

It follows from (2.2) that

$$v^{(+)}(\omega, \sigma^2) = 0. \quad (2.3)$$

This is shown as follows: comparing (2.2) with (1.13) we see that only the term with factor $B^{(+)}$ in (2.2) can contribute to $v^{(+)}$. This term contains γ -matrices only through the factor $i\gamma Q$, and depends on Q only through $i\gamma Q$ and $(p_1 + p_2)Q$. Taking a particular Lorentz frame in which $k_2 - k_1 = 0$, we carry out the integration over Q_0 . Then Q_0 in the integrand becomes zero on account of the delta-function $\delta[(k_1 - k_2)Q]$, hence $i\gamma Q$ and $(p_1 + p_2)Q$ become $i\gamma \mathbf{Q}$ and $(\mathbf{p}_1 + \mathbf{p}_2)\mathbf{Q}$ respectively. Therefore, on carrying out the integration over directions of \mathbf{Q} , the term under consideration contains γ -matrices only through the factor $i\gamma(\mathbf{p}_1 + \mathbf{p}_2)$, which becomes $i\gamma(p_1 + p_2)$ in the general Lorentz frame, because we have $p_{10} + p_{20} = 0$ in the Lorentz frame where $k_2 - k_1 = 0$ on account of the relation $(k_2 - k_1)(p_1 + p_2) = 0$. However, $i\gamma(p_1 + p_2)$ reduces to $-2M$ because of the Dirac equations satisfied by $u(p_1)$ and $\bar{u}(p_2)$. Thus the term in question does not contain γ -matrices ultimately, and does not contribute to $v^{(+)}$. This proves (2.3).*

As was mentioned in §1, (2.2) has been obtained under the following approximations: contributions of all the intermediate states other than the two-pion states are neglected in F_- and all the partial waves higher than the p -wave in the pion-pion scattering amplitude are neglected. Therefore (2.3) is a consequence of these approximations. Since (2.3) makes the integral containing $v^{(+)}$ in (1.15)

* Though we are considering the unphysical region $(k_1 - k_2)^2 < 0$, the "real" Lorentz frame in which $k_2 - k_1 = 0$ does not exist, unless ω and σ^2 satisfy a certain condition. However, we have verified for the first few terms in the perturbation expansion that the formal procedure used here always gives the correct results.

vanish, the numerical difference between $\text{Re}B^{(+)}$ and a sum of the Born terms and the integral of $\text{Im}B^{(+)}$ in (1.15) will give an idea about the error caused by our approximation.

In order to proceed further, we must calculate $A^{(+)}$ and $B^{(+)}$ in (2.1). Perturbation theory is not adequate for our purpose, because this theory cannot describe the behavior of s -wave pions correctly, and consequently it will yield incorrect results particularly for $A^{(+)}$ and $B^{(+)}$. We therefore adopt the following two methods, the one is the use of the dispersion relations, and the other, the static theory.

a) Calculation with the Dispersion Relations

The matrix element (2.1) can be regarded as the amplitude for an unphysical pion-nucleon scattering, in which a pion with energy-momentum $-k_1'$ is scattered into the one with energy-momentum k_2' . Therefore, it is formally proved that $A^{(\pm)}$ and $B^{(\pm)}$ in (2.1) satisfy the dispersion relations for the pion-nucleon scattering given by Chew et al.¹⁾ By means of these dispersion relations $A^{(\pm)}$ and $B^{(\pm)}$ can be expressed as the integrals of their imaginary parts in the region where the energy is physical but the scattering angle is unphysical. (When the variables ν' and κ'^2 are transformed into w and $\cos \theta$ by the use of (1.5) and (1.6) with ν' and κ'^2 in place of ν and κ^2 , this region is in which $w \geq 0$ but $|\cos \theta| > 1$.) In spite of the unphysical scattering angle, however, these imaginary parts can be written in terms of the phase shifts in the physical region by means of the partial-wave expansion of the former. In this way we can compute $A^{(\pm)}$ and $B^{(\pm)}$ using experimental phase shifts. This method has already been applied to the problem of the electromagnetic structure of nucleon by Chew et al. and Federbush et al.⁴⁾ In the present case one should be more careful about the convergence of the dispersion integrals than in the case of the nucleon structure, because we must calculate all of $A^{(\pm)}$ and $B^{(\pm)}$, while the form factors of nucleon involve $A^{(-)}$ and $B^{(-)}$ only. Therefore, we use here the subtracted dispersion relations:

$$A^{(\pm)}(\nu', \kappa^2) = A^{(\pm)}(\nu_0, \kappa^2) + (1/\pi) \int_{\nu_0}^{\infty} d\nu'' \text{Im} A^{(\pm)}(\nu'', \kappa^2) \times \left\{ 1/(\nu'' - \nu') \pm 1/(\nu'' + \nu') - 1/(\nu'' - \nu_0) \mp 1/(\nu'' + \nu_0) \right\}; \quad (2.4)$$

$$B^{(\pm)}(\nu', \kappa^2) = B^{(\pm)}(\nu_0, \kappa^2) + (g^2/2M) \left\{ 1/(\nu_B - \nu') \mp 1/(\nu_B + \nu') - 1/(\nu_B - \nu_0) \pm 1/(\nu_B + \nu_0) \right\} + (1/\pi) \int_{\nu_0}^{\infty} d\nu'' \text{Im} B^{(\pm)}(\nu'', \kappa^2) \times \left\{ 1/(\nu'' - \nu') \mp 1/(\nu'' + \nu') - 1/(\nu'' - \nu_0) \pm 1/(\nu'' + \nu_0) \right\} \quad (2.5)$$

where

$$\nu_0 = \mu^2 - \kappa^2/M, \quad \nu_B = -\mu^2/2M - \kappa^2/M.$$

In the present case, it is not necessary to add an infinitesimal imaginary part to each of the denominators in these formulas, because, on substituting them into (2.2) and combining terms appropriately, all the denominators become positive definite. In fact, this is true only when $\kappa^2 + M^2 > 0$, i.e., $\omega > (\sigma^2 - 2M^2)/M$. This condition is the one under which the intermediate states including nucleon-antinucleon pair does not contribute to F_- . We may consider this condition to hold, since in later calculations we shall cut off the integrals of $u^{(\pm)}$ and $v^{(\pm)}$ at $\omega' = (\sigma^2 - 2M^2)/M$.

It should be noticed that the meaning of Re in (2.2) must be reinterpreted as to take real part of the whole expression in order to obtain correct results; otherwise $u^{(\pm)}$ and $v^{(\pm)}$ would become imaginary. It is verified that in the lowest order perturbation theory correct results are obtained, indeed, by this treatment. Then, it is found that $u^{(\pm)}$ and $v^{(\pm)}$ contain only the real parts of l_0 and l_1 .

We expand $\text{Im } A^{(\pm)}$ and $\text{Im } B^{(\pm)}$ in (2.4) and (2.5) in the partial waves using the formulas given in reference 1), and neglect all the partial-wave amplitudes other than the (3.3) amplitude, for which, following Federbush et al., we make the narrow-resonance approximation:

$$\begin{aligned} \text{Im } f_{33}(\nu'', \kappa^2) &= \frac{g^2}{6M} \cdot \frac{(M\nu'' + \kappa^2)^2 - M^2\mu^2}{2M\nu'' + 2\kappa^2 + M^2 + \mu^2} \\ &\times \{1 + (w_r/M)\} \delta(2M\nu'' + 2\kappa^2 + \mu^2 - 2Mw_r - w_r^2), \end{aligned}$$

where w_r is the resonance energy in the center-of-mass system. We take $w_r = 2\mu$.

We also expand $A^{(\pm)}(\nu_0, \kappa^2)$ and $B^{(\pm)}(\nu_0, \kappa^2)$ in (2.4) and (2.5) in the partial waves. Because of $\kappa^2 \neq 0$, all partial waves appear in these expansions. However, we neglect all of them other than s - and p -waves, and for the latter phase shifts we use experimental values reported by Puppi at the CERN Conference in 1958⁵⁾.

Here the final expressions of $u^{(\pm)}$ and $v^{(\pm)}$ are not written, but the numerical results will be given in § 3.

b) Calculation in the static theory

Chew and Low⁶⁾ and Miyazawa⁷⁾ have developed techniques by means of which various matrix elements are related to the pion-nucleon phase shifts under the assumption that nucleon can be regarded as a fixed and extended source of pion field and that pion-nucleon interaction is described by a particular Hamiltonian. Extending the basic idea of these authors, we now calculate the matrix element of $\eta(0)$ appearing in (1.16).

From the definition of $\eta(x)$ we have in the nonrelativistic approximation

$$\bar{u}(p_2, f)\eta(\mathbf{x}, t) = \chi_f^*(-i\partial/\partial t + M)\psi(\mathbf{x}, t),$$

where f is the spin index which has been suppressed up to this time, and χ_f is the two-component spin eigenfunction of the final nucleon. From this and

$$\phi(\mathbf{x}, t) = \exp(iHt)\phi(\mathbf{x}, 0)\exp(-iHt)$$

(H is the total Hamiltonian), it follows that

$$\begin{aligned} \bar{u}(\mathbf{p}_2, f)\langle k_1', k_2' \text{ out} | \eta(0) | \mathbf{p}_1, i \rangle \\ = (k_{10}' + k_{20}') \chi_f^* \langle k_1', k_2' \text{ out} | \phi(0) | \mathbf{p}_1, i \rangle, \end{aligned} \quad (2.6)$$

where i is the spin index of the initial nucleon, and the charge indices of pions have been suppressed.

We expand the Heisenberg field operators at $t=0$ in the Fourier series as follows:

$$\begin{aligned} \phi(\mathbf{x}, 0) &= \sum_{p,j} A_{p,j} \chi_j \exp(i\mathbf{p}\mathbf{x}), \\ \phi(\mathbf{x}, 0) &= \sum_k (2\omega_k)^{-1/2} [a_k \exp(i\mathbf{k}\mathbf{x}) + a_k^* \exp(-i\mathbf{k}\mathbf{x})], \\ \dot{\phi}(\mathbf{x}, 0) &= -i \sum_k (\omega_k/2)^{1/2} [a_k \exp(i\mathbf{k}\mathbf{x}) - a_k^* \exp(-i\mathbf{k}\mathbf{x})]. \end{aligned}$$

We call the eigenstates of the occupation numbers $A_{p,j}^*$ and a_k^* as the bare-particle states and write them as $|\cdots \text{bare}\rangle$. For example,

$$\begin{aligned} |p, j \text{ bare}\rangle &= A_{p,j}^* |0\rangle, \quad |p, j; k \text{ bare}\rangle = A_{p,j}^* a_k^* |0\rangle, \\ |k_1, k_2 \text{ bare}\rangle &= a_{k_1}^* a_{k_2}^* |0\rangle \quad \text{and so on,} \end{aligned}$$

where $|0\rangle$ is the vacuum state for the bare particles.

By the substitution of the Fourier expansion of ϕ the matrix element in the right-hand side of (2.6) is rewritten as

$$\chi_f^* \langle k_1', k_2' \text{ out} | \phi(0) | \mathbf{p}_1, i \rangle = \langle k_1', k_2' \text{ out} | A_{pf} | \mathbf{p}_1, i \rangle,$$

where $\mathbf{p} = \mathbf{p}_1 - \mathbf{k}_1' - \mathbf{k}_2'$. We approximate the right-hand side of this by

$$(1/2) \sum_{k_1'', k_2''} \langle k_1' k_2' \text{ out} | k_2'' k_2'' \text{ bare} \rangle \langle k_1'' k_2'' \text{ bare} | A_{pf} | \mathbf{p}_1, i \rangle.$$

Then, because of the commutativity between A_{pf} and a_k^* the second matrix element becomes $\langle pf; k_1'', k_2'' \text{ bare} | \mathbf{p}_1, i \rangle$, which is more simply written as $\langle f; k_1'', k_2'' \text{ bare} | i \rangle$ in the static theory. Thus we have

$$\begin{aligned} \bar{u}(\mathbf{p}_2, f)\langle k_1', k_2' \text{ out} | \eta(0) | \mathbf{p}_1, i \rangle \\ = (k_{10}' + k_{20}') \langle f; k_1', k_2' \text{ bare} | i \rangle \\ + (k_{10}' + k_{20}') (1/2) \sum_{k_1'', k_2''} \left\{ \langle k_1', k_2' \text{ out} | k_1'', k_2'' \text{ bare} \rangle \right. \\ \left. - \delta_{k_1' k_1''} \delta_{k_2' k_2''} - \delta_{k_1' k_2''} \delta_{k_2' k_1''} \right\} \langle f; k_1'', k_2'' \text{ bare} | i \rangle. \end{aligned} \quad (2.7)$$

The appearance of the last term is due to pion-pion interaction. In the static theory, however, this term can be dropped for the following reason: It will be found later that the matrix element $\langle f; k_1', k_2' \text{ bare} | i \rangle$ is proportional to $(k_{10}' + k_{20}')^{-1}$,

so that the factor $(k_{10}' + k_{20}')$ in the right-hand side of (2.7) disappears in the first term but survives in the second term. After the substitution of (2.7) into (1.16), this factor becomes $k_{20} - k_{10}$ on account of the delta function. But in the static theory the energy conservation yields $k_{20} - k_{10} = 0$. Therefore, we may write

$$\begin{aligned} \bar{u}(p_2, f) \langle k_1', k_2' \text{ out} | \eta(0) | p_1, i \rangle \\ = (k_{10}' + k_{20}') \langle f; k_1', k_2' \text{ bare} | i \rangle. \end{aligned} \quad (2.8)$$

For the matrix element in the right-hand side we further make the following approximation:

$$\begin{aligned} \langle f; k_1', k_2' \text{ bare} | i \rangle &= \langle f; k_2' \text{ bare} | a_{k_1'} | i \rangle \\ &\cong \sum_j \langle f; k_2' \text{ bare} | j \rangle \langle j | a_{k_1'} | i \rangle \\ &+ \sum_{j, k} \langle f; k_2' \text{ bare} | j; k \text{ in} \rangle \langle j; k \text{ in} | a_{k_1'} | i \rangle, \end{aligned} \quad (2.9)$$

where $|j\rangle$ is the single physical-nucleon state, and $|j; k \text{ in}\rangle$ is the scattering state of a nucleon with spin j and a pion with momentum k . The sum in the last term should be understood as half the sum over "in" states and "out" states, though we have not written this explicitly. (2.9) is so-called the one-meson approximation.

We assume that only s - and p -wave pions interact with the nucleon. To avoid too lengthy an explanation, however, we leave the s -wave pions out of consideration, until their effects are added to the final results.

We begin with the consideration of the first matrix element in the last line of (2.9). We denote the total Hamiltonian by H , and the free Hamiltonian at $t=0$ by H_0 . Then $|j; k \text{ in}\rangle$ and $|f; k_2' \text{ bare}\rangle$ are eigenstates of H and H_0 respectively. From this fact it follows that

$$\begin{aligned} \langle f; k_2' \text{ bare} | j; k \text{ in} \rangle \\ = \langle f; k_2' \text{ bare} | j; k \text{ bare} \rangle + (k_0 - k_{20}' + i\epsilon)^{-1} \\ \times \langle f; k_2' \text{ bare} | [H' + H'(k_0 - H + i\epsilon)^{-1} H'] | j; k \text{ bare} \rangle, \end{aligned} \quad (2.10)$$

where H' is the interaction Hamiltonian. On the other hand the corresponding S -matrix element has the form:

$$\begin{aligned} \langle f; k_2' \text{ out} | j; k \text{ in} \rangle \\ = \langle f; k_2' \text{ bare} | j; k \text{ bare} \rangle - 2\pi i \delta(k_0 - k_{20}') \\ \times \langle f; k_2' \text{ bare} | [H' + H'(k_0 - H + i\epsilon)^{-1} H'] | j; k \text{ bare} \rangle. \end{aligned} \quad (2.11)$$

In terms of the phase shifts this is also written as

$$\begin{aligned} \langle f; k_2' \text{ out} | j; k \text{ in} \rangle &= \delta_{f,j} \delta_{k_2', k} + 2\pi i \delta(k_0 - k_{20}') \\ &\times 2\pi (k_0 k_{20}')^{-1/2} \chi_f^* \sum_{\alpha, \beta} h_{\alpha\beta}(k_0) P_{\alpha\beta}(k_2', k) \chi_j, \end{aligned} \quad (2.12)$$

with the abbreviation

$$\begin{aligned} h_{\alpha\beta}(k_0) &= (k_0^2 - \mu^2)^{-3/2} \exp[i\delta_{\alpha\beta}(k_0)] \sin \delta_{\alpha\beta}(k_0), \\ P_{11}(k', k) &= (1/3) \tau_{k'} \tau_k (\boldsymbol{\sigma} \cdot \mathbf{k}') (\boldsymbol{\sigma} \cdot \mathbf{k}), \\ P_{13}(k', k) &= (1/3) \tau_{k'} \tau_k [3(\mathbf{k}' \cdot \mathbf{k}) - (\boldsymbol{\sigma} \cdot \mathbf{k}') (\boldsymbol{\sigma} \cdot \mathbf{k})], \\ P_{31}(k', k) &= (\delta_{k'k} - (1/3) \tau_{k'} \tau_k) (\boldsymbol{\sigma} \cdot \mathbf{k}') (\boldsymbol{\sigma} \cdot \mathbf{k}), \\ P_{33}(k', k) &= (\delta_{k'k} - (1/3) \tau_{k'} \tau_k) [3(\mathbf{k}' \cdot \mathbf{k}) - (\boldsymbol{\sigma} \cdot \mathbf{k}') (\boldsymbol{\sigma} \cdot \mathbf{k})]. \end{aligned}$$

It is seen from (2.10) that the matrix element in the last term depends on k_2' only through the factor \mathbf{k}_2'/k_{20}' so long as $|\mathbf{k}_2'|^{-1}$ are much larger than the range of the pion-nucleon interaction. Therefore, if we compare (2.10) with (2.11), and (2.11) with (2.12), we have, to a good approximation,

$$\begin{aligned} \langle f; k_2' \text{ bare} | j; k \text{ in} \rangle &= \delta_{f,j} \delta_{k_2',k} + (k_{20}' - k_0 - i\epsilon)^{-1} \\ &\times 2\pi (k_{20}' k_0)^{-1/2} \chi_f^* \sum_{\alpha,\beta} h_{\alpha\beta}(k_0) P_{\alpha\beta}(k_2', k) \chi_j. \end{aligned} \quad (2.13)$$

In the same way it is found that, if $|j; k \text{ in}\rangle$ is replaced with $|j; k \text{ out}\rangle$ in the left-hand side of (2.13), $h_{\alpha\beta}$ is replaced with $h_{\alpha\beta}^*$ in the right-hand side.

Next we turn to the last matrix element in (2.9). This is written by means of the well known technique⁶⁾ as

$$\langle j; k \text{ out} | a_{k_1'} | i \rangle = - (k_0 + k_{10}')^{-1} \langle j; k \text{ out} | V_{k_1'}^* | i \rangle \quad (2.14)$$

with

$$V_k^* = -[H', a_k].$$

Here we assume that H' is the same as in the Chew-Low theory⁶⁾. Then we have $V_k^* = -V_k$, so that the matrix element in the right-hand side of (2.14) becomes essentially the scattering amplitude. Thus we have

$$\begin{aligned} \langle j; k \text{ out} | a_{k_1'} | i \rangle &= - (k_0 + k_{10}')^{-1} 2\pi (k_0 k_{10}')^{-1/2} \chi_j^* \sum_{\alpha,\beta} h_{\alpha\beta}(k_0) P_{\alpha\beta}(k, k_1') \chi_i. \end{aligned} \quad (2.15)$$

It follows from the invariance under time reversal that, if $\langle j; k \text{ out} |$ is replaced with $\langle j; k \text{ in} |$ in the left-hand side, $h_{\alpha\beta}$ is replaced with $h_{\alpha\beta}^*$ in the right-hand side.

The first term in the right-hand side of (2.9) can be written as

$$\sum_j \langle f \text{ bare} | a_{k_2'} | j \rangle \langle j | a_{k_1'} | i \rangle.$$

Here we approximately replace the bare-nucleon state $\langle f \text{ bare} |$ with the corresponding physical-nucleon state $\langle f |$. Then this term is easily calculated by the well-known method⁶⁾, and we obtain the result

$$\sum_j \langle f; k_2' \text{ bare} | j \rangle \langle j | a_{k_1'} | i \rangle \\ = -2\pi (k_{20}' k_{10}')^{-3/2} (f^2/\mu^2) \chi_f^* (\boldsymbol{\sigma} \cdot \mathbf{k}_2') (\boldsymbol{\sigma} \cdot \mathbf{k}_1') \tau_{k_2'} \tau_{k_1'} \chi_i, \quad (2.16)$$

where f is the renormalized and nonrationalized pseudovector coupling constant.

Thus we have expressed all terms in the right-hand side of (2.9) in terms of the quantities known experimentally. If the Chew-Low equation⁶⁾ for $h_{\alpha\beta}$ is used, the result obtained by substituting (2.13), (2.15) and (2.16) into (2.9) is found to be proportional to $(k_{10}' + k_{20}')^{-1}$ as was mentioned previously. Furthermore, this result is found to be symmetric in k_1' and k_2' , though they have not been treated symmetrically during the calculation. We substitute this result into (2.8), and neglect all $h_{\alpha\beta}$'s other than h_{33} . Further, we add the terms due to the s -wave pions. Then we have finally

$$\bar{u}(p_2, f) \langle k_1' i_1', k_2' i_2' \text{ out} | \gamma(0) | p_1, i \rangle \\ = -2\pi (k_{10}' k_{20}')^{-1/2} \chi_f^* \left\{ \partial_{i_2' i_1'} \left[- (1/3) (a_1 + 2a_3) \right. \right. \\ + (\mathbf{k}_2' \cdot \mathbf{k}_1') (B_+ (k_{10}', k_{20}') + 4H_+ (k_{10}', k_{20}')) \\ + i\boldsymbol{\sigma} \cdot (\mathbf{k}_2' \times \mathbf{k}_1') (B_- (k_{10}', k_{20}') - 2H_- (k_{10}', k_{20}')) \left. \right] \\ + (1/2) [\tau_{i_2'}, \tau_{i_1'}] \left[(1/6) (a_1 - a_3) (k_{10}' - k_{20}') \right. \\ + (\mathbf{k}_2' \cdot \mathbf{k}_1') (B_- (k_{10}', k_{20}') - 2H_- (k_{10}', k_{20}')) \\ + i\boldsymbol{\sigma} \cdot (\mathbf{k}_2' \times \mathbf{k}_1') (B_+ (k_{10}', k_{20}') + H_+ (k_{10}', k_{20}')) \left. \right] \left. \right\} \chi_i, \quad (2.17)$$

with the abbreviations

$$B_{\pm} (k_{10}', k_{20}') = (f^2/\mu^2) (1/k_{10}' \pm 1/k_{20}'),$$

$$H_{\pm} (k_{10}', k_{20}') = (1/3\pi) \int_{\mu}^{\omega_{\text{max}}} d\omega_k k^3 |h_{33}(\omega_k)|^2 \left\{ 1/(k_{10}' + \omega_k) \pm 1/(k_{20}' + \omega_k) \right\},$$

where a_1 and a_3 are the s -wave scattering lengths.

In order to treat the s -wave pions in the similar way as for the p -wave ones, we have used the generalized Chew-Low theory given by Drell et al.⁸⁾ We have therefore assumed that the interaction between s -wave pions and nucleon is described by the Hamiltonian used in that theory, namely,

$$H_s' = \lambda_0^0 \bar{\boldsymbol{\phi}} \cdot \bar{\boldsymbol{\phi}} + \lambda^0 \boldsymbol{\tau} \cdot (\bar{\boldsymbol{\phi}} \times \bar{\boldsymbol{\pi}})$$

where $\bar{\boldsymbol{\phi}}$ is the the integral of $\boldsymbol{\phi}$ multiplied by the form factor of nucleon, and λ_0^0 and λ^0 are adjustable parameters. We have neglected all terms quadratic in the s -wave amplitudes.

We substitute (2.17) and (1.17) with (1.22) into (1.16), and carry out the integrations over \mathbf{k}_1' and \mathbf{k}_2' . Comparing the result thus obtained with (1.13), we have

$$\begin{aligned}
 u^{(+)}(\omega, \sigma^2) &= -4\pi \operatorname{Re} l_0(-\kappa^2) [(\kappa^2 + \mu^2)/\kappa^2]^{1/2} \left\{ (1/3)(a_1 + 2a_3) + 2\pi \frac{f^2}{\mu^2} \frac{-\kappa^2 - \mu^2/2}{\sqrt{-\kappa^2 - \mu^2}} \right. \\
 &\quad \left. - (8/3\pi) \int_{\mu}^{\omega_{\max}} d\omega_k k^3 \omega_k |h_{33}(\omega_k)|^2 \left(\frac{\omega_k^2 - 2\kappa^2 - \mu^2}{\omega_k \sqrt{-\kappa^2 - \mu^2}} \tan^{-1} \frac{\sqrt{-\kappa^2 - \mu^2}}{\omega_k} - 1 \right) \right\}, \\
 v^{(-)}(\omega, \sigma^2) &= -4\pi M \operatorname{Re} l_1(-\kappa^2) [(\kappa^2 + \mu^2)/\kappa^2]^{1/2} \\
 &\quad \times \left\{ \pi (f^2/\mu^2) \sqrt{-\kappa^2 - \mu^2} + (2/3\pi) \int_{\mu}^{\omega_{\max}} d\omega_k k^3 \omega_k |h_{33}(\omega_k)|^2 \right. \\
 &\quad \left. \times \left(\frac{\omega_k^2 - \kappa^2 - \mu^2}{\omega_k \sqrt{-\kappa^2 - \mu^2}} \tan^{-1} \frac{\sqrt{-\kappa^2 - \mu^2}}{\omega_k} - 1 \right) \right\}, \\
 u^{(-)}(\omega, \sigma^2) &= -(2M)^{-1} (M\omega + 3\sigma^2 - M^2 - \mu^2) v^{(-)}(\omega, \sigma^2), \\
 v^{(+)}(\omega, \sigma^2) &= 0,
 \end{aligned}$$

where

$$\kappa^2 = (M\omega - \sigma^2)/2.$$

We have carried out the integration over ω_k making the narrow resonance approximation for $|h_{33}(\omega_k)|^2$ as in the case a). It should be noticed that in the present case one must cut off the integrals containing $u^{(\pm)}$ and $v^{(\pm)}$ in (1.14) and (1.15) at the lower limit $(\sigma^2 - 2\omega_{\max}^2)/M$ in accordance with the cut-off in the static theory. We take $\omega_{\max} = 6\mu$. But the results are quite insensitive to the cut-off energy, since we use the subtracted dispersion relations.

Numerical values of the parameters are taken to be

$$a_1\mu = 0.173, \quad a_3\mu = -0.110, \quad f^2 = 0.08.$$

The numerical results will be given in § 3.

§ 3. Numerical results and discussions

In this section we attempt to estimate the scattering amplitudes l_0 and l_1 for the pion-pion scattering, using our dispersion relations (1.14) and (1.15). We first introduce the following abbreviations:

$$\operatorname{Re}[A^{(+)}, A^{(-)}, B^{(+)}, B^{(-)}(\omega, \sigma^2)] = R_1, R_2, R_3, R_4(\omega, \sigma^2); \quad (3.1)$$

$$\begin{aligned}
 (P/\pi) &= \int_{[(M+\mu)^2 - 2\sigma^2]/2M}^{\infty} d\omega' / (\omega' - \omega) \operatorname{Im}[A^{(+)}, A^{(-)}, B^{(+)}, B^{(-)}(\omega', \sigma^2)] \\
 &= J_1, J_2, J_3, J_4(\omega, \sigma^2); \quad (3.2)
 \end{aligned}$$

$$\begin{aligned}
 (1/\pi) &= \int_{-\infty}^{(\sigma^2 - 2\mu^2)/M} d\omega' / (\omega' - \omega) [u^{(+)}, u^{(-)}, v^{(+)}, v^{(-)}(\omega', \sigma^2)] \\
 &= Q_1, Q_2, Q_3, Q_4(\omega, \sigma^2); \quad (3.3)
 \end{aligned}$$

$$-g^2 \left[1/(2M\omega + 2\sigma^2 - M^2) \pm 1/(4\sigma^2 - M^2 - 2\mu^2) \right] = \begin{pmatrix} B_3(\omega, \sigma^2) \\ B_4(\omega, \sigma^2) \end{pmatrix}; \quad (3.4)$$

$$S_i(\omega, \sigma^2) = R_i(\omega, \sigma^2) - J_i(\omega, \sigma^2) - B_i(\omega, \sigma^2), \quad (3.5)$$

where

$$B_1(\omega, \sigma^2) = B_2(\omega, \sigma^2) = 0.$$

Then (1.14) and (1.15) can be written compactly as

$$S_i(\omega, \sigma^2) = Q_i(\omega, \sigma^2). \quad (i=1, \dots, 4) \quad (3.6)$$

To get the subtracted dispersion relations, we differentiate both sides of (3.6) with respect to ω . Denoting partial derivatives of R_i, J_i, Q_i, B_i and S_i with respect to ω by R'_i, J'_i, Q'_i, B'_i and S'_i respectively, we have

$$S'_i(\omega, \sigma^2) = Q'_i(\omega, \sigma^2). \quad (i=1, \dots, 4). \quad (3.7)$$

S'_i can be computed by using the experimental pion-nucleon coupling constant and phase shifts. On the other hand, Q'_i is the integral over energies containing $\text{Re } l_0$ or $\text{Re } l_1$ in the integrand. So using the dispersion relation (3.7), we cannot determine the energy dependence of l_0 and l_1 precisely. Owing to the denominator $(\omega' - \omega)^2$ in the integrand of Q'_i , however, small values of ω' contribute mainly to Q'_i , so only the l_0 and l_1 in a low energy region are significant in Q'_i . At low energies the behaviors of l_0 and l_1 can, in general, be characterized by a few parameters, i.e., scattering lengths and effective ranges, etc., which provide us with good measures for the strength and sign of the pion-pion interaction. By means of (3.7), we can estimate the values of these parameters.

ω and σ^2 in (3.7) are related to the total kinetic energy w and the scattering angle θ for the pion nucleon scattering through (1.5) and (1.6). In numerical calculations, we set $w=0$ and $\cos \theta=1$. If we took w different from zero, we would have to calculate the differentiated principal-value integrals, and large errors would be unavoidable. On choosing $w=0$, and $\cos \theta=1$, R'_i is found to contain only s - and p -wave phase shifts, for which we use experimental values reported by Puppi at the CERN Conference in 1958.⁵⁾ Because of inaccuracy of the small p -wave phase shifts, considerably large errors appear in R'_i . In computing J'_i , we first expand $\text{Im } A^{(\pm)}$ and $\text{Im } B^{(\pm)}$ in the partial waves, and drop all the partial waves higher than the p -wave. For the (3,3) phase shift, we use the empirical formula given by Anderson.⁹⁾ In evaluating B'_i , we take $g^2/4\pi=14.4$.

The results thus obtained are shown in Table 1. From this table we see that S'_3 is small owing to the cancellation of two large terms, and this is not the case for the other S'_i . This is a satisfactory fact, since we have $v^{(+)}=0$, and hence $Q'_3=0$. However, the result $Q'_3=0$ has been obtained under the assumption that contributions of all the intermediate states other than two-pion states are negligible in F_- , and that all the partial waves higher than the p -wave are negligible in the

Table 1. Numerical values of R_i' , J_i' , B_i' and S_i' at $w=0$ and $\cos\theta=1$. Units are μ^{-2} for $i=1,2$ and μ^{-3} for $i=3,4$.

i	R_i'	$-J_i'$	$-B_i'$	S_i'
1	-2.7	- 7.2	0	- 9.9
2	-5.4	- 3.6	0	- 9.0
3	-2.4	-11.5	-11.7	- 2.6
4	-1.5	- 5.7	-11.7	-15.9

pion-pion scattering amplitude. Therefore, the nonvanishing value of S_3' must be due to the effects neglected by this assumption. It is natural to think that, in S_4' also there is a contribution of these neglected effects whose magnitude is about the same as that of S_3' since S_4' is of the same dimension as that of S_3' . When we use (3.7), this contribution should be regarded as an error in S_4' , since Q_4' has been calculated under the same assumption as above. This error is estimated to be ± 5 , i.e. about 30 %. (The value of $|S_3'|$ can be as large as ~ 5 on account of error in R_3' .) Errors of this kind will appear also in S_1' and S_2' . Though we have no way to estimate them, we will assume these to be 30 % as in S_4' . Furthermore, there is another error in each S_i' . This is mainly due to the inaccuracy of R_i' , and is estimated to be ± 3 . Thus the total error is estimated to be ± 6 for S_1' and S_2' , and ± 8 for S_3' and S_4' .*

Next we turn to the right-hand side of (3.7). Q_i' is given by the expression obtained from (3.3) by replacing the denominator $\omega' - \omega$ by its square. $u^{(\pm)}$ and $v^{(\pm)}$ in the integrand have been calculated by two methods; the one is explained in § 2 a) and the other, in § 2 b). The former will be referred to as the relativistic case and the latter, as the static case.

We found that Q_i' cannot be calculated with sufficient reliability, since this is a difference of two terms, both of which contain slowly converging integrals. Here we will calculate Q_1' and Q_4' only.

First we consider Q_1' . To perform the integration over ω' , it is necessary to assume the functional form of $\text{Re } l_0(-\kappa^2)$ appropriately. In terms of the phase shift for the pion-pion scattering, $\text{Re } l_0$ is written as

$$\text{Re } l_0(\omega_c^2) = (\omega_c/2k_c) \sin 2\delta_0, \quad (3.8)$$

where ω_c and k_c are respectively energy and momentum of either pion in the center-of-mass system, and δ_0 is the phase shift for the state with $l=0$ and $I=0$. Since the range of the pion-pion interaction is thought to be at most $(2\mu)^{-1}$, $\text{Re } l_0$ will not change very rapidly for $k_c \lesssim 2\mu$. On the other hand, we have actually found that the region of ω' corresponding to $k_c > 2\mu$ gives only negligible contribution to Q_1' .

* Such large errors are possible, but not so probable.

Taking these facts into consideration, we first set

$$(\omega_c/2k_c) \sin 2\delta_0 = \lambda_0 \quad (3.9)$$

where λ_0 is a real constant. Then we have these results

$$\begin{aligned} Q_1' &= -5.8 \lambda_0 \mu^{-2} && \text{for the relativistic case;} \\ &= -15.2 \lambda_0 \mu^{-2} && \text{for the static case.} \end{aligned}$$

Equating these values to the value of S_1' found above, we have

$$\left. \begin{aligned} \lambda_0 &= 1.7 \pm 1.0 && \text{for the relativistic case;} \\ &= 0.7 \pm 0.4 && \text{for the static case.} \end{aligned} \right\} \quad (3.10)$$

From this we can say that λ_0 is definitely positive. This means that the pion-pion interaction is attractive in the state $I=0$. The reason for this is as follows: If the pion-pion interaction is repulsive, δ_0 is negative and its magnitude is smaller than $\pi/2$, and hence $\text{Re } l_0$ is negative for small values of k_c . Value of k_c at which $\text{Re } l_0$ changes its sign is at least about the reciprocal of the range of pion-pion interaction, which is 2μ in the present case. As was mentioned above the region $k_c > 2\mu$ gives no appreciable contribution to Q_1' . Thus, with the repulsive interaction, we would necessarily have a positive Q_1' , which definitely contradicts the negative S_1' .

To get a more adequate measure of the strength of the interaction than λ_0 , we next make the scattering-length approximation; namely, we set

$$(k_c/\omega_c) \cot \delta_0 = 1/\alpha_0, \quad (3.11)$$

where α_0 is a real constant and is regarded as the scattering length, in units of μ^{-1} , of the pion-pion scattering in the states $I=0$. The functional form of $\text{Re } l_0$ is determined by (3.8) and (3.11), and Q_1' can be computed for any given value of α_0 . The results are shown in Table 2. All the values of Q_1' in this table are

Table 2. Numerical values of Q_1' in the scattering-length approximation.

α_0	Q_1' in units of μ^{-2}	
	relativistic case	static case
0.5	-4.2	-6.4
1	-5.6	-9.0
2	-5.4	-8.8
3	-3.8	-7.4

smaller in magnitude than S_1' in Table 1, but, within the error S_1' mentioned previously, any of them is in agreement with S_1' . The best agreement is obtained for $\alpha_0=1\sim 2$. If $\alpha_0 > 1$, it is possible to increase the magnitude of Q_1' to some extent by adding the effective-range term to the right-hand side of (3.11),

and thus to improve the agreement. On the contrary, if $\alpha_0 < 1$, the addition of the effective-range term will decrease $\text{Re } l_0$, and the agreement will become worse.* Therefore, if α_0 is much smaller than unity, Q_1' can never agree with S_1' . Thus we are led to the conclusion that, for the pion-pion scattering in the state $I=0$, the scattering length is of the same order as the Compton wave length of pion.

Next we consider Q_4' . We can compute this if $\text{Re } l_1$ is given. This is written as

$$\text{Re } l_1(\omega_c^2) = (\mu^2 \omega_c / 2k_c^3) \sin 2\delta_1, \quad (3.12)$$

where δ_1 is phase shift for the pion-pion scattering in the state with $I=1$ and $l=1$. Analogously to the previous case, we first set

$$(\mu^2 \omega_c / 2k_c^3) \sin 2\delta_1 = \lambda_1, \quad (3.13)$$

where λ_1 is a real constant. Then we have

$$\begin{aligned} Q_4' &= -60.4 \lambda_1 \mu^{-3} && \text{for the relativistic case;} \\ &= -79.8 \lambda_1 \mu^{-3} && \text{for the static case.} \end{aligned}$$

From this and the value of S_4' given in Table 1, taking account of the error ± 8 in S_4' , we get

$$\begin{aligned} \lambda_1 &= 0.27 \pm 0.13 && \text{for the relativistic case;} \\ &= 0.20 \pm 0.10 && \text{for the static case.} \end{aligned}$$

Thus λ_1 is seen to be definitely positive. From the same reason as the previous case, this means that the pion-pion interaction is attractive in the state $I=1$. Therefore, it is possible that the resonance scattering occurs in this state as in the $(3, 3)$ state in the pion-nucleon scattering.

For the purpose of comparison, we next make the effective-range approximation by setting

$$(\mu^2 \omega_c / k_c^3) \tan \delta_1 = \alpha_1 (1 - \omega_c / \omega_r)^{-1}, \quad (3.14)$$

where α_1 is a real constant, and ω_r is the resonance energy. Then we can compute Q_4' for any given values of α_1 and ω_r . The results are shown in Table 3. Any value of Q_4' in this table is in agreement with S_4' in Table 1 within the error. It is seen by the extrapolation that the lower limit of α_1 giving the agreement is about 0.1 for $\omega_r = 3\mu$, and is about 0.2 for $\omega_r = \infty$. Good agreement is, however, obtained for $\alpha_1 = 0.5 \sim 1.0$. These latter values of α_1 are much larger than the corresponding values of the $(3, 3)$ state in the pion-nucleon scattering, because in the latter case we know that the value of the quantity corresponding to α_1 is

* Since the effective range is in general positive for the attractive interaction, the addition of the effective-range term to the right-hand side of (3.11) increases $\cot \delta_0$, and decreases δ_0 . If $\alpha_0 < 1$, δ_0 determined by (3.11) never exceeds $\pi/4$, therefore $\text{Re } l_0$ given by (3.8) decreases with δ_0 .

Table 3. Numerical values of Q_4' in the effective-range approximation

α_1	Q_4' in units of μ^{-3}			
	relativistic case		static case	
	$\omega_r=3\mu$	$\omega_r=\infty$	$\omega_r=3\mu$	$\omega_r=\infty$
0.25	-10.0	-7.8	-12.0	-10.0
0.50	-12.0	-12.0	-15.0	-13.8
0.75	-13.6	-14.0	-17.4	-16.2
1.00	-14.4	-15.2	-19.4	-17.6

about 0.1. In the present case it is possible that the smaller values of α_1 are sufficient for good agreement between Q_4' and S_4' , if more appropriate values of ω_r are chosen. Though we can say nothing quantitatively, we can conclude that the pion-pion interaction in the state $I=0$ is the strength at least comparable to that of the pion-nucleon interaction in the $(3, 3)$ state.

Finally we add one remark. As was mentioned previously, we set $w=0$ and $\cos\theta=1$ in computing both sides of the dispersion relations. Then the path of integration over ω' is a horizontal straight line passing through the point A in Fig. 1, and consequently it is seen that the partial-wave expansion of $\text{Im } A^{(\pm)}$ and $\text{Im } B^{(\pm)}$ made in computing J_i' can be justified by the Lehmann's theorem only below the resonance energy. We see also that, if we shift this path slightly above, the energy region, in which the partial wave expansion is allowed, is far more increased. Indeed, if we set $w=63$ Mev and $\cos\theta=1$, this energy region covers almost all energies which contribute appreciably to J_i' . However, we have not made calculations with $w=63$ Mev, because, as was mentioned previously, there is a difficulty in computing J_i' accurately, unless w is zero. Instead of doing this, we have attempted to justify our calculations with $w=0$ in the following way: in contrast with J_i' , J_i can be computed with sufficient accuracy even if $w \neq 0$. We therefore compute both sides of (3.6), the dispersion relations without subtraction, for $w=0$ and $w=63$ Mev keeping $\cos\theta=1$. If both of the two choices of w yield the same result, we may consider that the calculations using (3.7) at $w=0$ have been justified. For that purpose it is convenient to take $i=1$ in (3.6), since S_1 does not contain the ambiguous Born term, and Q_1 can be computed with much more reliability than Q_2 . In the static case Q_1 turns out to be independent of w if $\cos\theta$ is kept to unity. In the relativistic case, on assuming (3.9), we have $Q_1=23.4 \lambda_0 \mu^{-1}$ for $w=0$, and $Q_1=27.0 \lambda_0 \mu^{-1}$ for $w=63$ Mev. On the other hand we have $S_1=26.6 \mu^{-1}$ for $w=0$ and $S_1=25.4 \mu^{-1}$ for $w=63$ Mev. From these results it can be said our treatment has been justified.

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Effect of Strange Particles on the S-Wave Pion-Nucleon Scattering

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Effect of strange particles on the S-wave pion-nucleon scattering at low energies is examined qualitatively. It is shown that this effect is too small to play an essential role in the suppression of the isospin-even part of the phase shift, while this effect is very important in the isospin-odd part.

§ 1. Introduction

As is well known the iso-even part of the S-wave pion-nucleon scattering phase shifts at low energies is too large in the Born approximation of the pseudo-scalar pion theory, while as for the iso-odd part the order of magnitude fits the experimental value in the same approximation. Provided that the traditional pseudoscalar pion theory works in this case, it is expected that higher order effects will suppress the iso-even part. Recently one of us (R. S.)¹⁾ investigated this problem using the Chew-Low method. Essential points of his result are as follows. As for the iso-even part, the fourth order pion effect (which differs from the fourth order effect of the usual perturbation calculation) is opposite in sign with the second order one and is so large that the order of magnitude of the experimental value can be reproduced by an appropriate cut-off factor (e. g. straight cut-off with $\omega_{\text{max}} = 8\mu$ for S-wave pion and 6μ for P-wave one). By the same recipe of calculation, however, a similar cancellation unfortunately occurs also for the iso-odd part, which then becomes too small. As a result of this the order of magnitude of the phase shifts, $\delta_{1/2}$ and $\delta_{3/2}$, can be reproduced but their signs become both negative in contradiction with the experimental values.

Since the fourth order pion effect cancels the second order one leaving their small difference, the sixth and higher order pion effects as well as the effect of strange particles become important. The purpose of this note is to examine the effect of strange particles.

The effect of strange particles was previously investigated by Langer²⁾ and

Nakayama.³⁾ However, their treatments do not seem satisfactory, particularly as to the higher order pion effects. Namely, they took into account, as the pion effect, only the Born term and its iteration and discarded contributions from multi-nucleon-pair states. But as was emphasized in I, the virtual states in which two pairs of nucleon-antinucleon are contained are very important. Moreover, the dependence of the result on the cut-off factor is rather sensitive and should be examined deliberately. In the following, new lights will be shed on these point at issue.

§ 2. Method of calculation

We assume that the baryons N , Λ and Σ , have same intrinsic parities and the kaon has odd parity relative to the baryons. The parity, isospin and strangeness are assumed to be conserved in the strong interactions. We consider the following sets of interactions with the pseudoscalar coupling: $NN\pi$, $\Sigma\Lambda\pi$, $\Sigma\Sigma\pi$, NAK and $N\Sigma K$. Their coupling constants (unrationalized and renormalized) are denoted by f_N , f_Λ , f_Σ , g_Λ and g_Σ , respectively.

The method of calculation is the same as in I. In the static approximation the T -matrix element for the S -wave pion-nucleon scattering with isospin $I(=1/2, 3/2)$ is written as $-2\pi(\omega_p\omega_q)^{-1/2}h_I(\omega_p)$. Here \mathbf{q} and \mathbf{p} are the initial and final pion momenta, respectively, and $\omega_p=(\mu^2+p^2)^{1/2}$, μ being the pion mass. The Low equation for $h_I(\omega)$ is

$$h_I(\omega) = V_I(\omega) + \frac{1}{\pi} \int_{\mu}^{\infty} d\omega_p p v_p^2 \left\{ \frac{|h_I(\omega_p)|^2}{\omega_p - \omega - i\epsilon} + \sum_{I'} A_{II'} \frac{|h_{I'}(\omega_p)|^2}{\omega_p + \omega} \right\}, \quad (2.1)$$

with

$$A_{II'} = \frac{1}{3} \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix},$$

where v_p is a cut-off factor. The natural unit ($c=\hbar=1$) is used. The term $V_I(\omega)$ will be called MBT (modified Born term), of which the "experimental value" was estimated in I by the effective range approach. The calculation of MBT was made in I up to the order of f_N^4 . We would like to emphasize again the difference between our calculation and the ordinary perturbation theory. In the calculation of MBT such a time ordered diagram as shown in Fig. 1 should be excluded, because it is automatically included in the integral term of Eq. (2.1) as a repetition of the second order scattering.

Now we calculate the effect of strange particles by the lowest order ($f_{\Lambda,\Sigma}^2 g_{\Lambda,\Sigma}^2$) diagrams. We take only the diagram of Fig. 2. Other diagrams of this order, i. e. vertex and self-



External pion lines attach the vertexes with \bigcirc .

— : N , ~ : π

Fig. 1 (time ordered)

energy corrections, are considered as absorbed in the renormalized coupling constants and baryon masses, though this approximation may quantitatively be very crude. Contrary to the fourth order diagram of the pion effect, the diagram of Fig. 2 does not contain the $\pi+N$ scattering state as an intermediate one. Therefore the calculation in this case is the same as in the ordinary perturbation theory.

In the calculation of MBT, we do not use the static approximation but include the recoil effect because it is not negligible compared with the small difference between the second and fourth order terms.

Expanding the final result with respect to ω/M (M is the nucleon mass), we have, up to the order of ω/M

$$\begin{aligned} V_{1/2}(\omega) &= 2f_N^2(\alpha_e - 2\beta_o\omega), \\ V_{3/2}(\omega) &= 2f_N^2(\alpha_e + \beta_o\omega), \end{aligned} \quad (2.2)$$

with

$$\begin{aligned} \left(\frac{2M}{\mu}\right)\alpha_e &= -1 + \frac{3f_N^2}{\pi}(I_\pi^S + I_\pi^P) + \frac{1}{\pi f_N^2}\{f_\Lambda^2(g_\Lambda^2 + g_\Sigma^2) + f_\Sigma^2 g_\Sigma^2\}(I_K^S + I_K^P), \\ \left(\frac{2M}{\mu}\right)\beta_o &= \frac{-1}{2M} + \frac{f_N^2}{\pi}(J_\pi^S + J_\pi^P) + \frac{1}{\pi f_N^2}\{f_\Lambda^2 g_\Sigma^2 + 2f_\Lambda f_\Sigma g_\Lambda g_\Sigma + f_\Sigma^2 g_\Sigma^2\}(J_K^S + J_K^P). \end{aligned} \quad (2.3)$$

The suffices e and o mean the iso-even and iso-odd parts, respectively. I 's and J 's denote energy integrals of the fourth order diagrams. The suffices S , P and π , K of I 's and J 's denote the S -wave, P -wave effects and the pion, kaon effects, respectively.

§ 3. Result and discussion

In the numerical evaluation the masses of Λ and Σ are put to be their average value, and a straight cut-off is used, though the integrals converge. As for the coupling constants we take for simplicity $f_N^2 = f_\Lambda^2 = f_\Sigma^2 = 15$, $g_\Lambda^2 = g_\Sigma^2 = 3 \sim 5$,* and $f_\Lambda f_\Sigma g_\Lambda g_\Sigma > 0$. The result is shown in Table I and Fig. 3. The experimental values were given in I as $\alpha_e \approx \beta_o \approx -0.01\pi/f_N^2$, or $(2M/\mu)\alpha_e \approx (2M/\mu)\beta_o \approx -0.03$. From these results we have the following conclusions.

i) *Iso-even part* The kaon effect is small compared with the pion one, though its sign is favorable, hence it cannot be an important factor in the problem of the S -wave suppression. Therefore the situation of the iso-even part is almost unchanged from that described in I where only the pion effect was considered, except



— : Λ, Σ
 ~ : K

Fig. 2 (not time ordered)

* This value is consistent with the result of the dispersion theoretical analysis of the $K^+ + p$ scattering. For example, Igi⁴⁾ estimated this as $g_\Lambda^2 + g_\Sigma^2 \approx 8$.

Table. I. Experimental values are $10^3 \cdot (2M/\mu)\alpha_e \sim 10^3 \cdot (2M/\mu)\beta_o \sim -3 \times 10$.

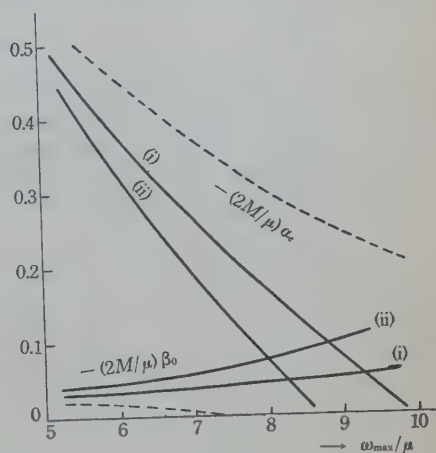
	$\frac{\omega_{\max}}{\mu}$	2nd order	4th order					
			pion effect		kaon effect $g_\Lambda^2 = g_\Sigma^2 = 3$		kaon effect $g_\Lambda^2 = g_\Sigma^2 = 5$	
			S-wave	P-wave	S-wave	P-wave	S-wave	P-wave
$\frac{2M}{\mu} \alpha_e \times 10^3$	6	-1000	1218	-674	99	-21	165	-35
	7		1469	-841	147	-39	245	-65
	8		1695	-997	169	-58	281	-97
$\frac{2M}{\mu} \beta_o \times 10^3$	6	-75	41	18	-4	-15	-7	-25
	7		48	22	-7	-26	-12	-44
	8		54	26	-9	-30	-15	-65

a slight change in the choice of the cut-off energy.

ii) *Iso-odd part* The second order term is largely cancelled out by the fourth order pion term, but recovered by the kaon term. If $g_\Lambda^2 \approx g_\Sigma^2 \approx 3$ the experimental value is reproduced. Thus an embarrassing conclusion of I is dissolved. Contrary to the iso-even part, the iso-odd part including kaon effect is rather insensitive to the cut-off energy. As long as the approximation up to fourth order with respect to the strong interactions is considered, the kaon effect seems to be very important for the iso-odd part.

iii) *Common cut-off for S- and P-waves* The values of α_e and β_o with common cut-off energies for the S- and P-waves are shown in Fig. 3. We see that the experimental values of the iso-even and iso-odd parts are both reproduced if $g_\Lambda^2 \approx g_\Sigma^2 \approx 2$ and the cut-off energy $\approx 10\mu$ are employed.

Our conclusion i) differs from that of Langer,²⁾ who stated that the iso-even part of the kaon effect is very large and almost cancels the second order term. It should be noted, however, that the kaon effect is sensitive to the cut-off, and is indeed very small, if a cut-off energy near the baryon mass is chosen, as shown in Table I. Nakayama³⁾ investigated the kaon effect assuming interaction scheme different from ours, but his conclusion is similar to ours as far as the kaon effect is concerned. Using a cut-off energy \approx hyperon mass, he obtained a large splitting of the phase shifts due to the kaon effect but no appreciable suppression of the iso-even part.



..... : pion effect only
 — : including kaon effect,
 (i) $g_\Lambda^2 = g_\Sigma^2 = 3$, (ii) $g_\Lambda^2 = g_\Sigma^2 = 5$
 Fig. 3 $-(2M/\mu)\alpha_e$ and $-(2M/\mu)\beta_o$ vs cut-off energy ω_{\max}

We have throughout neglected radiative corrections, namely vertex and self-energy corrections, since we have no reliable method of estimation of these corrections. According to the ordinary fourth order perturbation calculation these radiative corrections are of comparable order and of the same sign with the contribution of the diagram of the so-called convergent type (e. g. Fig. 2).⁵⁾ For a quantitative analysis these radiative corrections will not be negligible and must be subjected to further investigation. We think, however, that the qualitative character of the kaon effect is clarified by our calculation.

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On the Strong Interaction

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A model for the strong baryon-meson interaction is proposed (eq. (9)), and its consequence is discussed.

There have been presented many ideas to assume that the strong interactions are of a high symmetry concerning the Baryon states¹⁾. To assume the complete symmetry for the both of the π - and K -strong interactions, however, cannot be successful, because this assumption forbids the Σ^+ -production due to the K^-p collision²⁾. Assuming the complete symmetry only for π -strong interactions, we are led to the theory of Global symmetry, where K -strong interactions are assumed to be much weaker than π -strong interactions³⁾. The lack of symmetry in K -strong interactions, however, throws slight doubt on the assumption of the complete symmetry in π -strong interactions. Furthermore, the theory of global symmetry has not succeeded in describing phenomena due to strong interactions. It has not been clear what sorts of symmetry laws hold in the world of the baryons and mesons. This is a situation similar to that in the case of electromagnetic interactions. As for the latter interactions, we do not know why the μ -meson and electron interact with the electromagnetic field and the neutrino does not; we know only that all the particles interacting with the electromagnetic field (i.e. the charged particles) have the same electromagnetic coupling constant. This situation can be expressed in a way of saying that the electromagnetic coupling constant belongs to the photon rather than to the charged particles. Thus, it is interesting to assume that the π -strong coupling constant g_π^0 belongs to the pion and K -strong coupling constant g_K^0 does to the K -meson. In other words, we assume that

$$\begin{aligned} |\pi\text{-coupling constant}| &= g_\pi^0 \quad \text{or} \quad \text{zero}, \\ |K\text{-coupling constant}| &= g_K^0 \quad \text{or} \quad \text{zero}. \end{aligned} \tag{1}$$

It is to be noted this kind of universality also holds for weak interactions. If we assume intermediary bosons for this interaction, the situation is analogous to the present case.

Let us write the strong interaction in the following form,

$$H = H(NN\pi) + H(\Sigma\Sigma\pi) + H(\Sigma\Lambda\pi) + H(\Xi\Xi\pi) + H(NAK) \\ + H(N\Sigma K) + H(\Lambda\Xi K) + H(\Sigma\Xi K), \quad (2)$$

where

$$H(NN\pi) = ig_1^0 \bar{N} \gamma_5 \tau_k N \pi_k, \\ H(\Sigma\Lambda\pi) = ig_2^0 \bar{\Sigma}_k \gamma_5 \Lambda \pi_k + \text{h. c.}, \\ H(\Sigma\Sigma\pi) = -g_3^0 (\Sigma \times \gamma_5 \Sigma \cdot \pi), \\ H(\Xi\Xi\pi) = ig_4^0 \bar{\Xi} \gamma_5 \tau_k \Xi \pi_k.$$

The symbols Σ and π denote the isotopic vectors $(\Sigma_1, \Sigma_2, \Sigma_3)$ and (π_1, π_2, π_3) respectively, and the last four terms in H are the K -strong interactions. Here is assumed the charge independence of the strong interactions.

The theory of global symmetry is obtained when we assume $g_1^0 = g_2^0 = g_3^0 = g_4^0 = g_\pi^0$. This theory, however, has not succeeded in explaining phenomena due to strong interactions. Applying the perturbation calculation to the K -strong interaction, we are led, in the theory of global symmetry, to wrong branching ratios of the Σ^- and Λ -productions due to K^-p collision processes⁴⁾ and to the wrong angular distributions for the K -production by π -nucleon collisions. The conclusion is independent of the assumption of the weakness of the K -interaction if the interaction is of the γ_5 -type. The branching ratios of the reaction $K^- + p \rightarrow \Sigma^{\pm 0} + \pi^{\mp 0}$ with K^- nearly at rest tells us that $|\delta_1 - \delta_0| = 70^\circ$, where δ_1 and δ_0 are phase shifts of the π - Σ scattering in the final state of isotopic spin 1 and 0, respectively. It is shown that g_2 and g_3 should be much different from each other in order to give the large phase shift difference $|\delta_1 - \delta_0| = 70^\circ$.⁵⁾

It is also difficult for the theory of global symmetry to understand the K -nucleon scattering: this process suggests us that the K -interaction is not necessarily weaker than the π -interaction because the cross section $\sigma(K + N \rightarrow K + N)$ is about equal to that of the pion-nucleon scattering.⁶⁾

Another objection to the global symmetry comes from the mass difference among Σ 's, because the large mass difference between Σ^+ and Σ^- is very hard to understand in this model.⁷⁾

Let us now come to our assumption (1). Since the existence of Λ -hyperfragments shows that g_2 cannot be much smaller than g_1 , our assumption (1) leads us to the following assumption,

$$|g_1^0| = |g_2^0| = g_\pi^0, \\ g_3^0 = 0 \quad (\text{i. e. } H(\Sigma\Sigma\pi) = 0). \quad (3)$$

The g_3^0 is put equal to zero in order to get rid of the theory of global symmetry. This is consistent with the fact that no Σ -hyperfragment has been observed. The renormalized value g_3 is not zero because of corrections due to the strong interactions. However, in view of the near equality of the Gamow-Teller coupling constant to the Fermi coupling constant, we may suppose that this renormalization is rather

small. We may thus expect that

$$\begin{aligned} g_3 &\lesssim 0.3 g_1, \\ g_1 &\approx g_2 \approx \sqrt{15} \end{aligned} \quad (4)$$

for the renormalized constants.

Taking account of the large cross section for the process $K+N \rightarrow K+N$ and small cross section for $\gamma+p \rightarrow \Lambda+K^+$, we assume that

$$H(N\Lambda K)=0, \quad \text{coupling constant of } H(N\Sigma K)=g^0. \quad (5)$$

Under this assumption the observed value $g_{N\Lambda K}$ of the $(N\Lambda K)$ -coupling constant is given by effects due to the π -meson cloud, and therefore, it is expected that

$$g_{N\Lambda K} \lesssim 0.3 g_{N\Sigma K}. \quad (6)$$

It has been reported that a differential cross section for $\gamma+p \rightarrow \Sigma^0+K^+$ is about equal to that for $\gamma+p \rightarrow \Lambda+K^+$.⁸⁾ This, however, does not necessarily contradict with the relation (5), because the total cross section for the former process has not been measured. Experiments have shown that the cross section for $\Sigma^-+p \rightarrow \Lambda+n$ is slightly larger than that for $\Sigma^-+p \rightarrow \Sigma^0+n$.⁹⁾ It seems that the relations (4) and (6) lead us to the former cross section much larger than the latter, because the process $\Sigma^-+p \rightarrow \Lambda+n$ is induced by exchange of one pion and $\Sigma^-+p \rightarrow \Sigma^0+n$ does by exchange of two pions or of one K -meson. We shall leave this problem to further research on the nucleon-hyperon potentials.

We have so far mentioned nothing about the Ξ particle, because we do not know much about its interactions. It is, however, remarkable that the well-known relation

$$\frac{M_N+M_\Xi}{2} = \frac{3M_\Sigma+M_\Lambda}{4} \quad (7)$$

for the masses comes out when we make use of the lowest order perturbation calculation under the following assumption:

$$\begin{aligned} H(\Xi\Xi\pi) &= 0, \\ H(\Sigma\Xi K) &= 0, \end{aligned} \quad (8)$$

coupling constant of $H(\Lambda\Xi K)=g^0$.

Indeed, it is an easy matter to show that the self energies of baryons are of the following forms:

$$\begin{aligned} M_\Xi &= g^{02} I_K, \\ M_\Sigma &= 2g^{02} I_K + g_\pi^{02} I_\pi, \\ M_\Lambda &= 2g^{02} I_K + 3g_\pi^{02} I_\pi, \\ M_N &= 3g^{02} I_K + 3g_\pi^{02} I_\pi. \end{aligned}$$

The relation (7) has been once obtained in the theory of global symmetry.¹⁰⁾ We now see that the assumptions (4), (6) and (8) also lead us to (7).

The integrals I_κ and I_π must be negative in order to reproduce the observed mass levels while the straightforward perturbation calculation gives positive I if the coupling is of γ_5 -type. However, we have some evidences that the occurrence of nucleon-antinucleon pairs which gives positive contribution to self-energies are suppressed, and negative I 's would not be unreasonable.

Although we have not attempted the calculation of the mass difference $M_\Sigma^- - M_\Sigma^+$, we may point out that the difficulty inherent to the global symmetry is removed here.

Summarizing the results, we have

$$\begin{aligned} g_1^0 &= g_2^0 = g_\pi^0, \\ g_3^0 &= g_4^0 = 0, \\ g_{N\Sigma\kappa}^0 &= g_{\Xi\Lambda\kappa}^0 = g_\kappa^0, \\ g_{N\Lambda\kappa}^0 &= g_{\Xi\Sigma\kappa}^0 = 0. \end{aligned} \tag{9}$$

This choice of constants seems very promising for describing phenomena concerning strong interactions.

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On the Theory of Superexchange Interaction

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The superexchange interaction for the spin coupling in certain magnetic compounds is treated on the basis of an approach which takes into account the deformation of the electron orbitals of the intervening ions. Quantitative calculations are carried out for three centre and four electron system. The polarization of the intervening ion is considered by assuming slightly different orbitals for the two electrons with anti-parallel spins. This corresponds to the procedure of taking certain excited configurations into considerations. The effect of the excited states in which one of the anion electrons changes its spin is also considered by using the perturbation method. If one regards the triplet and singlet states of the system as the representatives of the ferromagnetic and antiferromagnetic states respectively, their energy difference arises as a result of the latter effect, while the same occurs by merely considering the former deformation if one uses the single determinant approximation. A rough numerical estimate gives reasonable agreement with the observed Néel temperature for MnO.

§ 1. Introduction

Since the suggestion of Kramers¹⁾ various coupling schemes invoking the role of the intervening non-magnetic ions in the indirect exchange mechanism have been proposed by several authors.²⁾ Because of the inherent difficulties involved in determining the antiferromagnetic ground state of the crystal, most of the works have been concerned mainly with the simplest model for the problem, i.e. the three centre and four electron system, which involves one electron each in the two magnetic ions and two coupled electrons occupying the same orbital in the intervening ion. In order to get the spin dependent energy difference, certain excited configurations of the system have to be taken into account as perturbing states to the zeroth order ground state which is assumed to be completely ionic.

In Anderson's well-known work,³⁾ which was further clarified by Van Vleck,⁴⁾ the states in which one electron of the intervening ion is transferred to one of the magnetic ions are considered as such excited configurations. As is strongly criticized by Slater,⁵⁾ the non-orthogonality of the orbitals is neglected in this treatment.

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Pratt⁶⁾ studied the problem using appropriately orthogonalized orbitals and concluded that the configurations considered by Anderson are not adequate and highly excited configurations where two electrons are transferred from the central ion to the magnetic ions give rise to the principal effect.

Yamashita and Kondo⁷⁾⁽⁸⁾⁽⁹⁾ have considered the non-orthogonality by expanding the energy expressions in power series of the overlap integral. This method, however, is not very appropriate to see the physical meaning of the mechanism. As is pointed out by Slater,⁵⁾ the use of the non-orthogonal orbitals not only gives rise to the mathematical difficulties in computing the matrix elements, but also obscures the concept of "*configuration*". If we use the term "*configuration*" only for those constructed from orthogonal orbitals, then the Heitler-London type wave-function obtained by accommodating electrons in non-orthogonal atomic orbitals is shown to be a mixture of several configurations in general. Conversely, if we construct Heitler-London like wave-function of a system, say a hydrogen molecule, by accommodating one electron each in the two localized orbitals, which are artificially orthogonalized in the same way as Wannier functions, it is easily seen that some amount of ionic structures are automatically admixed by the orthogonalization procedure in the case of the singlet state.⁵⁾ In this sense, therefore, the calculation taking account of the effect of overlap is equivalent to those in which two-electron transfer between the central ion and the magnetic ions is taken into consideration in part.

The extension of the quantitative treatment to the crystal was first attempted by Yamashita and Kondo.⁷⁾⁽⁸⁾⁽⁹⁾ Since the real antiferromagnetic spin state is entirely unknown, they estimated the energy of the "*ordered antiparallel spin state*", which is not an eigenstate of the total spin of the crystal. Although Kondo⁹⁾ has recently shown that the energy eigenvalue of the ordered antiparallel spin state lies in between those of the singlet and triplet states in the case of the three centre four electron system for some special models of the interaction, it is a little dangerous to assume the validity of this relationship in general.

Quite recently, Anderson¹⁰⁾ has suggested a new formal approach to the theory of superexchange interaction of the antiferromagnetic crystals. He introduces the new word "*spolaron*" for a single *d*-electron in the presence of the diamagnetic lattice. The spolaron represents an electron associated with its spin polarization and is regarded as an in principle exact solution of the one-electron problem plus the diamagnetic lattice. He assumes that the energy of the crystal is expressed as the sum of the energies of the running spolarons. By applying the Wannier transformation, he expresses the Hamiltonian of the system in terms of the operators associated with "*localized spolarons*". Then, as a matter of course, the off-diagonal terms, which represent the migration of the localized quasi particles from site to site, turn out to be of essential importance. Thus he claims that the virtual transfer of the electrons between the magnetic ions plays the dominant role in the indirect coupling of the spins. We may, however, say that this is the

natural conclusion derived from the *ad hoc* assumption.

Since each of the magnetic ions has an incomplete d -shell and is separated from others by intervening non-magnetic ions, it is impossible to express the state of the whole crystal, except for the completely ferromagnetic one, as a simple linear combination of the Slater determinants constructed from Bloch functions. The antiferromagnetic state must be a mixture of an enormous number of configurations expressed in terms of the Bloch functions; on the other hand, apart from the spin orientation, the same state can be represented by a single configuration in the Heitler-London scheme to a good approximation. Thus the adoption of the band-scheme is quite inadequate to describe the state of zeroth approximation of the insulator crystal under consideration. Therefore, even if we accept Anderson's assumption that the state is expressed as an assembly of the running spinlons, the character of the constituent quasi particles might be entirely different from that of the electrons expressed by Bloch functions of the d -band. Thus his quantitative estimate cannot be accepted to give a correct criterion for the most important mechanism of the superexchange interaction.

As a possible mechanism of the indirect spin coupling, Slater¹¹⁾ proposed that the deformation or polarization of the intervening ion stabilizes the energy of the antiferromagnetic state. This indirect exchange coupling is classified as a different mechanism from that of Anderson-Van Vleck in many articles. However, the latter may be regarded as a special case of Slater's idea in which the deformation is expressed as the transfer of the electron from the intervening ion to one of the magnetic ions. In fact, if one takes account of the mechanism suggested by Anderson and Hasegawa (cf. § 8.2 of ref. 2), it can be generalized and comes nearer to Slater's effect of deformation.

The deformation, however, can be formulated in a different manner by considering the excited states in which the electrons of the intervening ion are excited to its higher orbitals. Yamashita and Kondo⁸⁾ treated this kind of model in their paper as a possible mechanism. Instead of evaluating many integrals that appeared in the calculation, they expressed their result by making use of the dielectric properties of the crystal. Their investigation is, however, not very satisfactory in that they used some rather arbitrary assumptions and the result is merely qualitative.

In view of these circumstances, it seems necessary to carry out more quantitative calculation for the three centre four electron system taking the excited orbitals other than the occupied cation orbitals into account.

In what follows, the formulation will be made by using determinantal wavefunctions, because the use of the effective Hamiltonian expressed, for instance, in terms of Dirac's exchange operator, is less general and sometimes rather dangerous.⁵⁾ In fact, one of Anderson's conclusions³⁾ that the second order term is independent of the spin configurations is to be related to the artificial restrictions imposed on the various parts of the perturbing Hamiltonian which render the orbital transition spin independent and diagonalize the spin dependent part with respect to the orbital

state. There does not seem to be any sound justification for these assumptions. Actually, Nesbet¹²⁾ (as well as Pratt⁶⁾) shows the lowering of the anti-ferromagnetic state which appears in the second order term by taking account of the excited state in which two electrons are transferred from the intervening ion to the magnetic ones.

As seen in many calculations concerning the energy eigenvalues of the simplest molecules, especially for the triplet ground state of the oxygen molecule, the quantitative calculation is quite difficult for such a delicate problem as the energy difference between ferro- and antiferromagnetic states even for the simplest model as the three centre and four electron system. The purpose of the present paper is not to claim that the mechanism suggested here is the most important one, but to point out in detail that it constitutes an important spin dependent effect which seems to have been overlooked hitherto in discussing the magnetic properties of certain compounds.

§ 2. Single determinant approximation

It can be said in a qualitative sense that the indirect spin coupling is a consequence of the spatial correlation between the electrons of the central ion. In fact, this aspect of the interaction was the reason for the preference of the Heitler-London method by several authors, because the molecular orbital approach is not adequate for this purpose unless the configuration interaction is taken into account. In the present paper an attempt is made to describe the situation by using an approach intermediate between the HL and the MO schemes, which is similar to the method proposed by Müller and Eyring¹³⁾ for the calculation of electronic states of molecules and extended by Löwdin¹⁴⁾ to solids. This method of semi-localized or alternant orbitals leads to improved results for the calculation of energies because the correlation effect between electrons with antiparallel spins can be taken care of by this method. In what follows we shall consider two mechanisms for the deformation of the intervening ion; one is that pointed out by Slater and the other due to the Coulomb repulsion between two electrons in the ion.

We shall consider the symmetrical case $M^{2+}-X^{2-}-M^{2+}$. According to the idea of polarization or deformation of the intervening ion, it appears that the charge density of the two anion electrons is modified in such a manner that they interact differently with the two magnetic electrons if the spins of the magnetic ions are antiparallel. The np_σ -function of the intervening anion X , which is an odd function with respect to the plane intersecting the molecular axis normally at the centre, will be denoted by ϕ_0 . In order to bring about the deformation of this anion orbital by available excited orbitals, we consider the effect of orbitals having odd and even symmetry which are respectively denoted by ϕ_u and ϕ_g . Since the usual superexchange mechanism is out of our consideration in the present paper, the singly occupied d -orbitals of the magnetic ions are not taken into account as the excited orbitals.

The modified charge density of the two anion electrons is expected to be such that if the one points towards the first cation, the other points towards the second. To describe this physical situation the following symmetry relationship between the modified anion wave-function, i.e., ϕ_1 and ϕ_2 , is tentatively introduced. Under the reflection with respect to the symmetry plane defined above, these transform as $\phi_1 \rightarrow \phi_2^*$ and $\phi_2 \rightarrow \phi_1^*$. As a general case, we choose the following form of ϕ_1 and ϕ_2 , which have the desired property :

$$\phi_1 = c_0 \phi_0 + c_u \phi_u + c_g \phi_g \quad (1)$$

$$\phi_2 = -c_0^* \phi_0 - c_u^* \phi_u + c_g^* \phi_g, \quad (2)$$

where the coefficients are complex numbers, namely $c_0 = |c_0|e^{i\theta_0}$, $c_u = |c_u|e^{i\theta_u}$ and $c_g = |c_g|e^{i\theta_g}$. It is assumed that ϕ_0 and ϕ_u are appropriately orthogonalized to each other. Then the normalization conditions and the non-orthogonality for ϕ_1 and ϕ_2 are given by

$$\langle \phi_1 | \phi_1 \rangle = \langle \phi_2 | \phi_2 \rangle = |c_0|^2 + |c_u|^2 + |c_g|^2 = 1 \quad (3a)$$

$$S = \langle \phi_1 | \phi_2 \rangle = -c_0^{*2} - c_u^{*2} + c_g^{*2} \quad (3b)$$

$$\langle \phi_1 | \phi_1^* \rangle = \langle \phi_2^* | \phi_2 \rangle = c_0^{*2} + c_u^{*2} + c_g^{*2} \quad (3c)$$

$$\langle \phi_1^* | \phi_2 \rangle = \langle \phi_1 | \phi_2^* \rangle = -|c_0|^2 - |c_u|^2 + |c_g|^2. \quad (3d)$$

The orbital wave-functions of the electrons in the two magnetic ions are denoted by u_1 and u_2 respectively. We assume the occupation of each orbital by one electron. Then, including the spin functions, we shall have sixteen independent Slater determinants.

If we disregard the resultant total spin of the system, the ferromagnetic and antiferromagnetic states may be represented by the following single determinantal functions.

$$\text{Ferromagnetic : } [u_1 \phi_1 \bar{\phi}_2 u_2] \quad (4a)$$

$$\text{Antiferromagnetic : } [u_1 \phi_1 \bar{\phi}_2 \bar{u}_2]. \quad (4b)$$

Here the bracket notation represents the usual Slater determinants multiplied by $(4!)^{-1/2}$. The orbitals without bar include up spin and with bar down spin functions of the electrons in them.

The Hamiltonian (in atomic units; $e = \hbar = m = 1$) is expressed as

$$H = -\frac{1}{2} \sum_i \Delta_i + \sum_i V(r_i) + \sum_{i < j} \frac{1}{r_{ij}}, \quad (5)$$

where $V(r_i)$ is the potential acting on the i th electron due to the three nuclei and other electrons except the four under consideration. The last term is the usual Coulomb repulsion between the four electrons.

If we use the approximation that the direct exchange between the electrons

of the magnetic ions is negligible, the energy expectation values of these states are given by

$$E_f = 2\varepsilon_u + 2\varepsilon_\phi + 2K(u_1\phi_1) + 2K(u_1\phi_2) + K(u_1u_2) + K(\phi_1\phi_2) - J(u_1\phi_1) - J(u_1\phi_2) \quad (6a)$$

$$E_{af} = 2\varepsilon_u + 2\varepsilon_\phi + 2K(u_1\phi_1) + 2K(u_1\phi_2) + K(u_1u_2) + K(\phi_1\phi_2) - 2J(u_1\phi_1), \quad (6b)$$

where orthogonality between u 's and ϕ 's has been assumed. Hence the Coulomb and exchange integral, denoted by K and J respectively, involve only the Coulomb repulsion term r_{12}^{-1} of the Hamiltonian. The one electron terms involved in ε 's are expressed as

$$\varepsilon^u = \langle u_1 | -\frac{1}{2}\mathcal{A} + V | u_1 \rangle \quad \varepsilon_\phi = \langle \phi_1 | -\frac{1}{2}\mathcal{A} + V | \phi_1 \rangle. \quad (7)$$

Thus we get

$$E_f - E_{af} = J(u_1\phi_1) - J(u_1\phi_2). \quad (8)$$

Since the overlap between u_1 and ϕ_1 is greater than that between u_1 and ϕ_2 , $J(u_1\phi_1)$ is expected to be larger than $J(u_1\phi_2)$. Therefore the antiferromagnetic state would be lower than ferromagnetic one. A similar treatment has been given by Yamashita and Kondo.⁸⁾

It is, however, a little dangerous to approve of this result as our decisive conclusion. If we construct states with definite total spins, and consider that the lowest singlet and triplet with appropriate combinations of the constituent spins correspond to the antiferromagnetic and ferromagnetic states respectively, then the situation is not so simple as discussed above. The energy difference between the lowest triplet and singlet appears as the result of the second order perturbation treatment. Although it may not be justifiable that the real states of the three centre unit in crystal are described by those with definite total spins of the isolated four electron system to a better approximation than by the single determinantal functions such as (4a) and (4b), we may at least maintain that calculations based on the simplification as above are in danger of giving wrong results. In this sense we are also suspicious of the treatment using the ordered antiparallel spin arrangement as an approximation to the antiferromagnetic state. In what follows, we shall consider the states of definite total spins in detail.

§ 3. Energies of the lowest triplet and singlet

From the sixteen determinants mentioned above, we can construct one quintet, three triplets and two singlets. For the present purpose the quintet is of no interest, and accordingly we omit this. The representative states are given below:

Triplets

$$\begin{aligned} {}^3T_2 &= {}^3[{}^3(u_1u_2)(\phi_1\phi_2)] \\ &= \{[\bar{u}_1\phi_1\phi_2u_2] + [u_1\phi_1\phi_2\bar{u}_2] + [u_1\bar{\phi}_1\phi_2u_2] - [u_1\phi_1\bar{\phi}_2u_2]\} / 2(1 - |S|^2)^{1/2} \end{aligned} \quad (9a)$$

$${}^3\mathcal{F}_1 = {}^3[{}^1(u_1 u_2) {}^3(\phi_1 \phi_2)] = \{[\bar{u}_1 \phi_1 \phi_2 u_2] - [u_1 \phi_1 \phi_2 \bar{u}_2]\} / \{2(1 - |S|^2)\}^{1/2} \quad (9b)$$

$${}^3\mathcal{F}_0 = {}^3[{}^3(u_1 u_2) {}^1(\phi_1 \phi_2)] = \{[u_1 \phi_1 \bar{\phi}_2 u_2] - [u_1 \bar{\phi}_1 \phi_2 u_2]\} / \{2(1 + |S|^2)\}^{1/2}. \quad (9c)$$

These are the states with $M_s=1$. The others with lower M_s values can be easily derived from these.

Singlets

$${}^1\mathcal{F}_1 = {}^1[{}^3(u_1 u_2) {}^3(\phi_1 \phi_2)] = \{2[\bar{u}_1 \phi_1 \phi_2 \bar{u}_2] + 2[u_1 \bar{\phi}_1 \bar{\phi}_2 u_2] - [\bar{u}_1 \bar{\phi}_1 \phi_2 u_2] - [u_1 \phi_1 \bar{\phi}_2 \bar{u}_2] - [\bar{u}_1 \phi_1 \bar{\phi}_2 u_2] - [u_1 \bar{\phi}_1 \phi_2 \bar{u}_2]\} / \{12(1 - |S|^2)\}^{1/2}, \quad (10a)$$

$${}^1\mathcal{F}_0 = {}^1[{}^1(u_1 u_2) {}^1(\phi_1 \phi_2)] = [\bar{u}_1 \bar{\phi}_1 \phi_2 u_2] - [\bar{u}_1 \phi_1 \bar{\phi}_2 u_2] - [u_1 \bar{\phi}_1 \phi_2 \bar{u}_2] + [u_1 \phi_1 \bar{\phi}_2 \bar{u}_2]\} / 2\{ (1 + |S|^2) \}^{1/2}. \quad (10b)$$

The various states enumerated above, however, do not possess the requisite symmetry properties with respect to the reflection in the symmetry plane defined earlier. It is convenient to take such a linear combination of each state with its mirror image that satisfies the desired symmetry requirements. These are obtained as follows. Let R be the reflection operator; then we have

$$\left. \begin{aligned} R {}^3\mathcal{F}_2 &= {}^3\mathcal{F}_2^* \\ R {}^3\mathcal{F}_1 &= -{}^3\mathcal{F}_1^* \\ R {}^3\mathcal{F}_0 &= -{}^3\mathcal{F}_0^* \end{aligned} \right\} \quad \left. \begin{aligned} R {}^1\mathcal{F}_1 &= {}^1\mathcal{F}_1^* \\ R {}^1\mathcal{F}_0 &= {}^1\mathcal{F}_0^* \end{aligned} \right\} \quad (11)$$

Thus the new normalized states having the appropriate property are given as described below:

$$\left. \begin{aligned} {}^3\mathcal{F}_2 &= (2N_2)^{-1/2} ({}^3\mathcal{F}_2 - {}^3\mathcal{F}_2^*) \\ {}^3\mathcal{F}_1 &= (2N_1)^{-1/2} ({}^3\mathcal{F}_1 + {}^3\mathcal{F}_1^*) \\ {}^3\mathcal{F}_0 &= (2N_0)^{-1/2} ({}^3\mathcal{F}_0 + {}^3\mathcal{F}_0^*) \end{aligned} \right\} \quad (12a)$$

$$\left. \begin{aligned} {}^1\mathcal{F}_1 &= (2N_1)^{-1/2} ({}^1\mathcal{F}_1 + {}^1\mathcal{F}_1^*) \\ {}^1\mathcal{F}_0 &= (2N_0)^{-1/2} ({}^1\mathcal{F}_0 + {}^1\mathcal{F}_0^*) \end{aligned} \right\} \quad (12b)$$

where

$$\left. \begin{aligned} N_0 &= 1 + \{|\langle \phi_1 | \phi_1^* \rangle|^2 + \langle \phi_1 | \phi_2^* \rangle^2\} / (1 + |S|^2) \\ N_1 &= 1 + \{|\langle \phi_1 | \phi_1^* \rangle|^2 - \langle \phi_1 | \phi_2^* \rangle^2\} / (1 - |S|^2) \\ N_2 &= 1 - \{|\langle \phi_1 | \phi_1^* \rangle|^2 - \langle \phi_1 | \phi_2^* \rangle^2\} / (1 - |S|^2) \end{aligned} \right\} \quad (13)$$

In addition to these, there are even triplet and odd singlet states which, however, are not of any interest in that the ground triplet and singlet states have odd and even symmetry respectively.

The matrix elements of the Hamiltonian (5) necessary for the following calculations are given below:

Diagonal elements

$$\begin{aligned}
{}^3E_2 &= \langle {}^3\phi_2 | H | {}^3\phi_2 \rangle = \{ \langle {}^3\mathcal{F}_2 | H | {}^3\mathcal{F}_2 \rangle - \text{Re} \langle {}^3\mathcal{F}_2 | H | {}^3\mathcal{F}_2^* \rangle \} / N_2 \\
{}^3E_1 &= \langle {}^3\phi_1 | H | {}^3\phi_1 \rangle = \{ \langle {}^3\mathcal{F}_1 | H | {}^3\mathcal{F}_1 \rangle + \text{Re} \langle {}^3\mathcal{F}_1 | H | {}^3\mathcal{F}_1^* \rangle \} / N_1 \\
{}^3E_0 &= \langle {}^3\phi_0 | H | {}^3\phi_0 \rangle = \{ \langle {}^3\mathcal{F}_0 | H | {}^3\mathcal{F}_0 \rangle + \text{Re} \langle {}^3\mathcal{F}_0 | H | {}^3\mathcal{F}_0^* \rangle \} / N_0 \\
{}^1E_1 &= \langle {}^1\phi_1 | H | {}^1\phi_1 \rangle = \{ \langle {}^1\mathcal{F}_1 | H | {}^1\mathcal{F}_1 \rangle + \text{Re} \langle {}^1\mathcal{F}_1 | H | {}^1\mathcal{F}_1^* \rangle \} / N_1 \\
{}^1E_0 &= \langle {}^1\phi_0 | H | {}^1\phi_0 \rangle = \{ \langle {}^1\mathcal{F}_0 | H | {}^1\mathcal{F}_0 \rangle + \text{Re} \langle {}^1\mathcal{F}_0 | H | {}^1\mathcal{F}_0^* \rangle \} / N_0.
\end{aligned} \tag{14}$$

We can write down the matrix elements of the type $\langle {}^1\mathcal{F}_0 | H | {}^1\mathcal{F}_0 \rangle$ in the following compact form:

$$\begin{aligned}
& [1 + (-)^{S_x} |\langle \phi_1 | \phi_2 \rangle|^2] \langle {}^{2S+1}\mathcal{F}(S_m S_x) | H | {}^{2S+1}\mathcal{F}(S_m S_x) \rangle \\
& = [2\varepsilon_u + K(u_1 u_2) + (-)^{S_m} J(u_1 u_2)] [\langle \phi_1 | \phi_1 \rangle \langle \phi_2 | \phi_2 \rangle + \langle \phi_1 | \phi_2 \rangle \langle \phi_2 | \phi_1 \rangle (-)^{S_x}] \\
& + \langle \phi_1 \phi_2 | h_1(Q) + h_2(Q) + g_{12} | \phi_1 \phi_2 \rangle + (-)^{S_x} \langle \phi_1 \phi_2 | h_1(Q) + h_2(Q) + g_{12} | \phi_2 \phi_1 \rangle,
\end{aligned} \tag{15}$$

where S_m and S_x stand for the resultant spins of the two magnetic electrons in u_1 and u_2 and of those in the central anion X respectively; e.g. $S_m=1$ and $S_x=0$ for ${}^3\mathcal{F}_0$ given by (9c). Other various symbols are explained below:

$$Q = 1 + [S(S+1) - S_m(S_m+1) - S_x(S_x+1)]/2, \tag{16}$$

$$\langle \phi_a | h(Q) | \phi_b \rangle = \langle \phi_a | -\frac{1}{2} J + V | \phi_b \rangle + \sum_i \{ \langle u_i \phi_a | g_{12} | u_i \phi_b \rangle - \frac{Q}{2} \langle u_i \phi_a | g_{12} | \phi_b u_i \rangle \}, \tag{17}$$

and

$$\langle \phi_a \phi_b | g_{12} | \phi_c \phi_d \rangle = \iint \phi_a^*(\mathbf{r}_1) \phi_b^*(\mathbf{r}_2) \frac{1}{r_{12}} \phi_c(\mathbf{r}_1) \phi_d(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2. \tag{18}$$

The matrix elements of the type $\text{Re} \langle {}^1\mathcal{F}_0 | H | {}^1\mathcal{F}_0^* \rangle$ are obtained by replacing all the ϕ 's occurring in the kets by the corresponding ϕ^* 's. It is to be noted that with the definition given above the diagonal matrix elements of all the states can be obtained with the help of (15).

Off-diagonal elements

$$\left. \begin{aligned}
\langle {}^3\phi_0 | H | {}^3\phi_2 \rangle &= i \text{Im} \{ \langle {}^3\mathcal{F}_0 | H | {}^3\mathcal{F}_2 \rangle - \langle {}^3\mathcal{F}_0 | H | {}^3\mathcal{F}_2^* \rangle \} / (N_0 N_2)^{1/2} \\
\langle {}^3\phi_0 | H | {}^3\phi_1 \rangle &= \text{Re} \{ \langle {}^3\mathcal{F}_0 | H | {}^3\mathcal{F}_1 \rangle + \langle {}^3\mathcal{F}_0 | H | {}^3\mathcal{F}_1^* \rangle \} / (N_0 N_1)^{1/2} \\
\langle {}^1\phi_0 | H | {}^1\phi_1 \rangle &= \text{Re} \{ \langle {}^1\mathcal{F}_0 | H | {}^1\mathcal{F}_1 \rangle + \langle {}^1\mathcal{F}_0 | H | {}^1\mathcal{F}_1^* \rangle \} / (N_0 N_1)^{1/2},
\end{aligned} \right\} \tag{19}$$

where

$$\left. \begin{aligned}
\langle {}^3\mathcal{F}_0 | H | {}^3\mathcal{F}_2 \rangle &= [J(u_1; \phi_1 \phi_2) - J(u_2; \phi_2 \phi_1)] / \{2(1 - |S|^4)\}^{1/2} \\
\langle {}^3\mathcal{F}_0 | H | {}^3\mathcal{F}_1 \rangle &= [J(u_1; \phi_1 \phi_2) + J(u_2; \phi_2 \phi_1)] / 2(1 - |S|^4)^{1/2} \\
\langle {}^1\mathcal{F}_0 | H | {}^1\mathcal{F}_1 \rangle &= [J(u_1; \phi_1 \phi_2) + J(u_2; \phi_2 \phi_1)] / \{4(1 - |S|^4)/3\}^{1/2},
\end{aligned} \right\} \tag{20}$$

and

$$J(u_1; \phi_1 \phi_2) = \langle u_1 \phi_1 | g_{12} | \phi_1 u_1 \rangle \langle \phi_2 | \phi_2 \rangle - \langle u_1 \phi_1 | g_{12} | \phi_2 u_1 \rangle \langle \phi_2 | \phi_1 \rangle \\ - \langle u_1 \phi_2 | g_{12} | \phi_2 u_1 \rangle \langle \phi_1 | \phi_1 \rangle + \langle u_1 \phi_2 | g_{12} | \phi_1 u_1 \rangle \langle \phi_1 | \phi_2 \rangle. \quad (21)$$

If we use the approximation that the direct exchange between the electrons of the magnetic ions is negligible, i.e. the term $J(u_1 u_2)$ in (15) is always zero, we notice that, in contrast with (6a, b), the diagonal elements 3E_0 and 1E_0 are degenerate. The parameters occurring in (1) and (2) should be determined by minimizing the energy expectation value of the triplet or singlet ground states with respect to these. The same values will be used for the matrix elements of the higher states and the off-diagonal elements. For the present purpose it is sufficient to consider the energy depression of the lowest triplet ${}^3\Phi_0$ due to the interaction with ${}^3\Phi_1$ and ${}^3\Phi_2$ and of the lowest singlet ${}^1\Phi_0$ as a result of the interaction with ${}^1\Phi_1$. We shall use the perturbation method for this. Thus using the usual perturbation procedure, the energy depression of the lowest triplet and the lowest singlet are respectively given as:

$$\Delta E \text{ (triplet)} = |\langle {}^3\Phi_0 | H | {}^3\Phi_1 \rangle|^2 / ({}^3E_1 - {}^3E_0) + |\langle {}^3\Phi_0 | H | {}^3\Phi_2 \rangle|^2 / ({}^3E_2 - {}^3E_0) \\ \Delta E \text{ (singlet)} = |\langle {}^1\Phi_0 | H | {}^1\Phi_1 \rangle|^2 / ({}^1E_1 - {}^1E_0). \quad (22)$$

The dominant terms for the diagonal elements 3E_1 and 1E_1 are common, hence it is expected that their values would not be very different from each other. We may accordingly use the approximation

$${}^3E_1 \simeq {}^1E_1 = E_g', \quad (23a)$$

and use the notations

$$E_g = E_g' - E_0 \quad \text{and} \quad E_u = {}^3E_2 - E_0, \quad (23b)$$

where

$$E_0 = {}^3E_0 = {}^1E_0.$$

Thus the energy difference between the triplet and singlet states can be expressed as

$${}^3E - {}^1E = \{ {}^3E_0 - \Delta E \text{ (Triplet)} - ({}^1E_0 - \Delta E \text{ (Singlet)}) \} \\ = \frac{2}{E_g} |\langle {}^3\Phi_0 | H | {}^3\Phi_1 \rangle|^2 - \frac{1}{E_u} |\langle {}^3\Phi_0 | H | {}^3\Phi_2 \rangle|^2, \quad (24)$$

where use has been made of the relationship

$$|\langle {}^1\Phi_0 | H | {}^1\Phi_1 \rangle|^2 = 3 |\langle {}^3\Phi_0 | H | {}^3\Phi_1 \rangle|^2. \quad (25)$$

The nature of spin coupling between the electrons of the magnetic ions thus depends on the relative magnitude of the two terms on the right-hand side of (24). If the first term is dominant, then the coupling would be anti-ferromagnetic and if the second term is much greater than the first, it would be ferromagnetic.

In the foregoing sections we have given an analysis of super-exchange interaction in terms of semi-localized orbitals. In what follows, we shall express the details of the calculations in terms of integrals involving atomic or molecular orbitals, so as to get a clearer insight into the mechanism.

§ 4. Mechanism of the deformations

It would be convenient to visualize the various states given in §§ 2 and 3 when written in terms of ϕ_0 , ϕ_u and ϕ_g . They are given below

$${}^3\phi_2 = -i[{}^3(u_1 u_2)^3(\phi_0 \phi_u)] \quad (26a)$$

$${}^{1,3}\phi_1 = C_1(0g) {}^{1,3}[{}^{3,1}(u_1 u_2)^3(\phi_0 \phi_g)] + C_1(ug) {}^{1,3}[{}^{3,1}(u_1 u_2)^3(\phi_u \phi_g)] \quad (26b)$$

$${}^{1,3}\phi_0 = \sum_{i=0, u, g} C_0(ii) {}^{1,3}[{}^{1,3}(u_1 u_2)^1(\phi_i \phi_i)] + C_0(0u) {}^{1,3}[{}^{1,3}(u_1 u_2)^1(\phi_0 \phi_u)], \quad (26c)$$

where

$$\left. \begin{aligned} C_1(0g) &= |c_0| \cos(\theta_0 - \theta_g) / (\text{normalization}) \\ C_1(ug) &= |c_u| \cos(\theta_u - \theta_g) / (\text{normalization}) \\ C_1(0g)^2 + C_1(ug)^2 &= 1, \end{aligned} \right\} \quad (27)$$

and

$$\left. \begin{aligned} C_0(00) &= -|c_0|^2 / (\text{normalization}) \\ C_0(uu) &= -|c_u|^2 / (\text{normalization}) \\ C_0(gg) &= +|c_g|^2 / (\text{normalization}) \\ C_0(0u) &= -2|c_0||c_u| \cos(\theta_0 - \theta_u) / (\text{normalization}) \\ \sum_{i=0, u, g} C_0(ii)^2 + C_0(0u)^2 &= 1. \end{aligned} \right\} \quad (28)$$

The matrix elements in terms of integrals involving the orbitals u_1 , u_2 , ϕ_0 , ϕ_u and ϕ_g are as follows:

Diagonal elements

$${}^3E_2 = \langle \phi_0 | h(0) | \phi_0 \rangle + \langle \phi_u | h(0) | \phi_u \rangle + K(0u) - J(0u) \quad (29a)$$

$$\begin{aligned} {}^{1,3}E_1 &= \sum_{i=0, u} C_1(ig)^2 [\langle \phi_i | h(\pm 1) | \phi_i \rangle + \langle \phi_g | h(\pm 1) | \phi_g \rangle + K(ig) - J(ig)] \\ &\quad + 2C_1(0g)C_1(ug) [\langle \phi_0 | h(\pm 1) | \phi_u \rangle + K^g(0u) - J^g(0u)] \\ &\quad (+\text{for triplet, } -\text{for singlet}) \end{aligned} \quad (29b)$$

$$\begin{aligned} {}^{1,3}E_0 &= C_0(ii)^2 [2\langle \phi_i | h(1) | \phi_i \rangle + K(ii)] \\ &\quad + C_0(0u)^2 [\langle \phi_0 | h(1) | \phi_0 \rangle + \langle \phi_u | h(1) | \phi_u \rangle + K(0u) + J(0u)] \\ &\quad + \sum_{i < j} 2C_0(ii)C_0(jj)J(ij) + 2C_0(00)C_0(0u) [\langle \phi_0 | h(1) | \phi_u \rangle + K^0(0u)] \\ &\quad + 2C_0(uu)C_0(0u) [\langle \phi_u | h(1) | \phi_0 \rangle + K^u(u0)] + 2C_0(gg)C_0(0u)J^g(0u), \end{aligned} \quad (29c)$$

where the various symbols stand for the following :

$$\left. \begin{aligned} K(ij) &= \langle \phi_i \phi_j | g_{12} | \phi_i \phi_j \rangle, & J(ij) &= \langle \phi_i \phi_j | g_{12} | \phi_j \phi_i \rangle, \\ K^k(ij) &= \langle \phi_i \phi_k | g_{12} | \phi_j \phi_k \rangle, & J^k(ij) &= \langle \phi_i \phi_k | g_{12} | \phi_k \phi_j \rangle. \end{aligned} \right\} \quad (30)$$

Off-diagonal elements

$$\begin{aligned} \langle {}^3\phi_0 | H | {}^3\phi_2 \rangle &= i[2C_0(00) \langle u_1 \phi_0 | g_{12} | \phi_u u_1 \rangle - 2C_0(0u) \langle u_1 \phi_u | g_{12} | \phi_0 u_1 \rangle \\ &\quad - 2C_0(0u) \langle u_1 \phi_0 | g_{12} | \phi_0 u_1 \rangle - \langle u_1 \phi_u | g_{12} | \phi_u u_1 \rangle] \end{aligned} \quad (31a)$$

$$\begin{aligned} \langle {}^3\phi_0 | H | {}^3\phi_1 \rangle &= 3^{1/2} \langle {}^1\phi_0 | H | {}^1\phi_1 \rangle \\ &= -C_1(0g) C_0(00) 2^{1/2} \langle u_1 \phi_0 | g_{12} | \phi_g u_1 \rangle + C_0(0u) C_1(0g) 2^{1/2} \langle u_1 \phi_g | g_{12} | \phi_0 u_1 \rangle \\ &\quad - C_0(0u) C_1(0g) \langle u_1 \phi_u | g_{12} | \phi_g u_1 \rangle - C_0(0u) C_1(ug) 2^{1/2} \langle u_1 \phi_u | g_{12} | \phi_g u_1 \rangle \\ &\quad + C_0(gg) C_1(ug) 2^{1/2} \langle u_1 \phi_g | g_{12} | \phi_u u_1 \rangle - C_0(0u) C_1(ug) \langle u_1 \phi_0 | g_{12} | \phi_g u_1 \rangle. \end{aligned} \quad (31b)$$

The various parameters occurring in the above equations should be determined for each states by minimizing the respective expectation values with respect to the parameters. However, it may be a reasonable approximation to calculate them so as to minimize the value of the zeroth order ground triplet or singlet states.

Though the merit of the present formalism is that in terms of the single configuration $(u_1 u_2)(\phi_1 \phi_2)$ all the relevant transitions are taken into account, it would be of interest to understand the calculations in terms of the original atomic orbitals. It has been shown that the deformation of the anion functions does not introduce any change in the zeroth order ground ferromagnetic (triplet) and antiferromagnetic (singlet) state in contrast with the simple calculation shown in § 2. The ground triplet and singlet are, in fact, degenerate so long as the direct exchange interaction between the magnetic ions is neglected. As is easily seen in (26a, b, c), the deformation described by $\phi_0 \rightarrow \{\phi_1 \text{ and } \phi_2\}$ is equivalent to the mixing of the configurations in which anion electrons are promoted to the excited orbitals ϕ_u and ϕ_g without changing their spins. On the other hand, the energy difference between the lowest triplet and singlet appears due to the spin dependent transitions, i.e. due to the mixing of the states whose anion electrons have parallel spins. Namely, in Slater's polarization mechanism one of the anion electrons and one of the magnetic electrons simultaneously change their spin directions, the former being pushed up to the excited orbital due to the exclusion principle. This spin dependent transition takes place even when $c_0=1$ or $\phi_1=\phi_2=\phi_0$. Although the spin-independent deformation visualized by the non-equivalent anion orbitals ϕ_1 and ϕ_2 facilitates the spin-dependent polarization mechanism, the effect is not so direct as expected from the simple calculation based on the single determinantal wave-functions. One must bear this fact in mind in discussing the mechanism of the indirect spin coupling of this type.

§ 5. Numerical estimate for MnO

In this section, we shall evaluate the energy difference between the lowest triplet and singlet states for MnO crystal. Although the determination of the expectation value of (29c) by minimizing with respect to the coefficients is expected to yield better result, it may, however, suffice for the present purpose to study the quantitative aspect of the coupling by using the approximation where $c_0=1$.

The expression (24) now reduces to

$${}^3E-{}^1E = \frac{4}{E_g} |\langle u_1 \phi_g | g_{12} | \phi_0 u_1 \rangle|^2 - \frac{4}{E_u} |\langle u_1 \phi_u | g_{12} | \phi_0 u_1 \rangle|^2, \quad (32)$$

where E_g can be taken to be equal to the excitation energy to ϕ_g orbital and E_u that to ϕ_u . The amount of deformation would therefore depend on the magnitudes of the matrix elements appearing in this expression as well as E_g and E_u . The quantities E_g^{-1} and E_u^{-1} may be called 'deformability' of the anion electrons common to both types of deformations. In this sense the spin-dependent deformation responsible for the energy difference (32) is not quite independent of that caused by the correlation.

It is thus of great importance to understand the nature and the relative energies of the various excited orbitals available to the anion electrons. There are two possibilities in this description. First, the lowlying available orbitals are the appropriate excited orbitals of the central ion itself. Then there may be suitable orbitals arising out of a linear combination of the available cation orbitals. It must be noted that there is no clear distinction between these two. For studying the exchange coupling, all these orbitals should be taken into account. In what follows, we shall study the antiferromagnetic case for the system MnO in relation to the ideas presented above.

For MnO the lowest orbital ϕ_0 of the anion appropriate to the molecular unit chosen may be taken to be the $2p$ orbital. Let us first consider the excited orbitals of the anion. Since an adequate knowledge is not available even for the lowest $2p$ orbitals¹⁵⁾, it is very difficult to know about the excited orbitals of the O^{2-} ion in the crystal. So we must restrict ourselves to qualitative discussion. For this purpose we may consider the relative importance of $3s$, $3p$ and $3d$ orbitals on an O^{2-} ion in a crystal. The influence of the environment is often represented by an electric field acting on the ion in question. As is well known, octahedrally situated six adjacent cations (e.g. in MnO, which is of rocksalt structure, the six Mn^{2+} ions are surrounding the O^{2-} ion) give rise to the crystal field with cubic symmetry, which does not affect the $3s$ and $3p$ but would lower the $3d\gamma$ orbitals. A rough estimate shows that the $3d\gamma$ orbitals of O^{2-} are lowered relative to $3s$ and $3p$ by an amount of the order of $10^5 \text{ cm}^{-1} \simeq \frac{1}{2}$ a.u. Therefore, it seems that $3d\gamma$ is lower than $3p$, further, of the two $3d\gamma$ orbitals only $3d\gamma_e$ with Σ_g symmetry is of importance for the linear model under consideration.

The above considerations indicate that for the excited anion orbitals the contribution of the second term in (32) is relatively negligible compared with the first. The term to be considered in this case is $4|\langle u_1\phi_g|g_{12}|\phi_0u_1\rangle|^2/E_g$ where ϕ_g stands for $3d\gamma_\sigma$ on O^{2-} . This would favour antiferromagnetic coupling.

Let us now consider the excited orbitals of the cations. On the Mn^{2+} ion the lowlying available orbitals that may be considered are $4s$, $4p$ and, to a certain extent, $4d$. For the present purpose, we shall take χ_1 and χ_2 as one of the hybridized $4s4p^34d^2$ orbitals around the respective cations having the appropriate directionality (i.e. pointing towards the central ion). We consider the following linear combinations of these orbitals having the appropriate symmetry and orthogonality properties.

$$\phi_g^c = (\chi_1 + \chi_2)/\sqrt{2} \quad (33)$$

$$\phi_u^c = (\chi_1 - \chi_2 - 2S_0\phi_0)/\sqrt{2(1-2S_0^2)}, \quad (34)$$

where

$$S_0 = \langle \chi_1 | \phi_0 \rangle.$$

It is to be noted here that the orbital energies of ϕ_g^c and ϕ_u^c may not be the same. This difference arises due to the fact that the energy of $\chi_1 - \chi_2$ is expected to be pushed up because of mixing with ϕ_0 orbital of the anion which has the same symmetry. Even if one assumed that the orbital energies are not very different, the exchange integrals $\langle u_1\phi_g^c|g_{12}|\phi_0u_1\rangle$ and $\langle u_1\phi_u^c|g_{12}|\phi_0u_1\rangle$ will differ from each other because of the factor S_0 coming in (34). Thus we see that we cannot neglect the non-orthogonality of the orbitals ϕ_0 and $(\chi_1 - \chi_2)/\sqrt{2}$.

The foregoing analysis shows that we should consider the effect of the low orbitals of the anions and cations. That the effect of the lowest excited orbital would be the most is only to be expected. It is difficult, at present, to ascertain as to which of the even orbitals, $3d\gamma_\sigma$ on O^{2-} and $(\chi_1 + \chi_2)/\sqrt{2}$, is the lowest. We may rather say that these two express the same molecular orbital in different ways of approximation, because they not only have the same symmetry but also closely resemble each other in their charge distributions.

We shall, here, give the tentative calculations in terms of (33) and (34). The energy difference between the triplet and singlet is now expected as

$${}^3E - {}^1E = 4(J_g^2 - J_u^2)/(E_m - E_0), \quad (35)$$

where E_m represents the mean orbital energy of ϕ_g^c and ϕ_u^c , and

$$J_g = \langle u_1\phi_g^c|g_{12}|\phi_0u_1\rangle \simeq \langle u_1\chi_1|g_{12}|\phi_0u_1\rangle/\sqrt{2}$$

$$J_u = \langle u_1\phi_u^c|g_{12}|\phi_0u_1\rangle \simeq \{\langle u_1\chi_1|g_{12}|\phi_0u_1\rangle - 2S_0\langle u_1\phi_0|g_{12}|\phi_0u_1\rangle\}/\sqrt{2(1-2S_0^2)}.$$

Now the values of the above integrals are estimated as (See appendix.)

$$\langle u_1\chi_1|g_{12}|\phi_0u_1\rangle \simeq 0.0082 \text{ a. u.}$$

$$\langle u_1 \phi_0 | g_{12} | \phi_0 u_1 \rangle \simeq 0.013 \text{ a. u.}$$

$$S_0 \simeq 0.3.$$

Using the above values it turns out that $J_u^2 \simeq J_g^2/100$. Hence the second term in (35) is negligible compared with the first.

For a rough estimate the value of $E_m - E_0$ is taken from ionization potential of Mn^{2+} and electron affinity data of $\text{O}^- \rightarrow \text{O}^{2-}$. This turns out to be of the order of 1 a.u. Accordingly, ${}^3E - {}^1E \simeq 1.2 \times 10^{-4}$ a.u.

According to the molecular field model the strength of superexchange interaction between two sublattices can be defined as

$$kT_s = \frac{1}{2} k(T_n + \vartheta) = \frac{1}{3} zS(S+1)\delta E, \quad (36)$$

where δE is the energy difference between ferromagnetic and antiferromagnetic states, z is the numbers of magnetic ions correlated to a given magnetic ion. T_n is the Néel temperature and ϑ is the Curie-Weiss constant and S is the spin quantum number and in this case 5/2. For MnO $z=6$, and if we use the above value for $\delta E = {}^3E - {}^1E \simeq 1.2 \times 10^{-4}$ a.u., we get

$$T_s = (T_n + \vartheta)/2 \simeq 650^\circ\text{K}.$$

The observed mean value is about 370°K . In spite of the crude approximation the above estimate gives the right order of magnitude.

§ 6. Concluding remarks

In order to clarify the mechanism of the indirect exchange interaction, one must first know about the lowest excited states of the system. At present, however, it is extremely difficult to get a reliable information about the excited states. In the present work, therefore, we have tentatively used such excited orbitals which have maxima of the charge distribution in between the central and the magnetic ions. A rough numerical estimate is made on the assumption that these orbitals can be expressed, to a good approximation, by a linear combination of the empty cation orbitals having appropriate directionalities. This formalism can be regarded to incorporate a quantitative formulation of Goodenough's idea of semicovalent model⁽¹⁰⁾ for such compounds where some of the cation magnetic orbitals are empty and are available for making up appropriate hybridized orbitals mentioned above.

The use of the occupied cation orbitals has been intentionally avoided in the present calculation simply because it has been already discussed in detail by several authors. In addition to these two, there is another possibility of the interaction mechanism, i.e. virtual migration of the electrons from one magnetic cation to another, which is the approach adopted by Anderson in his new work.⁽¹⁰⁾ Though he has concluded that the last mechanism is dominantly important compared with the others, the conclusion depends upon the starting model and, as is emphasized in § 1, his assumption seems to be open to question. The truth may be a compromise of all

these mechanisms in order of merit. We have no intention of deciding in favour of our model at this stage, since to get a final decision is beyond the scope of the present work.

Our treatment is analogous to the Ruderman-Kittel mechanism of the hyperfine interaction in metals.¹⁷⁾ The role of the intervening ion corresponds to that of the conduction electrons in metals. This type of calculations has been applied by several authors to the quantitative treatment of the Zener model of ferromagnetism and to the interpretation of the magnetic properties of some alloys.¹⁸⁾

The extension of our model to the crystal is to be carried out in the next step; however, as emphasized in § 1, difficulty lies in determining the spin eigenstate of the whole crystal. As shown in the calculation of § 2, simplification such as the single determinant approximation may not represent the real state. The present situation of the theories appears to be too incomplete and confused and needs to be tidied up by a theory which takes into account all the mechanisms in their right perspective. However, one must be careful lest the formal approach, in all its elegance, should be in danger of being not very different from that of the phenomenological one.

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Appendix

Numerical evaluation of the integrals

Since the calculation of the two centre integrals is very complicated, Mulliken's approximation¹⁹⁾ was used to reduce the following complicated integrals into simpler forms.

$$J(\phi_0 u_1) = \langle \phi_0 u_1 | g_{12} | u_1 \phi_0 \rangle = \frac{1}{4} S_1^2 \{ K(\phi_0 \phi_0) + 2K(u_1 \phi_0) + K(u_1 u_1) \} \quad (\text{A} \cdot 1)$$

$$\langle u_1 \chi_1 | g_{12} | \phi_0 u_1 \rangle = \frac{1}{2} S_1 \{ \langle u_1 u_1 | g_{12} | u_1 \chi_1 \rangle + \langle u_1 \phi_0 | g_{12} | \chi_1 \phi_0 \rangle \}, \quad (\text{A} \cdot 2)$$

where S_1 is the overlap integral between u_1 and ϕ_0 .

The integral $K(\phi_0 \phi_0)$ is evaluated by using the Slater $2p$ function for O^{2-} . For u_1 , which is assumed to be a $3d\gamma_e$ orbital of Mn^{2+} , the Slater function is not very appropriate. The values of the quantities $\langle r^2 \rangle$, $\langle r^4 \rangle$ and $\langle r^6 \rangle$ turn out to be much larger than those obtained from Hartree-Fock functions.²⁰⁾ A slightly modified Slater type function has therefore to be taken. This is achieved in the following manner. The modified exponent for the Slater function is derived with the help of the ratio $\langle r^n \rangle_{\text{Slater}} / \langle r^n \rangle_{\text{HF}}$ for $n=0, 2, 4$ and 6 . By extrapolating to $n=-1$, a correction factor $\gamma=1.5$ for Coulomb integral $K(u_1 u_1)$ is obtained as

$$K(u_1 u_1)_{\text{modified}} = 1.5 K(u_1 u_1)_{\text{Slater}}.$$

Thus noting that $K(u_1 u_1) \propto Z_{\text{eff}}$, we employ the usual Slater type function of u_1 except that the exponent is multiplied by the factor $\gamma=1.5$.

For evaluating the two centre integral $K(u_1 \phi_0)$, the uniformly charged sphere approximation²¹⁾ was adopted. The $2p_\sigma$ charge cloud is approximated by two spheres in contact and each having the charge $e/2$ and radius 1.1 a.u. This value of the radius was so taken as to conform with the computed value of $K(\phi_0 \phi_0)$ for the Slater $2p$ function. Similarly, the charge cloud of $3d$ function u_1 was approximated by two spheres with radius 1.0 a.u. in contact and a ring with radius 0.7 a.u. encircling normally near the region of their contact. The charge distribution was taken $0.308e$ for each sphere and $0.384e$ for the ring. This charge distribution was taken in accordance with the angular dependence of the $3d\gamma_\sigma$ function. The charge distribution for the sp^3d^2 hybridized function χ_1 (or χ_2) turned out to be $0.95e$ in the direction pointing towards the intervening ion and only $0.05e$ away from it if we assume a common radial part for the s , p and d functions. This distribution was also approximated by two charged spheres one larger (with radius 2.1 a.u.) and the other very small, the effect of which was neglected.

Integrals	Numerical values in atomic units
$K(u_1 \phi_0)$	0.267
$K(\phi_0 \phi_0)$	0.753
$K(u_1 u_1)$	0.78
$J(u_1 \phi_0)$	0.013
$\langle u_1 u_1 g_{12} u_1 u_1 \rangle$	0.0720
$\langle u_1 \phi_0 g_{12} u_1 \phi_0 \rangle$	0.0302
$\langle u_1 u_1 g_{12} \phi_0 u_1 \rangle$	0.0082

In order to evaluate the integrals on the right-hand side of (A-2), the factor $u_1 \chi_1$ was replaced by a uniformly charged sphere of radius 1.0 a.u. with density $(\rho_u \rho_\chi)^{1/2}$. The overlap integrals $S_1=0.16$ was determined from the table of Jaffé and Doak²²⁾ for the interatomic distance of 4.2 a.u. The values of these integrals for the unit Mn-O-Mn thus determined are set out in the left table.

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On the Roles of Effective Interactions in Nuclear Collective Motion

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An attempt to investigate the nuclear collective motion from the standpoint of particle excitations is outlined. In our method it is possible to investigate the roles of the effective inter-particle interactions in the nuclear collective motions and clarify the mechanism of the collective excitation. The fundamental idea is illustrated by taking the simplified two-dimensional harmonic oscillator shell model with the quadrupole-quadrupole effective interactions between particles. It is shown that this system successfully describes both the surface vibrational and the rotational collective motions consistently, depending upon the assumed configurations of particles in the shells.

§ 1. Introduction

The striking success of the nuclear collective model in explaining the large variety of low-energy data has stimulated the interest of many workers in finding the method of quantum mechanical description of the nuclear collective motion. Various methods¹⁾ have been developed by many authors and have served us for understanding the various aspects of the nuclear collective motion.

Most of these methods have been constructed on the supposition that some appropriate collective coordinates can be introduced in anticipation or that some specific collective modes of motion can be extracted from "kinematical" considerations. In these methods, it seems that, the role of the effective inter-particle interactions as the origin of the collective motion has not been clarified satisfactorily. This problem will, however, be one of the most important problems to be investigated urgently, at the present stage of the study of the nuclear unified model.

It is our main purpose to outline an approach going beyond the limit of applicability of the above mentioned methods. According to our method developed in this paper, it is possible to investigate the role of the effective interactions in the nuclear collective motions and clarify the mechanism of the collective excitations. This possibility allows us to get a further insight into the various aspects of the unified model.

In this paper we shall illustrate the fundamental ideas of our method by taking

the simplified harmonic oscillator shell model with the quadrupole-quadrupole effective interactions. In the course of the discussion an explicit correspondence to the Bohr-Mottelson theory²⁾ and a relation to the theories of Elliott³⁾ and Moszkowski⁴⁾ will be given. To avoid unnecessary complications, the discussion will be made with the two-dimensional model.

In § 2 some preliminary discussions will be made. In § 3, by extending the theory of Sawada et al.⁵⁾ of plasma oscillations to our case, we shall determine the collective variables which correspond to the collective coordinates describing the surface deformation in the phenomenological theory of Bohr and Mottelson.²⁾ Then, the role of the effective interactions in the collective surface oscillation will be clarified. It will be shown that a restriction to the strength of the quadrupole-quadrupole interactions is to be required as the condition that the collective surface oscillation can be excited in our system. By using the collective coordinates thus obtained in § 3, the method of "auxiliary variables"⁶⁾ will be applied to the original Hamiltonian and will be discussed in § 4. This method makes it easy to understand the separation of the original system into the collective and the internal ones, and consequently to understand the mechanism of the coupling between these two modes of motions and also the role of the effective interactions in the intrinsic motions. In § 5, the method will be further extended to the case of the strongly deformed nuclear shape, and the equilibrium deformation and the rotational motion of our system will be discussed. Concluding remarks will be given in § 6.

§ 2. The model Hamiltonian and the quadrupole-quadrupole interactions

Our discussions on the nuclear many-body system will be developed in the model space in the sense of Eden and Francis.^{7),8)} We may expect that the short range singular parts of nuclear interactions (including the repulsive cores) are removed at lower excited states in the model space. This is, so to speak, the basis of the nuclear shell model. The Schrödinger equation in the model space can be obtained by adopting an appropriate model operator M as follows:

$$\Psi = M^{-1} \Phi, \quad (2.1a)$$

$$(H - E)\Psi = (M^{-1} \mathcal{H} M - E)\Psi = 0, \quad (2.1b)$$

where H and \mathcal{H} are respectively the model and the actual nuclear Hamiltonian, Ψ and Φ being the eigenfunctions of H and \mathcal{H} belonging to the same energy eigenvalue E . In the case of infinite nuclear matter, the operator M which transforms the actual system into the model space has been discussed from several points of view.^{9),10)} Here we shall not discuss this problem in detail, but only assume that the operator M exists also in the finite nuclear bound system. Further we shall assume that the operator M is approximately unitary in the low energy region near the ground state.

In the model space the main part of the nuclear interactions may be replaced

by a sum of a spherically symmetric self-consistent potential and some remaining effective interactions between particles. Probably these effective interactions play a very important role in the region of the nuclear surface, because in this region the Pauli principle is less effective in cutting down the correlations between nucleons.

We shall assume that the effective interaction is of the quadrupole-quadrupole type and the spherically symmetric self-consistent potential is of the form of isotropic harmonic oscillator. Of course, the form of the quadrupole-quadrupole interactions which was already used by Elliott³⁾ and Moszkowski⁴⁾ is a too simplified one. The realistic effective interactions would be much more complicated. However, it would be natural to suppose that the quadrupole-quadrupole interactions reflect the characteristic feature of such a part of the realistic effective interactions that is responsible for the nuclear collective motion corresponding to the surface deformation of order 2.*

In the two-dimensional approximation, which we take for simplicity as was discussed in § 1, our model Hamiltonian is

$$H = H_0 + H_{Q-Q}, \quad (2.2a)$$

$$H_0 = \sum_{i=1}^A \frac{1}{2m} (p_{x_i}^2 + p_{y_i}^2) + \sum_{i=1}^A \frac{m}{2} \omega_0^2 (x_i^2 + y_i^2), \quad (2.2b)$$

$$H_{Q-Q} = -\frac{1}{2} \sum_{i,j}^A V(x_i, y_i; x_j, y_j) = -\frac{1}{2} \sum_{i,j}^A \sum_{k=1,2} G^2 f_k(x_i, y_i) f_k(x_j, y_j), \quad (2.2c)$$

where the explicit forms of $f_k(x, y)$'s are

$$f_1(x, y) = \frac{m\omega_0}{2\hbar} (x^2 - y^2) = \frac{m\omega_0}{2\hbar} r^2 \cos 2\varphi, \quad (2.3a)$$

$$f_2(x, y) = \frac{m\omega_0}{2\hbar} 2xy = \frac{m\omega_0}{2\hbar} r^2 \sin 2\varphi, \quad (2.3b)$$

and G^2 is a constant characterizing the strength of the quadrupole-quadrupole interaction.

The single-particle states of the harmonic oscillator potential are characterized by the principal quantum number n and orbital angular momentum λ .** For the sake of simplicity we shall not consider the spin variable hereafter. The single-particle energies are given by

$$\epsilon_n = \hbar\omega_0(n+1), \quad (2.4)$$

and the possible values of λ are $\pm n, \pm(n-2), \dots, \pm 1$ or 0 . The single-particle wave function takes the form:

* Let us assume that the realistic effective two-body interaction is expanded in spherical harmonics. Then, the quadrupole-quadrupole interaction may be considered to correspond to the term with $l=2$. As discussed by Belyaev¹⁰⁾, the effects of the higher multipole terms may be important for the pairing correlations. In this paper we shall not consider such effects.

** In the two-dimensional case there is only one possible direction of angular momentum.

$$\phi_{n\lambda}(x, y) = R_{n\lambda}(r) \phi_{\lambda}(\varphi), \quad (2.5a)$$

where $R_n(r)$ and $\phi_{\lambda}(\varphi)$ are

$$R_{n\lambda}(r) = \left\{ 2 \left(\frac{n+\lambda}{2} \right)! \left(\frac{n-\lambda}{2} \right)! \right\}^{1/2} \sum_{a=|\lambda|}^n \frac{(-1)^{(a-|\lambda|)/2} \rho^a e^{-\rho^2/2}}{((a-\lambda)/2)!((a+\lambda)/2)!((n-a)/2)!}, \quad (2.5b)$$

$$(\rho \equiv (m\omega_0/\hbar)^{1/2} r; \quad (-1)^a = (-1)^n)$$

$$\phi_{\lambda}(\varphi) = (2\pi)^{-1/2} \exp(i\lambda\varphi). \quad (2.5c)$$

In order to take into account the exclusion principle easily we shall adopt the second quantization procedure. Then, our Hamiltonian can be written as follows:

$$H = H_0 + H_{Q-Q}, \quad (2.6a)$$

$$H_0 = \sum_{\alpha} \epsilon_{\alpha} a_{\alpha}^* a_{\alpha}, \quad (2.6b)$$

$$H_{Q-Q} = -\frac{1}{2} \sum_{\alpha\alpha', \beta\beta'} V(\alpha\alpha', \beta\beta') a_{\alpha}^* a_{\alpha'} a_{\beta}^* a_{\beta'}, \quad (2.6c)$$

where

$$V(\alpha\alpha', \beta\beta') = G^2 \sum_{k=1,2} f_k(\alpha\alpha') f_k(\beta\beta'), \quad (2.6d)$$

$$f_k(\alpha\alpha') = \int dx \int dy \phi_{\alpha}^*(x, y) f_k(x, y) \phi_{\alpha'}(x, y), \quad (2.6e)$$

and α denotes a set of two quantum numbers, (n, λ) . a_{α}^* and a_{α} are respectively the operators which create and annihilate a fermion in the state α . They satisfy the commutation relations of Fermi particles:

$$a_{\alpha} a_{\alpha'}^* + a_{\alpha'}^* a_{\alpha} = \delta_{\alpha\alpha'},$$

$$a_{\alpha} a_{\alpha'} + a_{\alpha'} a_{\alpha} = a_{\alpha}^* a_{\alpha'}^* + a_{\alpha'}^* a_{\alpha}^* = 0. \quad (2.7)$$

Now the interaction Hamiltonian (2.6c) can be decomposed as follows:

$$H_{Q-Q} = H_I + H_{II} + H_{III}, \quad (2.8a)$$

$$H_I = -\frac{1}{2} \sum_{\alpha\alpha'}^{nd} \sum_{\beta\beta'}^{nd} V(\alpha\alpha', \beta\beta') a_{\alpha}^* a_{\alpha'} a_{\beta}^* a_{\beta'}, \quad (2.8b)$$

$$H_{II} = -\frac{1}{2} \sum_{\alpha\alpha'}^{nd} \sum_{\beta\beta'}^d V(\alpha\alpha', \beta\beta') a_{\alpha}^* a_{\alpha'} a_{\beta}^* a_{\beta'} \\ - \frac{1}{2} \sum_{\alpha\alpha'}^d \sum_{\beta\beta'}^{nd} V(\alpha\alpha', \beta\beta') a_{\alpha}^* a_{\alpha'} a_{\beta}^* a_{\beta'}, \quad (2.8c)$$

$$H_{III} = -\frac{1}{2} \sum_{\alpha\alpha'}^d \sum_{\beta\beta'}^d V(\alpha\alpha', \beta\beta') a_{\alpha}^* a_{\alpha'} a_{\beta}^* a_{\beta'}. \quad (2.8d)$$

$\sum_{\alpha\alpha'}^{nd}$ means the summation over the pairs of states (α, α') differing from each other with respect to the principal quantum numbers (i.e., $n_{\alpha} \neq n_{\alpha'}$). $\sum_{\alpha\alpha'}^d$ means the summation over the pairs of states with the same principal quantum number, (i.e. $n_{\alpha} = n_{\alpha'}$).

H_{III} , which has the condition that all matrix elements of the f 's between single-particle states of different n vanish, is just what was used by Elliott.³⁾

Moszkowski⁴⁾ asserts that such an interaction is responsible not only for the rotational motion but also for the collective vibrational motion. In the next section, however, we shall show that H_I , rather than H_{III} , is mainly responsible for the collective oscillations. Contrary to H_{III} , H_I has only the matrix elements of the f 's between different shells.

§ 3. Derivation of surface oscillation variables

(I) Particle excitations and the quadrupole-quadrupole interactions

Let us examine the collective oscillation in our simplified two-dimensional nuclear model. On this point, the theory of Sawada et al.⁵⁾ of the plasma oscillation is of interest. Since the idealized nuclear surface motion in the theory of Bohr and Mottelson²⁾ corresponds, in its essence, to the density oscillation near the nuclear surface, Sawada's theory based on the "density approximation" can also be applied to our bound system.* This point of view was already adopted by Hayakawa¹³⁾ and one of the present authors¹⁴⁾ in the case of the description of collective motion in the closed shell nuclei.

In this section we shall consider, for a moment, the system with the Hamiltonian

$$H = H_0 + H_I. \quad (3.1)$$

This corresponds to adopting the quadrupole-quadrupole interactions with the extra condition that all matrix elements between pairs of states with the same principal quantum numbers vanish.**

If the interaction are absent, each particle in our system occupies, in the ground state, each individual level of harmonic oscillator in regular sequence. We shall tentatively call this state "free vacuum". Since the value of single-particle energy does not depend on the magnetic quantum number λ , such a definition of vacuum has some ambiguity for assigning the configuration in the uppermost shell which is partly filled. Though a discussion concerning this ambiguity will be given later, we shall, for a moment, arbitrarily choose one of the configurations with the lowest energy. By making use of this free vacuum, we now introduce the following operators:

$$\begin{aligned} \theta_\alpha a_\alpha &= b_\alpha^*, & \theta_\alpha a_\alpha^* &= b_\alpha, \\ (1 - \theta_\alpha) a_\alpha &= a_\alpha, & (1 - \theta_\alpha) a_\alpha^* &= a_\alpha^*, \end{aligned} \quad (3.2)$$

* In this connection, it is interesting to see that the usual method¹⁾ of quantum mechanical description of the nuclear collective motion, which leads the original Hamiltonian into one assumed in Bohr-Mottelson's collective model²⁾, just corresponds to the generalization of the Bohm-Pines theory¹²⁾ of plasma oscillations, and further to see that the approximations used in the method are closely related to the "density approximation".

** As a result of the Pauli principle, the particle transitions within the same shell, which are caused by H_{II} and H_{III} , effectively arise only when the shell is partly filled, and in the case of the closed shell nuclei H_{II} and H_{III} vanish effectively. The effects of particle transitions through H_{II} and H_{III} will be discussed in the next section.

where θ_α is defined as follows:

$$\theta_\alpha = \begin{cases} 1 & \text{if } \alpha \text{ is an occupied single-particle state in the free vacuum.} \\ 0 & \text{if } \alpha \text{ is an unoccupied single-particle state in the free vacuum.} \end{cases} \quad (3.3)$$

Then b_α^* and b_α mean the creation and annihilation operators of a hole in state α below the Fermi surface, and a_α^* and a_α the creation and annihilation operators of a particle in state α above the Fermi surface.

The commutation relations among these operators are

$$\begin{aligned} a_\alpha a_{\alpha'}^* + a_{\alpha'}^* a_\alpha &= \delta_{\alpha\alpha'} (1 - \theta_\alpha), \\ b_\alpha b_{\alpha'}^* + b_{\alpha'}^* b_\alpha &= \delta_{\alpha\alpha'} \theta_\alpha. \end{aligned} \quad (3.4)$$

With the use of (3.2), we decompose H_I in (3.1) into two parts:

$$H_I = H_I^{(\text{pair})} + H_I^{(\text{res})} \quad (3.5a)$$

where

$$H_I^{(\text{pair})} = -\frac{1}{2} \sum_{\alpha\alpha'}^{md} \sum_{\beta\beta'}^{md} V(\alpha\alpha', \beta\beta') (a_\alpha^* b_{\alpha'}^* + b_\alpha a_{\alpha'}) (a_\beta b_{\beta'}^* + b_\beta a_{\beta'}). \quad (3.5b)$$

$H_I^{(\text{pair})}$ contains only the terms representing the creation and annihilation of two particle-hole pairs and the transformations of the pairs. $H_I^{(\text{res})}$ means the rest of the terms of H_I .

Now, in our case the particle-hole pair must have a definite orbital angular momentum, $m = \pm 2$, because of the nature of the quadrupole-quadrupole interactions. We define, therefore, the following operators which create and annihilate the pairs with the definite angular momentum m :

$$\begin{aligned} C_m^*(\alpha\alpha') &= \delta_{\lambda\alpha', \lambda_\alpha - m} a_\alpha^* b_{\alpha'}^*, \\ C_m(\alpha\alpha') &= \delta_{\lambda\alpha', \lambda_\alpha - m} b_{\alpha'} a_\alpha. \end{aligned} \quad (3.6)$$

Commutation relations between these operators are

$$\begin{aligned} [C_m(\alpha\alpha'), C_{m'}^*(\beta\beta')] &= \delta_{mm'} \delta_{\alpha\beta} \delta_{\alpha'\beta'} (1 - \theta_\alpha) \theta_{\alpha'} \\ &\quad - \delta_{\lambda\alpha', \lambda_\alpha - m} \delta_{\lambda\beta', \lambda_\beta - m} (\delta_{\alpha'\beta'} \theta_{\alpha'} a_\beta^* a_\alpha + \delta_{\alpha\beta} (1 - \theta_\alpha) b_{\beta'}^* b_{\alpha'}), \end{aligned} \quad (3.7a)$$

$$[C_m(\alpha\alpha'), C_{m'}(\beta\beta')] = [C_m^*(\alpha\alpha') C_{m'}^*(\beta\beta')] = 0. \quad (3.7b)$$

The last term on the right-hand side of (3.7a) vanishes when operated on free vacuum. In the low energy region which we are now considering, the contribution from these terms would be small. Therefore we may drop these terms. Further a similar approximation will be made hereafter on the interaction Hamiltonian H_I , i.e., we shall neglect $H_I^{(\text{res})}$.

Then our system is reduced to that which is described by the following Hamiltonian:

$$H_c = H_0 + H_I^{(\text{pair})} \quad (3.8)$$

with Boson-like commutation relations:

$$\begin{aligned} [C_m(\alpha\alpha'), C_m^*(\beta\beta')] &= \delta_{m\alpha\beta} \delta_{\alpha'\beta'} (1 - \theta_\alpha) \theta_{\alpha'}, \\ [C_m(\alpha\alpha'), C_m(\beta\beta')] &= [C_m^*(\alpha\alpha'), C_m^*(\beta\beta')] = 0. \end{aligned} \quad (3.7)'$$

The approximations made in obtaining (3.8) and (3.7)' are very similar to the so-called "random phase approximation" of Bohm-Pines.¹²⁾

(II) Eigenvalue equation for the one-pair state and a solution of the collective oscillation

In order to construct the eigenenergy equations of H_c for the one-pair state we shall follow the procedure similar to that used by Sawada.⁵⁾

First we shall note the following relations:

$$\begin{aligned} [C_m^*(\alpha\alpha'), H_0] &= -(\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}}) C_m^*(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'}, \\ [C_{-m}(\alpha\alpha'), H_0] &= (\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}}) C_{-m}(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'}. \end{aligned} \quad (3.9)$$

$$\begin{aligned} [C_m^*(\alpha\alpha'), H_1^{(\text{pair})}] &= \frac{1}{2} G^2 \sum_{\beta\beta'}^{nd} v(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'} v(\beta\beta') \\ &\quad \times \{C_m^*(\beta\beta') + C_{-m}(\beta\beta')\}, \\ [C_{-m}(\alpha\alpha'), H_1^{(\text{pair})}] &= -\frac{1}{2} G^2 \sum_{\beta\beta'}^{nd} v(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'} \\ &\quad \times v(\beta\beta') \{C_m^*(\beta\beta') + C_{-m}(\beta\beta')\}, \end{aligned} \quad (3.10)$$

$v(\alpha\alpha')$ being defined by

$$v(\alpha\alpha') = \frac{m\omega_0}{2\hbar} \int R_{n_\alpha \lambda_\alpha}(r) r^2 R_{n_{\alpha'} \lambda_{\alpha'}}(r) r dr.$$

In obtaining (3.10) we used the commutation relations (3.7)'.

Next we shall remark the following identity:

$$\begin{aligned} 0 &\equiv (\Psi, (E_c - H_c) C_m^*(\alpha\alpha') \Psi_0) \\ &= (E_c - E_0) (\Psi, C_m^*(\alpha\alpha') \Psi_0) - (\Psi, [H_c, C_m^*(\alpha\alpha')] \Psi_0), \end{aligned} \quad (3.11)$$

where Ψ is some eigenstate of H_c with eigenvalue E_c , while Ψ_0 is the ground state of H_c with energy E_0 . By making use of (3.9) and (3.10), we get from (3.11) the following equations:

$$\begin{aligned} \{E_c - E_0 - (\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}})\} (\Psi, C_m^*(\alpha\alpha') \Psi_0) \\ + \frac{1}{2} G^2 v(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'} (\Psi, \sum_{\beta\beta'}^{nd} v(\beta\beta') \{C_m^*(\beta\beta') + C_{-m}(\beta\beta')\} \Psi_0) = 0, \end{aligned} \quad (3.12a)$$

$$\begin{aligned} \{E_c - E_0 + (\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}})\} (\Psi, C_{-m}(\alpha\alpha') \Psi_0) \\ - \frac{1}{2} G^2 v(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'} (\Psi, \sum_{\beta\beta'}^{nd} v(\beta\beta') \{C_m^*(\beta\beta') + C_{-m}(\beta\beta')\} \Psi_0) = 0. \end{aligned} \quad (3.12b)$$

From (3.12a) and (3.12b) the eigenvalue equation for the one-pair state is obtained:

$$1 = -G^2 \sum_{\alpha\alpha'}^{nd} \frac{(\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}}) v(\alpha\alpha') v(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'}}{(\Delta E)^2 - (\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}})^2}, \quad (3.13)$$

with

$$\Delta E \equiv E_c - E_0.$$

The equation (3.13) has a root ΔE which is real and positive and is smaller than the smallest single-particle excitation energy $(\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}})$, provided that the condition

$$G^{-2} > \sum_{\alpha\alpha'}^{nd} \frac{(v(\alpha\alpha'))^2 (1 - \theta_\alpha) \theta_{\alpha'}}{\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}}} \quad (3.14)$$

is satisfied. The state with such lowest excitation energy, $\Delta E \equiv \hbar\omega$, can be regarded as that corresponding to the collective surface oscillation in Bohr-Mottelson's unified model.²⁾

It should be noted that (3.14) represents a restriction to the strength of the quadrupole-quadrupole interactions. When this condition is not satisfied, the collective oscillation mode will dissipate and our approximation constructing this mode becomes meaningless.

(III) *Derivation of variables of the collective oscillation*

It has been made clear from the discussion in (II) that the state of the collective oscillation is the state with one-pair over the ground state Ψ_0 . Therefore, it may be anticipated that the collective eigenstate Ψ_c and its creation operator B_m^* have the forms:

$$\Psi_c = B_m^* \Psi_0, \quad (3.15a)$$

$$B_m^* = \sum_{\alpha\alpha'}^{nd} a(\alpha\alpha') C_m^*(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'} + \sum_{\alpha\alpha'}^{nd} b(\alpha\alpha') C_{-m}(\alpha\alpha') (1 - \theta_\alpha) \theta_{\alpha'}. \quad (3.15b)$$

According to the similar procedure to that used by Sawada et al.,⁵⁾ the coefficient $a(\alpha\alpha')$ and $b(\alpha\alpha')$ must be chosen so that Ψ_c is a properly normalized eigenfunction of H_c , i.e.,

$$H_c \Psi_c = (E_0 + \hbar\omega) \Psi_c \quad (3.16)$$

with

$$H_c \Psi_0 = E_0 \Psi_0.$$

Thus we obtain the following operator equations:

$$[H_c, B_m^*] = \hbar\omega B_m^*, \quad (3.17a)$$

$$[H_c, B_m] = -\hbar\omega B_m, \quad (3.17b)$$

Since Ψ_0 is the ground state,

$$B_m \Psi_0 = 0.$$

This condition allows us to fix the normalization, since

$$\begin{aligned} (\Psi_c, \Psi_c) &= (\Psi_0, B_m B_m^* \Psi_0) = (\Psi_0, [B_m, B_m^*] \Psi_0) \\ &= \sum_{\alpha\alpha'}^{nd} |a(\alpha\alpha')|^2 (1 - \theta_\alpha) \theta_{\alpha'} - \sum_{\alpha\alpha'}^{nd} |b(\alpha\alpha')|^2 (1 - \theta_\alpha) \theta_{\alpha'} = 1. \end{aligned} \quad (3.18)$$

The coefficients $a(\alpha\alpha')$ and $b(\alpha\alpha')$ can now be determined by inserting (3.15b) into (3.17a) and (3.17b) and by using the commutation relations (3.9) and (3.10). The result is that

$$a(\alpha\alpha') = \frac{Nv(\alpha\alpha')}{(\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}}) - \hbar\omega}, \quad (3.19a)$$

$$b(\alpha\alpha') = \frac{-Nv(\alpha\alpha')}{(\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}}) + \hbar\omega}, \quad (3.19b)$$

where N is the normalization constant which is determined by Equation (3.18), i.e.,

$$N^2 \left\{ \sum_{\alpha\alpha'} \frac{v(\alpha\alpha')^2 (1 - \theta_\alpha) \theta_{\alpha'}}{(\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}} - \hbar\omega)^2} - \sum_{\alpha\alpha'} \frac{v(\alpha\alpha')^2 (1 - \theta_\alpha) \theta_{\alpha'}}{(\epsilon_{n_\alpha} - \epsilon_{n_{\alpha'}} + \hbar\omega)^2} \right\} = 1. \quad (3.18)'$$

Thus we can obtain the explicit form of the creation and annihilation operators of the collective oscillation, B_m^* and B_m .

In order to clarify the correspondence to the collective variables used in the Bohr-Mottelson theory,²⁾ let us introduce the following variables:

$$\xi_m = \sqrt{\frac{\hbar}{2\omega}} (B_m + B_{-m}^*), \quad \xi_m^* = \xi_{-m}, \quad (3.20a)$$

$$\pi_m = i\sqrt{\frac{\hbar\omega}{2}} (B_m^* - B_{-m}), \quad \pi_m^* = \pi_{-m}, \quad (3.20b)$$

which satisfy the commutation relations:

$$[\pi_m, \xi_{m'}] = -i\hbar\delta_{mm'}. \quad (3.21)$$

It is clear that the collective variables (3.20) are closely connected with those of the surface oscillation in the Bohr-Mottelson model.²⁾

§ 4. Separation of the model Hamiltonian into collective and internal parts

(I) The reduced Hamiltonian

In the preceding section, by explicitly using the effective interactions, we have found the collective oscillation variables in our system. The next task is to separate the Hamiltonian (2.6) into the collective and internal parts.

Prior to this manipulation, we shall reduce our interaction Hamiltonian (2.8) to the following form:

$$H_{Q-Q}^{(\text{red})} = H_I^{(\text{pair})} + H_{II}^{(\text{red})} + H_{III}^{(\text{red})}, \quad (4.1)$$

$$H_I^{(\text{pair})} = -\frac{1}{2} \sum_{\alpha\alpha'} \sum_{\beta\beta'}^{m\bar{d}} V(\alpha\alpha' \beta\beta') (a_\alpha^* b_{\alpha'}^* + b_\alpha a_{\alpha'}) (a_\beta^* b_{\beta'}^* + b_\beta a_{\beta'}), \quad (4.1a)$$

$$\begin{aligned} H_{II}^{(\text{red})} = & -\frac{1}{2} \sum_{\alpha\alpha'} \sum_{\alpha_0\alpha_0'}^{m\bar{d}} V(\alpha\alpha', \alpha_0\alpha_0') (a_\alpha^* b_{\alpha'}^* + b_\alpha a_{\alpha'}) a_{\alpha_0}^* a_{\alpha_0'} \\ & -\frac{1}{2} \sum_{\beta\beta'} \sum_{\alpha_0\alpha_0'}^{m\bar{d}} V(\alpha_0\alpha_0', \beta\beta') a_{\alpha_0}^* a_{\alpha_0'} (a_\beta^* b_{\beta'}^* + b_\beta a_{\beta'}), \end{aligned} \quad (4.1b)$$

$$H_{\text{III}}^{(\text{red})} = -\frac{1}{2} \sum_{\alpha_0 \alpha_0'} \sum_{\beta_0 \beta_0'} V(\alpha_0 \alpha_0', \beta_0 \beta_0') a_{\alpha_0}^* a_{\alpha_0'} a_{\beta_0}^* a_{\beta_0'}, \quad (4.1c)$$

where the suffix zero stands for the outermost shell which is partly filled. Here we have adopted the approximations used in § 3 for H_I . As for H_{III} , we have picked up only terms which cause particle transitions within the outer partly-filled shell. As a result of the Pauli principle, the particle transitions within a shell can arise only when the shell is partly-filled. In the low energy region near the ground state, therefore, the above reduction of H_{III} may be a plausible approximation. The approximations made for H_I and H_{III} have also been implied for H_{II} .

(II) *Use of the "method of auxiliary variables"*⁶⁾

We shall take the reduced Hamiltonian, $H_0 + H_{Q-Q}^{(\text{red})}$, and start with the Schrödinger equation:

$$H^{(\text{red})} \Psi^{(0)} \equiv (H_0 + H_{Q-Q}^{(\text{red})}) \Psi^{(0)} = E \Psi^{(0)}. \quad (4.2)$$

In order to separate the reduced Hamiltonian into collective and internal parts, we shall employ "the method of auxiliary variables".⁶⁾ Following this method, we introduce, at first, the canonical auxiliary variables, α_m and β_m , which satisfy the canonical commutation relations:

$$[\alpha_m, \beta_{m'}] = i\hbar \delta_{mm'}. \quad (4.3)$$

Corresponding to $\hat{\xi}_m$ and π_m in (3.20), they have the conjugate relations:

$$\alpha_m^* = \alpha_{-m}, \quad \beta_m^* = \beta_{-m}.$$

In compensation for introducing the auxiliary variables, we impose on $\Psi^{(0)}$ the supplementary conditions:

$$\alpha_m \Psi^{(0)} = 0. \quad (4.4)$$

Now we shall successively perform the following unitary transformations:

$$\Psi^{(1)} = U_1^{-1} \Psi^{(0)}, \quad U_1 = \exp(i \sum_m \beta_m \hat{\xi}_m / \hbar), \quad (4.5a)$$

$$\Psi^{(2)} = U_2^{-1} \Psi^{(1)}, \quad U_2 = \exp(-i \sum_m \pi_m \alpha_m / \hbar), \quad (4.5b)$$

$$\Psi^{(t)} = U_1^{-1} \Psi^{(2)}, \quad U_1 = \exp(i \sum_m \beta_m \hat{\xi}_m / \hbar). \quad (4.5c)$$

Then the transformed Hamiltonian and the supplementary conditions become

$$H^{(t)} = (U_1 U_2 U_1)^{-1} H^{(\text{red})} U_1 U_2 U_1, \quad (4.6a)$$

$$\hat{\xi}_m \Psi^{(t)} = 0. \quad (4.6b)$$

This representation is the so-called collective representation, where the collective variables $\hat{\xi}_m$ and π_m in the original Hamiltonian are completely replaced by the auxiliary variables, α_m and β_m . Therefor, the collective motion of our system can be visualized in this representation.

To clarify the physical significance of the representation, we first perform the

successive transformations $U \equiv U_1 U_2 U_1$ to the Hamiltonian $H_c = H_0 + H_1^{(pair)}$. Then the transformed Hamiltonian becomes

$$H_c^{(t)} = U^{-1} H_c U = \left(\sum_{m=\pm 2} \frac{1}{2} \beta_m \beta_m^* + \sum_{m=\pm 2} \frac{1}{2} \omega^2 \alpha_m \alpha_m^* \right) + (H_c - \sum_{m=\pm 2} \frac{1}{2} \pi_m \pi_m^* - \sum_{m=\pm 2} \frac{1}{2} \omega^2 \xi_m \xi_m^*). \quad (4.7a)$$

In the calculation of this transformation we employed the operator identity:

$$\exp(iS) O \exp(-iS) = O + i[S, O] - \frac{1}{2}[S, [S, O]] + \dots$$

and used the commutation relations (3.17a) and (3.17b). The first bracket of the right-hand side of (4.7a) represents the collective oscillation. The second part commutes with ξ_m and π_m , so that it does not contain the collective variables and represents the internal motion.

Next we transform the remaining parts of the reduced Hamiltonian, i. e., $H_{II}^{(red)}$ and $H_{III}^{(red)}$. The result is

$$U^{-1} H_{II}^{(red)} U = -kG^2(\omega/2\hbar)^{1/2} \sum_{\substack{m \\ \alpha_0 \alpha_0'}} \alpha_m v(\alpha_0 \alpha_0') A_m(\alpha_0 \alpha_0') + (H_{II}^{(red)} + kG^2(\omega/2\hbar)^{1/2} \sum_{\substack{m \\ \alpha_0 \alpha_0'}} \xi_m v(\alpha_0 \alpha_0') A_m(\alpha_0 \alpha_0')), \quad (4.7b)$$

$$U^{-1} H_{III}^{(red)} U = H_{III}^{(red)}, \quad (4.7c)$$

where

$$A_m(\alpha_0 \alpha_0') \equiv \delta_{\lambda \alpha_0' \lambda \alpha_0 - m} \alpha_{\alpha_0}^* \alpha_{\alpha_0'},$$

$$k \equiv \sum_{\alpha \alpha'}^{m2} v(\alpha \alpha') a(\alpha \alpha') (1 - \theta_\alpha) \theta_{\alpha'} - \sum_{\alpha \alpha'}^{m2} v(\alpha \alpha') b(\alpha \alpha') (1 - \theta_\alpha) \theta_{\alpha'},$$

$a(\alpha \alpha')$ and $b(\alpha \alpha')$ being given by (3.19a) and (3.19b). In obtaining (4.7b) and (4.7c), we have made a new approximation that the operators $A_m(\alpha_0, \alpha_0')$'s commute with the collective variables ξ_m and π_m . Such an approximation is similar to that made in § 3, and means that only the terms in $A_m(\alpha_0 \alpha_0')$ which excite particle-hole pairs are significant in our energy region near the ground state.

Collecting the results of (4.7a), (4.7b) and (4.7c) we have the transformed reduced Hamiltonian:

$$H^{(t)} = H_{coll} + H_{coupl} + H_{intr}, \quad (4.8)$$

$$H_{coll} = \sum_{m=\pm 2} \left(\frac{1}{2} \beta_m \beta_m^* + \frac{1}{2} \omega^2 \alpha_m \alpha_m^* \right), \quad (4.8a)$$

$$H_{coupl} = -kG^2(\omega/2\hbar)^{1/2} \sum_{\substack{m \\ \alpha_0, \alpha_0'}} \alpha_m v(\alpha_0 \alpha_0') A_m(\alpha_0 \alpha_0'), \quad (4.8b)$$

$$H_{intr} = H_{intr}^{(0)} + H_{III}^{(red)}, \quad (4.8c)$$

$$H_{intr}^{(0)} = \left(H_c - \frac{1}{2} \sum_m \pi_m \pi_m^* - \frac{1}{2} \sum_m \omega^2 \xi_m \xi_m^* \right) + \{ H_{II}^{(red)} + kG^2(\omega/2\hbar)^{1/2} \sum_{\substack{m \\ \alpha_0, \alpha_0'}} \xi_m v(\alpha_0 \alpha_0') A_m(\alpha_0 \alpha_0') \}. \quad (4.8d)$$

Under our approximations, the Hamiltonian $H^{(t)}$ does no longer contain π_m and ξ_m , i.e.,

$$\begin{aligned}[H^{(t)}, \pi_m] &= [H^{(t)}, \xi_m] = 0, \\ [H_{\text{intr}}, \pi_m] &= [H_{\text{intr}}, \xi_m] = 0.\end{aligned}$$

In (4.8), H_{coll} represents the collective motion corresponding to the surface oscillation in the Bohr-Mottelson model.²⁾ H_{coupl} means the interaction between collective and particle motions, the particles being in the outer unfilled shell. $H_{\text{II}}^{(\text{red})}$ is nothing but the restricted quadrupole-quadrupole interactions between particles in outer partly-filled shell, with which Elliott⁸⁾ has obtained the rotational levels. Then the remaining terms $H_{\text{intr}}^{(0)}$ can be interpreted as the energy of the internal motion.

(III) *Coupled system of the collective oscillation and the particle motion in the partly-filled shell*

Now let us consider the Schrödinger equation with Hamiltonian (4.8):

$$\begin{aligned}H^{(t)}\Psi^{(t)} &= E\Psi^{(t)}, \\ \xi\Psi^{(t)} &= 0.\end{aligned}\tag{4.9}$$

Since

$$[H_{\text{intr}}^{(0)}, \xi_m] = [H_{\text{intr}}^{(0)}, \pi_m] = 0,$$

the following equations hold for the "free internal motion":

$$H_{\text{intr}}^{(0)}\chi_{\text{intr}} = E_{\text{intr}}^{(0)}\chi_{\text{intr}},\tag{4.10a}$$

$$\xi_m\chi_{\text{intr}} = 0,\tag{4.10b}$$

where χ_{intr} is the wave function for the internal motion. It would be quite difficult to solve (4.10). In this paper we shall not attempt to solve this equation, but assume, for a moment, that the intrinsic energy $E_{\text{intr}}^{(0)}$ can be approximated by that of the harmonic oscillator shell model. Properly speaking, the "free vacuum" state defined tentatively in § 3 should be replaced by the solution of (4.10) in the Hartree approximation. In this case, the ambiguity on configurations in the uppermost partly-filled shell might be removed.

Now, the low energy properties of our system (4.9) may be characterized by the following equation:

$$H_{\text{eff}}\phi = E_{\text{eff}}\phi\tag{4.11}$$

with the effective Hamiltonian:

$$H_{\text{eff}} = H_{\text{coll}} + H_{\text{coupl}} + H_{\text{II}}^{(\text{red})} + H_p,\tag{4.12}$$

where

$$H_p = \sum_{\alpha_0} \epsilon_{\alpha_0} a_{\alpha_0}^* a_{\alpha_0}.$$

Then the total energy of our system is given by

$$E = (E_{\text{intr}}^{(0)} - \sum_{\alpha_0}^{\text{occ.}} \epsilon_{\alpha_0}) + E_{\text{eff}}.\tag{4.13}$$

The physical picture which underlies the reduction of (4.9) into (4.11) may be summarized as follows: In the low energy region near the ground state the particles in the closed shells manifest themselves only through the collective surface oscillation. The interactions between these particles and the particles in the unfilled shell are completely replaced by the interaction between the above collective oscillation and the latter particles. The interparticle interactions remain effectively only between particles in the unfilled shell.

In the next section we shall investigate the properties of Equation (4.11).

§ 5. Rotational motion and equilibrium deformation

In the previous section we have shown that, in the low energy region near the ground state, our system with the Hamiltonian (2.6) can be reduced to that with the effective Hamiltonian (4.12). The main part of the effective Hamiltonian has the same form as that assumed in the Bohr-Mottelson model.²⁾ In our case, however, there remain a special additional interaction $H_{\text{III}}^{(\text{red})}$ which acts only between the particles in the outer unfilled shell.

Now let us investigate the system (4.11) in the so-called "strong coupling case". By the transformation

$$\alpha_m = \frac{\partial}{\sqrt{2}} \exp(-im\theta), \quad (5.1)$$

$$\alpha_m^* = \alpha_{-m}, \quad (m = \pm 2)$$

H_{coll} in (4.12) becomes

$$H_{\text{coll}} = T_{\text{rot}} + H_{\text{vib}}, \quad (5.2)$$

$$T_{\text{rot}} = \frac{\hbar^2}{2\mathcal{I}_0} L_\theta^2 = -\frac{\hbar^2}{8\delta^2} \frac{\partial^2}{\partial \theta^2}, \quad (5.2a)$$

$$H_{\text{vib}} = T_{\text{vib}} + V_{\text{vib}} = -\frac{\hbar^2}{2\delta} \frac{\partial}{\partial \delta} \left(\delta \frac{\partial}{\partial \delta} \right) + \frac{1}{2} \omega^2 \delta^2, \quad (5.2b)$$

$$L_\theta = \frac{1}{i} \frac{\partial}{\partial \theta}, \quad \mathcal{I}_0 = 4\delta^2. \quad (5.2c)$$

Here the quantity δ can be regarded as a deformation parameter and θ means the angle between the major axis of ellipse and the space-fixed axis.

Next consider $H_{\text{III}}^{(\text{red})}$ and H_{coupl} . They can be rewritten as follows:

$$H_{\text{III}}^{(\text{red})} = -\frac{1}{2} G^2 \sum_{k=1,2} T_k^2, \quad (5.3)$$

$$H_{\text{coupl}} = -kG^2 (\omega/\hbar)^{1/2} \delta (\cos 2\theta \cdot T_1 + \sin 2\theta \cdot T_2), \quad (5.4)$$

where T_k are defined by

$$T_k \equiv \sum_{\alpha_0 \alpha_0'} f_k(\alpha_0 \alpha_0') \alpha_{\alpha_0}^* \alpha_{\alpha_0'}, \quad (k=1, 2) \quad (5.5)$$

$f_k(\alpha_0\alpha_0')$ being the matrix elements of $f_k(x, y)$ as defined in (2.3).

Note that the $f_k(\alpha_0\alpha_0')$ are the matrix elements between the states in the degenerate outer partly-filled shell in the isotropic harmonic oscillator. Under this restriction they are equivalent to the matrix elements of the following quantities, $t_k(xy, p_x p_y)$:

$$t_1(xy, p_x p_y) = \frac{m\omega_0}{2\hbar} \cdot \frac{x^2 - y^2}{2} + \frac{1}{2\hbar m\omega_0} \cdot \frac{p_x^2 - p_y^2}{2},$$

$$t_2(xy, p_x p_y) = \frac{m\omega_0}{2\hbar} \cdot xy + \frac{1}{2\hbar m\omega_0} \cdot p_x p_y.$$

As was discussed by Elliott³⁾ and Moszkowski,⁴⁾ T_k 's satisfy the following commutation relations which are the same as those of the components of the three-dimensional angular momentum :

$$[T_i, T_j] = iT_k, \quad (5.6)$$

where i, j and k are the cyclic permutation of 1, 2 and 3. T_3 is defined by

$$T_3 = \frac{1}{2} L_\varphi, \quad (5.7)$$

$\hbar L_\varphi$ being the angular momentum operator of the particles in the outer unfilled shell :

$$\hbar L_\varphi = \hbar \sum_{\alpha_0} \lambda_{\alpha_0} \mathbf{a}_{\alpha_0}^* \mathbf{a}_{\alpha_0}. \quad (5.7a)$$

With use of (5.6), we can rewrite (5.3) in the form

$$H_{\text{III}}^{(\text{red})} = -\frac{1}{2} G^2 T^2 + \frac{1}{8} G^2 L_\varphi^2, \quad (5.8)$$

in which

$$T^2 = T_1^2 + T_2^2 + T_3^2.$$

With the aid of (5.2), (5.4) and (5.8) the effective Hamiltonian (4.12) becomes

$$H_{\text{eff}} = \frac{\hbar}{2\mathcal{J}_0} L_\theta^2 + \frac{G^2}{8} L_\varphi^2 - kG^2 (\omega/\hbar)^{1/2} \partial (\cos 2\theta \cdot T_1 + \sin 2\theta \cdot T_2) + h^{(0)} + H_{\text{vib}}, \quad (5.9)$$

where $h^{(0)}$ is the energy of the particle motion :

$$h^{(0)} = \epsilon_{n_{\alpha_0}} N_0 + \frac{G^2}{2} T^2, \quad (5.10)$$

N_0 being the occupation number of the particles in the outer partly-filled shell :

$$N_0 = \sum_{\alpha_0} \mathbf{a}_{\alpha_0}^* \mathbf{a}_{\alpha_0}.$$

The lowest eigenvalue of $h^{(0)}$ is easily found to be

$$\epsilon^{(0)} = \epsilon_{n_{\alpha_0}} N_0' - \frac{G^2}{2} T_0' (T_0' + 1) \quad (5.11)$$

where $T'_0(T'_0+1)$ is the maximum possible eigenvalue of T^2 and N'_0 is the eigenvalue of N_0 .

In order to make clear the physical picture of (5.9), let us define ϑ and V , through

$$\begin{aligned} T_1 &= \frac{1}{2}(V \cdot \cos 2\vartheta + \cos 2\vartheta \cdot V), \\ T_2 &= \frac{1}{2}(V \cdot \sin 2\vartheta + \sin 2\vartheta \cdot V). \end{aligned} \quad (5.12)$$

Then we can easily find that

$$[L_\vartheta, \vartheta] = -i, \quad [L_\vartheta, V] = 0 \quad (5.13)$$

with the aid of the relations

$$[L_\vartheta, T_1] = 2iT_2, \quad [L_\vartheta, T_2] = -2iT_1,$$

which are derived from (5.6). (5.13) shows that ϑ can be regarded as a rotational orientation angle of particles. In the case of the configuration with the lowest energy (5.11), the value of V can be regarded as T'_0 because of its definition. Making use of this value, we can rewrite the third term in (5.9), which is the coupling term between the collective and the particle motions, as follows:

$$H_{\text{comp}} = -kG^2(\omega/\hbar)^{1/2} \delta \cdot T'_0 \cos 2(\theta - \vartheta).$$

Then (5.9) becomes

$$\begin{aligned} H_{\text{eff}} &= \frac{\hbar^2}{2\mathcal{J}_0} L_\theta^2 + \frac{\hbar^2}{2\mathcal{J}_1} L_\vartheta^2 - kG^2(\omega/\hbar)^{1/2} \delta \cdot T'_0 \cos 2(\theta - \vartheta) \\ &+ \text{terms not depending on the rotational motions,} \end{aligned} \quad (5.14)$$

where

$$\hbar^2/2\mathcal{J}_1 \equiv G^2/8.$$

The form of this Hamiltonian is nothing but that of "the coupled two rotors", which was proposed by Tomonaga¹⁵⁾ from his ingenious intuitive insight and was developed by Miyazima and Wada.¹⁶⁾ Very similar argument to Tomonaga's has been given also by Bohr and Mottelson¹⁷⁾ independently.

Following Tomonaga's prescription, we introduce two angles:

$$\begin{aligned} \theta &= (\mathcal{J}_0 \theta + \mathcal{J}_1 \vartheta) / (\mathcal{J}_0 + \mathcal{J}_1), \\ \Omega &= \theta - \vartheta, \end{aligned} \quad (5.15)$$

which stand for the angular center of mass and the relative angle of the rotors, respectively. Then, the total angular momentum of the system is given by

$$\hbar L_\Omega = \frac{\hbar}{i} \frac{\partial}{\partial \theta},$$

and the Hamiltonian (5.14) is rewritten as

$$H_{\text{eff}} = \frac{\hbar^2}{2(\mathcal{J}_0 + \mathcal{J}_1)} L_{\odot}^2 - \frac{\hbar^2(\mathcal{J}_0 + \mathcal{J}_1)}{2\mathcal{J}_0\mathcal{J}_1} \frac{\partial^2}{\partial \Omega^2} - kG^2(\omega/\hbar) \delta \cdot T_0' \cos 2\Omega$$

+ terms not depending on the rotational motions. (5.16)

In the "strong coupling" case under consideration, the value of $kG^2(\omega/\hbar) \delta \cdot T_0'$ is larger than the rotational energy of the two rotors. The angle Ω will, therefore, perform a small vibration around an equilibrium point, $\Omega=0$ provided $k>0$, so that we may expand the third term in (5.16) in the power series in Ω and may take the series up to Ω^2 . Then the third term becomes

$$-kG^2(\omega/\hbar)^{1/2} \delta \cdot T_0' + 2kG^2(\omega/\hbar)^{1/2} \delta \cdot T_0' \Omega^2. \quad (5.17)$$

Before obtaining the eigenvalue of (5.16) (i.e., that of (5.9)), we shall discuss the equilibrium deformation of our system. From (5.16) and (5.17) it can be easily seen that the terms not depending on the rotational motions in (5.9) is expressed as

$$h = h^{(0)} - kG^2(\omega/\hbar)^{1/2} \delta \cdot T_0' + H_{\text{vib}}. \quad (5.18)$$

If we make use of the adiabatic approximation for the vibrational motion, (i.e., if we fix the deformation variable δ), the lowest energy is given by

$$E(\delta, T_0', N_0') = \epsilon_{n_{\alpha_0}} N_0' - \frac{G^2}{2} T_0'(T_0' + 1) - kG^2(\omega/\hbar)^{1/2} \delta \cdot T_0' + \frac{1}{2} \omega^2 \delta^2. \quad (5.19)$$

Then, the equilibrium deformation which gives the minimum value of E can be obtained through

$$\frac{\partial}{\partial \delta} E(\delta, T_0', N_0') = 0$$

and becomes

$$\delta_{\text{eq}} = kG^2 \omega^{-2} (\omega/\hbar)^{1/2} T_0'. \quad (5.20)$$

This shows that the equilibrium deformation of our system depends not only on the strength G^2 of effective interactions and on the characteristic frequency of collective motion, but also on the configuration of the outer-shell-particles.

With the aid of (5.16), (5.17) and (5.20), we can finally find the energy eigenvalue of (5.9)

$$E_{\text{eff}} = \frac{\hbar^2 I^2}{2(\mathcal{J}_0 + \mathcal{J}_1)} + \hbar \omega_{\Omega} \left(\nu + \frac{1}{2} \right) - kG^2(\omega/\hbar)^{1/2} \delta_{\text{eq}} T_0'$$

$$+ \epsilon_{n_{\alpha_0}} N_0' - \frac{G^2}{2} T_0'(T_0' + 1) + \epsilon_{\text{vib}}, \quad (5.21)$$

where

$$\omega_{\Omega}^2 = \frac{4(\mathcal{J}_0 + \mathcal{J}_1)}{\mathcal{J}_0 \mathcal{J}_1} kG^2(\omega/\hbar)^{1/2} \delta_{\text{eq}} T_0'.$$

$\hbar I$ is the magnitude of the total angular momentum and ν is zero or positive integer. ϵ_{vib} is an energy eigenvalue of the vibrational motion around the equilibrium deformation δ_{eq} . In the strong coupling case, of course, the second term in (5.21) is much larger than the first term. Remember that the total energy of our original system is given by (4.13)

$$E = E_{\text{eff}} + (E_{\text{intr}}^{(0)} - \epsilon_{n\alpha_0} N_0').$$

§ 6. Summary and discussion

Adopting the simplified two-dimensional harmonic oscillator shell model with the quadrupole-quadrupole interaction, we investigated the nuclear collective motion from the standpoint of particle excitations.

It was shown that in the low energy region near the ground state the transitions of particles between the different shells contribute mainly to the collective surface oscillation and the transitions of particles within the same shells affect the intrinsic motion mainly and the rotational motion partially. Thus the coupling between the collective surface oscillation and intrinsic motion originates in such a part of the effective interactions that has the matrix elements linking the above two types of transitions. As a result of the Pauli principle, the particle transitions within a shell can arise only when the shell is unfilled; therefore in the low energy region (near the ground state) the particles in the closed internal shells manifest themselves only through the collective surface oscillation, apart from their contribution to the determination of the ground state of the intrinsic motion.

Thus, it was made clear that, in so far as the low energy phenomena are concerned, our system reduces to a coupled system of the collective surface oscillation and the motion of particles in the outer unfilled shell. The effective Hamiltonian of this coupled system has the same form as that assumed in the Bohr-Mottelson theory,²⁾ except for that in our case there remain special residual interactions, which act between the particles within the outer unfilled shell and partly contribute to the rotational motion.

The rotational motion discussed by Elliott³⁾ is just what such residual interactions make themselves responsible for. In the so-called strong coupling case, there is another rotational motion due to the collective surface motion which was introduced by Bohr and Mottelson²⁾ phenomenologically. Thus it was shown that the total rotational motion of our system is obtained by the resultant motion of these two rotors, which are coupled with each other. The situation is very similar to "the model of coupled two rotors" for nuclear rotation which was developed by Tomonaga,¹⁵⁾ Miyazima and Wada,¹⁶⁾ and independently by Bohr and Mottelson.¹⁷⁾

The equilibrium deformation of our system has also been discussed in detail. Then, it has been shown that the magnitude of equilibrium deformation depends

not only on the strength of the quadrupole-quadrupole interaction and on the character of collective surface motion, but also on the configurations and the states of the outer-shell-particles.

Although our discussions are restricted to the simplified two-dimensional harmonic oscillator shell model with the quadrupole-quadrupole interactions, our treatment may be generalized to more realistic cases. Our method seems superior to that of the usual methods¹⁾ of quantum mechanical description of the nuclear collective motion in the sense that it is developed by explicitly using the effective interactions between particles. In the generalization to more realistic cases, however, it will further be necessary to take into account the effects of higher multipole terms of the realistic effective interactions as was discussed by Belyaev.¹¹⁾

In conclusion, the following must be emphasized. In this paper we only intended to outline an approach to the further investigation of the nuclear collective motion, and the detailed investigation on the applicability of various approximations has been left out. So one must be content only with the qualitative understanding of the mechanism of collective motions and the role of the effective interactions.

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Electron Interaction in Very Long Linear Conjugated Molecules. II

—Elementary Excitations in a System with Small Energy Gap—

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Lowlying excited states in a one-dimensional system which has a small energy gap (ϵ_g) in the spectra of H_0 , the one-electron Hamiltonian, are investigated. Plasmons and excitons are our main interests. These states are investigated in the framework of the tight-binding approximation. In usual one-dimensional systems, such as very long linear conjugated molecules, the plasmon levels sink into the level continuum given by excitation energies of one-pair states. However, it is shown that the plasma oscillations are stable when their energies are sufficiently larger than the energy gap. On the other hand, the plasma oscillations with sufficiently small momentum whose energies in case of $\epsilon_g=0$ were smaller than the present energy gap seem to be dissolved away into the level continuum.

A formulation to derive exciton solutions is given. The screening effect for the attractive force between the electron and the hole is investigated by means of the Gell-Mann and Brueckner technique and is found to grow larger as the energy gap becomes smaller. By this screening effect the possibility of getting the exciton-like bound state diminishes as $\epsilon_g \rightarrow 0$. As a corollary we have found that the potential energy of two electric charges $z_1 e$ and $z_2 e$ separated by a distance r in a three dimensional system with small energy gap becomes $z_1 z_2 e^2 \exp(-\alpha r)/r$ if $r \ll |B/\Delta B|$, where $|B|$ is a measure of the Fermi energy and ΔB is a measure of the gap, while it becomes $z_1 z_2 e^2/\epsilon r$ if $r \gg |B/\Delta B|$, where $\epsilon \propto r_s^{-1}(|B|/(\Delta B)^2) \times (e^2/a_B)$.

§ 1. Introduction

In the previous paper¹⁾ (hereafter referred to as I) low-energy electronic states with small wave numbers in the very long linear conjugated molecules with equal bond lengths and equal bond angles are investigated by means of Tomonaga's sound wave method²⁾ within the framework of Pariser-Parr's π electron approximation. The formulation employed may be regarded as the tight-binding approximation for plasma waves.

It has been an interesting problem to see whether very long linear conjugated molecules have alternating shorter and longer bonds even in their middle parts or they have equal bonds in the parts sufficiently far apart from the ends. Many physicists believe the existence of the alternation.³⁾ Accordingly, it will be interesting to investigate low energy electronic excitations in the molecule with alternating shorter and longer bonds. In order to simplify our problem, we here investigate the π electrons moving in an infinitely long molecular skeleton with

fixed alternating bonds shown in Fig. 1. Then the one-electron energy of these π electrons has a Brillouin gap. In case of an infinitely long molecule the difference between the longer bond length and the shorter one will be very small. Then the gap should be very small. Thus our problem is similar to that of electronic excitations in graphites or intrinsic semiconductors. Qualitative aspects of our discussions seem to prevail in such three dimensional systems, though from quantitative point of view we should modify our treatment in each case. We will call the long linear molecules with equal bonds the N -molecules and those with alternating bonds the S -molecules.

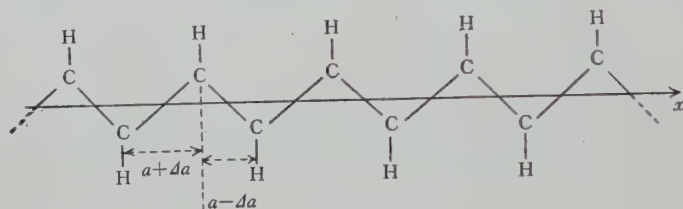


Fig. 1

§ 2 Review of the treatment for N -molecules

As shown in I, the Hamiltonian for the π electron system of an N -molecule is given by

$$\left. \begin{aligned} H &= H_0 + H_I, \\ H_0 &= \sum_k^\sigma \epsilon(k) N_{k\sigma}, \quad H_I = \frac{1}{2} \sum J_k \rho_k \rho_{-k}, \\ N_{k\sigma} &= a_{k\sigma}^* a_{k\sigma}, \quad \rho_k = \sum_l^\sigma a_{l+k\sigma}^* a_{l\sigma}, \end{aligned} \right\} \quad (2.1)$$

in the framework of Pariser-Parr's π electron approximation which has got a great success in the interpretation of optical spectra of organic molecules. In the above, a_k is the destruction operator for the Bloch orbital

$$\phi_k = \frac{1}{\sqrt{N}} \sum_{-N/2 < R \leq N/2} \exp(ikR) \varphi_R \quad (2.2)$$

where φ_R is the atomic orbital (or Wannier function) belonging to the R -th atom, k is given by

$$k = \frac{2\pi}{N} n \quad (n = \text{integer}),$$

and σ denotes the spin function α or β .

(2.2) may be stated as follows,

$$a_{k\sigma} = N^{-1/2} \sum_{-N/2 < R \leq N/2} \exp(-ikR) b_{R\sigma}, \quad (2.3)$$

where b_R is the destruction operator for φ_R . Further, J_k , the Fourier component of the interaction, is given by

$$NJ_k = \frac{2e^2}{a} K_0(\gamma|k|)$$

in our one-dimensional system, K_0 being the modified Bessel function of the second kind, and γ being some dimensionless constant introduced in I. In Pariser-Parr's π electron approximation, the one-electron energy $\epsilon(k)$ is given by

$$\epsilon(k) = 2B \cos k \quad (2.4)$$

where B is the resonance integral between two adjacent carbon atoms.

If the Tomonaga method of one-dimensional sound waves is applied to (2.1), lowlying excited states with small momenta are described by a Boson assembly which consists of two different groups of Bosons. One of the groups is composed of plasmons, each of which has the following energy quantum

$$W = |n| \sqrt{\epsilon'(n_F) [\epsilon'(n_F) + 4J_n]} \quad (2.5)$$

where $k = (2\pi/N)n$ ($\ll \pi/2$) is the wave vector of this Boson, n being an integer. Another group is described by Bosons corresponding to spin-density waves. The energy quantum of a Boson belonging to this group is given by

$$W = |n| \epsilon'(n_F). \quad (2.5')$$

Same answer can be obtained by Sawada's useful method.⁴⁾ In this method only the interactions of the type shown in Fig. 2 are taken into account, the remaining interactions being neglected. Then the Coulomb interaction is replaced by

$$H_I \rightarrow H_C = \frac{1}{2} \sum J_n \zeta_n \zeta_{-n}$$

where

$$\zeta_n = \sum_{\substack{|n'| \leq n_F \\ |n' + n| > n_F}}^{\sigma} (a_{n'+n\sigma}^* b_{n'\sigma}^* + b_{-n'\sigma} a_{-n'-n\sigma})$$

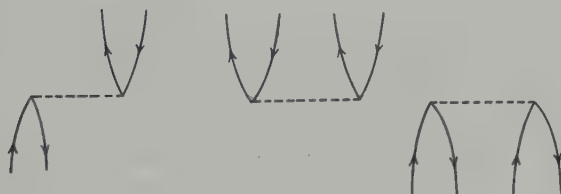


Fig. 2

in which $b_{n\sigma}^* = a_{n\sigma}$ ($n \leq n_F$). Further in the commutator

$$[H_C, a_{m\sigma}^* b_{n\sigma}^*] \text{ and } [H_C, b_{n\sigma} a_{m\sigma}]$$

one may use the following commutation relation

$$[b_{n'\sigma'}, a_{m'\sigma'}, a_{m\sigma}^* b_{n\sigma}^*] = \delta_{\sigma\sigma'} \delta_{nn'} \delta_{mm'}$$

because the interaction diagrams are restricted as mentioned above. Then, one can find an excitation energy W given by

$$1 = 2J_n \sum_{\substack{|n'| \leq n_F \\ |n' + n| > n_F}} \left\{ \frac{1}{W - \epsilon(n' + n) + \epsilon(n')} - \frac{1}{W + \epsilon(n' + n) - \epsilon(n')} \right\}. \quad (2.6)$$

In the one-dimensional system, for sufficiently small n compared with n_F , we may use

$$\epsilon(n' + n) - \epsilon(n') = |n| \epsilon'(n_F) \quad (2.7)$$

in the denominator of the above eigenvalue equation. Then we get Eq. (2.5).

In Sawada's theory the plasma excitation connects with only the one pair excitation $\Psi_0 \rightarrow a_{n_F+n}^* b_{n_F}^* \Psi_0$ when the electron interaction is adiabatically switched off, where Ψ_0 represents the ground state when $e=0$ and n is here assumed to be positive. There are also "scattering solutions" whose excitation energies are almost equal to $\epsilon(n' + n) - \epsilon(n')$ and connect with those of one-pair excitations $\Psi_0 \rightarrow a_{n'+n}^* b_{n'}^* \Psi_0$ ($n' = n_F - n + 1, n_F - n + 2, \dots, n_F - 1$) when the interaction is adiabatically switched off. According to Tomonaga's theory, on the other hand, the one-pair excitations which connect with the plasmon state are not limited to $\Psi_0 \rightarrow a_{n_F+n}^* b_{n_F}^* \Psi_0$ and are large in number. However, as mentioned in I, Tomonaga's proof to show the equivalence between the Boson Hamiltonian and the original Fermion Hamiltonian is not sufficient, though his theory as a whole is very ingenious and instructive. However, we must not jump to the conclusion that Tomonaga's result is incomplete, for there are no definite reason to show that the correspondence between one-pair states in the original Fermion system and the Boson states in Tomonaga's formulation is incomplete and, further, Sawada's method also contains a questionable point especially in one-dimensional systems. In our later discussions the difference between both theories is not important and we will not enter into this problem in more detail.

If the above scattering states could really exist, Araki's interpretation⁵⁾, according to which the excitation of a plasmon state with the lowest momentum is assigned to the first absorption band of a Carotenoid, would be very questionable. One might think at first that such scattering states could not absorb light quanta because of the conservation laws of energies and momenta. However, light waves can be absorbed by end effects, and dimensions of actual molecules are much smaller than the wave length of the light to be absorbed, that is, the conservation of momenta is out of question.

There are also low energy states of another type omitted in Tomonaga's theory, as mentioned by Nakajima⁶⁾: Configurations each of which is given by $a_n^* b_m^* \bar{\Psi}_0$, n being slightly larger than n_F and m being slightly larger than $-n_F$, will not give plasmons and would give low-energy scattering states whose energies are nearly equal to $\epsilon(n) - \epsilon(m)$. These situations are illustrated in Fig. 3. These scattering states can absorb light quanta in actual molecules. Thus, a kind of energy-gap model in Kuhn's

sense³⁾ or so is required, in order to explain the convergency of the first absorption wave-length of the conjugated molecules.¹⁾

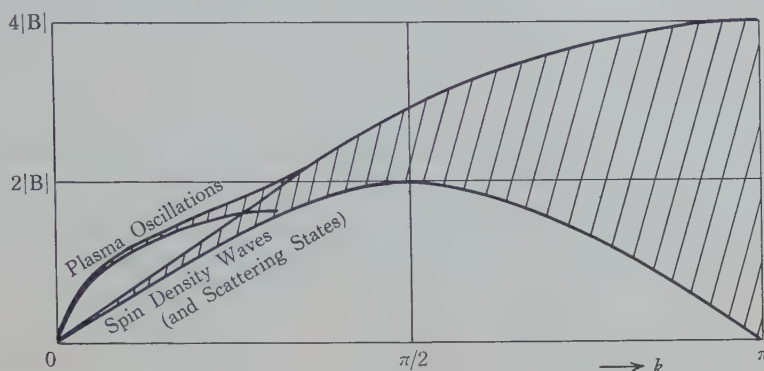


Fig. 3. Excitation energies of one-pair states in an N -molecule

§ 3. The Hamiltonian of the π electrons in a long molecule with alternating bonds

In our π electron Hamiltonian of an S -molecule the resonance integral defined in I is not merely B but $B + \Delta B$ or $B - \Delta B$, and further the Coulomb integral $J(R_1 - R_2)$ defined in I depends not only on $R = |R_1 - R_2|$ but on R_1 and R_2 . Here we assume that these differences are not large. Adopting Pariser-Parr's π -electron approximation and neglecting the end effects as in I, we get the Hamiltonian $H = H_0 + H_I$ given by

$$\begin{cases} H_0 = H_0^0 + \Delta H_0 \\ H_I = H_I^0 + \Delta H_I, \end{cases}$$

where

$$\begin{aligned} H_0^0 &= B \sum_{-N/2 < R \leq N/2}^{\sigma} (b_{R+1,\sigma}^* b_{R,\sigma} + b_{R-1,\sigma}^* b_{R,\sigma}) = \sum_{-\pi < k \leq \pi}^{\sigma} \epsilon(k) a_{k,\sigma}^* a_{k,\sigma}, \\ \Delta H_0 &= \Delta B \sum_{-N/4 < R \leq N/4}^{\sigma} (b_{2R,\sigma}^* b_{2R+1,\sigma} + b_{2R+1,\sigma}^* b_{2R,\sigma}) \\ &\quad - \Delta B \sum_{-N/4 \leq R < N/4}^{\sigma} (b_{2R,\sigma}^* b_{2R-1,\sigma} + b_{2R-1,\sigma}^* b_{2R,\sigma}), \\ H_I^0 &= \frac{1}{2} \sum_{-N/2 < R_1, R_2 \leq N/2} \rho(R_1) \rho(R_2) J(R_1 - R_2) = \frac{1}{2} \sum_k J_k \rho_k \rho_{-k} + \text{const}, \end{aligned} \quad (3.1)$$

and ΔH_I consists of the interaction between electric dipoles, each of which has dipole moment $(-1)^R e^2 \cdot \Delta a \cdot \rho(R)$. Hence ΔH_I is a short-range interaction as well as weak one. Therefore, we neglect this.

We will express ΔH_0 in terms of a_k^* and a_k . Hereafter, we will omit the spin suffix σ in order to save notations, unless it is necessary. Using (2.3), we get

$$\begin{aligned} & \sum_{-N/4 < R \leq N/4} (b_{2R}^* b_{2R-1} + b_{2R-1}^* b_{2R}) \\ &= \sum_{|k| \leq \pi} (\cos k) a_k^* a_k + i \sum_{|k|, |k+\pi| \leq \pi} (\sin k) a_{k+\pi}^* a_k + i \sum_{|k|, |k-\pi| \leq \pi} (\sin k) a_{k-\pi}^* a_k, \end{aligned}$$

and

$$\begin{aligned} & \sum_{-N/4 \leq R < N/4} (b_{2R}^* b_{2R+1} + b_{2R+1}^* b_{2R}) \\ &= \sum_{|k| \leq \pi} (\cos k) a_k^* a_k - i \sum_{|k|, |k+\pi| \leq \pi} (\sin k) a_{k+\pi}^* a_k - i \sum_{|k|, |k-\pi| \leq \pi} (\sin k) a_{k-\pi}^* a_k. \end{aligned}$$

Substituting these into (3.1), we obtain

$$\Delta H_0 = 2i\Delta B \sum_{|k|, |k+\pi| \leq \pi} (\sin k) a_{k+\pi}^* a_k + 2i\Delta B \sum_{|k|, |k-\pi| \leq \pi} (\sin k) a_{k-\pi}^* a_k.$$

Therefore

$$\begin{aligned} H_0 &= 2B \sum_{-\pi < k \leq \pi} (\cos k) a_k^* a_k \\ &+ 2i\Delta B \sum_{|k|, |k+\pi| \leq \pi} (\sin k) a_{k+\pi}^* a_k + 2i\Delta B \sum_{|k|, |k-\pi| \leq \pi} (\sin k) a_{k-\pi}^* a_k. \end{aligned} \quad (3.2)$$

This is easily diagonalized by means of the following transformation

$$\left. \begin{aligned} a_{\pi/2-k} &= \left(\frac{1+\gamma_k}{2} \right)^{1/2} a_f(-k) - i \left(\frac{1-\gamma_k}{2} \right)^{1/2} a_c(-k) \\ a_{-\pi/2-k} &= -i \left(\frac{1-\gamma_k}{2} \right)^{1/2} a_f(-k) + \left(\frac{1+\gamma_k}{2} \right)^{1/2} a_c(-k) \end{aligned} \right\} \quad (3.3)$$

$$\left. \begin{aligned} a_{-\pi/2+k} &= -i \left(\frac{1+\gamma_k}{2} \right)^{1/2} a_f(k) + \left(\frac{1-\gamma_k}{2} \right)^{1/2} a_c(k) \\ a_{\pi/2+k} &= \left(\frac{1-\gamma_k}{2} \right)^{1/2} a_f(k) - i \left(\frac{1+\gamma_k}{2} \right)^{1/2} a_c(k) \end{aligned} \right\} \quad (3.3')$$

where $\pi/2 > k > 0$ and

$$\gamma_k = \frac{|B \sin k|}{\sqrt{B^2 \sin^2 k + (\Delta B)^2 \cos^2 k}}. \quad (3.4)$$

Then (3.2) is written as

$$H_0 = \sum_{-\pi/2 < k \leq \pi/2} E_c(k) a_c^*(k) a_c(k) + \sum_{-\pi/2 < k \leq \pi/2} E_f(k) a_f^*(k) a_f(k), \quad (3.5)$$

where

$$E_c(k) = -E_f(k) = 2\sqrt{B^2 \sin^2 k + (\Delta B)^2 \cos^2 k}. \quad (3.6)$$

Thus we have a Brillouin gap $\epsilon_g = 4\Delta B$, where we have chosen $\Delta B > 0$. These spectra of H_0 are shown in Fig. 4.

Now we will express the Coulomb interaction

$$H_I = \sum_{k>0} J_k \rho_k \rho_{-k}$$

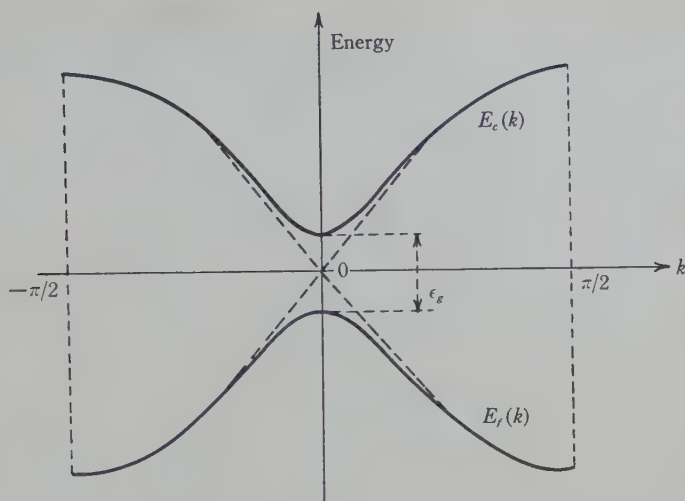


Fig. 4

in terms of $a_f(k)$ and $a_c(k)$. The result is as follows,

$$\begin{aligned} \rho_k = & \sum_{l>0} A_l^{l+k} \{ a_f^*(l+k) a_f(l) + a_c^*(l+k) a_c(l) \\ & + a_f^*(-l) a_f(-l-k) + a_c^*(-l) a_c(-l-k) \} \\ & + \sum_{\kappa>l>0} B_l^{k-l} \{ a_f^*(k-l) a_f(-l) + a_c^*(k-l) a_c(-l) \} \\ & + \sum_{l>0} i f_l^{l+k} \{ a_c^*(l+k) a_f(l) + a_f^*(l+k) a_c(l) \\ & + a_f^*(-l) a_c(-l-k) + a_c^*(-l) a_f(-l-k) \} \\ & + \sum_{k>l>0} i g_l^{k-l} \{ a_f^*(k-l) a_c(-l) + a_c^*(k-l) a_f(-l) \}, \end{aligned} \quad (3.7)$$

$$\rho_{-k} = \rho_k^*,$$

where

$$\left. \begin{aligned} A_l^{l+k} & \equiv \frac{1}{2} \{ (1-\gamma_{l+k})^{1/2} (1-\gamma_l)^{1/2} + (1+\gamma_{l+k})^{1/2} (1+\gamma_l)^{1/2} \} \\ B_l^{k-l} & \equiv \frac{1}{2} \{ (1+\gamma_{k-l})^{1/2} (1-\gamma_l)^{1/2} + (1-\gamma_{k-l})^{1/2} (1+\gamma_l)^{1/2} \} \\ f_l^{l+k} & \equiv \frac{1}{2} \{ (1+\gamma_{l+k})^{1/2} (1-\gamma_l)^{1/2} - (1-\gamma_{l+k})^{1/2} (1+\gamma_l)^{1/2} \} \\ g_l^{k-l} & \equiv \frac{1}{2} \{ (1+\gamma_{k-l})^{1/2} (1+\gamma_l)^{1/2} - (1-\gamma_{k-l})^{1/2} (1-\gamma_l)^{1/2} \} \end{aligned} \right\}. \quad (3.8)$$

Here we adopt the following conventional notations,

$$a_c(k) \equiv a(k),$$

$$a_f(k) \equiv b^*(k),$$

$b^*(k)$ corresponding to the creation operator for a hole. Further, in order to simplify the notations in (3.7), we introduce the following quantities,

$$C_l^{l+k} \begin{cases} \equiv A_l^{l+k}(l > 0), \\ \equiv B_{-l}^{k+l}(0 > l > -k), \\ \equiv A_{-l-k}^{-l}(-k > l), \end{cases} \quad e_l^{l+k} \begin{cases} \equiv f_l^{l+k}(l > 0), \\ \equiv g_{-l}^{k+l}(0 > l > -k), \\ \equiv f_{-l-k}^{-l}(-k > l), \end{cases} \quad (3.9)$$

where $k > 0$. Then we have

$$\rho_k = \sum_l C_l^{l+k} \{b(l+k)b^*(l) + a^*(l+k)a(l)\} - i\zeta_k \quad (3.10)$$

where

$$\zeta_k \equiv \sum_l e_l^{l+k} \{a^*(l+k)b^*(l) + b(l+k)a(l)\} \quad (3.11)$$

for $k > 0$ and

$$\zeta_{-k} \equiv \zeta_k^*.$$

For our later discussions we give values of e_l^{l+k} in Table I.

Table I

k	$\ll \Delta B/ B $		$\gg \Delta B/ B $	
	$\ll \Delta B/ B $	$\gg \Delta B/ B $	$\ll \Delta B/ B $	$\gg \Delta B/ B $
g_l^{k-l}	$\frac{k}{2} \frac{\Delta B}{ B }$		$1/\sqrt{2}$	1
f_l^{l+k}	$\frac{k}{2} \frac{\Delta B}{ B }$	$\frac{k}{2} \left(\frac{\Delta B}{ B } \right) \text{cosec}^2 l$	$1/\sqrt{2}$	$\frac{\Delta B}{2 B } (\cot(l+k) - \cot l)$

§ 4. Plasma oscillations

Though our main purpose is to study the one-dimensional system, we will give some qualitative argument on the collective motions in a three-dimensional system. In general, if NJ_k is replaced by $4\pi e^2/ak^2$, our discussions on the one-dimensional model seem to be in essential points the ones on a three-dimensional system.

Excited states which may be interpreted as oscillating states of density waves might ordinarily be found even in insulators, if we seek such solutions among high energy states. However, many of them would be much different in character from the "usual" plasmon states in metals, and investigation of such high energy states is beyond our task. Our problem is to ascertain whether plasma oscillations similar to those in N -molecules ($\epsilon_g = 0$) exist in S -molecules ($\epsilon_g > 0$) or not.

Now the plasmon states in N -molecules are given by the diagrams of Gell-Mann and Brueckner's type⁷⁾ as shown in Fig. 5 which we call a plasmon diagram. Therefore, as in § 2, we seek for solutions within the framework of the plasmon diagrams and investigate the solutions to see

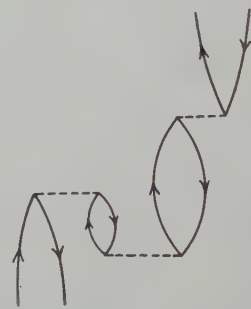


Fig. 5

whether much difference can be found between these solutions and those of N -molecules. Then, according to Sawada,⁴⁾ we may replace the Coulomb interaction H_I by

$$H_I \rightarrow H_c = \sum_{k>0} J_k \zeta_k \zeta_{-k}$$

where ζ_k is given by (3.11) or

$$\zeta_k = \sum_l e_l^{l+k} \{ \vartheta^*(l, l+k) + \vartheta(l+k, l) \}. \quad (4.1)$$

In the above

$$\vartheta(l, k) = b(l) a(k). \quad (4.2)$$

Then, in the framework of the plasmon diagrams, we get

$$\begin{aligned} [H_c, \vartheta^*(l, l+k)] &= J_k \cdot e_l^{l+k} \cdot \zeta_k, \\ [H_c, \vartheta(l+k, l)] &= -J_k \cdot e_l^{l+k} \cdot \zeta_k, \\ [H_0, \vartheta^*(l, l+k)] &= E(l, l+k) \vartheta^*(l, l+k), \\ [H_0, \vartheta(l+k, l)] &= -E(l, l+k) \vartheta(l+k, l), \end{aligned}$$

where

$$E(l, k) \equiv E_c(k) - E_f(l). \quad (4.3)$$

From these, we get the following eigenvalue equation for excitation energy W :

$$1 = J_k \sum_l^\sigma |e_l^{l+k}|^2 \left\{ \frac{1}{W - E(l, l+k)} - \frac{1}{W + E(l, l+k)} \right\} \quad (4.4')$$

or

$$1 = 4J_k \sum_l \frac{E(l, l+k) |e_l^{l+k}|^2}{W^2 - E^2(l, l+k)}. \quad (4.4)$$

We seek for solutions of (4.4) for some extreme values of k .

Case I. $|k| \ll 4B/|B|$

In this case (4.4) may be approximated as

$$1 = 8J_k \sum_{l>0} \frac{E(l, l+k) |f_l^{l+k}|^2}{W^2 - E^2(l, l+k)} = \frac{4}{\pi} N J_k \int_0^{\pi/2} \frac{E(l, l+k) |f_l^{l+k}|^2}{W^2 - E^2(l, l+k)} dl. \quad (4.5)$$

We calculate $E(l, l+k) |f_l^{l+k}|^2$ up to the lowest order term with respect to k . Then we get

$$E(l, l+k) |f_l^{l+k}|^2 = k^2 \frac{B^2 4B^2}{(4B^2 \cos^2 l + B^2 \sin^2 l)^{3/2}} + O(k^3). \quad (4.6)$$

(Case I.1) Three-dimensional system ($|k| \ll 4B/|B|$);

$$N \cdot J_k = \frac{4\pi e^2}{a} \frac{1}{k^2}.$$

At first we assume that the solution of (4.5) satisfies

$$W > \text{Max}\{E(l, l+k)\}.$$

Now $NJ_k \cdot E(l, l+k) |f_{l+k}'|^2$ in the numerator of (4.5) is extremely large only when $l \ll \Delta B/|B|$ and is practically zero when $l \gg \Delta B/|B|$, as is easily shown from (4.6). Further, for $l \ll \Delta B/|B|$,

$$W \gg E(l, l+k).$$

Then we get from (4.5)

$$W^2 = 8J_k \sum_{l>0} E(l, l+k) |f_{l+k}'|^2.$$

Substituting (4.6) into the above and replacing the sum by the integral, we obtain

$$\begin{aligned} W^2 &= \frac{16e^2}{a} B^2 \Delta B^2 \int_0^{\pi/2} \frac{dl}{(B^2 \sin^2 l + \Delta B^2 \cos^2 l)^{3/2}} \\ &= \frac{16e^2}{a} |B| \left(\frac{\Delta B}{B} \right)^2 \int_0^{\pi/2} \frac{dl}{[1 - \{1 - (\Delta B/B)^2\} \cos^2 l]^{3/2}}. \end{aligned}$$

In case $\Delta B \ll |B|$ we get, as shown in the Appendix,

$$\int_0^{\pi/2} \frac{dl}{[1 - \{1 - (\Delta B/B)^2\} \cos^2 l]^{3/2}} = \left(\frac{B}{\Delta B} \right)^2.$$

Therefore we obtain

$$W = 4 \sqrt{\frac{e^2}{a} \cdot |B|} \equiv W_p.$$

This is nothing but the excitation energy of a plasmon given by substituting $NJ_k = (4\pi e^2/a) \cdot (1/k^2)$ into (2.5), i.e. the plasmon energy in the "normal state" ($\epsilon_g = 0$). For a very small k , the maximum value of $E(l, l+k)$ is almost equal to $4|B|$, as shown in Fig. 7. Therefore, $W_p > 4|B|$ has now been required. This means $e^2/a > |B|$. If this condition is satisfied, the plasmon level lies separately from the level continuum given by $E(l, l+k)$. On the other hand, if $e^2/a < |B|$, the plasmon level sinks into the level continuum. Even in this case, we cannot say that the plasma oscillation disappears when the gap appears. In this case the dispersion relation is written as⁸⁾

$$1 = \frac{4}{\pi} NJ_k \cdot P \int \frac{E \cdot |e_{l+k}'|^2}{W^2 - E^2} \mathcal{Q}(E) dE \quad (4.7)$$

where P means the principal value,

$$E = E(l, l+k)$$

and

$$dl = \mathcal{Q}(E) dE.$$

Then we get again the plasma frequency which is nearly equal to the one in the "normal state" ($\epsilon_g=0$). The electron excitations $(f, l) \rightarrow (c, l+k)$ in which $E(l, l+k)$'s are nearly equal to W should be treated separately. These excitations give the level width of our plasma oscillation. We estimate this width by means of the following formula,

$$\hbar\Gamma = \frac{1}{N} \frac{4\pi e^2}{ak^2} \sum_l E(l, l+k) |e_l^{l+k}|^2 \delta(E(l, l+k) - W)$$

which has been derived by Kanazawa⁹⁾ in the framework of the time-dependent perturbation illustrated by Fig. 6. In order that

$$E(l, l+k) = W_p$$

is satisfied, $|l|$ should be very large compared with $\Delta B/|B|$. Then we get

$$E(l, l+k) |e_l^{l+k}|^2 = k^2 \cdot \Delta B/|B| \cdot \Delta B/\sin^3 l.$$

Therefore $\hbar\Gamma/B \propto (\Delta B/B)^2$. Consequently, if the gap is very small, the energy width of the plasmon level lying in the level continuum is very small.

(Case I·2) One-dimensional system ($|k| \ll \Delta B/|B|$)

In this case $NJ_k = (2e^2/a) K_0(\gamma|k|)$, then the dispersion relation (4·7) is written as

$$1 = \frac{4e^2}{a} k^2 K_0(\gamma|k|) P \int_{l=0}^{l=\pi/2} \frac{B^2 \Delta B^2}{(B^2 \sin^2 l + \Delta B^2 \cos^2 l)^{3/2}} \frac{\Omega(E) dE}{W^2 - E^2(l, l+k)}.$$

Now $\lim_{k \rightarrow 0} k^2 K_0(\gamma|k|) = 0$. Therefore, W should be equal to $E(l, l+k) = E$, at which the above integrand rapidly changes. Then, it is very doubtful whether this solution does correspond to a real plasmon state. Indeed, one obtains the conclusion that there are no plasmons in the region of k satisfying $\Delta B/|B| \gg k$, if one is allowed to use the "existence criterion" of Bohm and Pines mentioned below. This criterion is stated as follows: For any k , the plasma oscillation exists if $I \equiv W_k/(\Phi |J_k \rho_k \rho_{-k}| \Phi) < 1$, but it does not exist if $I > 1$. In the above, Φ is a Slater determinant composed of a single particle state belonging to a degenerate Fermi sphere. In our case $W_k \cong \epsilon_g$. Further, noting that we are now considering the the case of $k \ll \Delta B/|B|$, we get

$$\begin{aligned} (\Phi |J_k \rho_k \rho_{-k}| \Phi) &= J_k (\sum_l |e_l^{l+k}|^2) \\ &\cong 2J_k \left(\frac{k}{2} \left/ \frac{\Delta B}{B} \right. \right)^2 \times \frac{N}{2\pi} \left(\frac{\Delta B}{|B|} \right) = \frac{1}{16} (W_k^{(0)})^2 / \Delta B = \frac{1}{4} (W_k^{(0)})^2 / \epsilon_g, \end{aligned}$$

where $W_k^{(0)} = \sqrt{(4/\pi) k^2 |B| NJ_k}$ is the energy of a plasmon in the case of $\epsilon_g = 0$. Then we obtain

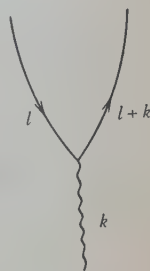


Fig. 6

$$I \cong 4(\epsilon_g/W_k^{(0)})^2 \gg 1.$$

Thus, the plasmon levels which exist in N -molecules under the energy gap completely disappear in S -molecules and seem to be dissolved away into the level continuum above the gap.

Case II. $\pi/2 \gg k \gg 4B/|B|$

In the dispersion formula

$$1 = \frac{2NJ_k}{\pi} \cdot P \int_{l=-\pi/2}^{l=\pi/2} \frac{E(l, l+k) |e_l^{l+k}|^2}{W^2 - E^2(l, l+k)} \Omega(E) dE$$

we may use

$$e_l^{l+k} \begin{cases} = g_{-l}^{k+l}, & \text{if } -k < l < 0. \\ \cong 0, & \text{otherwise.} \end{cases}$$

Therefore, we get

$$1 = \frac{2NJ_k}{\pi} \int_{-k}^0 \frac{E(l, l+k)}{W^2 - E^2(l, l+k)} dl.$$

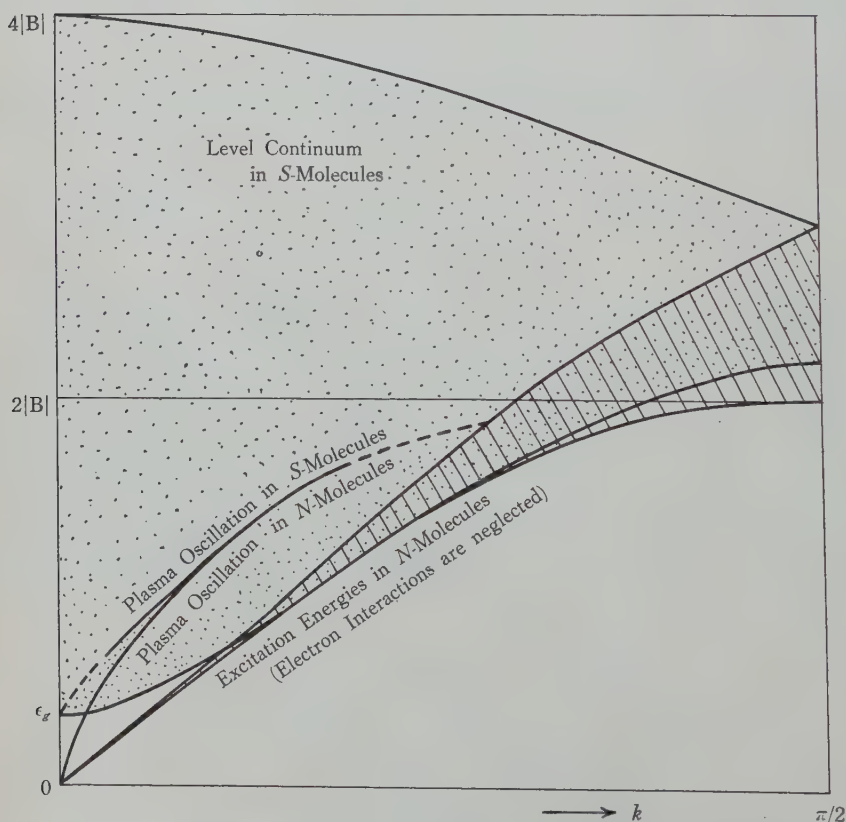


Fig. 7. Excitation energies of one-pair states

We can find a solution satisfying

$$W \gg E(l, l+k) \quad \text{for} \quad -k < l < 0.$$

Further in the integrands, we may use

$$E(l, l+k) = 2|B|k.$$

Then the above eigenvalue equation is nothing but the one in an N -molecule. This is true in the one-dimensional system as well as in the three-dimensional one. Especially, in the former the plasmon level always lies in the level continuum (unless $k^2 N J_k > 4|B|$). The level width of this plasma oscillation is given by

$$\hbar\Gamma = J_k \sum_l E(l, l+k) |e_l^{l+k}|^2 \delta(E(l, l+k) - W).$$

l should be very large compared with $\Delta B/|B|$ in order to satisfy $E(l, l+k) = W$. Then,

$$e_l^{l+k} \rightarrow f_l^{l+k} \propto \frac{\Delta B}{|B|} (\cot(l+k) - \cot l).$$

Thus, $\hbar\Gamma$ is sufficiently small and hence stable plasma oscillations may exist in S -molecules when $\Delta B \ll |B|$.

§ 5. Excitons

As is well known, the plasma oscillations come from the high mobility of electrons which leads to the screening of the Coulomb interaction between electrons. If the mobility were very large, the attractive force between an electron and a hole will also be screened and the possibility that the electron-hole pair has a bound state will be diminished. Thus excitons and plasmons are mutually exclusive. The exciton solution is given by the diagrams shown in Fig. 8. In the usual derivation^{(10),(11),(12)} of the equation describing the hydrogen-like internal motion of an exciton, one implicitly adopts only these diagrams and neglects all the other ones. Thus

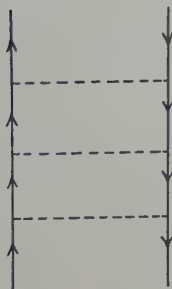


Fig. 8

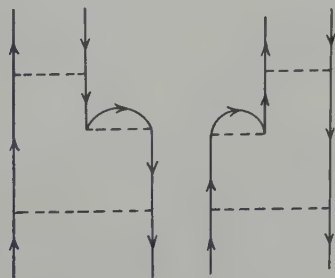


Fig. 9

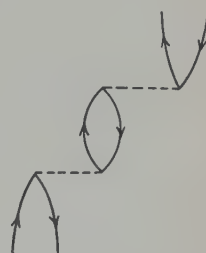




Fig. 10

we will call the diagrams shown in Fig. 8 the exciton diagrams. If the energy gap is very large as in case of insulators, the diagrams for an electron-hole pair which seem important at first sight are the exciton diagrams, exchange diagrams¹³⁾ (Fig. 9), and Macke's diagrams¹⁴⁾ (Fig. 10) (and of course, the ones given by mixing these diagrams). The remaining diagrams may be neglected, because they should be accompanied by "vacuum polarizations" which require a large energy in case of a large energy gap. The exchange diagrams will also play a minor role because they can be included in $E_c(k)$ or $E_f(k)$. One neglects Macke's diagrams, because in the off-diagonal element $(a^*(l'+k)b^*(l')\Psi_0|H_I|b(l)a(l+k)\Psi_0)$ the term  is generally small compared with . However, the reason why other diagrams than exciton ones may be neglected in actual insulators would not be very clear. Indeed, the polarization waves in Mott-Pines' sense¹⁵⁾ which can never be given by exciton diagrams might exist in insulators.

At first we take only the exciton diagrams. Then

$$\rho_k \rightarrow \sum_l C_l^{l+k} \{b(l+k)b^*(l) + a^*(l+k)a(l)\}.$$

Further the Coulomb interaction

$$H_I = \sum_{k>0} J_k \rho_k \rho_{-k}$$

may be replaced by

$$H_I \rightarrow H_c = - \sum_{\kappa>0} \sum_{\lambda, \mu} J_{\kappa} \cdot C_{\mu}^{\mu+\kappa} C_{\lambda}^{\lambda+\kappa} \{a^*(\lambda)b^*(\mu)b(\mu+\kappa)a(\lambda+\kappa) + a^*(\lambda+\kappa)b^*(\mu+\kappa)b(\mu)a(\lambda)\}.$$

Introducing $\vartheta(\mu, \lambda) \equiv b(\mu)a(\lambda)$, we rewrite this as

$$H_c = - \sum_{\kappa \neq 0} \sum_{\lambda, \mu} J_{\kappa} C_{\mu}^{\mu+\kappa} C_{\lambda}^{\lambda+\kappa} \vartheta^*(\mu, \lambda) \vartheta(\mu+\kappa, \lambda+\kappa).$$

As we limit ourselves in the exciton diagrams, we may use the commutation relation

$$[\vartheta(\lambda, \mu), \vartheta^*(\lambda', \mu')] = \delta_{\lambda\lambda'} \delta_{\mu\mu'}$$

in the evaluation of $[H_c, \vartheta^*(h-k, h)]$. Then we get

$$[H_c, \vartheta^*(h-k, h)] = - \sum_{\kappa \neq 0} J_{\kappa} C_{h-k-\kappa}^{h-k} C_{h-\kappa}^{\kappa} \vartheta^*(h-k-\kappa, h-\kappa). \quad (5.1)$$

Eq. (5.1) together with

$$[H_0, \vartheta^*(h-k, h)] = E(h-k, h) \vartheta^*(h-k, h)$$

determines the motion of our electron-hole pair.

Here we introduce the following quantity,

$$\mathcal{L}^*(k, R) = N^{-1/2} \sum_{\lambda} \exp(ihR) \vartheta^*(h-k, h).$$

Then we have

$$\sum_{R'} \mathcal{H}_c(R'-R) \cdot U_k(n, R') = E_c \left(\frac{1}{i} \frac{\partial}{\partial R} \right) U_k(n, R),$$

and

$$\sum_{R'} \exp[ik(R'-R)] \cdot \mathcal{H}_f(R'-R) \cdot U_k(n, R') = E_f \left(-k + \frac{1}{i} \frac{\partial}{\partial R} \right) U_k(n, R).$$

Therefore, Eq. (5.7) becomes

$$\begin{aligned} & \left[E_c \left(\frac{1}{i} \frac{\partial}{\partial R} \right) - E_f \left(-k + \frac{1}{i} \frac{\partial}{\partial R} \right) \right] U_k(n, R) \\ & - \int V_k(R, R') U_k(n, R') dR' = W U_k(n, R). \end{aligned}$$

We here perform the following transformation,¹⁰⁾

$$\exp \left(-\frac{i}{2} kR \right) U_k(n, R) = \Gamma_k(n, R).$$

Then, we get

$$\mathfrak{L} \cdot \Gamma_k(n, R) - \int V_k(R, R') \Gamma_k(n, R') dR' = W \cdot \Gamma_k(n, R), \quad (5.8)$$

where

$$\begin{aligned} \mathfrak{L} &= \exp \left(-\frac{i}{2} kR \right) \left[E_c \left(\frac{1}{i} \frac{\partial}{\partial R} \right) - E_f \left(-k + \frac{1}{i} \frac{\partial}{\partial R} \right) \right] \exp \left(\frac{i}{2} kR \right) \\ &= E_c \left(\frac{k}{2} + \frac{1}{i} \frac{\partial}{\partial R} \right) - E_f \left(-\frac{k}{2} + \frac{1}{i} \frac{\partial}{\partial R} \right). \end{aligned} \quad (5.9)$$

As we are now concerned with the very low energy states, we here consider the case $k \rightarrow 0$. Further, we replace $E_c(p)$ and $E_f(p)$ by

$$\begin{aligned} E_c(p) &\rightarrow \tilde{E}_c(p) = 2\Delta B + \frac{\hbar^2}{2m^*} (p/a)^2 \\ -E_f(p) &\rightarrow -\tilde{E}_f(p) = 2\Delta B + \frac{\hbar^2}{2m^*} (p/a)^2, \end{aligned} \quad (5.10)$$

where

$$\frac{\hbar^2}{m^* a^2} = \left[\frac{d^2}{dp^2} E_c(p) \right]_{p=0} = \left[\frac{d^2}{dp^2} E_f(p) \right]_{p=0} = 2 \frac{B^2 - \Delta B^2}{\Delta B}. \quad (5.11)$$

Then we obtain

$$\mathfrak{L} = 4\Delta B + \frac{\hbar^2 k^2}{2 \cdot (2m^* a^2)} - \frac{\hbar^2}{m^* a^2} \frac{d^2}{dR^2}.$$

Omitting the kinetic energy due to the translational motion of the center of mass of the electron-hole pair, we get the following wave equation for the internal motion of our electron-hole pair,

$$-\frac{\hbar^2}{m^*a^2} \cdot \frac{d^2}{dR^2} \Gamma_k(n, R) - \int V_k(R, R') \Gamma_k(n, R') dR' = (W - \epsilon_g) \Gamma_k(n, R). \quad (5.12)$$

If $4B \ll |B|$, (5.11) becomes

$$m^* = \frac{4B\hbar^2}{2B^2a^2} = \frac{1}{2} \left(\frac{4B}{|B|} \right) \cdot \left(\frac{e^2/a}{|B|} \right) \left(\frac{a_B}{a} \right) m, \quad (5.13)$$

where a_B is the Bohr radius and m denotes the usual electron mass. Therefore $4B \ll |B|$ means $m^* \ll m$, for $a \sim a_B$ and $|B| \gtrsim (e^2/a)$. In this case, it is very difficult to localize the internal motion of our electron-hole pair, because of the uncertainty principle. Then, the bound state of this pair, if it could exist, should have a widely spreading orbital for the internal motion. Then, the long range part of the attracting force between the electron and the hole becomes important, while the short range part plays a less important role. Therefore, among the Fourier components of this potential, J_κ 's with small κ play important roles.

One may think that such a widely spreading character of the internal motion is completely due to our "ad hoc" assumption (5.10). Indeed, $\tilde{E}(p)$ gives an unreasonably high energy for the pair configuration $\vartheta^*(p-k, p)$ when $|p| \gg 4B/|B|$ (cf. Fig. 11). Hence, the mixing of the configuration $\vartheta^*(p-k, p)\Psi_0$ (with $|p| \gg 4B/|B|$) might be undeservedly suppressed. However, the mixing of such a configuration will actually be small, for J_κ (with $\pi/2 \gg \kappa \gg 4B/|B|$) should be replaced by $\tilde{J}_\kappa (\ll J_\kappa)$, as shown in our later discussions, and then the matrix element between the above configuration and $\vartheta^*(p-k, p)\Psi_0$ (with $|p| \ll 4B/|B|$) will become very small. Now, we will examine the potential $V_k(R, R')$. When $k \rightarrow 0$, we get

$$C_\kappa^k(R-R') = \frac{1}{N} \sum_h |C_{h-\kappa}^h|^2 \exp[ih(R-R')].$$

For small κ , $C_{h-\kappa}^h = 1$. Then we obtain

$$C_\kappa^k(R-R') = \delta(R-R').$$

Therefore, for sufficiently large R , we get

$$\begin{aligned} V_k(R, R') &= \sum_{\kappa \neq 0} J_\kappa \cdot \exp(i\kappa R) \delta(R-R') \\ &= \frac{e^2}{Ra} \cdot \delta(R-R'). \end{aligned}$$

This assures the existence of the bound state solution for Eq. (5.12). The dissociation energy of this bound state is roughly estimated as follows.

$$W - \epsilon_g \cong -\frac{1}{4} \frac{e^2}{a_B} \frac{4B\hbar^2}{B^2a^2} \frac{1}{m} = -\frac{1}{4} 4B \left(\frac{e^2}{a} \frac{1}{|B|} \right)^2 = -\frac{1}{16} \epsilon_g \cdot \left(\frac{e^2}{a} \frac{1}{|B|} \right)^2 \quad (5.14)$$

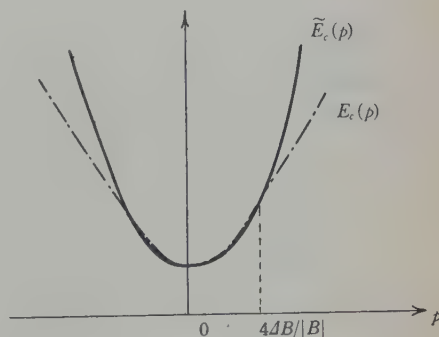
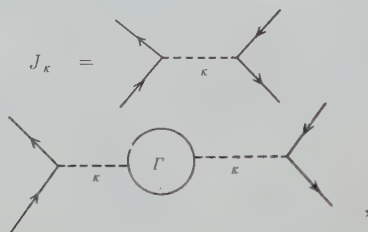


Fig. 11

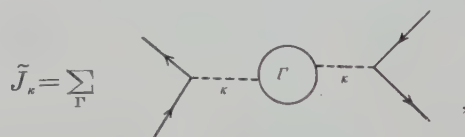
It should be noted that $|B|$ is of the same order as that of e^2/a , though $|B| > e^2/a$. However, we must not jump to the conclusion that we can find an exciton level appreciably lower than the energy gap in an S -molecule.

The reason why we have got such an exciton level is that we have taken into consideration only the exciton diagrams as shown in Fig. 8 or, in other words, we have completely neglected the screening effects for the long range Coulomb interaction between the electron and the hole. This interaction is not only



but also

where Γ is some polarization diagram.¹⁰⁾ Actually, for a scattering process with momentum transfer κ , the effective interaction should be given by



where the summation includes $\Gamma=0$. This \tilde{J}_κ will be much smaller than J_κ .

Among various Γ 's plasmon diagrams will be most important in case of a small gap. At first we consider the simplest case as shown in Fig. 12. The contribution from these diagrams is estimated as follows,

$$J^{(1)}(\kappa) \equiv -2J_\kappa^2 \sum_{\mu}^{\sigma} \frac{|e_{\mu}^{\mu+\kappa}|^2}{E(\mu, \mu+\kappa)} = -\frac{2NJ_\kappa^2}{\pi} \int_{-\pi/2}^{\pi/2} \frac{|e_{\mu}^{\mu+\kappa}|^2 d\mu}{E(\mu, \mu+\kappa)}. \quad (5.15)$$

This expression is not correct from the quantitative point of view, for in the energy denominator we have neglected the energy change induced by the intra-band transition of our electron-hole pair. For $\kappa \ll \Delta B/|B|$, however, (5.15) is a

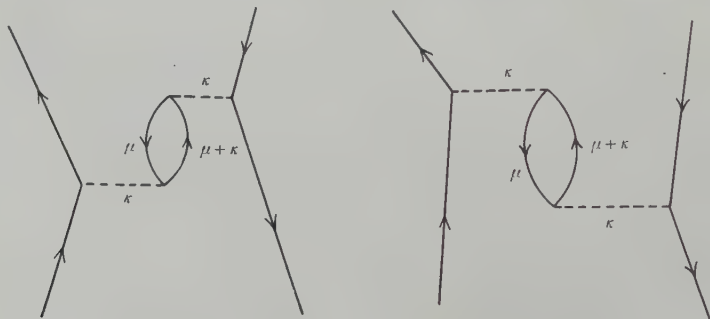


Fig. 12

sufficiently good approximation, for such an energy change is usually smaller than the polarization energy $E(\mu, \mu + \kappa)$. For $\pi/2 \gg \kappa \gg \Delta B/|B|$, on the other hand, the use of (5.15) can never be justified from the quantitative point of view. However, from the qualitative point of view, our result seems to remain unchanged, even if we take into account such an energy change. It should be noted that by using (5.15) we have replaced the screening for the interaction between the electron and the hole by the one for the interaction between two motionless charges placed in our many-electron system. In other words, we have neglected the retardation corresponding to the time spent by the interaction in polarizing the many-electron system. Though we can take this retardation into account by means of Hubbard's method¹⁶⁾ or else, we here neglect this, for the result including the retardation will complicate our later discussions. Here we adopt a similar technique to the one adopted by Gell-Mann and Brueckner⁷⁾, i.e. Feynman's path-integral method; we introduce the following function,

$$F_{\kappa}(t) \equiv \int_{-\pi/2}^{\pi/2} |e_{\mu}^{\mu+\kappa}|^2 \cdot \exp[-|t|E(\mu, \mu + \kappa)] d\mu.$$

Then (5.15) is written as

$$J^{(1)}(\kappa) = -\frac{NJ_{\kappa}^2}{\pi} \int_{-\infty}^{\infty} F_{\kappa}(t) dt.$$

This is further expressed as

$$J^{(1)}(\kappa) = -\frac{NJ_{\kappa}^2}{\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt_1 \cdot F_{\kappa}(t_1) \delta(t_1 - t).$$

Next, we consider all plasmon diagrams shown in Fig. 13. Further, for our present qualitative arguments, we neglect all the other diagrams in the summation over Γ . Then we get

$$\tilde{J}_{\kappa} = \sum_{n=0}^{\infty} J^{(n)}(\kappa),$$

$$J^{(n)}(\kappa) \equiv J_{\kappa} \cdot (-1)^n \left(\frac{NJ_{\kappa}}{\pi} \right)^n \cdot A_n(\kappa)$$

$$A_n(\kappa) \equiv \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \cdot F_{\kappa}(t_1) \cdots F_{\kappa}(t_n) \cdot \delta(t_1 + \cdots + t_n - t).$$

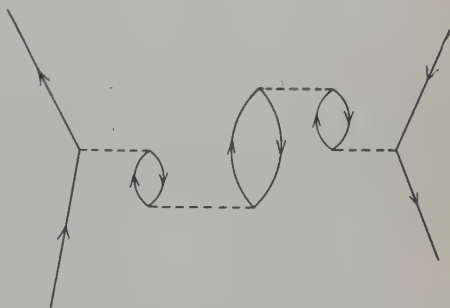


Fig. 13

Expanding the above δ -function into the Fourier series, we get

$$A_n(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dt [Q_\kappa(u)]^n \exp(-iut) = [Q_\kappa(0)]^n,$$

where

$$Q_\kappa(u) \equiv \int_{-\infty}^{\infty} F_\kappa(t) \exp(itu) dt.$$

Therefore

$$\tilde{J}_\kappa = J_\kappa \sum_{n=0}^{\infty} (-1)^n \left(\frac{NJ_\kappa}{\pi} \right)^n [Q_\kappa(0)]^n = J_\kappa \frac{1}{1 + (NJ_\kappa/\pi) Q_\kappa(0)}, \quad (5.16)$$

where

$$Q_\kappa(0) = \int_{-\pi/2}^{\pi/2} d\mu \int_{-\infty}^{\infty} dt \cdot \exp[-|t| \cdot E(\mu, \mu + \kappa)] \cdot |e_\mu^{\mu + \kappa}|^2 = 2 \int_{-\pi/2}^{\pi/2} d\mu \frac{|e_\mu^{\mu + \kappa}|^2}{E(\mu, \mu + \kappa)}. \quad (5.17)$$

(Case I) $\kappa \ll \Delta B/|B|$.

(5.17) may be safely replaced by

$$Q_\kappa(0) = 4 \int_0^{\pi/2} d\mu \frac{|f_\mu^{\mu + \kappa}|^2}{E(\mu, \mu + \kappa)}.$$

By means of (4.6) and (3.6), the above quantity is written as

$$\begin{aligned} Q_\kappa(0) &= \kappa^2 B^2 (\Delta B)^2 \int_0^{\pi/2} \frac{d\mu}{[(\Delta B)^2 \cos^2 \mu + B^2 \sin^2 \mu]^{5/2}} \\ &= \frac{\kappa^2 (\Delta B)^2}{|B|^3} \int_0^{\pi/2} \frac{d\mu}{[1 - \{1 - (\Delta B/B)^2\} \cos^2 \mu]^{5/2}}. \end{aligned}$$

This integral is evaluated in the Appendix, and is approximately equal to $\frac{2}{3}(B/\Delta B)^4$. Using this, we obtain

$$Q_\kappa(0) = \frac{2\kappa^2 |B|}{3(\Delta B)^2}.$$

Then from (5.16) we obtain

$$\tilde{J}_\kappa = J_\kappa \cdot \frac{1}{1 + 2(NJ_\kappa) \cdot \kappa^2 |B| / 3\pi (\Delta B)^2}. \quad (5.18)$$

(Case II) $\pi/a \gg \kappa \gg \Delta B/|B|$.

(5.16) may be safely replaced by

$$Q_{\kappa}(0) = 2 \int_{-\kappa}^0 \frac{d\mu}{E(\mu, \mu + \kappa)}. \quad (\text{cf. Table I})$$

Using $E(\mu, \mu + \kappa) = 2|B|\kappa$, we get

$$Q_{\kappa}(0) = 1/|B|.$$

Substituting the above into (5.16), we obtain

$$\tilde{J}_{\kappa} = J_{\kappa} \cdot \frac{1}{1 + NJ_{\kappa}/\pi|B|}. \quad (5.19)$$

In Case II, the screened interaction in the S -molecule is equal to the one in the N -molecule. Note that $NJ_{\kappa} \gg |B|$, when $|\kappa| \ll \pi/2$. Especially, in three dimensional systems ($NJ_{\kappa} = (1/a) \cdot 4\pi e^2/\kappa^2$), \tilde{J}_{κ} is nothing but the Fourier coefficient of the potential $e^2 \cdot \exp(-\alpha r)/r$. Then, the attractive potential between the electron and the hole separated by a distance r , $a|B|/\Delta B > r \gg a$, is not $-(e^2/r)$ but $-e^2 \cdot \exp(-\alpha r)/r$. In Case I ($\kappa \ll \Delta B/|B|$), the screening effect is weaker, though this effect grows stronger as $\Delta B \rightarrow 0$. Especially in three dimensional systems $\tilde{J}_{\kappa} = J_{\kappa} \cdot \frac{1}{1 + (8/3)(e^2/a)|B|/\Delta B^2}$. Therefore, for $r \gg (|B|/\Delta B)a$, the interaction becomes $-e^2/\epsilon r$, where $\epsilon \cong 8/3 \cdot e^2/a \cdot |B|/\Delta B^2$. Though this dielectric constant is extremely larger than unity, the interaction $-e^2/\epsilon r$ assures the existence of the excitons. Namely, the exciton solution is certainly obtained whenever the energy gap may arise and be compatible with the plasmon solution. However, the dissociation energy of the exciton in a three-dimensional system is extremely small compared with the gap, i.e., $(W - \epsilon_g/\epsilon_g)$ tends rapidly to zero in proportion to ϵ_g^2 as $\epsilon_g \rightarrow 0$, for the dielectric constant is inversely proportional to ϵ_g^2 . Thus, the exciton cannot be observed if ϵ_g is sufficiently small, for the exciton level should have the energy width.¹¹⁾

In our one-dimensional system, however, $\tilde{J}_{\kappa}/J_{\kappa}$ for $\kappa \ll \Delta B/|B|$ is not so small as in the three-dimensional case. It should be noted that $\lim_{\kappa \rightarrow 0} \tilde{J}_{\kappa} = J_{\kappa}$. Namely, the interaction reduces to the completely unscreened form e^2/r , as $r \rightarrow \infty$. Accordingly, the possibility of observing the exciton in our one-dimensional system is much larger than that in the three-dimensional one. However, $(W - \epsilon_g/\epsilon_g)$ will tend to zero as $\epsilon_g \rightarrow 0$ even in this one-dimensional system because of the following reason. The classical distance of separation between the electron and the hole in their bound state whose dissociation energy is indicated by (5.14) is roughly equal to $(|B|/\Delta B)a_R$. Then \tilde{J}_{κ} 's with $\kappa \sim \Delta B/|B|$ should play important roles. The value of $\tilde{J}_{\kappa}/J_{\kappa}$ at $\kappa = \Delta B/|B|$ may be estimated by extrapolating the formula (5.17) or (5.18). Both formulae give about the same value for $\tilde{J}_{\kappa}/J_{\kappa}(\kappa = \Delta B/|B|)$, i.e. $\pi|B|/NJ_{\kappa}$. Note that $NJ_{\kappa} \gg |B|$. Hence $\tilde{J}_{\kappa}/J_{\kappa} \ll 1$ even for $\kappa \sim \Delta B/|B|$. Then the classical distance of separation should be much larger than $(|B|/\Delta B)a_R$, and so the dissociation energy $W - \epsilon_g$ of this exciton, if it could exist, would be much smaller than ϵ_g .

There are some possibilities of obtaining a large energy gap³⁾ in an S -molecule. Thus we could not reject the possibility that the lowest excited state corresponding to the first optical absorption would be the exciton state.

§ 6. Summary and discussions

One can never get the convergency, if one attributes its origin to the plasma oscillation of the one-dimensional electron gas in a long tube. Thus, a kind of energy-gap model in Kuhn's sense or the like is required, in order to explain the convergency.

When the energy-gap ϵ_g arises in the spectra of H_0 , the plasma oscillations whose energy quanta $\hbar\omega$ are sufficiently large compared with ϵ_g remains stable even if their energies sink into the level-continuum of the pair excitation energies. On the other hand, the excited states which are represented in N -molecules by the plasmons with $\hbar\omega < \epsilon_g$ are much altered. The solution of Eq. (4.7) is always larger than ϵ_g , and in case $|k| \ll \Delta B/|B|$ it is almost equal to the gap. Moreover, it is very questionable to regard the solution as the one corresponding to a real plasma oscillation. Indeed, one obtains the conclusion that there are no plasmons in the region of k satisfying $|k| \ll \Delta B/|B|$, if one is allowed to use the "existence criterion" of Bohm and Pines. It should be noted that the above conclusions are limited to the one-dimensional system. Indeed, the plasma oscillations in the three-dimensional system remain unchanged when the small energy-gap arises.

Our discussions about the exciton are rather rough. Especially the use of (5.10) would be open to questions. However, we may reasonably conclude that the screening effect for the attractive force between the electron and the hole grows larger as the energy gap becomes smaller. By this screening effect the possibility of observing the exciton diminishes as ϵ_g becomes smaller. Further, our very rough arguments give the conclusion that the excitation energy of the lowest exciton-like bound state, if it could exist, would be nearly equal to the gap. Then, such a state would not be observed, for the exciton level should have energy width. Thus the lowest excitation energy of our many electron system with small energy gap in the spectra of H_0 seems to be practically equal to the gap.

§ 7. Acknowledgements

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Appendix

At first we calculate $\int_0^{\pi/2} \frac{d\mu}{[1 - (1-k)\cos^2\mu]^{3/2}} \equiv I$ in case $k \ll 1$. The integrand

is overwhelmingly large only when $0 \lesssim \mu \ll 1$. Then we may use the following approximation,

$$I \cong \int_0^{\pi/2} \frac{\cos \mu d\mu}{[1 - (1-k) \cos^2 \mu]^{5/2}}$$

$$= \int_0^{\pi/2} \left[\cos \mu + \frac{3}{2} (1-k) \cos^3 \mu + \frac{3 \cdot 5}{2 \cdot 4} (1-k)^2 \cos^5 \mu + \dots \right] d\mu.$$

Using the well-known formula,

$$\int_0^{\pi/2} \cos^{2n+1} \mu d\mu = \frac{(2n) \cdot \dots \cdot 4 \cdot 2}{(2n+1) \cdot \dots \cdot 5 \cdot 3},$$

we get

$$I \cong 1 + (1-k) + (1-k)^2 + \dots = \frac{1}{1 - (1-k)} = \frac{1}{k}.$$

Next we calculate $\int_0^{\pi/2} \frac{d\mu}{[1 - (1-k) \cos^2 \mu]^{5/2}} \equiv I'$ in case $k \ll 1$. By the same reason as mentioned above the following technique may be used,

$$I' \cong \int_0^{\pi/2} \frac{\cos^3 \mu d\mu}{[1 - (1-k) \cos^2 \mu]^{5/2}}$$

$$= \int_0^{\pi/2} \left[\cos^3 \mu + \frac{5}{2} (1-k) \cos^5 \mu + \frac{5 \cdot 7}{2 \cdot 4} (1-k)^2 \cos^7 \mu + \dots \right] d\mu$$

$$= \frac{2}{3} [1 + 2(1-k) + 3(1-k)^2 + \dots]$$

$$= \frac{2}{3} \frac{1}{[1 - (1-k)]^2} = \frac{2}{3} k^{-2}.$$

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On the Relative Parity of Charged and Neutral K Mesons

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Standing on a view-point that the mass difference between the particles belonging to the same charge multiplet is mainly due to the oddness of the relative parity between charged and neutral K mesons, i. e. $p(K) = \text{odd}$, it is shown, on the basis of the perturbation calculation, that this view-point is permissible under certain conditions. One of them is the equality of the relative parities of Λ , Σ 's and Ξ 's. This perhaps does not suffer any change even when any higher order contribution of the perturbation expansion is taken into account.

In order to obtain some information about the parities of charged and neutral K mesons, various effects due to $p(K) = \text{odd}$, such as the mass difference between charged and neutral K mesons, are investigated. The result thus obtained is not inconsistent with $P(K^\pm) = \text{odd}$, which may be in agreement with the result obtained from the K^\pm -nucleon scattering experiment; the most favorable parity assignment to K mesons is $P(K^+) = \text{odd}$ and $P(K^0) = \text{even}$.

Further, a comment standing on the view-point of $p(K) = \text{odd}$ is made about the implicit assumption that the baryon spectrum known at present is closed.

§ 1. Introduction

Two years ago, Hiida and Sawamura¹⁾ calculated the neutron-proton mass difference by using the measured values of charge and magnetic moment distributions of a nucleon²⁾ and showed that the sign of the neutron-proton mass difference is contrary to the observed one. Similar results have been obtained by many authors.³⁾

Recently Kato and Takeda⁴⁾ showed by semi-quantitative discussions that the electromagnetic interaction, being thought to be the only origin which violates the charge independence in the strong interaction, cannot give both the large mass splittings among Σ particles and the finite decay probability of a charged K meson into two pions, the latter being zero under the $|J| = 1/2$ rule. Moreover, the fact that the mass of a neutral K meson is larger by about 4 Mev than that of a charged one⁵⁾ contradicts with what may be expected by the analogy with pions. It seems also to be hard to obtain the order of magnitude for these mass differences, apart from the sign.⁶⁾

Considering these circumstances, it seems difficult to understand the mass differences among the particles belonging to the same charge multiplet* as due to

* Hereafter we write this mass difference as "the MD " for brevity.

the electromagnetic interaction. This difficulty seems to be of much significance. If it is the case that the MD cannot be understood by the electromagnetic origin, there should be some origin which violates the charge independence other than the electromagnetic one. Here we take a view-point that *the MD is mainly due to the oddness of the relative parity between charged and neutral K mesons, i.e. $p(K)=\text{odd}$* . Although there has been no reliable calculation for the self energy, it may be allowable to examine whether this view-point is possible or not. This view-point might be the most drastic one among various view-points in which the MD is attributed in any way to violation of the charge independence in the strong interaction. However, this view-point seems to be very interesting because it may be possible to give some clue to the future theory. In the last part of the present paper, these points will be discussed.

Here it is worthy to notice that we do not consider the mass splittings among the four kinds of baryons as due to the strong K interaction. Gell-Mann and Takeda⁷⁾ proposed a model in which, at the stage of the strong K interaction being switched off, all baryons are degenerate, and the K interaction causes the mass splittings among the four kinds of baryons. However, if we assume $p(K)=\text{odd}$, we cannot adopt such a model. In fact, using Eqs. (1) and (2) in § 2 and requiring the MD to be approximately vanishing, we can obtain the relation for self energies⁷⁾

$$2\{\delta m(N) + \delta m(\Xi)\} \approx \delta m(\Lambda) + 3 \cdot \delta m(\Sigma),$$

but at the same time we cannot get the large mass splittings among the four kinds of baryons. That is to say, we hold a view that there exists some more fundamental origin for the mass splittings among the four kinds of baryons other than the strong K interaction; we take the view that the bare masses of baryons with respect to the strong K interaction are different from each other in accordance with the order of observed masses. Thus, it seems allowable to use the observed masses in calculating the MD on the basis of the perturbation argument.

Pais discussed the possibility of $p(K)=\text{odd}$ from the view-point of his global symmetry concerning the strong baryon-meson interactions.⁸⁾ From the dispersion theoretical analysis of angular distributions for associated productions, Taylor obtained the result that it is not inconsistent with the available experimental data that $p(K)$ equals odd and a charged K meson has the odd parity relative to the (ΛN) and the (ΣN) system.⁹⁾ His conclusion seems to be too premature to derive such a definite conclusion.¹⁰⁾

In the following, by the argument based on the perturbation argument, we will point out that it may be possible to explain the MD under the assumption of $p(K)=\text{odd}$. For the purpose of obtaining some information about the parities of K mesons, such effects due to $p(K)=\text{odd}$ as the mass difference between charged and neutral K mesons are investigated, and the result thus obtained is shown to be consistent with the K^+ -nucleon scattering experiment.

These considerations are based on the assumption that the mass spectrum known at present is closed. Therefore it will be investigated, from the view-point of self energies, whether the particles, which are not excluded to exist in the Nishijima-Gell-Mann rule¹¹⁾ but have not yet been observed, are permitted to exist under the assumption of $p(K)=\text{odd}$.

§ 2. The mass differences among the particles belonging to the same charge multiplet

We will calculate at first the self energy by using the lowest order perturbation, the reliability of which is an open question but we assume its result to be reliable at least with respect to the order of magnitude. The self energy contribution from Fig. 1 is obtained by using the Feynman cutoff.¹²⁾ The result is as follows:

When $\Gamma=i\gamma_5$,

$$\delta m_a = -\frac{g^2}{(4\pi)^2} \int_0^1 dx (m_b - m_a x) \cdot F_a(b; c), \quad (1)$$

and when $\Gamma=I$,

$$\delta m_a = -\frac{g^2}{(4\pi)^2} \int_0^1 dx (m_b + m_a x) \cdot F_a(b; c), \quad (2)$$

where

$$F_a(b; c) \equiv \ln \frac{m_a^2(1-x)^2 + (m_b^2 - m_a^2)(1-x) + (m_c^2 - \lambda^2)x}{m_a^2(1-x)^2 + (m_b^2 - m_a^2)(1-x) + m_c^2 x}.$$

Assuming $p(K)=\text{odd}$ and the parity of Λ relative to nucleons to be even, we obtain the following results*:

(i) The mass difference between a neutron and a proton: in the case of $P(K^\pm)=\text{odd}$ (even) and $P(\Sigma)=\text{even}$,

$$\delta m(n) - \delta m(p) \approx +(-) \frac{g_{\Sigma NK}^2}{4\pi} m_\Sigma \cdot I_N(\Sigma) - (+) \frac{g_{\Lambda NK}^2}{4\pi} m_\Lambda \cdot I_N(\Lambda), \quad (3)$$

and in the case of $P(K^\pm)=\text{odd}$ (even) and $P(\Sigma)=\text{odd}$,

$$\delta m(n) - \delta m(p) \approx - (+) \frac{g_{\Sigma NK}^2}{4\pi} m_\Sigma \cdot I_N(\Sigma) - (+) \frac{g_{\Lambda NK}^2}{4\pi} m_\Lambda \cdot I_N(\Lambda). \quad (4)$$

Here $P(K^\pm)=\text{odd}$ and $P(\Sigma)=\text{even}$ mean charged K mesons are pseudoscalar and the parity of Σ particles relative to nucleons is even, respectively. Because

*For simplicity, we assume the form of the interaction Hamiltonian to be charge independent in usual sense apart from γ_5 . For example, see Eqs. (11)~(14) in ref. 8). To get the chief conclusion in the following, it is sufficient but not necessarily essential to assume the validity of the relations among various coupling constants required by the charge independence hypothesis.

$I_a(b) \equiv \frac{1}{2\pi} \int_0^1 dx \cdot F_a(b; K)$ is of the order of $1/10$ when λ is taken to be of the order of m_N , Eqs. (3) and (4) show that the case of $P(\Sigma) = \text{odd}$ should be excluded. We can easily understand from Fig. 2 and Eqs. (1) and (2) the reason why the case of $P(\Sigma) = \text{even}$ is allowable.

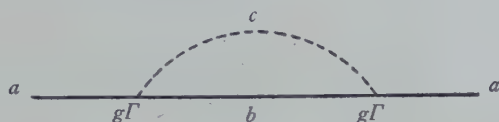


Fig. 1. $\Gamma = i\gamma_5$ or I .

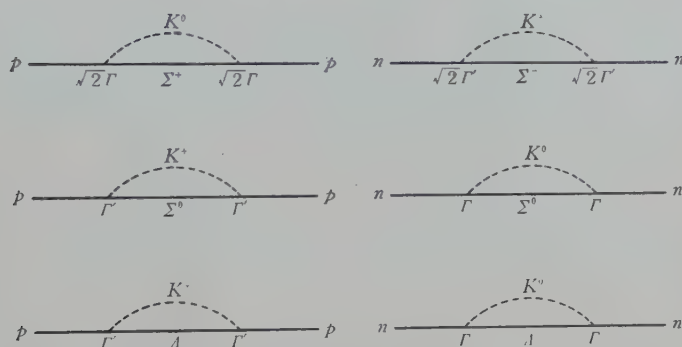


Fig. 2. $\Gamma = I$ (or $i\gamma_5$) and $\Gamma' = i\gamma_5$ (or I).

(ii) The mass difference among Σ particles: in the case of $P(\Xi) = \text{even}$ and $P(K^\pm) = \text{odd}$ (even),

$$\begin{aligned} \delta m(\Sigma^0) - \delta m(\Sigma^+) &\approx +(-) \frac{g_{\Sigma N K}^2 m_N}{4\pi} I_\Sigma(N) - (+) \frac{g_{\Sigma \Xi K}^2 m_\Xi}{4\pi} I_\Sigma(\Xi) \\ &\approx \delta m(\Sigma^-) - \delta m(\Sigma^0), \end{aligned} \quad (5)$$

and in the case of $P(\Xi) = \text{odd}$ and $P(K^\pm) = \text{odd}$ (even),

$$\begin{aligned} \delta m(\Sigma^0) - \delta m(\Sigma^+) &\approx \delta m(\Sigma^-) - \delta m(\Sigma^0) \\ &\approx +(-) \frac{g_{\Sigma N K}^2 m_N}{4\pi} I_\Sigma(N) + (-) \frac{g_{\Sigma \Xi K}^2 m_\Xi}{4\pi} I_\Sigma(\Xi). \end{aligned} \quad (6)$$

Here we assumed $P(\Sigma) = \text{even}$ according to the result obtained in (i). From (5) and (6) we see immediately $P(\Xi)$ must be even.

(iii) The mass difference between Ξ particles: in the case of $P(K^\pm) = \text{odd}$ (even),

$$\delta m(\Xi^0) - \delta m(\Xi^-) \approx +(-) \frac{g_{\Sigma \Xi K}^2 m_\Sigma}{4\pi} I_\Xi(\Sigma) - (+) \frac{g_{\Lambda \Xi K}^2 m_\Lambda}{4\pi} I_\Xi(\Lambda). \quad (7)$$

Here we assumed $P(\Sigma)$ and $P(\Xi)$ are both even, and this is consistent.

From these considerations the following conclusion is derived: Under the

hypothesis of $p(K)=\text{odd}$ it may be possible to explain the MD as mainly due to the strong baryon- K meson interaction; then permissible parity assignment is $P(\Lambda)=P(\Sigma)=P(\Xi)$.

To derive this conclusion, it is implicitly assumed that the higher order contributions to the MD are smaller than the lowest one. However, as shown in the following, this assumption seems to be true because the above conclusion does not suffer any change for any higher order contribution to the MD .

As we can easily see by drawing all of the fourth order graphs such as Fig. 3, the condition to obtain the small MD is that $P(\Lambda)=P(\Sigma)=P(\Xi)$ and the baryon- K meson interaction coupling constants are nearly equal to each other. This situation is not changed even when the vertices in lower order graphs are substituted by ones with black circles as shown in Fig. 4. For instance, if $P(\Lambda)=P(\Sigma)=P(\Xi)$ holds and the baryon- K meson interaction coupling constants are nearly equal, it will be possible to expect a relation $\Gamma(\Sigma NK) \approx \Gamma(\Lambda NK)$ to be true.

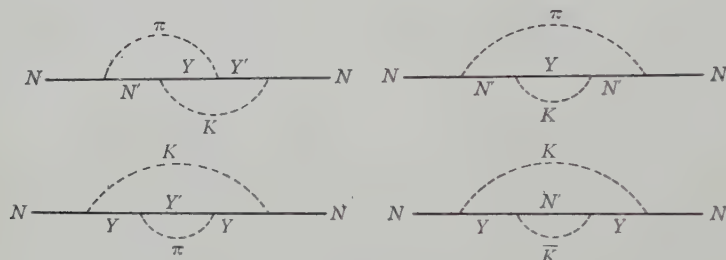


Fig 3.

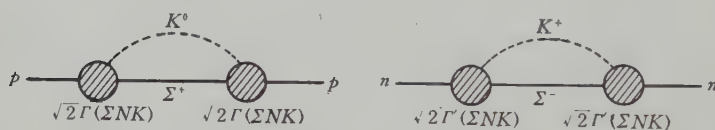


Fig 4

Owing to the argument based on the perturbation calculation, it is difficult to obtain any definite information concerning the signs of the MD . But remembering the discussion given in § 1, it is not meaningless to calculate $I_a(b)$'s numerically (where the mass values are taken to be the observed ones) and see how the situation is. The results are given in Tables I and II. From Table I it is easy to see that we can get a reasonable value for $\delta m(n) - \delta m(p)$ by using an appropriate cutoff momentum.

Table I

$(\lambda/m_N)^2$	$(m_\Sigma/m_N) \cdot I_N(\Sigma)$	$(m_\Lambda/m_N) \cdot I_N(\Lambda)$
1/4	0.045	0.008
1	0.142	0.127
2	0.185	0.193

Table II

$(\lambda/m_N)^2$	$(m_N/m_\Sigma) \cdot I_\Sigma(N)$	$(m_\Xi/m_\Sigma) \cdot I_\Xi(\Xi)$
1/4	0.053	0.199
1	0.132	0.276
2	0.183	0.340

Thus, on the basis of the perturbation calculation, the view-point that the MD is mainly due to $p(K)=\text{odd}$ has been shown to be consistent under certain conditions.

§ 3. Parities of charged and neutral K mesons

For the purpose of obtaining some information about the parities of K mesons, various effects due to $p(K)=\text{odd}$ must be investigated. $P(\Lambda)=P(\Sigma)=P(N)$ is assumed to hold in accordance with the conclusion obtained above.

At first, we calculate the MD between charged and neutral K mesons by the lowest order perturbation, which we think, gives a reasonable result with respect to the sign of the self energy. Here the bare masses of K mesons are assumed to be equal. The contributions from Fig. 5 are given by the following:

For the scalar K meson,

$$\delta m(K(s)) = \frac{1}{m_K} \left\{ 3 \left(\frac{g_{\Sigma NK}^2}{4\pi} \right) I_s(\Sigma) + \frac{g_{\Lambda NK}^2}{4\pi} I_s(\Lambda) \right\}, \quad (8)$$

and for the pseudoscalar K meson,

$$\delta m(K(ps)) = \frac{1}{m_K} \left\{ 3 \left(\frac{g_{\Sigma NK}^2}{4\pi} \right) I_{ps}(\Sigma) + \frac{g_{\Lambda NK}^2}{4\pi} I_{ps}(\Lambda) \right\}, \quad (9)$$

where

$$I_s(a) \equiv \frac{i}{2\pi^3} \int_0^1 dx \int d^4k \frac{k^2 + x(1-x)m_K^2 - m_N m_a}{(k^2 + L_a(x) - i\varepsilon)^2}$$

and $L_a(x) \equiv -m_K^2 \cdot x(1-x) + (1-x)m_N^2 + xm_a^2$; $I_{ps}(a)$ is obtained by substituting $-m_a$ for m_a in $I_s(a)$. If the k_0 -integration is performed, we obtain

$$I_s(a) = \frac{1}{(2\pi)^2 m_K} \int_0^1 dx \int d^3k \frac{-2k^2 - (1-x)m_N^2 - xm_a^2 + m_N m_a}{(k^2 + L_a(x))^{3/2}}.$$

Therefore, we can get

$$\delta m(K(s)) - \delta m(K(ps)) = \frac{2 \cdot m_N}{\pi m_K} \left\{ 3 \left(\frac{g_{\Sigma NK}^2}{4\pi} \right) m_\Sigma \cdot J(\Sigma) + \frac{g_{\Lambda NK}^2}{4\pi} m_\Lambda \cdot J(\Lambda) \right\} > 0,$$

where

$$J(a) \equiv \int_0^1 dk \int_0^1 dx \frac{k^2}{(k^2 + L_a(x))^{3/2}}.$$

For the cutoff momentum $\lambda \sim (1/10)m_a$, we get $J(a) \sim 10^{-4}$, a favorable value of $J(a)$. Certainly this value of λ seems small. But the perturbation method may not be reliable for processes involving virtual pair creation and annihilation. The signs of the self energies of K mesons obtained above seem correct, for they are in agreement with the result derived by Lehman.¹³⁾ On the other hand, it may be allowable to expect that the effective Y - N - K coupling constants become smaller due to pair damping effect. Thus, we expect that a more reliable calculation will give the correct sign and more favorable cutoff momentum. Therefore, we think the small value of λ does not necessarily show invalidity of our standpoint. In conclusion, it may be said that the charged K meson should be pseudoscalar.

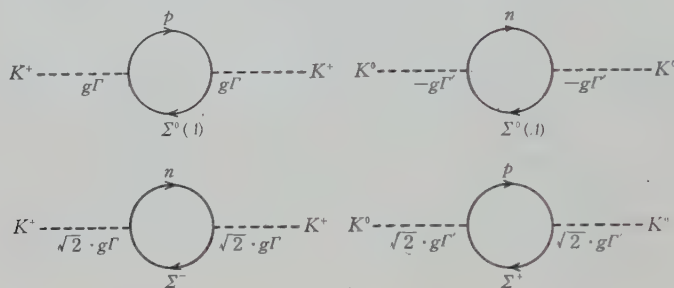


Fig. 5

Next, we will investigate the effect due to $p(K)=\text{odd}$ to the anomalous magnetic moment (AMM) of the nucleon. Using the perturbation argument given by Gupta¹⁴⁾ and assuming $P(\Sigma)=\text{even}$, the AMM's of the nucleon are given as follows: In the case of $p(K)=P(K^+)=\text{odd}$,

$$\begin{aligned}\mu_p &= 0.035 \left(\frac{g_{\pi N}^2}{4\pi} \right) + \frac{g_{\Sigma NK}^2}{4\pi} \left[\frac{1}{2} \{ I_1(\Sigma) + \frac{g_{\Lambda NK}^2}{g_{\Sigma NK}^2} I_1(\Lambda) \} + 2 \cdot J_2(\Sigma) \right], \\ \mu_n &= -0.261 \left(\frac{g_{\pi N}^2}{4\pi} \right) + \frac{g_{\Sigma NK}^2}{4\pi} \left[I_1(\Sigma) + 2 \cdot I_2(\Sigma) \right];\end{aligned}\quad (10)$$

in the case of $p(K)=\text{odd}$ and $P(K^\pm)=\text{even}$,

$$\begin{aligned}\mu_p &= 0.035 \left(\frac{g_{\pi N}^2}{4\pi} \right) - \frac{g_{\Sigma NK}^2}{4\pi} \left[\frac{1}{2} \{ J_1(\Sigma) + \frac{g_{\Lambda NK}^2}{g_{\Sigma NK}^2} J_1(\Lambda) \} + 2 \cdot I_2(\Sigma) \right], \\ \mu_n &= -0.261 \left(\frac{g_{\pi N}^2}{4\pi} \right) - \frac{g_{\Sigma NK}^2}{4\pi} \left[J_1(\Sigma) + 2 \cdot J_2(\Sigma) \right].\end{aligned}\quad (11)$$

Here $I_1(Y)$ and $I_2(Y)$ denote the contributions due to the boson current and the fermion current respectively, when the virtual baryon is a hyperon Y and a Y - N - K interaction is γ_5 -coupling. $J_1(Y)$ and $J_2(Y)$ are counterparts when a Y - N - K interaction is I -coupling. I 's and J 's are all positive and of the order of 1. We take $g_{\Sigma NK}^2 \approx g_{\Lambda NK}^2$ according to the argument in § 2. Then, from the above equations we see the only case that tends to bring the theoretical results closer to the experi-

mental values of both μ_p and μ_n is the case of $P(K^\pm) = p(K) = \text{odd}$.* This result is consistent with that obtained at first in this section. But, in so far as only the vector part of the AMM is concerned, it is hard for us to say definitely to which parity assignment to the K meson should be preferred.

There are also various effects due to $p(K) = \text{odd}$ on leptonic and non-leptonic weak interactions. One of the most important effects may be the ratio $R \equiv P(K^+ \rightarrow \pi^+ \pi^0) / P(K_1^0 \rightarrow \pi^+ \pi^- \text{ and } 2\pi^0)$. Recently, Kato and Takeda⁴⁾ showed that under the $| \Delta I | = 1/2$ rule, we have to use an anomalously large cutoff momentum in order to explain the experimental value of R as the renormalization effect of the coupling constants due to the electromagnetic interaction, and also showed that, even if we take into account the deviation from the exact $| \Delta I | = 1/2$ rule caused by the neutron-proton and $\pi^\pm - \pi^0$ mass differences, it is difficult to obtain the experimental value of R , contrary to the conclusion obtained by Sugano.¹⁵⁾ In order to get the conclusion, Kato and Takeda assumed the strong interactions to be of direct coupling; they critically discussed the result obtained by Sugano and showed it to be doubtful. We can calculate in a similar way to that of Kato and Takeda the contributions from Fig. 6 and Fig. 7, and examine, under the

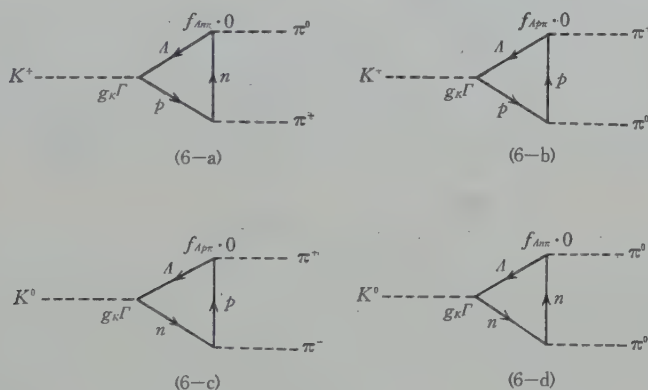


Fig. 6

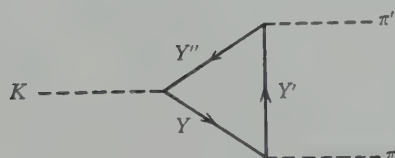


Fig. 7 ($YY'Y''$) denotes (NNN) , $(\Xi\Xi\Xi)$, $(\Sigma\Sigma\Sigma)$, $(\Lambda\Sigma\Sigma)$, etc.; therefore KYY'' vertex is a weak one.

hypothesis of $p(K) = \text{odd}$, whether it is possible to explain the experimental value of R , without any recourse to the $| \Delta I | = 1/2$ rule, in other words, by assuming that there does not occur any cancellation between the matrix elements (6-a) and

* The equations corresponding to (10) and (11) in the case of the scalar or pseudoscalar K meson theory are given in ref. 14). Substituting the experimental values of μ_p and μ_n into (10), we get $g^2_{\pi N} / 4\pi \approx 10.5$ and $g^2_K / 4\pi \approx 4.8$.

(6-b) in Fig. 6. Assuming that all the weak interactions are of the vector type, we can obtain $R \ll 1$ only for $P(K^+) = \text{even}$. However, this theoretical result is sensitive for the cutoff momentum. If we take all the weak interactions to be of the scalar type, i.e.

$$H(\Lambda N \pi) = f_{\Lambda N \pi} \cdot \bar{n}(\alpha + \beta \cdot \gamma_5) \Lambda \cdot \pi^0 + f_{\Lambda p \pi} \cdot \bar{p}(\alpha + \beta \gamma_5) \Lambda \cdot \pi^+ + h.c.,$$

no effects of $p(K) = \text{odd}$ appear in the contributions from Fig. 7, in other words, the matrix elements contain only α and not β . Calculating the contributions from Fig. 6, we obtain a relation $P(K(ps))/P(K(s)) \approx \alpha^2/\beta^2$, being almost independent of a value of the cutoff momentum. If $|\alpha| \approx (1/10) \cdot |\beta|$, as required to explain the asymmetry factor of Λ decay, $P(K^+)$ should be odd. Further, it would be necessary to investigate the consistency among various decay modes of charged and neutral K mesons. However, it seems to be difficult to obtain some decisive conclusion about the parities of K mesons by investigating decay processes, then it will be sufficient to confine ourselves to mentioning the work carried out by Fujii and Kawaguchi,¹⁶⁾ the result of which is consistent with $P(K^+) = \text{odd}$.

It has been pointed out that $P(K^\pm) = \text{odd}$ is preferable, by investigating the K^\pm -nucleon scattering experiment.¹⁷⁾

In conclusion of this section, it may be possible to say that no unlikely results for the present experimental data are derived at all from the parity assignment $p(K) = P(K^\pm) = \text{odd}$. Rather our parity assignment $P(K^\pm) = \text{odd}$ and $P(K^0) = \text{even}$ seems to give reasonable results not only for the MD of baryons and K mesons but also for the AMM of the nucleon and the decay probability of the K meson.

§ 4. Final remarks

As has been pointed out by many authors, it seems difficult to get any reasonable explanation of the MD as due to the electromagnetic correction to the charge independence. We think this difficulty to be noteworthy and that the main part of the MD is due to some deviation from the charge independence of the strong interaction. In this paper we took the view-point that the major cause for the MD is the oddness of $p(K)$ and have investigated whether this view-point is allowable or not. By the perturbation calculation, it has been showed that our view-point is possible only when $P(\Lambda) = P(\Sigma) = P(\Xi)$. Perhaps this does not change even if any higher order contributions are taken into account. The order of magnitude of the fourth order correction on the MD is expected to be smaller than the second order one as it is easy to see. The positive sign for $m(K^0) - m(K^+)$ may be obtained for a pseudoscalar charged K meson. It can be said that $P(K^\pm) = \text{odd}$ has overall consistency. Anyhow, a further analysis of experimental data along the method proposed by Taylor⁹⁾ is expected.

Throughout the argument given above, it is implicitly assumed that the Nishijima-Gell-Mann rule¹¹⁾ is valid and the baryon spectrum known at present is closed. Therefore, we must investigate how the argument given in the preceding

sections should be changed, if the particles which are not excluded in the Nishijima-Gell-Mann rule are taken into account. They are as the following: A boson π_0' , a charged boson B^+ , and two fermions V^+ and W^- , which are all isotopic singlets and have the strangeness $S=0, 2, 1$ and -3 , respectively. The lowest order contributions from the graphs shown in Fig. 8 are as follows:

$$\left| \delta m(n) - \delta m(p) \right| \approx \frac{g_{V NK}^2}{4\pi} m_V \cdot I_N(V),$$

and

$$\left| \delta m(\Xi^0) - \delta m(\Xi^-) \right| \approx \frac{g_{W NK}^2}{4\pi} m_W \cdot I_N(W).$$

Comparing them with Eqs. (5) and (7), the reason why $\delta m(n) - \delta m(p)$ and $\delta m(\Xi^0) - \delta m(\Xi^-)$ are small will be ascribed to $g_{\Lambda NK}^2/4\pi \ll g_{\Sigma NK}^2/4\pi$ (or $g_{\Lambda NK}^2/4\pi$) or the non-existence of V^+ and W^- particles. The former is consistent with the fact that they have not been found. It is important to remember that the assumption of $p(K)=\text{odd}$ can be confirmed to a certain extent by considering the abundance of V^+ and W^- .

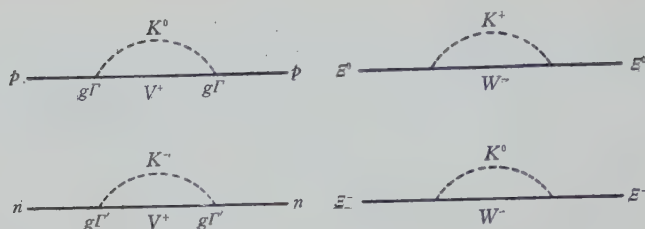
If the hypothesis of $p(K)=\text{odd}$ is true, it is a very mysterious problem that there is a mass degeneracy between charged and neutral K mesons. In order to solve it, it may be required to reconsider the concept of the isotopic spin. There are various effects of $p(K)=\text{odd}$ to the weak interaction: The problem of the decay probabilities of the K meson is certainly one of them. However, we rather hope that this problem might give a clue to the reconsideration of the isotopic spin.*

Nowadays all kinds of elementary particles are described in the direct product space of the Lorentz and the charge spaces. But we think this situation is temporary and in the future theory all particles should be described in the Lorentz space only. From this view-point, it may be expected that some properties of elementary particles ought to be found which are not invariant under individual transformation in the Lorentz or the charge space. We expect this property may be found in the processes in which K mesons take part.** Anyhow, from our view-point it seems to be very important to establish phenomenologically a correlation between the degree of freedom belonging to the Lorentz space and that belonging to the charge space.

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* In this sense, it is important to remember the work carried out by S. Okubo, et al.,¹⁸⁾ and the experimental result by Boldt, et al.¹⁹⁾

** It may be permitted to say that the K meson is a charge doublet even if $p(K)=\text{odd}$ is true. But, then, the meaning of the doublet is different from the ordinary one.



Fgi. 8

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A Possible Symmetry in Sakata's Model for Bosons-Baryons System

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In this paper we study a possible symmetry in Sakata's model for the strongly interacting particles. In the limiting case in which the basic particles, proton, p , neutron, n and Λ -particle, Λ , have an equal mass, our theory holds the invariance under the exchange of p and Λ or n and Λ in addition to the usual charge independence and the conservation of electrical and hyperonic charge.

From our theory the following are obtained: (a) iso-singlet π_0' -meson state, which is a pseudo-scalar, exists, (b) the spin of Ξ -particle may be $(3/2)^+$ and (c) several resonating states in K - and π -nucleon scattering are anticipated to exist.

§ 1. Introduction

Through the analysis of the various particles existing in nature and mutual interactions among them, we have obtained the useful concepts of family¹⁾ and universality²⁾ of the interactions to clarify the complicated situation of the particle physics. For the Boson- and baryon-families which have a kind of universal interaction, e. g., the strong interaction, the well-known rule of Nakano, Nishijima and Gell-Mann³⁾ is valid. The complete understanding of the more fundamental origin of this rule is far from us at present, but a possible way of its realistic grasp has been proposed by Sakata.⁴⁾ Although many objections will be brought against this theory, we shall in this paper follow the idea of Sakata for its prospective insight on the present situation of the theory of elementary particles.

Now, following this idea we assume proton, p , neutron, n and Λ -particle, Λ to be the basic particles which compose other baryons and Bosons in Fermi-Yang's sense⁵⁾. The strong interaction is characterized by the following selection rule,

$$\Delta N_p = \Delta N_n = \Delta N_\Lambda = 0, \quad (A)$$

where ΔN_i means the change of i -th particle's number.

According to the relation (A) and from the similarity of the nature of these three particles (mass, spin, etc.) and of their role in the strong interaction, we are tempted to regard the three particles as standing on an equal footing. In fact when

the mass difference is neglected and the electromagnetic interaction is switched off, we may not find any difference among these three basic particles. Thus we can reasonably expect that a certain symmetry is also realized in their mutual interactions. This view has once been stressed by one of the authors (S.O.)⁶⁾ and later, independently, by Yamaguchi.

In this paper we have investigated a possible framework of such a theory. We shall give the mathematical construction of the theory in § 2 and some physical results derived from this theory in § 3.

§ 2. Mathematical construction

We propose, here, a framework which explicitly assures the equivalence of the three basic particles, p , n and Λ , in the limiting case in which they are of equal mass. This means that our theory guarantees the invariance under the exchange of Λ and p or Λ and n in addition to the usual charge independence and the conservation of hyperonic charges. Our statement on the nature of particle state, e. g., spin and parity but except mass (or energy level), still holds with the finite mass difference between Λ and nucleon, when the mass of Λ is adiabatically increased from its original value (equal to the nucleon mass) to the actual one.

Now, denote the basic particles by the generic symbol χ^* ($\chi^* = \delta_1^*$, for p , etc.). Then the above mentioned symmetry is expressed by the invariance under transformations of the 3-dimensional unitary group $U(3)$:

$$\chi^{\kappa'} = A_{\kappa}^{\kappa'} \chi^{\kappa}; \quad \bar{A}_{\kappa}^{\kappa'} = A_{\kappa}^{\kappa'}, \quad (1)$$

where the matrix $(A_{\kappa}^{\kappa'})$ is the inverse of $(A_{\kappa'}^{\kappa})$ and $\bar{A}_{\kappa}^{\kappa'}$ denote the complex conjugates of $A_{\kappa}^{\kappa'}$. An infinitesimal transformation has the form

$$\chi^{\kappa'} = (\delta_{\kappa}^{\kappa'} + i\epsilon X_{\kappa}^{\kappa'}) \chi^{\kappa}; \quad i^2 = -1, \quad (2)$$

where $(X_{\kappa}^{\kappa'})$ is an Hermitian matrix and can be expressed linearly in terms of nine independent matrices X_{ij} ($i, j=1, 2, 3$): e. g.,

$$(X_{ij})_{\kappa}^{\kappa'} = \frac{1}{2} \delta_{i\kappa} \delta_{j\kappa'} (1-i) + \frac{1}{2} \delta_{i\kappa} \delta_{j\kappa'} (1+i). \quad (3)$$

They satisfy the commutation relations

$$[X_{ij}, X_{kl}] = X_{ij} X_{kl} - X_{kl} X_{ij} = i(\delta_{ik} X_{[jl]} - \delta_{lj} X_{[ki]} - \delta_{jk} X_{(il)} + \delta_{il} X_{(kj)}), \quad (4)$$

where $()$ and $[]$ for indices denote the ordinary processes of symmetrization and alternation respectively.

In a continuous representation of $U(3)$ of degree n , X_{ij} are represented by $n \times n$ matrices M_{ij} which satisfy the same commutation laws as X_{ij} . For the sake of physical understanding it is convenient to introduce the following quantities:

$$\begin{aligned} I_1 &= M_{(12)}, \quad I_2 = M_{[12]}, \quad I_3 = \frac{1}{2}(M_{11} - M_{22}), \\ I_+ &= I_1 + iI_2, \quad I_- = I_1 - iI_2, \quad \mathbf{I}^2 = \sum_i (I_i)^2, \end{aligned} \quad (5)$$

$$N_B = \sum_i M_{ii}, \quad Q = M_{11}, \quad S = -M_{33},$$

$$N_{ij} = M_{(ij)} + iM_{[ij]}; \quad (ij = 13, 23, 31, 32). \quad (6)$$

There are two other quantities which are important from the standpoint of representation, i. e.,

$$M = \sum_{ij} (M_{ij})^2,$$

$$M' = \sum_{ijk} (M_{ij}\{M_{jk}, M_{ik}\} + M_{ij}\{M_{ik}, M_{kj}\} + M_{ij}\{M_{kj}, M_{ki}\} \\ - M_{ij}\{M_{jk}, M_{ki}\}) ; \quad \{A, B\} = AB + BA. \quad (7)$$

Their commutation relations are given in the Appendix.

Three quantities N_B , M and M' are commutable with any M_{ij} and their eigenvalues specify each irreducible representation. On the other hand, since N_B , M , M' , Q , S , I^2 and I_3 are commutable with each other, there will be a basis in terms of which all these matrices are of the diagonal form. In fact, from the configuration (p, n, A) we are informed that I_3 , Q , S and N_B are isospin, charge number, strangeness quantum number and baryon number, respectively. If v is a simultaneous eigenstate of Q , S and I_3 , so is $N_{ij}v$, and the corresponding eigenvalues change as in Table I. M and M' are the new quantum numbers which are characteristic to our theory. Their eigenvalues can be easily calculated in the following manner.

Table I

	$4Q$	$4I_3$	$4S$
N_{13}	+1	$+\frac{1}{2}$	+1
N_{23}	0	$-\frac{1}{2}$	+1
N_{31}	-1	$-\frac{1}{2}$	-1
N_{32}	0	$+\frac{1}{2}$	-1

We consider an irreducible representation of $U(3)$. Let s_0 be the maximum eigenvalue of S , and denote by v_0 the simultaneous eigenstate of Q , S , N_B and I_3 which corresponds to the maximum I_3 among those eigenstates with $S=s_0$. Then we have $I_+v_0=N_{13}v_0=N_{23}v_0=0$. It follows from this that the values of M and M' for v_0 are given by

$$M = q_0^2 + s_0^2 + (n_B - q_0 + s_0)^2 + 2(q_0 + s_0),$$

$$M' = 4[q_0^3 + s_0^3 + (n_B - q_0 + s_0)^3 + 3(q_0^2 - s_0^2) - (q_0 + s_0) + 4l_0], \quad (8)$$

where n_B , q_0 and l_0 are the eigenvalues of N_B , Q and I_3 corresponding to the v_0 , respectively. Since M and M' have the same values for states of an irreducible representation, (8) is the desired result.

Physical meanings of M and M' are not so obvious. In addition, we shall be

confronted with some difficulty when dealing with these quantities. So, it seems better to use s_0 and l_0 rather than the values of M and M' . That is, a set of n_B , s_0 and l_0 can specify an irreducible representation of $U(3)$, which is of the degree $\frac{1}{2}(2l_0+1)[\frac{1}{2}(n_B+3s_0)+l_0+2][\frac{1}{2}n_B+3s_0-l_0+1]$ and is composed of $(2l_0+1) \cdot [\frac{1}{2}(n_B+3s_0)-l_0+1]$ irreducible representations of isospin. Moreover, it is also easy to obtain the law of decomposing a product of representations. The details of this approach will be published elsewhere.

We next consider the $(m+n)$ -body system of m baryons and n anti-baryons, and denote its Salpeter-Bethe amplitude by

$$\bar{\chi}_{x_1} \cdots \bar{\chi}_{x_n} \chi^{\lambda_1} \cdots \chi^{\lambda_m} \\ = \langle B | T(\bar{\chi}_{x_1}(x_1) \cdots \bar{\chi}_{x_n}(x_n) \chi^{\lambda_1}(y_1) \cdots \chi^{\lambda_m}(y_m)) | \mathcal{Q} \rangle, \quad (9)$$

where $|B\rangle$ is an eigenstate of the total Hamiltonian and $|\mathcal{Q}\rangle$ the true vacuum. It is to be noted that the index of a baryon χ^κ is written as a superscript and that of an anti-baryon $\bar{\chi}_\kappa$ as a subscript. Such a notation is useful because an anti-baryon behaves like a covariant vector under the transformation (1) :

$$\bar{\chi}_{\kappa'} = A_{\kappa'}^{\kappa'} \bar{\chi}_\kappa = A_{\kappa'}^{\kappa} \bar{\chi}_\kappa. \quad (10)$$

The amplitude (9) is a mixed tensor $T_{x_1 \cdots x_n}^{\lambda_1 \cdots \lambda_m}$ of contravariant valence m and covariant valence n . So the decomposition of the system into its irreducible constituents is reduced to that of the corresponding tensor space. For this purpose, we have only to decompose the tensor space according to Young's diagram with respect to the upper and lower indices separately, and then to apply the "contraction operation" for an upper and a lower index or for several such pairs of indices. The latter process is similar to the trace operation in the case of the orthogonal group. In what follows we take the cases $m=n=1$ and $m=2, n=1$ for illustration.

(i) Two-body system of a baryon and an anti-baryon

The corresponding tensor T_κ^λ can be decomposed into two irreducible constituents, by means of contraction operation, thus

$$T_\kappa^\lambda = T_{\underset{1}{\kappa}}^\lambda + T_{\underset{2}{\kappa}}^\lambda; \quad T_{\underset{2}{\kappa}}^\lambda \stackrel{\text{def}}{=} \delta_\kappa^\lambda T_\alpha^\alpha / 3, \quad T_{\underset{1}{\kappa}}^\lambda \stackrel{\text{def}}{=} T_\kappa^\lambda - T_{\underset{2}{\kappa}}^\lambda.$$

$T_{\underset{2}{\kappa}}^\lambda$ has the only one independent component and the corresponding Salpeter-Bethe amplitude is $\chi^\kappa \bar{\chi}_\kappa$. $T_{\underset{1}{\kappa}}^\lambda$ is characterized by $T_{\underset{1}{\alpha}}^\alpha = 0$ and has eight independent components. We present the basis vectors in Table II explicitly together with the quantum numbers. Each basis vector belonging to an irreducible representation must have the same eigenvalues of energy (=mass), spin and parity.

(ii) Three-body system of two baryons and an anti-baryon

We first decompose a tensor $T_\kappa^{\lambda\mu}$ with respect to the upper indices as follows :

$$T_\kappa^{\lambda\mu} = T_\kappa^{(\lambda\mu)} + T_\kappa^{[\lambda\mu]}.$$

Next by applying the contraction operation we have

$$T_{\kappa}^{[\lambda\mu]} = T_{\kappa}^{\lambda\mu} + T_{\kappa}^{\lambda\mu}; \quad T_{\kappa}^{\lambda\mu} \stackrel{\text{def}}{=} \frac{1}{2}(\partial_{\kappa}^{\lambda} T_{\alpha}^{[\alpha\mu]} - \partial_{\kappa}^{\mu} T_{\alpha}^{[\alpha\lambda]}), \quad T_{\kappa}^{\lambda\mu} \stackrel{\text{def}}{=} T_{\kappa}^{[\lambda\mu]} - T_{\kappa}^{\lambda\mu},$$

$$T_{\kappa}^{(\lambda\mu)} = T_{\kappa}^{\lambda\mu} + T_{\kappa}^{\lambda\mu}; \quad T_{\kappa}^{\lambda\mu} \stackrel{\text{def}}{=} \frac{1}{4}(\partial_{\kappa}^{\lambda} T_{\alpha}^{(\alpha\mu)} + \partial_{\kappa}^{\mu} T_{\alpha}^{(\alpha\lambda)}), \quad T_{\kappa}^{\lambda\mu} \stackrel{\text{def}}{=} T_{\kappa}^{(\lambda\mu)} - T_{\kappa}^{\lambda\mu}.$$

We thus obtain four irreducible constituents. Each of $T_{\kappa}^{\lambda\mu}$ and $T_{\kappa}^{\lambda\mu}$ has three independent components and the corresponding representation is equivalent to the

Table II

Class I	$M=6, M'=8$	Note
$S=-1, I=1/2$	$-(A\bar{n})$ $(A\bar{p})$	\bar{K}^0 \bar{K}^-
$S=0, I=0$	$(p\bar{p} + n\bar{n} - 2A\bar{A})/\sqrt{6}$	$\pi^{0'}$
$S=0, I=1$	$(p\bar{n})$ $(p\bar{p} - n\bar{n})/\sqrt{2}$ $(n\bar{p})$	π^+ π^0 π^-
$S=1, I=1/2$	$(p\bar{A})$ $(n\bar{A})$	K^+ K^0
Class II	$M=0, M'=0$	Note
$S=0, I=0$	$(p\bar{p} + n\bar{n} + A\bar{A})/\sqrt{3}$	$\pi^{0''}$

Table III $[A, B] = A(x)B(y) - B(x)A(y), (A, B) = A(x)B(y) + B(x)A(y)$

Class I	$M=3, M'=20$	Note
$S=-1, I=0$	$\{\bar{p}[p, A] + \bar{n}[n, A]\}/2$	$K^- + p \rightarrow$
$S=0, I=1/2$	$\{\bar{A}[A, p] - \bar{n}[p, n]\}/2$ $\{\bar{A}[A, n] + \bar{p}[p, n]\}/2$	
Class III	$M=3, M'=20$	Note
$S=-1, I=0$	$\{2\bar{A}AA + \bar{p}(p, A) + \bar{n}(n, A)\}/2\sqrt{2}$	$K^- + p \rightarrow$
$S=0, I=1/2$	$\{\bar{A}(A, p) + 2\bar{p}pp + \bar{n}(pn)\}/2\sqrt{2}$ $\{\bar{A}(A, n) + 2\bar{n}nn + \bar{p}(pn)\}/2\sqrt{2}$	
Class II	$M=7, M'=4$	Note
$S=-1, I=1$	$\bar{n}[p, A]/\sqrt{2}$ $\{\bar{p}[p, A] - \bar{n}[n, A]\}/2$ $\bar{p}[n, A]/\sqrt{2}$	Σ^+ Σ^0 Σ^-
$S=0, I=1/2$	$\{\bar{A}[A, p] + \bar{n}[p, n]\}/\sqrt{2}$ $\{\bar{A}[A, n] - \bar{p}[p, n]\}/\sqrt{2}$	
$S=1, I=0$	$\bar{A}[n, p]/\sqrt{2}$	$K^+ + n \rightarrow$

Class IV $M=11, M'=76$		Note
$S=-2, I=1/2$	$-\bar{n}\Lambda\Lambda$	Ξ^0
	$\bar{p}\Lambda\Lambda$	Ξ^-
$S=-1, I=0$	$\{2\bar{\Lambda}\Lambda\Lambda - \bar{p}(p, \Lambda) - \bar{n}(n, \Lambda)\}/2\sqrt{2}$	$K^- + p \rightarrow$
$S=-1, I=1$	$-\bar{n}(p, \Lambda)/\sqrt{2}$	$K^- + p \rightarrow$ $K^- + n \rightarrow$
	$\{\bar{p}(p, \Lambda) + \bar{n}(n\Lambda)\}/2$	
	$\bar{p}(n, \Lambda)/\sqrt{2}$	
$S=0, I=1/2$	$\{2\bar{p}pp + \bar{n}(p, n) - 3\bar{\Lambda}(A, p)\}/2\sqrt{6}$	$\pi^+ + n \rightarrow$
	$\{2\bar{m}nn + \bar{p}(p, n) - 3\bar{\Lambda}(A, n)\}/2\sqrt{6}$	$\pi^- + p \rightarrow$
$S=0, I=3/2$	$-\bar{n}pp$	$(I=3/2, J=3/2)$ resonance state in π - N scattering
	$\{\bar{p}pp - \bar{n}(n, p)\}/\sqrt{3}$	
	$\{\bar{p}(n, p) - \bar{n}nn\}/\sqrt{3}$	
	$\bar{p}nn$	
$S=1, I=1$	$\bar{\Lambda}pp$	$K^+ + p \rightarrow$
	$\bar{\Lambda}(p, n)/\sqrt{2}$	$K^+ + n \rightarrow$
	$\bar{\Lambda}nn$	

original one (1). $T_{\kappa}^{\lambda\mu}$ and $T_{\kappa}^{\lambda\mu}$ are characterized by

$$T_{\kappa}^{(\lambda\mu)} = T_{\alpha}^{\alpha\mu} = 0 \quad \text{and} \quad T_{\kappa}^{[\lambda\mu]} = T_{\alpha}^{\alpha\mu} = 0,$$

so they give representations of degree six and fifteen respectively. The basis vectors of each representation are given in Table III.

§ 3. Application of the theory

Before referring to the results obtained from our theory, we should like to describe explicitly our equation of motion and the interaction Hamiltonian. The free field equation is the Dirac equation with spin 1/2:

$$(i\gamma_{\mu}\partial_{\mu} + \kappa)\chi = 0, \quad \chi = \begin{pmatrix} p \\ n \\ \Lambda \end{pmatrix}. \quad (11)$$

For the interaction which acts so as to compose other particles we take the four field interaction. Then the following expression is a sole one,*

$$H' = \lambda(\bar{\chi}O\chi)^2, \quad (12)$$

where O is the usual Dirac matrix. We do not enter the problem how to construct the composite particle from (11) and (12), and our approach here is quite phenomenological.

* Apparently the more general expression (B) in the preceding letter⁶⁾ can be reduced to (12) by using Fierz's formula for the ordering exchange of particle.

Now our theory has so far assumed the complete equivalence between Λ and nucleon N while the real case has the asymmetry due to the existing mass difference between them, so we should note the modification of the theory. If the true situation is attained by adiabatically increasing the mass of Λ from its original value (equal to the nucleon mass) to the actual one, then the change of Hamiltonian is expressed by the addition of such a term as

$$H'' = \Delta\kappa(\bar{\Lambda}\Lambda). \quad (13)$$

By including the term (13) the complete symmetry of our theory is broken and M and M' are no more good quantum numbers. But we notice that the parity (P), spin (J), iso-spin (I) and the strangeness quantum number (S) of each state still do not change with the inclusion of (13), namely,

$$\Delta P = \Delta J = \Delta I = \Delta S = 0; \Delta M \neq 0; \Delta M' \neq 0. \quad (14)$$

Thus we obtain the following conclusions:

(α) In the limiting case of equal mass for p , n and Λ , the corresponding particle states of an irreducible representation must have the same nature; equal mass (or energy) level, same parity and equal spin.

(β) When the finite mass difference is taken into account as (13), M and M' may not be good quantum numbers. Irreducible representation with different values of M and M' may become to mix, and the mass (or energy) level of each state will change. But the original value of spin, parity, iso-spin and strangeness of each state must still be preserved and the irreducible classes with different spin and parity will not mix with each other.

A possible reasoning for that the actual case is attained by the inclusion of such a term as (13) is as follows. We know another family—the lepton family the situation of which is very similar to our case. Within the energy region now available for us μ -meson and electron behave with close resemblance in the electromagnetic interaction as well as in the weak interaction in spite of their large mass splitting. Accordingly the origin of their mass difference, if it exists, must be confined in a far smaller region than the one where the usual interactions play a dominant role. We should like to think that the mass splitting between Λ and nucleon arises from a similar cause.

Now we shall enter into the concrete problem.

(i) Two-body system of baryon and anti-baryon.

From Table II it is informed that two neutral particle states π_0' and π_0'' are anticipated to exist in addition to the well established seven Bose particles ($\pi^+\pi^0\pi^-$, K^+K^0 , $K^-\bar{K}^0$).

π_0' belongs to the same class as that of the other seven particles and must be a pseudo-scalar particle with isotopic spin 0*. In the limiting case of equal

* A possible role of π_0' in decay process has been studied by Sawada and Yonezawa⁷⁾. We thank them for informing us of their results.

mass for Λ and N , the mass of π_0' is equal to that of usual π -meson (in this case the mass of K is also equal to that of π -meson).

π_0'' alone forms the other irreducible base and its nature is not known to us except its isotopic spin=0. But we should note that if we take the strong Fermi interaction (12) to be of well-known $S-T+P$ or $V-A$ or $S-A+P$ (invariant for the exchange of ordering) type, the sign of potential in Fermi-Yang's sense between baryon and anti-baryon is opposite for class I and for class II. If this is the case, π_0'' may not be a bound state.

K -meson must be pseudo-scalar. This means that when we make up the following Yukawa type of interaction

$$\Lambda ON \cdot K,$$

the Dirac matrix O must be γ_5 or $\gamma_5 \gamma_\mu$.

(ii) Three-body system of two baryons and one anti-baryon (see Table III).

Twenty-seven states appear in this case which are classified into four irreducible representations. Now we shall present a remark for each class separately.

Class IV: In this class we have the $I=1/2$ $S=-2$ state the bound level of which is Ξ -particle. There is also the $I=3/2$ $S=0$ state which corresponds to the $\pi+N$ system with $I=3/2$ in the lowest configuration of the usual theory. The $\pi+N$ system may be the free scattering state or the well-known $I=3/2$ resonant state. However, we may reasonably take the resonant state as that corresponding to the Ξ -particle state, because the scattering state (continuous spectrum) will not go into the bound (discrete spectrum) state by adiabatically changing the mass. Thus the nature of Ξ particle will be same as that of the $I=3/2$ $J=3/2$ resonant state in $\pi-N$ scattering, that is, the spin of Ξ is $J=(3/2)^+$.

Our reasoning here is rather phenomenological and not logically strict. For instance, if there exist other discrete (but metastable) levels which are yet unobserved and the correspondence is such as that indicated by the arrows in Fig. 1,

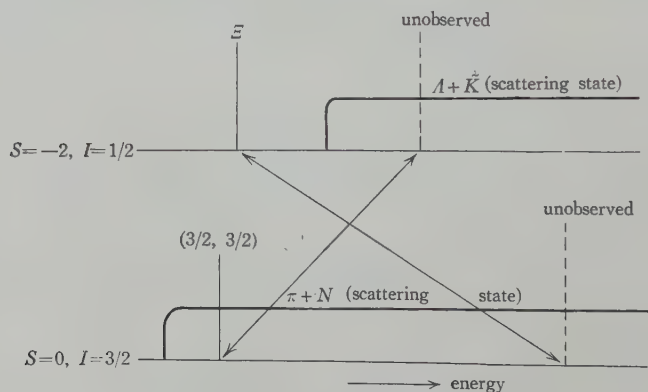


Fig. 1

then our conclusion will fail. But our original intention rests on the conjecture that the symmetry will not be drastically destroyed in the actual case and the closeness of mass level of Ξ and the $(I=3/2, J=3/2)$ resonant state induces us to accept the correspondence between them. And in the experiment we have not any other $I=3/2, S=0$ resonance-like state in the energy interval of ~ 500 Mev from the mass of Ξ .

There are nine other corresponding states in class IV, whose spin is $J=3/2$. All of them are thought to be unstable states. But we may expect that some of them will be realized as the resonating states with $J=3/2$ in K -nucleon and π -nucleon scattering in not so high energy region (say < 1 Bev). Some possible channels leading to these states are presented in "note" of Table III.

Class II: There is the $(I=1, S=-1)$ state which we take as Σ -particle. Although Σ -like state appears in Class IV, we regard it rather as the excited state of Σ , because the ground level of Class IV is of spin $J=3/2$ as stated above and Σ is known to be of spin $1/2$. Now the other states of Class II corresponding to Σ have spin $J=1/2$ and the experimental check for this will be found in $K^+ + n$ ($I=0, S=+1$) scattering.

Class I and Class III: Both classes have the same quantum numbers as that of the one-body configuration. If they possess the spin, parity and other nature in common with the one-body configuration, these states will mix with p , n and Λ state correspondingly. The situation is also the same for $(I=1/2, S=0)$ state of Class (II). Of course the statement here is nothing beyond the speculation. Some of the states might be realized as the resonating states.

In this paper we have proposed a theory in which the systematical side of Sakata's theory is stressed, while the problem of dynamics such as a composition of the particles is left untouched. We hope, however, that if our theory is qualitatively supported by future experiments, then it will give some of clues to attack the dynamical side of composite model.

In conclusion we should like to express our deep gratitude to Profs. S. Sakata and K. Sakuma for their keen interest in this work. We should also like to thank Prof. Y. Yamaguchi at CERN who has sent us very stimulating information about his work in which the similar course to ours is developed. One of the authors (S. O.) thanks the colleagues of Sakuma laboratory for their helpful discussions.

Appendix

$$\begin{aligned}
 [I_2, I_3] &= iI_1, [I_3, I_1] = iI_2, [I_1, I_2] = iI_3, \\
 [Q, I_1] &= iI_2, [Q, I_2] = -iI_1, [Q, I_3] = 0, \\
 [S, I_1] &= [S, I_2] = [S, I_3] = [S, Q] = 0, \\
 [Q, I^2] &= [S, I^2] = [I_3, I^2] = 0,
 \end{aligned}$$

$$\begin{aligned}
& [I_+, I_3] = -I_+, [I_-, I_3] = I_-, [I_+, I_-] = 2I_3, \\
& [I_+, Q] = -I_+, [I_-, Q] = I_-, [I_+, S] = [I_-, S] = 0, \\
& [N_{13}, I_1] = -\frac{1}{2}N_{23}, [N_{23}, I_1] = -\frac{1}{2}N_{13}, [N_{31}, I_1] = \frac{1}{2}N_{32}, [N_{32}, I_1] = \frac{1}{2}N_{31}, \\
& [N_{13}, I_2] = -\frac{i}{2}N_{23}, [N_{23}, I_2] = \frac{i}{2}N_{13}, [N_{31}, I_2] = -\frac{i}{2}N_{32}, [N_{32}, I_2] = \frac{i}{2}N_{31}, \\
& [N_{13}, I_3] = -\frac{1}{2}N_{13}, [N_{23}, I_3] = \frac{1}{2}N_{23}, [N_{31}, I_3] = \frac{1}{2}N_{31}, [N_{32}, I_3] = -\frac{1}{2}N_{32}, \\
& [N_{13}, I_+] = 0, [N_{23}, I_+] = -N_{13}, [N_{31}, I_+] = N_{32}, [N_{32}, I_+] = 0, \\
& [N_{13}, I_-] = -N_{23}, [N_{23}, I_-] = 0, [N_{31}, I_-] = 0, [N_{32}, I_-] = N_{31}, \\
& [N_{13}, Q] = -N_{13}, [N_{23}, Q] = 0, [N_{31}, Q] = N_{31}, [N_{32}, Q] = 0, \\
& [N_{13}, S] = -N_{13}, [N_{23}, S] = -N_{23}, [N_{31}, S] = N_{31}, [N_{32}, S] = N_{32}, \\
& [N_{13}, N_{31}] = Q + S, [N_{13}, N_{32}] = I_+, [N_{13}, N_{23}] = 0, \\
& [N_{23}, N_{32}] = Q + S - 2I_3, [N_{23}, N_{31}] = I_-, [N_{31}, N_{32}] = 0.
\end{aligned}$$

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Coupling Constants in β -Decay and Nucleon Structure

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The difficulty concerning the renormalized axial vector coupling constant of β -decay with pionic corrections is analyzed from a simple model about the nucleon structure. In this proposed model, the nucleon is supposed to consist of the 'nucleon core' of spin $\frac{1}{2}$ and the pionic cloud, and the core is assumed to be composed of static bare N , \bar{N} ensemble. An attempt is made to determine what restrictions are imposed on the wave functional of the nucleon core by the known values of β -decay coupling constants. In the case in which the states of the bare particle are restricted to s- and p-state, it is found that the three-body configurations ($NN\bar{N}$) play the predominant role in order to yield the maximum value of $G_A/G_V (\cong 1.38)$, the ratio of the axial vector- to the vector-coupling constant of the nucleon core. From this analysis, $N\bar{N}$ configurations of iso-singlet and spin triplet are shown to be predominant.

§ 1. Introduction

In the problem of the nucleon structure, the pion theory has achieved a great success in the low energy phenomena with which the outer region of the pion cloud surrounding the nucleon is concerned. In the phenomena where the contributions from the inner region of the cloud and from the nucleon core must be taken into account, however, the pion theory has not yet given any satisfactory explanations. The problem of the coupling constants in β -decay is considered to belong to the latter phenomena.

Hitherto, for the β -interaction of the nucleon, the V - A type has been established from the experimental analysis of the β -decay in the various nuclei and the relative magnitude of the coupling has been obtained as¹⁾

$$G_A/G_V \cong 1.25.$$

The vector part of β -interaction is proved to suffer no renormalization effects from the strong interactions by the introduction of the conserved current which is proposed by Gelshtein, Zel'dovich,²⁾ Feynman and Gell-Mann,³⁾ and this agrees very well with the experimental fact that the coupling constants of the vector part in the μ -decay and the β -decay of O^{14} are of the equal magnitude. For the axial vector part, however, the coupling constant suffers in general the corrections due to the strong interactions since the appropriate conserved current cannot be constructed.

From the standpoint of universal V - A interaction, it should be remembered that the pion theory has not yet succeeded to explain the observed ratio G_A/G_V by

both the lowest perturbation^{2),4)} and the static approximation method.⁵⁾ For example, in the static approximation method the structure of the vertex of the Gamow-Teller interaction $(\sigma\tau)$ are same as that of the π - N interaction $(\sigma\tau)\nabla$, so the renormalizations for the vertices are

$$g_G^0 \langle p | \sigma\tau | n \rangle = g_G^r \langle p_0 | \sigma\tau | n_0 \rangle \quad \text{for } \beta\text{-decay,}$$

and

$$f^0 \langle p | \sigma\tau | n \rangle = f^r \langle p_0 | \sigma\tau | n_0 \rangle \quad \text{for } \pi\text{-}N \text{ interaction,}$$

then it holds

$$g_G^r / g_G^0 = f^r / f^0 \leq 1.$$

If the conserved current is introduced to the vector part following Feynman et al., the ratio of axial vector coupling to vector coupling is inevitably smaller than unity which definitely disagrees with the experimental evidence. The same situation is confirmed also by the lowest perturbation method. This suggests that if the β -interaction is considered to suffer the corrections from the pionic strong interaction, then it could not be sufficient to take into account only the one pion region of the cloud but the contributions from the inner region of cloud should be necessarily included.

Recently, this difficulty is analyzed by several authors. C. Iso⁶⁾ proposed the strong four-fermion-interaction between nucleons and Λ -particles obtained by the Sakata model and calculated the N - \bar{N} pair effects and Y. Fujii⁷⁾ calculated the effects of three-pion contribution by the point approximation for the nucleon loop, and R. J. Blin-Stoyle⁸⁾ introduced a σ -boson with spin 0 and iso-scalar interacting with a nucleon, so they showed the possibility to explain the ratio G_A/G_V . The essential feature of these theories lies in the fact that the N - \bar{N} pair modifying the ratio has different spacial and iso-spin configurations from the one constructed by a single pion.

In this paper, as an approach to explore the inner region of the nucleon structure, we will propose a simple model of the nucleon core with which we determine the possible N - \bar{N} configurations being able to offer a correction enhancing the axial vector coupling constant. This model consists of the following assumptions:

i) The main corrections for the observed coupling of β -interaction are due to the N, \bar{N} ensemble within 'nucleon core' inside the pion cloud.

ii) The physical nucleon is assumed to consist of the pion cloud (outer region) and of the 'nucleon core' (inner region) which is composed of the N, \bar{N} ensemble instead of pions. Generally it may be expected that pions in the inner region of the nucleon will 'diffuse' into N, \bar{N} pairs even within the ordinary pion theory as well as Sakata's compound model.⁹⁾

iii) At present stage, we have no exact knowledge about the dynamical law which governs the N, \bar{N} ensemble. The N, \bar{N} ensemble, however, is assumed to behave non-relativistically and the exchange of energy-momentum with each other

is negligible compared with their mass (at least to understand the character of the effects of N, \bar{N} ensemble to the ratio G_A/G_V).

In § 2, the wave functional of this model will be formulated. In § 3, we shall determine what restrictions are imposed on the wave functional of the nucleon core of our model by the observed values of β -decay coupling constants. This approach is similar to that of the Sachs model¹⁰⁾ which determined the wave functional of pion cloud by the anomalous magnetic moment of the nucleon and the electron-neutron interaction. The wave functional thus obtained will have to be dynamically reconstructed by the future theory. Therefore, if at present we obtain the successful model involving the static N, \bar{N} ensemble, it will offer a possible picture for the nucleon structure as a phenomenological theory.

§ 2. Static approximation for the N, \bar{N} ensemble of nucleon core

From the assumptions (ii), (iii) made in the preceding section, the state of the nucleon core is in general expanded by the occupation number of bare particles such as

$$\Psi = \sum_{n,n'} C_{n,n'} \Phi(n, n') = C_{1,0} \Phi(1,0) + C_{2,1} \Phi(2,1) + \cdots,$$

where Ψ is the wave functional of the total nucleon core and n, n' denote the number of N and \bar{N} , and $\Phi(n, n')$ is the wave functional containing n nucleons and n' anti-nucleons. Here we restrict the functional up to three-body configuration. Addition of angular momentum is as follows: l and L denote the relative angular momentum of N_1, N_2 , and the angular momentum of \bar{N} in total C. M system. We restrict either l or L up to p -wave. The two-body system of N_1, N_2 are restricted by the exclusion principle and its configurations are shown as Table I.

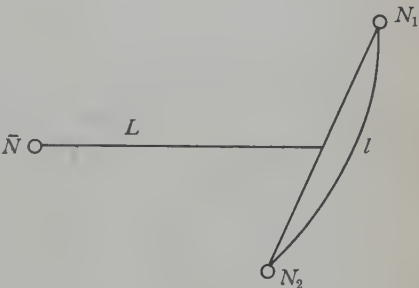


Fig. 1

Table I. s and a denote the state in which the radial function is symmetric and antisymmetric for the exchange of N_1 and N_2 .

$N_1 - N_2$ system		
$T^s(NN)=0, \quad T^a(NN)=1;$	3S_1	1P_1
$T^s(NN)=1, \quad T^a(NN)=0;$	1S_0	$^3P_{0,1,2}$

As \bar{N} has an opposite intrinsic parity to N , two cases of configuration are possible according to the following assumption about the parity of bare nucleon:

(Case I) The intrinsic parity of the bare nucleon is odd to the physical nucleon.

In this case only three-body configuration can be realized with $l=L=0$ under the restriction $l+L \leq 1$.

(Case II) The intrinsic parity of the bare nucleon is even to physical nucleon. In this case one-body and three-body configurations with $l=1$, $L=0$ and $l=0$, $L=1$ are realized and superposed on each other.

(i) *Wave functional.* For iso-spin part, the following two functionals of iso-spin $T=1/2$ are possible:

$$\begin{aligned}\phi_0 &= -\frac{1}{\sqrt{2}}(\eta(N_1)\eta(N_2))\eta(\bar{N}) & \text{for } T(N\bar{N})=0, \\ \phi_1 &= \frac{1}{\sqrt{6}}(\eta(N_1)\tau\eta(N_2))\tau\eta(\bar{N}) & \text{for } T(NN)=1,\end{aligned}$$

where $\eta(N) = \begin{bmatrix} p \\ n \end{bmatrix}$, $\eta(\bar{N}) = \begin{bmatrix} \bar{n} \\ -\bar{p} \end{bmatrix}$, $\tilde{\eta}^T = i\tau_2\eta$, and the numerical factors are normalization factors.

Constructing space-part similarly, we obtain the total wave functionals as follows.

(Case I) Only three-body configurations are realized where $N-N$ is in the relative s -state and \bar{N} in the s -state. The core state with total $J=\frac{1}{2}$ total $T=\frac{1}{2}$ is

$$\begin{aligned}\Psi\rangle &= \left[C(1, \tfrac{1}{2}; \mathcal{S})f_1(r_1, r_2, r_{\bar{N}}; \mathcal{S})\left(\frac{1}{\sqrt{6}}\right)(\tilde{\xi}(N_1)\sigma\xi(N_2))\sigma\xi(\bar{N})\cdot\phi_0 \right. \\ &+ C(0, \tfrac{1}{2}; \mathcal{S})f_2(r_1, r_2, r_{\bar{N}}; \mathcal{S})\left(-\frac{1}{\sqrt{2}}\right)(\tilde{\xi}(N_1)\xi(N_2))\xi(\bar{N})\cdot\phi_1 \\ &+ C(0, \tfrac{1}{2}; \mathcal{A})f_3(r_1, r_2, r_{\bar{N}}; \mathcal{A})\left(-\frac{1}{\sqrt{2}}\right)(\tilde{\xi}(N_1)\xi(N_2))\xi(\bar{N})\phi_0 \\ &\left. + C(1, \tfrac{1}{2}; \mathcal{A})f_4(r_1, r_2, r_{\bar{N}}; \mathcal{A})\left(\frac{1}{\sqrt{6}}\right)(\tilde{\xi}(N_1)\sigma\xi(N_2))\sigma\xi(\bar{N})\phi_1 \right] a_{N_1}^+ a_{N_2}^+ b_{\bar{N}}^+ \rangle_0, \end{aligned} \quad (1)$$

where $f(r_1, r_2, r_{\bar{N}}; S)$ are the normalized radial functions in which S denotes \mathcal{S} or \mathcal{A} when the radial function is symmetric or antisymmetric for the exchange of N_1 and N_2 . $C(j_1, j_2; S)$ is an amplitude of the state in which the N_1-N_2 system has an angular momentum j_1 and \bar{N} has j_2 , and S denotes also the symmetry (\mathcal{S}) or antisymmetry (\mathcal{A}) of the corresponding radial function. These amplitudes can be assumed to be real from the requirement of the time reversal invariance unless the ground state of the core has a degeneracy. a^+ , b^+ are the creation operators of N and \bar{N} , and \rangle_0 is the free vacuum.

$\xi(N) = \begin{bmatrix} \xi^+ \\ \xi^- \end{bmatrix}$ is spin function and $\tilde{\xi}^T = i\sigma_2\xi$.

(Case II) In this case, the one-body and three-body configurations are superposed in general and in the latter configuration $N-N$ is in the relative p -state and \bar{N} is in the s -state and vice versa. The core state with total $J=\frac{1}{2}$, total $T=\frac{1}{2}$ is then

$$|\Psi\rangle = C_0 f(x) \hat{\xi}(N) \eta(N) \mathbf{a}_N^+ |0\rangle + |\Psi^s\rangle + |\Psi^a\rangle, \quad (2)$$

$$\begin{aligned} |\Psi^s\rangle = & \left[C_1(1, 1, 0; \mathcal{S}) f_1(x, \mathcal{S}) \left(-\frac{1}{\sqrt{2}} \right) (\tilde{\xi}(N_1) \hat{\xi}(N_2)) \mathbf{r}_{12} \sigma \hat{\xi}(\bar{N}) \phi_0 \right. \\ & + C_2(0, 1, 0; \mathcal{S}) f_2(x, \mathcal{S}) \left(-\frac{1}{\sqrt{2}} \right) (\tilde{\xi}(N_1) \sigma \hat{\xi}(N_2)) \mathbf{r}_{12} \hat{\xi}(\bar{N}) \cdot \phi_1 \\ & + C_3(1, 1, 0; \mathcal{S}) f_3(x, \mathcal{S}) \left(\frac{1}{2i} \right) [(\tilde{\xi}(N_1) \sigma \hat{\xi}(N_2)) \times \mathbf{r}_{12}]_i \sigma_i \hat{\xi}(\bar{N}) \cdot \phi_1 \\ & + C_4(0, 0, 1; \mathcal{S}) f_4(x, \mathcal{S}) \left(-\frac{1}{\sqrt{2}} \right) (\tilde{\xi}(N_1) \hat{\xi}(N_2)) \mathbf{q}_3 \sigma \hat{\xi}(\bar{N}) \cdot \phi_1 \\ & + C_5(1, 0, 1; \mathcal{S}) f_5(x, \mathcal{S}) \left(-\frac{1}{\sqrt{2}} \right) (\tilde{\xi}(N_1) \sigma \hat{\xi}(N_2)) \mathbf{q}_3 \hat{\xi}(\bar{N}) \cdot \phi_0 \\ & \left. + C_6(1, 0, 1; \mathcal{S}) f_6(x, \mathcal{S}) \left(\frac{1}{2i} \right) [(\tilde{\xi}(N_1) \sigma \hat{\xi}(N_2)) \times \mathbf{q}_3]_i \sigma_i \hat{\xi}(\bar{N}) \cdot \phi_0 \right] \mathbf{a}_{N_1}^+ \mathbf{a}_{N_2}^+ \mathbf{b}_{\bar{N}}^+ |0\rangle, \end{aligned} \quad (2')$$

and

$$\begin{aligned} |\Psi^a\rangle = & \left[C_1(0, 0, 1; \mathcal{A}) f_7(x, \mathcal{A}) \left(-\frac{1}{\sqrt{2}} \right) (\tilde{\xi}(N_1) \hat{\xi}(N_2)) \mathbf{q}_3 \sigma \hat{\xi}(\bar{N}) \cdot \phi_0 \right. \\ & + C_2(1, 0, 1; \mathcal{A}) f_8(x, \mathcal{A}) \left(-\frac{1}{\sqrt{2}} \right) (\tilde{\xi}(N_1) \sigma \hat{\xi}(N_2)) \mathbf{q}_3 \hat{\xi}(\bar{N}) \cdot \phi_1 \\ & + C_3(1, 0, 1; \mathcal{A}) f_9(x, \mathcal{A}) \left(\frac{1}{2i} \right) [(\tilde{\xi}(N_1) \sigma \hat{\xi}(N_2)) \times \mathbf{q}_3]_i \sigma_i \hat{\xi}(\bar{N}) \cdot \phi_1 \\ & + C_4(1, 1, 0; \mathcal{A}) f_{10}(x, \mathcal{A}) \left(-\frac{1}{\sqrt{2}} \right) (\tilde{\xi}(N_1) \hat{\xi}(N_2)) \mathbf{r}_{12} \sigma \hat{\xi}(\bar{N}) \cdot \phi_1 \\ & + C_5(0, 1, 0; \mathcal{A}) f_{11}(x, \mathcal{A}) \left(-\frac{1}{\sqrt{2}} \right) (\tilde{\xi}(N_1) \sigma \hat{\xi}(N_2)) \mathbf{r}_{12} \hat{\xi}(\bar{N}) \phi_0 \\ & \left. + C_6(1, 1, 0; \mathcal{A}) f_{12}(x, \mathcal{A}) \left(\frac{1}{2i} \right) [(\tilde{\xi}(N_1) \sigma \hat{\xi}(N_2)) \times \mathbf{r}_{12}]_i \sigma_i \hat{\xi}(\bar{N}) \phi_0 \right] \mathbf{a}_{N_1}^+ \mathbf{a}_{N_2}^+ \mathbf{b}_{\bar{N}}^+ |0\rangle, \end{aligned} \quad (2'')$$

where $f_i(x, S) = f_i(r_1, r_2, r_{\bar{N}}; S)$ are the normalized radial functions in which S denotes \mathcal{S} or \mathcal{A} when the radial function is symmetric or antisymmetric for the exchange of N_1 and N_2 . C_0 is an amplitude of one-body configuration and $C_i(j_1, l_1, l_2; S)$ are amplitudes of the three-body configuration in which the N_1 - N_2 system has an angular momentum j_1 , orbital angular momentum l_1 and \bar{N} has an orbital angular momentum l_2 , and S denotes the symmetry (\mathcal{S}) or antisymmetry (\mathcal{A}) of the corresponding radial function. As above, the amplitudes can be real unless the ground state of the core has degeneracy and the normalization condition is fulfilled as

$$\sum_{i, j_1, l_1, l_2, S} |C_i(j_1, l_1, l_2; S)|^2 = 1. \quad (3)$$

\mathbf{r}_{12} is a dimensionless unit space vector along the relative coordinate between N_1 and N_2 , \mathbf{q}_3 is the same unit vector.

(ii) *Interaction.* In this model, β -decay of a nucleon core can be induced by \bar{N} , and also by the annihilation and creation of $N\bar{N}$ pair besides the normal process of N . These are the processes as shown by Fig. 2.

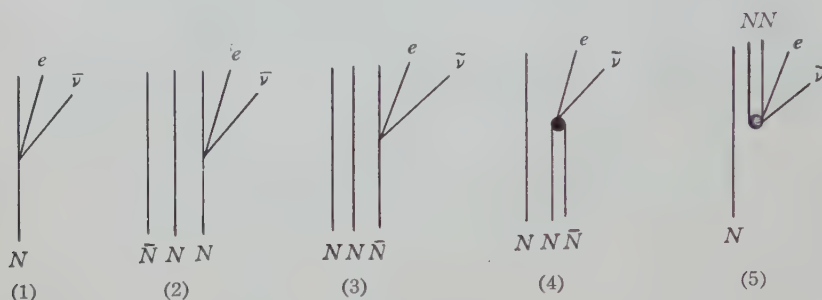


Fig. 2 Decay scheme of the core.

The interaction Hamiltonian is given, assuming the universal $V-A$ interaction, for vector and axial vector part, as

$$H_V = g_V (\bar{\psi}_N \gamma_\mu \tau_+ \psi_N) (\bar{\psi}_e \gamma_\mu \psi_\nu),$$

$$H_A = g_A (\bar{\psi}_N i \gamma_\mu \gamma_5 \tau_+ \psi_N) (\bar{\psi}_e i \gamma_\mu \gamma_5 \psi_\nu), \quad (4)$$

with obvious notations. Then the non-relativistic forms of (4) for each process are

(a) β -decay induced by N ,

$$H_V = g_V (\psi_N^\dagger \tau_+ \psi_N) (\psi_e^\dagger \psi_\nu),$$

$$H_A = g_A (\psi_N^\dagger \boldsymbol{\sigma} \tau_+ \psi_N) (\psi_e^\dagger \boldsymbol{\sigma} \psi_\nu). \quad (5)$$

(b) β -decay induced by \bar{N} ,

the charge conjugation for the bare nucleon field ψ is

$$\psi_{\bar{N}} = -i \tau_2 C \bar{\psi}_N^T,$$

where

$$C = i \rho_1 \sigma_2,$$

then

$$H_V = g_V (\psi_N^\dagger \tau_+ \psi_{\bar{N}}) (\psi_e^\dagger \psi_\nu),$$

$$H_A = -g_A (\psi_N^\dagger \boldsymbol{\sigma} \tau_+ \psi_{\bar{N}}) (\psi_e^\dagger \boldsymbol{\sigma} \psi_\nu). \quad (6)$$

(c) β -decay induced by annihilation of $N\bar{N}$ pair,

$$H_V = -g_V (\tilde{\psi}_{\bar{N}} \boldsymbol{\sigma} \psi_N) (\psi_e^\dagger \boldsymbol{\alpha} \psi_\nu),$$

$$H_A = g_A (\tilde{\psi}_{\bar{N}} \psi_N) (\psi_e^\dagger \gamma_5 \psi_\nu), \quad (7)$$

where

$$\tilde{\psi}^T = i \sigma_2 \psi.$$

(d) β -decay induced by creation of $N\bar{N}$ pair,

$$\begin{aligned}
 H_V &= -g_V (\phi_p^+ \boldsymbol{\sigma} (-i\sigma_2) \phi_n^*) (\phi_e^+ \boldsymbol{\alpha} \psi_\nu), \\
 H_A &= g_A (\phi_p^+ (-i\sigma_2) \phi_n^*) (\phi_e^+ \boldsymbol{\gamma}_5 \psi_\nu).
 \end{aligned}
 \tag{8}$$

(iii) *Matrix Elements.* The decay schemes of (1) (2) (3) in Fig. 2 can be treated as follows. Since each N , \bar{N} which forms a nucleon core behaves nonrelativistically in our model, the matrix elements can be calculated in the manner similar to the nuclear matrix elements for β -decay of nuclei. From the assumption that the N , \bar{N} system of a nucleon core is constructed by the charge independent interaction, the Fermi matrix element is given as analogous to the nuclear matrix element of mirror nuclei,

$$|M_F|^2 = \left| \int (\Psi_f^* \sum_{(i)} \tau_+^{(i)} \Psi_i) dv \right|^2 = T(T+1) - T_z^{\text{initial}} T_z^{\text{final}} = 1, \tag{9}$$

and the Gamow-Teller matrix element is given similarly from Eq. (5) and (6), as

$$|M_G|^2 = \frac{J+1}{J} \left| \left\langle \sum_{(i)}^N \sigma_z^{(i)} \tau_z^{(i)} - \sum_{(i)}^{\bar{N}} \sigma_z^{(i)} \tau_z^{(i)} \right\rangle_{J_z=J} \right|^2, \tag{10}$$

where \sum^N , $\sum^{\bar{N}}$ are the sum over only each N and \bar{N} , J and T denotes the total angular momentum and the total iso-spin, and the expectation value is to be calculated only for either initial or final state with $J_z=J$. This expression generally holds independently of the number of constituents N and \bar{N} . The processes accompanied with the annihilation or creation of a $N\bar{N}$ pair (4) (5) in Fig. 2 have no contributions in such a non-relativistic approximation case, because the space integral becomes

$$\iiint f_i(r_1, r_2, r_{\bar{N}}; S) \times (\mathbf{r}_{12} \text{ or } \mathbf{q}_3) d^3 r_1 d^3 r_2 d^3 r_{\bar{N}} = 0. \tag{11}$$

(iv) *Differences between β -decay of Mirror Nuclei and that of this model.* In order to see the essential feature of β -decay induced by this model we compare this process with β -decay of mirror nucleus induced by the transition between the states of an iso-multiplet. The latter concerns with only non-relativistic nucleon ensemble, but the former with nucleon-antinucleon ensemble. In the mirror nuclei, $f\tau$ -values and nuclear matrix elements are in general given by

$$(f\tau)^{-1} \propto g_F^2 |M_F|^2 + g_G^2 |M_G|^2, \tag{12}$$

$$[\text{for free neutron}] \quad |M_F|^2 = 1, \quad |M_G|^2 = 3,$$

$$[\text{for other mirror nuclei}] \quad |M_F|^2 = 1, \quad |M_G|^2 \leq 3.$$

Therefore, if mirror nuclei are supposed as hypothetical elementary particles, then their effective β -couplings suffer always the restriction

$$g_G^{\text{eff}}/g_F^{\text{eff}} \leq 1, \tag{13}$$

contrary to the actual tendency of nucleon. The reason for this is as follows. The

Gamow-Teller matrix element of mirror nucleus reads

$$|M_G|^2 = 3 \left| \left\langle \sum_{(i)} \sigma_z^{(i)} \tau_z^{(i)} \right\rangle_{J_z = J = 1/2} \right|^2,$$

where the total J is $\frac{1}{2}$. However, in this case the eigenvalues of the relevant operator has always values ± 1 because of the Pauli principle governing all the constituents. (see Fig. 3)

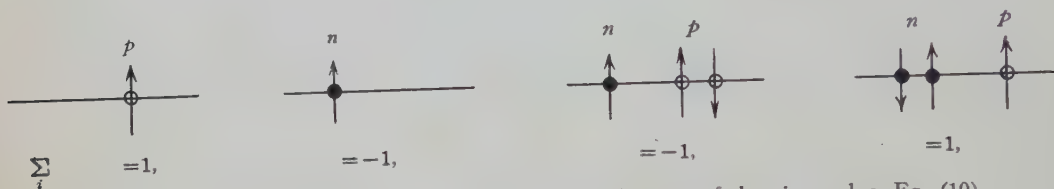


Fig. 3 Configurations of mirror nuclei. \sum_i denotes the sum of the eigenvalue, Eq. (10).

That is, we have

$$\sum_{(i)} \sigma_z^{(i)} \tau_z^{(i)} = \pm 1, \quad (14)$$

and obtain the result Eq. (13).

On the other hand, in N, \bar{N} ensemble, the Pauli principle imposes no restrictions on the $N\text{-}\bar{N}$ system (regardless of the charge conjugation parity). Then from Eq. (9), the Fermi matrix element is

$$|M_F|^2 = 1,$$

but for the Gamow-Teller matrix element, the eigenvalue of operator in Eq. (10) can have a maximum value $= \pm 3$. (see Fig. 4)

That is, in the case up to three bodies we have

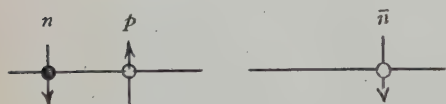


Fig. 4 In this case; $\sum_i = +3$.

$$1 \leq \left| \sum_{(i)}^N \sigma_z^{(i)} \tau_z^{(i)} - \sum_{(i)}^{\bar{N}} \sigma_z^{(i)} \tau_z^{(i)} \right| \leq 3. \quad (15)$$

Therefore, contrary to the case of mirror nuclei, the effective coupling constants take the ratio

$$g_G^{\text{eff}}/g_F^{\text{eff}} \geq 1.$$

This difference lies essentially in the failure of the exclusion principle between N and \bar{N} . If we choose an ideal case the maximum value for expectation value will be realized and one might have $g_G^{\text{eff}}/g_F^{\text{eff}} = 3$. Even when higher configurations such as five, seven...-body system are taken into consideration, if there are no holes and each particle occupies all the allowed orbits in turn, the basic configuration having the maximum eigenvalue is the one similar to Fig. 4, so Eq. (15) holds still for the upper limit of eigenvalues. For the general case, however, the configuration becomes more complicated and the upper limit seems to have

no definite values, but as the increase of the number of \bar{N} imposes the restriction on the configurations for \bar{N} 's themselves, the upper limit may also be saturated to some not very large value.

The Fermi part suffers no corrections in our model, because the vector current conserves automatically as the system is concerned only with N and \bar{N} .

§ 3. Results

The renormalized coupling constants in our model can be calculated straightforwardly. Here we show only the results:

Case I. We obtain

$$G_V/g_V=1,$$

$$G_A/g_A = \frac{1}{3} |C(1, \frac{1}{2}; \mathcal{S})|^2 + \frac{1}{3} |C(0, \frac{1}{2}; \mathcal{S})|^2 + \frac{4}{3} \text{Re} C^*(1, \frac{1}{2}; \mathcal{S}) C(0, \frac{1}{2}; \mathcal{S}) I_{12} \\ - |C(0, \frac{1}{2}; \mathcal{A})|^2 + \frac{7}{9} |C(1, \frac{1}{2}; \mathcal{A})|^2 + \frac{4}{3} \text{Re} C^*(1, \frac{1}{2}; \mathcal{A}) C(0, \frac{1}{2}; \mathcal{A}) I_{34}, \quad (16)$$

where g is the bare coupling constant and G is the effective coupling constant of the nucleon core, and

$$I_{ij} = \iiint f_i(r_1, r_2, r_{\bar{N}}; S) f_j(r_1, r_2, r_{\bar{N}}; S) d^3r_1 d^3r_2 d^3r_{\bar{N}}, \\ I_{ii}=1.$$

If we put $I_{ij}=1$, Eq. (16) shows that the ratio G_A/g_A is always smaller than $+1$ which is definitely inconsistent with experiment.

Case II. We obtain

$$G_V/g_V=1,$$

$$G_A/g_A = |C_0|^2 + \sum_S \left\{ \frac{1}{3} (|C_1(S)|^2 + |C_2(S)|^2 + |C_3(S)|^2) \right. \\ \left. - \frac{1}{9} |C_4(S)|^2 - |C_5(S)|^2 + \frac{1}{3} |C_6(S)|^2 \right. \\ \left. - \frac{4}{3\sqrt{3}} \text{Re} C_1^*(S) C_2(S) - \frac{8}{3\sqrt{6}} \text{Re} C_1^*(S) C_3(S) + \frac{8\sqrt{2}}{9} \text{Re} C_2^*(S) C_3(S) \right. \\ \left. - \frac{4}{3\sqrt{3}} \text{Re} C_4^*(S) C_5(S) - \frac{8}{3\sqrt{6}} \text{Re} C_4^*(S) C_6(S) \right\}, \quad (17)$$

where $C_i(S) \equiv C_i(j_1, l_1, l_2; S)$, and it is assumed $I_{ij}=1$. Then in this case, we obtain the maximum value for G_A/g_A as

$$G_A/g_A|_{\text{max}} \cong 1.38, \quad (18)$$

and the amplitude of this wave functional are

$$C_0=0,$$

$$C_1(S)/C_3(S) \cong -(1-1/7),$$

$$C_2(S)/C_3(S) \cong (1-1/12),$$

$$C_4(S) = C_5(S) = C_6(S) = 0,$$

$$\text{where} \quad \sum_S \{|C_1(S)|^2 + |C_2(S)|^2 + |C_3(S)|^2\} = 1. \quad (19)$$

Therefore in order to obtain $G_A/g_A = \max$, the one-body configuration must have zero contribution, and then the non-zero amplitudes have nearly equal weight.

If we decompose this configuration with respect to the sub-state of a $N\bar{N}$ pair, we obtain from Eq. (2), putting $C_1(S) = -C_2(S) = -C_3(S) \cong 1/\sqrt{6}$, and $f_i(x, S) = 1$,

$$\begin{aligned} \Psi^S \cong & (-1.50) \times (\tilde{\xi}_N \tilde{\xi}_{\bar{N}}) \rho \sigma \tilde{\xi}_N \cdot (\tilde{\eta}_N \eta_{\bar{N}}) \eta_N \\ & + (1.44) \times (\tilde{\xi}_N \tilde{\xi}_{\bar{N}}) \rho \sigma \tilde{\xi}_N \cdot (\tilde{\eta}_N \tau \eta_{\bar{N}}) \tau \eta_N \\ & + (-3.96) \times (\tilde{\xi}_N \sigma \tilde{\xi}_{\bar{N}}) \rho \tilde{\xi}_N \cdot (\tilde{\eta}_N \eta_{\bar{N}}) \eta_N \\ & + (0.02) \times (\tilde{\xi}_N \sigma \tilde{\xi}_{\bar{N}}) \rho \tilde{\xi}_N \cdot (\tilde{\eta}_N \tau \eta_{\bar{N}}) \tau \eta_N, \end{aligned} \quad (20)$$

(where ρ denotes \mathbf{r}_{12} or \mathbf{q}_3 with respect to the state in which the radial function is symmetric ($S=\mathcal{S}$) or antisymmetric ($S=\mathcal{A}$) for the exchange of N_1 and N_2) except the common factor and the normalization factors and these amplitudes are not altered by the exchange ($N_1 N_2$) in (N_1, N_2, N) .^{*} Therefore the probability with which \bar{N} couples to the other N with $T(N\bar{N})=0$, spin triplet becomes predominant (nearly with 16:2).

§ 4. Summary

Summarizing the above results, we can conclude as follows.

- (1) Assuming the nucleon core to be composed of the N, N ensemble, we have a possibility of obtaining $G_V/G_A \cong 3$ which is attributed to the core (where there are assumed to be no holes in the allowed orbits in case of any $2n+1$ -body system). In this case, for the physical nucleon, the final ratio G_A/G_V including the correction of pion cloud predicted by the static theory is nearly consistent with the experimental value.
- (2) When the parity of a bare nucleon is even (case II), and the state is considered up to three-body configuration, the effective coupling constants take $(G_A/G_V)_{\max} \cong 1.38$ only when one-body configuration has zero contribution.¹¹⁾
- (3) In the configuration yielding the maximum ratio we obtain, decomposing the

* This will be made clear when we antisymmetrize wave functional Eq. (2') and (2'') with respect to N_1 and N_2 .

For example, the first term of Eq. (20) is in exact form ($S=\mathcal{S}$)

$$\begin{aligned} & (\xi_N \tilde{\xi}_{\bar{N}}) \mathbf{r} \sigma \tilde{\xi}_N \cdot (\eta_N \eta_{\bar{N}}) \eta_N \\ & \equiv \frac{1}{2} [(\xi(N_1) (\xi(\bar{N}))) \mathbf{r}_{12} \sigma \tilde{\xi}(N_2) \cdot (\eta(N_1) \eta(\bar{N})) \eta(N_2) - (\xi(N_2) \xi(\bar{N})) \mathbf{r}_{21} \sigma \tilde{\xi}(N_1) \cdot (\eta(N_2) \eta(\bar{N})) \eta(N_2)]. \end{aligned}$$

wave functional with respect to the sub-state of a $N\bar{N}$ pair

$$\left[\begin{array}{l} \text{State in which } \bar{N} \text{ couples to other } N \\ \text{with } T(N\bar{N})=0, \text{ and } {}^3S_1 \text{ or } {}^3P_0. \end{array} \right] : [\text{Others}] \geq 16 : 2.$$

Of course these results ((1), (2) and (3)) are too crude to discuss quantitatively on account of static assumption but have only qualitative significance.

(4) Fermi-coupling always suffers no corrections in our model.

The author would like to express his cordial thanks to Dr. Z. Maki for many suggestive advices. His thanks are also due to Drs. Y. Fujii and Y. Ohnuki and the members of the Institute for many valuable discussions.

Addendum

For the other problem, e.g., the anomalous magnetic moment and the charge distribution of nucleon, this model yields no definite conclusions. From the wave functional obtained by Eq. (19), we obtain the magnetic moment ~ 1.25 for a proton core and ~ -0.09 for a neutron core. This is too insufficient but only has slight tendency to explain the scalar part of the magnetic moment $M_p + M_n \geq 1.5$ which has to be attributed to the core in order to obtain the observed value by the pion static theory as calculated by E. Yamada.¹²⁾ For the charge distribution, we cannot say anything unless the form factors are determined dynamically. These would be solved by future developments of the theory.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter will be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

On the Phase Transition of Barium Titanate Crystal

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August 12, 1959

Spontaneous deformation of barium titanate crystal was discussed in the previous article.¹⁾²⁾ The succeeding calculations concerning the variation of the ionic arrangement of the crystal due to an external field are reported in this note.

When an external field is applied in the direction perpendicular to the spontaneous polarization, the energy is expressed as

$$U_{E\perp}^{\text{clamp}} = U_{E0} - E_{Ti} n_{Ti} e y_{Ti} + (a_{Ti} + a'_{Ti} + a''_{Ti}) y_{Ti}^2 + b_{Ti} y_{Ti}^4, \quad (1)$$

where a clamped state without no shear is considered. In this equation, U_{E0} means the energy of the spontaneous polarization, and the succeeding terms are the energy due to small displacement of Ti ions y_{Ti} in the y -direction by the applied field, which is at right angles to the spontaneous polarization (x -direction), where only the shift of

Ti ions are taken into account. Now E_{Ti} and $n_{Ti}e$ are respectively the local field acting on Ti ions and the charge of Ti ions; a_{Ti} and b_{Ti} are coefficients of the second and the fourth powers of potential acting on Ti ions due to overlap and van der Waals' interactions, and a'_{Ti} is the dipole interaction coefficient between one Ti ion and other all ions. The term $a''_{Ti}y_{Ti}^2$ means the work necessary to bend a straight line of force that acts on Ti ions in the x -direction as local fields due to spontaneous polarization. In the language of a molecular model $a''_{Ti}y_{Ti}^2$ is the work necessary in order to shift one Ti ion and bend the electron clouds of Oxygen ions that lie on the x -axis.

The coefficients are numerically expressed as*

$$E_{Ti} = 3.246 E(1 - 3.61 \Delta_y + 0.269 \Delta_z + 0.269 \Delta_x),$$

$$a_{Ti} = 10^9 (0.51227 - 7.1302 \Delta_y$$

* In the first equation, the factor 3.246 was given in the Appendix of article II by the method of Slater³⁾, and the coefficients of Δ 's can be obtained by using the method of the modified Lorentz factor S_Z in article I, whose procedure is rather long and will be explained in the forthcoming article. The second and the third equations were already calculated in article I by the similar method of Devonshire.⁴⁾ In the fourth equation the factor 8.17 was given in the Appendix of article II.

$$+0.6954 \Delta_x + 0.6954 \Delta_x + 43.397 \Delta_y^2 \\ - 4.2605 \Delta_x^2 - 4.2605 \Delta_x^2),$$

$$b_{Ti} = 10^{23} (1.8952 - 23.738 \Delta_y - 0.576 \Delta_x \\ - 0.576 \Delta_x + 167.32 \Delta_y^2 + 4.0873 \Delta_x^2 \\ + 4.0873 \Delta_x^2),$$

$$a'_{Ti} = a'_{Ti0} (1 - 8.17 \Delta_y),$$

$$a''_{Ti} = a'_{Ti0} (1 - 8.17 \Delta_x) x_0 \cdot \frac{1}{l(1 + \Delta_x)}, \quad (2)$$

where E is an applied field, Δ_x 's are the relative ratios of lattice elongation compared with 4\AA in the x -direction and so on, a'_{Ti0} is the dipole interaction coefficient in the standard cube $(4\text{\AA})^3$, x_0 is the spontaneous shift of Ti ions, and l is 2\AA . Here $l(1 + \Delta_x)$ corresponds to the half lattice spacing in the x -direction, and in the expression

$$a''_{Ti} y_{Ti}^2 = a'_{Ti0} (1 - 8.17 \Delta_x) x_0 \\ \cdot \frac{1}{l(1 + \Delta_x)} \cdot y_{Ti}^2, \quad (3)$$

$a'_{Ti0} (1 - 8.17 \Delta_x) x_0$ means the half of the force acting on one Ti ion which is predominantly caused by the electron clouds of the oxygens that lie on the x -axis, and $y_{Ti}/l(1 + \Delta_x)$ corresponds to the angle between the x -axis and the direction of the electron clouds that are bent by the shift of Ti ions in the y -direction. Then $a''_{Ti} y_{Ti}^2$ will be understandable as a term which expresses the work done against the local force that exists spontaneously in the crystal.

Now inquiring into the value of $a_{Ti} + a'_{Ti} + a''_{Ti}$ in Eq. (1), it is found that it has a positive sign when the dipole interaction coefficient a'_{Ti0} takes a value between -4.98×10^5 and

Dipole Interaction Coefficients (c. g. s.)	Shift of Ti ions due to an Applied Field E (c. g. s.)	Dielectric Constant* $\epsilon_{\perp}^{\text{clump}}$	Spontaneous Polarization ¹⁾²⁾ (μ coulomb per cm^2)
$a'_{Ti0} \times 10^{-5}$	$\times n_{Ti} E \times 10^{-6} \text{\AA}$	($n_{Ti} = 4$)	
Cubic phase {	0	8.59	0
	-1	10.48	0
	-2	13.63	0
	-3	19.91	0
	-4	38.59	0
	-4.9	340.3	0
	-4.98	∞	0
Tetragonal phase {	-5.0	510.1	3.21
	-5.1	251.8	11.88
	-5.2	250.4	16.24
	-5.4	413.6	22.16
	-5.58	∞	32.16
Orthorhombic phase {	—	—	—
	—	—	—

* This is calculated by using the results of the previous article. ¹⁾²⁾ These values tabulated do not include "optical" part of the dielectric constant, which is estimated at about 6, so the real dielectric constants can be roughly obtained by adding the value 6 to each dielectric constant.

-5.58×10^5 , where the former* and the latter correspond to the transition points from a tetragonal phase to cubic and orthorhombic phases respectively.

The shift of Ti ions due to the applied field E is also obtained by solving the equation

$$O = \frac{\partial}{\partial y_{Ti}} U_{E \perp}^{\text{clamp}}, \quad (4)$$

and then the clamped dielectric constant in the direction perpendicular to the spontaneous polarization can be estimated.

The results are tabulated on page 737.

The more precise discussion about the dielectric constants including $\epsilon_{//}$, which is in the direction parallel to the spontaneous polarization, will be reported later.

The author is indebted to Professor H. Takahashi and the members of his laboratory for their useful discussions in the seminar.

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* This transition point -4.98×10^5 was already discussed in the previous article. ^{1,2)}

High Energy Scattering of Nucleons from Nuclei

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August 15, 1959

In the previous papers,^{1,2)} we showed that the angular distribution of the high energy n - d and n - He^4 scattering can be well reproduced by using the impulse approximation. In the present note, we examine whether the impulse approximation can well reproduce the elastic scattering of high energy nucleons from heavier nuclei, too.

Several authors^{3,4)} have analysed the angular distribution of the nuclear elastic scattering of high energy protons assuming a transparent nucleus with a uniform density. They used the optical model given by Fernbach, Serber and Taylor⁵⁾ or the WKB or the Born approximation for the optical potential. It has been found that the above nuclear models give only approximate accounts of such a scattering. It is more realistic to consider that the nuclear potential is rounded near the nuclear surface and it is expected that the effect of this diffused character of the nuclear surface is by no means negligible in high energy scattering. Employing the nuclear potential used by Woods and Saxon,⁷⁾ Sternheimer and others⁶⁾ calculated the polarization of nucleons elastically scattered from nuclei.

In the impulse approximation,^{1,8)} the scattering amplitude can be reduced to

the product of a Fourier component of the nuclear density times the sum of the amplitudes of the two-body scattering :

$$\sum_{p=1}^A (\Phi_b V_{0p} \Psi_a) = \sum_{p=1}^A \int e^{-i(k_f - k_i) \cdot r} |\psi(r)|^2 d\mathbf{r} \cdot (2\pi)^{-3/2} \int e^{-i(1/2)k(lab)} \cdot V(|0p|) \psi_{(1/2)k(lab)}(\mathbf{r}_{0p}) d\mathbf{r}_{0p}. \quad (1)$$

Here 0 means the incident particle. $|\psi(r)|^2$ is the density of the target nucleus. As for the other notations, see our previous paper.¹⁾ In the present note, we discuss only the first factor in the expression (1). This is because of the following two reasons. First, the phase shifts derived by meson theory can be used only up to 150 Mev.⁹⁾ (In the previous papers^{1),2)} we saw that the set of the phase shifts derived by meson theory gives the best fit among others for both $n-d$ and $n-He^4$ scatterings.) Secondly, the position of maxima and minima of the diffraction patterns is determined by the first factor and is scarcely changed if we take account of the second factor.

As the nuclear density distribution, we adopt the following form :

$$|\psi(r)|^2 = \rho_0 \left[\exp\left(\frac{r-R}{a}\right) + 1 \right]^{-1}, \quad (2)$$

with

$$a = 0.49 \times 10^{-13} \text{ cm},$$

$$R = 1.23 A^{1/3} \times 10^{-13} \text{ cm}.$$

First, we calculate the position of the first minima of the diffraction patterns of 340 Mev protons elastically scattered from nuclei ranging from C to Bi and compare the calculated values with the measured ones.³⁾ The result is shown

in Fig. 1. The agreement between the experimental and the theoretical values is very good.

Next, we compare the calculated values of the position of the first minima at 84 Mev neutrons scattered from Cu and Pb with the measured values.¹⁰⁾ The calculation predicts the minima at 24° for Cu and at 17° for Pb. The agreement with experimental values seems good.

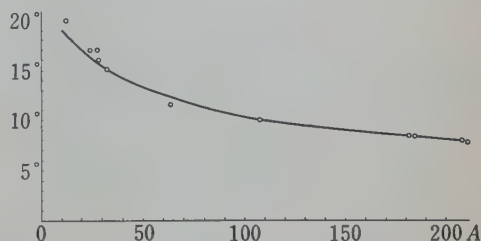


Fig. 1 The position of the first minima of 340 Mev protons scattered from 11 elements plotted vs the mass number. o: first minimum.

The above results may show the utility of the impulse approximation for heavy nuclei at high energy scatterings.¹¹⁾ (Of course, the Born approximation shows the same pattern, but clearly the impulse approximation is more realistic than the Born approximation.)

In closing this note, we examine the validity of the uniform density assumption of the nuclear matter adopted by others. For this purpose, the following expansion,

$$\begin{aligned} \int e^{-i\kappa r} |\psi(r)|^2 d\mathbf{r} = & \frac{4\pi\rho_0}{\kappa} \left[\left(\frac{1}{\kappa^2} \right. \right. \\ & \left. \left. + \frac{\pi^2}{6} a^2 - \dots \right) \sin \kappa R \right. \\ & \left. - (\kappa R) \left(\frac{1}{\kappa^2} - \frac{\pi^2}{6} a^2 + \dots \right) \cos \kappa R \right], \end{aligned} \quad (3)$$

is very useful, for the density distribution expressed in (2). Apparently, the first term gives the Fourier component for the uniform density distribution. When we assume the uniform density of the nucleus, the error caused by neglecting the diffuseness of the nuclear boundary is within 10%, if

$\kappa < \sqrt{(3/5)} \cdot (\pi a)^{-1} \simeq 0.5$. This condition can be expressed as follows.

If

$$E_{lab} \simeq 50 \text{ Mev, then } \theta \lesssim 20^\circ,$$

$$50 \text{ Mev} < E_{lab} < 100 \text{ Mev, then } \theta \lesssim 15^\circ,$$

$$100 \text{ Mev} < E_{lab} < 200 \text{ Mev, then } \theta \lesssim 10^\circ,$$

and

$$200 \text{ Mev} < E_{lab} < 350 \text{ Mev, then } \theta \lesssim 6^\circ.$$

(4)

Hence the uniform density assumption has the limited range of validity, especially for high energy scattering.

The author would like to express his thanks to Professor M. Kobayasi for his interest.

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- 11) T. Sasakawa. (The paper in preparation for presentation). In this paper, analysis is made for 96 Mev proton scattering from various nuclei. A value of R slightly different from the value adopted in the present note reproduces the experimental values of the first and the second minimum: $R=1.32 \text{ A}^{1/3} \times 10^{-13} \text{ cm}$. The optical model analysis was performed at this energy by Glassgold and Kellog. (*Phys. Rev.* **109** (1955), 1291)

Some Remarks on Feynman's Variational Method

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September 8, 1959

Feynman¹⁾ proposed a variational method, in which action functions are used as trial functions, and obtained good results by it in the problem of polaron. His method originates in the stationary expression

$$\langle \langle B_F | A_I \rangle \rangle = \langle B_F | A_I \rangle_0 \exp \{ (i/\hbar) \langle I - I_0 \rangle_0 \} \quad (1)$$

for the probability amplitude of the transition from an initial state A to a final state B , where the symbol $\langle Q \rangle$ is defined by

$$\langle Q \rangle = \langle B_F | A_I \rangle^{-1} \langle B_F | Q | A_I \rangle, \quad (2)^*$$

Q being an operator, and the quantities

* Note that the method can be used even for the definition of $\langle Q \rangle$ as $\langle Q \rangle = \text{Tr} \{ \rho Q \} / \text{Tr} \{ \rho \}$, ρ being the density matrix of the medium.

I and I_0 are respectively the exact and trial action function. In (1) we have denoted the quantities governed by the trial action I_0 by the subscript 0. The stationary character of (1) is based on the variational principle

$$\delta \langle B_F | A_I \rangle = (i/\hbar) \langle \delta I \rangle \langle B_F | A_I \rangle \quad (3)$$

for the amplitude governed by the action I . Inspection of (3) may lead us to the rather simple expression

$$\begin{aligned} \langle B_F | A_I \rangle = & \langle B_F | A_I \rangle_0 \{ 1 \\ & + (i/\hbar) \langle I - I_0 \rangle_0 \}, \end{aligned} \quad (4)$$

although this has a resemblance to the perturbation theory. Besides (1) and (4), we may use the expression

$$\begin{aligned} \langle B_F | A_I \rangle = & \langle B_F | A_I \rangle_0 \\ & \times \{ 1 - (i/\hbar) \langle I - I_0 \rangle_0 \}^{-1}. \end{aligned} \quad (5)$$

In fact, the expressions (1), (4) and (5) have the first order variation

$$\begin{aligned} \delta \langle B_F | A_I \rangle = & (i/\hbar)^2 \langle B_F | A_I \rangle_0 \\ & \times \langle T \{ (I - I_0) \delta I_0 \} \rangle_0 \end{aligned}$$

for infinitesimal variations of I_0 about I and become the exact amplitude at the stationary point $I_0 = I$. Here T stands for the time-ordering operator.

Now, similar expressions can also be formulated for a matrix element (2) of an operator, say Q . The matrix element of Q obeys the variational principle

$$\begin{aligned} \delta \langle Q \rangle = & \langle \delta_p Q \rangle + (i/\hbar) \langle T(Q \delta I) \rangle \\ & - (i/\hbar) \langle Q \rangle \langle \delta I \rangle, \end{aligned} \quad (6)$$

where δ_p is the partial variation with respect to explicit dependences of Q on the action function. In what follows, we shall exclusively consider the quantities without explicit dependences on the

action. We can write the stationary expression of $\langle Q \rangle$ first in the form

$$\begin{aligned} \langle Q \rangle = & \langle Q \rangle_0 + (i/\hbar) [\langle T \{ Q(I - I_0) \} \rangle_0 \\ & - \langle Q \rangle_0 \langle I - I_0 \rangle_0], \end{aligned} \quad (7)$$

corresponding to (4). The last term in the right-hand side of (7) represents subtraction of contributions of the non-connected graphs from the second term. Then (7) is rewritten as

$$\langle Q \rangle = \langle Q \rangle_0 + (i/\hbar) \langle T \{ Q(I - I_0) \} \rangle_0^c, \quad (8)$$

where the superscript c stands for the connected graphs. The expression (7) or (8) is stationary at $I_0 = I$, because

$$\delta \langle Q \rangle = (i/\hbar)^2 \langle T \{ Q(I - I_0) \delta I_0 \} \rangle_0^c,$$

and it becomes the exact $\langle Q \rangle$ at $I_0 = I$. Besides (7) or (8), we can also use the stationary expressions

$$\begin{aligned} \langle Q \rangle = & \langle Q \rangle_0 \exp \{ (i/\hbar) \langle T \\ & \times \{ Q(I - I_0) \} \rangle_0^c \langle Q \rangle_0^{-1} \}, \end{aligned} \quad (9)$$

$$\begin{aligned} \langle Q \rangle = & \langle Q \rangle_0 [1 - (i/\hbar) \langle T \{ Q(I - I_0) \} \rangle_0^c \\ & \times \langle Q \rangle_0^{-1}]^{-1}, \end{aligned} \quad (10)$$

corresponding to (1) and (5), respectively. At $I_0 = I$, both (9) and (10) have the stationary character similar to (8) and the value $\langle Q \rangle$. The approximation (9) corresponds to the summation of graphs in the exponential type, while (10) to that in the damping type. In addition to the variational expressions for the amplitude or the matrix element of an operator, we can consider the stationary expression

$$\langle Q \rangle = Q_0 - i/\hbar [I - I_0, Q_0] \quad (11)$$

for the operator itself. This is understood from the variational principle

$$\delta Q = -(i/\hbar) [\delta I, Q] \quad (12)$$

Finally, we present a simple application to calculation of the one-particle Green function in the meson-nucleon system with the ps -coupling, by putting $|A_I\rangle = |B_I\rangle = |\text{vacuum}\rangle$ and $Q = (i\hbar)^{-1} T(\psi(x)\bar{\psi}(x'))$. Now consider the trial action function

$$I_0 = I_{\text{free}} + i(g^2/2) \iint dx dx' A_F(x-x') \times \bar{\psi}(x) \gamma_5 \tau \psi(x) \bar{\psi}(x') \gamma_5 \tau \psi(x'), \quad (13)$$

under which the meson field is just the free operator and the nucleon field ψ obeys the equation

$$[\gamma \partial + M + g \int d^4 x' A_F(x-x') \bar{\psi}(x') \times \gamma_5 \tau \psi(x')] \psi(x) = 0. \quad (14)$$

Hence we get

$$\begin{aligned} \langle T\{Q(I-I_0)\} \rangle_0 &= (g^2/2\hbar) \\ &\times \iint A_F(x''-x''') \langle T\{Q\bar{\psi}(x'') \\ &\times \gamma_5 \tau \psi(x'') \bar{\psi}(x''') \gamma_5 \tau \psi(x''')\} \rangle_0 \\ &\times dx'' dx'''. \end{aligned} \quad (15)$$

It may be meaningful to calculate (15) by solving (14) in the perturbation theory, because, from (9) or (10), we can obtain an approximation of Green's function corresponding to a partial sum of the total perturbation series. (From (9) and (10), one may also get the relations between renormalized and unrenormalized coupling constants²⁾.) At any rate the approximation relies sensitively upon the choice of the trial action.

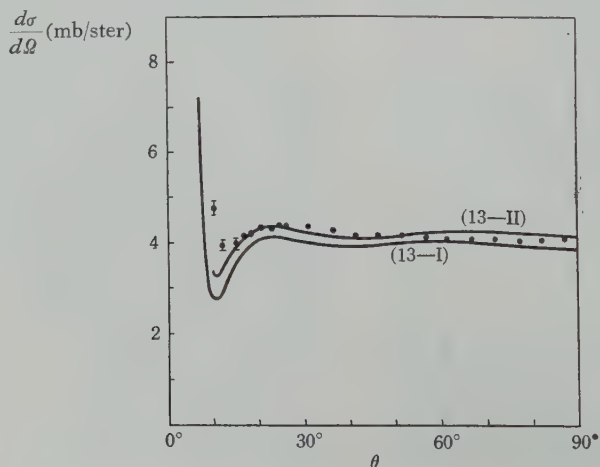
- 1) R. P. Feynman, Phys. Rev. **97** (1955), 660.
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Errata

Two-Nucleon Potential with "One-Pion-Exchange Tail"

T. Hamada, J. Iwadare, S. Otsuki, R. Tamagaki & W. Watari
Prog. Theor. Phys. **22** (1959), 566.

The Fig. 5 on page 577 should be replaced with the corrected one shown below.



Errata

On the Representations of Field Quantities. II

Hitoshi WAKITA

Prog. Theor. Phys. **21** (1959), 299

- 1) Lemma 2.2 on page 304 should be removed.
- 2) In Assumption 2.4 on page 304, "irreducible" should read "maximal and irreducible."
- 3) The last paragraph of § 2 on page 305 should read as follows:

In any physically meaningful representation $\mathfrak{A}_{\mathfrak{F}}$ we say that a sub- $*$ -algebra $\mathfrak{B}_{\mathfrak{F}}'$ of $\mathfrak{B}_{\mathfrak{F}}$ is full if $(\Psi_1, \overline{a_{\mathfrak{F}}}\Psi_1) = (\Psi_2, \overline{a_{\mathfrak{F}}}\Psi_2)$ for every $a_{\mathfrak{F}} \in \mathfrak{B}_{\mathfrak{F}}'$ imply $\Psi_1 = \alpha\Psi_2$ ($|\alpha|=1$) for any $\Psi_1, \Psi_2 \in \mathfrak{F}$.

Lemma 2.2. Let $\mathfrak{B}_{\mathfrak{F}}'$ be full in a physically meaningful representation $\mathfrak{A}_{\mathfrak{F}}$, and $\overline{\mathfrak{B}_{\mathfrak{F}}'}$ be a $*$ -algebra which is formed by the set $\{\overline{a_{\mathfrak{F}}}; a_{\mathfrak{F}} \in \mathfrak{B}_{\mathfrak{F}}'\}$. Then $(\overline{\mathfrak{B}_{\mathfrak{F}}'}\Psi)^c = \mathfrak{F}$ for any $\Psi \in \mathfrak{F}$, $\Psi \neq 0$, and the representation $\mathfrak{A}_{\mathfrak{F}}$ is irreducible.

Proof. Let $(\overline{\mathfrak{B}_{\mathfrak{F}}'}\Psi)^c = \mathfrak{F}_1 \neq \mathfrak{F}$, and $\mathfrak{F} = \mathfrak{F}_1 \oplus \mathfrak{F}_2$. This implies $\overline{\mathfrak{B}_{\mathfrak{F}}'}\mathfrak{F}_1 \subseteq \mathfrak{F}_1$ and $\overline{\mathfrak{B}_{\mathfrak{F}}'}\mathfrak{F}_2 \subseteq \mathfrak{F}_2$. Put $\Phi_{\pm} = \Psi_1 \pm \Psi_2$ for any $\Psi_1 \in \mathfrak{F}_1$, $\Psi_2 \in \mathfrak{F}_2$, $\Psi_1, \Psi_2 \neq 0$; then $(\Phi_+, \overline{a_{\mathfrak{F}}}\Phi_+) = (\Phi_-, \overline{a_{\mathfrak{F}}}\Phi_-)$ for all $a_{\mathfrak{F}} \in \mathfrak{B}_{\mathfrak{F}}'$. This is a contradiction. Q.E.D.

Let $\mathfrak{B}_{\mathfrak{F}}'$ is full in one representation; then it is full in any maximal and irreducible physically meaningful representation, and we may unambiguously say that \mathfrak{B}' is full.

- 4) On page 306, 10th line from the bottom, "any" should read "some."
- 5) On page 307, 2nd and 3rd lines, the sentence "As \mathfrak{U}_1 and $\mathfrak{U}_2 \dots$ " should read as follows:
Let \mathfrak{U}' be a subalgebra of \mathfrak{U} which is generated by $\{\mathfrak{U}_1, \mathfrak{U}_2\}$. Then, from Assumption 3.6 and 3.7, we can show that the direct product $\mathfrak{U}_1 \times \mathfrak{U}_2$ is isomorphic to \mathfrak{U}' , and in the following we can regard $\mathfrak{U}_1 \times \mathfrak{U}_2$ as \mathfrak{U}' .
- 6) In Assumption 3.6 on page 307, " $\mathfrak{U}_1 \times \mathfrak{U}_2$ " should read " $\mathfrak{B}_1 \times \mathfrak{B}_2$."
- 7) On page 309, 6th and 7th lines, "from Assumption 3.6..." should read "as $\mathfrak{B}_1 \times \mathfrak{B}_2$ is full in the V -system, it is obvious from Lemma 2.2 that $(\mathfrak{U}'\Psi)^c = \mathfrak{F}_V$ for any $\Psi \in (\mathfrak{F}_V)_0$."
- 8) On page 309, 5th line from the bottom, " \mathfrak{F}_{∞} " should read " $\times_{i=1}^{\infty} (\mathfrak{F}_i)_0$."
- 9) After Assumption 3.8 on page 310, the following should be inserted:

In Theorem 2 we have shown that there is a one-to-one correspondence between \mathfrak{F}_V and $\mathfrak{F}_1 \otimes \mathfrak{F}_2$, and that the correspondence is very natural from the physical point of view. Therefore, we can regard $\mathfrak{F}_1 \otimes \mathfrak{F}_2$ as \mathfrak{F}_V identifying the corresponding elements. In Assumption 3.8 $\Psi_V^0 = \Psi_1^0 \times \Psi_2^0$ should be regarded as an element of this $\mathfrak{F}_V = \mathfrak{F}_1 \otimes \mathfrak{F}_2$.

- 10) In Theorem 3 on page 310, "and irreducible" should be struck out.

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An Alternative Approach to the Ergodic Problem

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(Received May 19, 1959)

The usual approach to the ergodic problem makes use of the idea of "coarse-graining". That is, a macroscopic observer is supposed to be limited to "coarse-grained" experiments, and the resultant lack of complete information about the system gives rise to the irreversible increase of the (coarse-grained) entropy. It is shown that this approach is untenable, since macroscopic observers are not restricted in principle to coarse-grained experiments, and in fact one "fine-grained" experiment has already been carried out in practice. An alternative approach is presented which avoids these difficulties. The irreversible increase of entropy is due to molecules outside the system proper, which collide with the outside of the box enclosing the system; this leads to a truly random, in principle unpredictable perturbation, which can be treated only stochastically. The number of particles within the system is irrelevant for this purpose, and in particular need not be large.

§ 1. Introduction

In statistical mechanics, we consider ensembles of systems described by a density matrix U . If each system is enclosed in a perfectly reflecting wall, and has an internal Hamiltonian H , the time development of the ensemble is given by the Liouville equation

$$\frac{dU}{dt} = \frac{1}{i\hbar} [H, U]. \quad (1.1)$$

The entropy of the typical system of the ensemble is defined by

$$S = -k \operatorname{Tr}(U \ln U), \quad (1.2)$$

where k is Boltzmann's constant. The *thermal equilibrium state* is that unique density matrix U_e which leads to the largest entropy S , subject to whatever macroscopic constraints are imposed on the system (e.g., given average energy, given volume, given number of particles).¹⁾

If we start, at time $t=0$, from a density matrix U_0 , then the Liouville equation (1.1) leads to the time development

$$U(t) = \exp(-iHt/\hbar) U_0 \exp(+iHt/\hbar). \quad (1.3)$$

This is a unitary transformation, which preserves traces of functions of U . Hence we obtain immediately:

$$S(t) = S(0). \quad (1.4)$$

That is, the entropy S fails to increase with time, and the thermal equilibrium distribution U_e is not approached by the system.

Since actual systems do tend to equilibrium as time goes on, this result is unacceptable. The usual way out of this difficulty is by means of the introduction of "coarse-grained averages".^{2,3)} One argues that a macroscopic observer is not in a position to make sufficiently detailed "fine-grained" measurements to determine the complete density matrix U , and hence the entropy S of the system. The quantity S of equation (1.2) is called the "fine-grained entropy" and is in practice not measurable. Rather, macroscopic observers are restricted to "coarse-grained" measurements, i.e., measurements in which quantum states are lumped together into groups, and one determines which of these groups a given system of the ensemble falls into; no finer distinctions are made, and in particular the actual quantum state is not determined for any system on which measurements are carried out. One can allow for this coarse-graining process mathematically by introducing a mapping $U \rightarrow \tilde{U}$ of the actual, fine-grained density matrix U onto a coarse-grained density matrix \tilde{U} . The coarse-grained entropy is then defined by

$$\tilde{S} = -k \operatorname{Tr}(\tilde{U} \ln \tilde{U}). \quad (1.5)$$

It is shown in reference 2 that, under very general assumptions concerning the nature of the coarse-graining process $U \rightarrow \tilde{U}$, the coarse-grained entropy \tilde{S} increases with time, thereby approaching the thermal equilibrium value which is the largest value it can reach.

This way out of the difficulty of equation (1.4) retains the Liouville equation (1.1) but modifies the definition of entropy, from (1.2) to (1.5), and at the same time gives up the idea of a unique equilibrium distribution U_e . We now say that equilibrium has been reached whenever the coarse-grained entropy \tilde{S} has attained its equilibrium value; this condition is much too weak to pick out a unique statistical matrix U_e , and; in fact, the Liouville equation is inconsistent with approach to any one unique limiting distribution U_e .

It should be noted that the coarse-graining approach depends crucially upon the assertion that "fine-grained" measurements are impracticable, and thus the fine-grained entropy is a meaningless concept. We shall show in section 2 of this paper that such measurements are not only possible in principle, but at least one such measurement has already been carried out by an actual macroscopic observer: The spin-echo experiment of Hahn.⁴⁾ *It is therefore not permissible to base fundamental arguments in statistical mechanics on coarse-graining.* The failure of the fine-grained entropy, (1.2), to increase with time is not merely an unimportant

curiosity. Rather, it represents an important and measurable aspect of the physical situation. No theoretical approach which leads to constancy of S can constitute a satisfactory logical basis for statistical mechanics.

A different approach has already been considered in the literature.^{(5), (6), (7), (8)} It consists in retaining the definition of entropy, but modifying the Liouville equation (1.1) through introducing the random influence of the thermal motion of the wall of the system. It has been shown that this approach leads to thermal equilibrium under very general conditions.

Nevertheless, this approach has not found general acceptance. There is a common feeling that it should not be necessary to introduce the wall of the system in so explicit a fashion. For example, a system contained inside a calorimeter approaches internal thermal equilibrium, at some temperature T , long before it reaches thermal equilibrium with the world outside the calorimeter, at room temperature. Furthermore, it is considered unacceptable philosophically, and somewhat "unsporting", to introduce an *explicit* source of randomness and stochastic behaviour directly into the basic equations. Statistical mechanics is felt to be a part of mechanics, and as such one should be able to start from purely causal behaviour.

Section 3 of the present paper is devoted to a discussion of these arguments. We arrive at the conclusion that the objections listed above are invalid, and that there exists a sound logical and philosophic basis for introducing stochastic concepts directly into the basic equations of statistical mechanics. The difficulty about the calorimeter is only apparent, and disappears when one distinguishes carefully between the relevant relaxation times.

§ 2. Arguments against coarse-graining

On general, philosophical grounds, the coarse-graining procedure is somewhat disconcerting, in that it makes an actual physical process (the approach of the system to equilibrium) dependent on the accidental shortcomings of the observer who makes measurements on the system.* Close interaction between observer and observed system is of course a commonplace in quantum mechanics. But the ergodic problem already occurs in classical statistical mechanics, and (with the possible exception of von Neumann⁹⁾) there is general agreement that the transition from classical mechanics to quantum mechanics makes no essential difference to this problem. If we take a purely classical view, then there is no objection in principle to a fine-grained observation of each system of our ensemble of systems. Coarse-graining is not intrinsic to the problem under study, and it is dragged in artificially, so to speak, in order to save the principle of increasing entropy. The same is true in quantum mechanics, the only difference being that we mean

* A different interpretation of the meaning of the coarse-graining procedure is possible, and will be discussed at the end of section 3.

something different by a "measurement", namely the determination of the quantum state ψ rather than the determination of $3N$ coordinates and $3N$ momenta.

We shall now go on to demonstrate by means of a Gedanken-experiment that coarse-graining can be misleading, and we then show that an actual experiment closely analogous to this Gedanken-experiment has already been carried out.

Consider an ideal gas of particles constrained to move in the x -direction only, confined by walls at $x=a$ and $x=-a$. Initially, at time $t=0$, all N particles are located at the centre, at $x=0$, and they have a Maxwellian distribution of velocities. Thus the combined probability distribution in x and v at time $t=0$ is

$$U(x, v) = P_0 \delta(x) \exp(-\frac{1}{2}mv^2/kT) \quad \text{at } t=0 \quad (2.1)$$

where P_0 is a normalization constant.

As time goes on, the particles are reflected back and forth between the two walls. Under the usual assumption that the walls are perfectly reflecting mirrors, the speed of any one particle does not change. However, their initial non-uniform distribution in space is smeared out rather quickly. The characteristic time for this process is the time which the "average" particle takes to traverse the distance $2a$, i.e.,

$$T_0 = 2a/v_0, \quad (2.2)$$

where v_0 is some average speed, for instance the root-mean-square velocity. After, say, $10T_0$, the distribution of the particles in space will be sensibly uniform. Their distribution in momentum space is Maxwellian to start out with, and stays that way. Hence, any "coarse-grained" experiment to determine the combined distribution function $U(x, v)$ is bound to lead to the equilibrium result,

$$U(x, v) = \frac{P_0}{2a} \exp(-\frac{1}{2}mv^2/kT). \quad \text{Equilibrium} \quad (2.3)$$

This is the way in which coarse-graining leads to the thermal equilibrium distribution.

Actually, however, the true distribution function is not at all equal to (2.3). Rather, there are complicated correlations between the positions and momenta of all the particles. To see that this is true, consider the following Gedanken-experiment: At time $t=T$, reverse the velocities of all particles of the gas. It follows immediately that at time $t=2T$, the initial state (all particles at $x=0$) reoccurs. We shall call this the *reversal experiment*.

This kind of recurrence of the initial configuration has nothing to do with the Poincare recurrence cycle. The time for the latter is unimaginably long. Thus, if we carry out the reversal experiment on a system whose distribution function is truly (2.3), we shall indeed reach any unusual distribution (including the one with all particles at $x=0$) eventually, but the time involved is so long that it is of no practical importance. Conversely, if the reversal experiment, carried out on

an unknown system, leads to all particles congregating at $x=0$ some finite, small time T later, then we are perfectly safe in concluding that the unknown system was not really in equilibrium, i.e., did not really have the distribution function (2.3) in the fine-grained sense.

To distinguish between the true equilibrium distribution (2.3) and the distribution obtained in our Gedanken-experiment at time T , we shall use the term "quasi-equilibrium" for the latter. To an observer constrained to make coarse-grained measurements only, there is no difference between quasi-equilibrium and true equilibrium. If we accept the restriction that macroscopic observers can make only coarse-grained measurements, our reversal experiment is not a possible experiment for a macroscopic observer, and there is no difficulty. No macroscopic observer can distinguish between quasi-equilibrium and true equilibrium, and thus this distinction is simply sophistry of no consequence to physics.

However, macroscopic observers *can* carry out reversal experiments in principle. It is not necessary to *measure* a velocity in order to *reverse* it. Thus, the process of *reversing* all velocities does not imply *measurement* of all velocities (which latter would indeed be impossible for a macroscopic observer). Hence there is no difficulty *in principle* with our reversal experiment.

Even more striking is the fact that just such a reversal experiment *has actually been carried out* already. This is the "spin-echo" experiment of Hahn.⁴⁾ In this experiment, spins are aligned by a strong pulse at time $t=0$. As a result of small inhomogeneities in the "constant" magnetic field H_0 applied to the specimen, different nuclear spins undergo Larmor precession at slightly different rates. Hence the spins get out of alignment rather quickly, in a time T_2 determined principally by the inhomogeneities in H_0 . At some later time, $t=T \gg T_2$, quasi-equilibrium has been reached for the spin distribution. At this time, Hahn puts on another strong pulse, whose main effect is to reverse the direction of precession of every spin.* At time $t=2T$ all the spins are aligned once more, and this fact shows itself through the observation of a coherent magnetization at that time, known as the "spin-echo".

Since an experiment essentially equivalent to our hypothetical reversal experiment has actually been carried out by a macroscopic observer, we conclude that *macroscopic observers are not restricted to coarse-grained experiments. It is therefore not permissible to base fundamental arguments in statistical mechanics on coarse-graining.*

* Our description of this experiment is deliberately simplified. Actually, the pulse at $t=T$ does not reverse the directions of precession, but rather reverses the direction in which each spin points at this moment. In terms of our previous ideal-gas analogy, this amounts to leaving all velocities unaltered, but moving each particle i of the gas instantaneously from position x_i to position $-x_i$. In view of the symmetry around $x=0$, the ultimate result at time $t=2T$ is the same as in the velocity reversal experiment.

Some additional comments are desirable here:

(1) One may exclude the spin-echo experiment from consideration by saying that it is not a "thermodynamic measurement", and statistical mechanics is meant to apply only to results of thermodynamic measurements. We consider this attitude unacceptable, both philosophically and practically: From the practical point of view, one would need a clear-cut and easily applicable definition of just what constitutes a thermodynamic measurement, as opposed to non-thermodynamic measurements. We are not aware of such a definition, nor do we believe it would be easy to construct one. From the philosophical point of view, we feel that statistical mechanics should aim to describe as big a part of nature as possible; limiting the range of applicability of statistical mechanics to certain types of measurements requires its own philosophical justification; we see no valid philosophical justification for limiting statistical mechanics to the results of "thermodynamic measurements", whatever may be meant by this term. In the next section we shall discuss another limitation on the applicability of statistical mechanics, a limitation for which we feel there is adequate philosophical justification.

(2) In the discussion of the spin-echo experiment, we have ignored the influence of spin-spin interactions. The spin-spin interactions do decrease the size of the echo pulse, but this is from our present point of view accidental. The reversal pulse of Hahn produces, not a complete reflection of all spin directions, but rather a rotation of 180 degrees around the x -axis. As long as all spins remain in the x - y plane, this is equivalent to a reversal of all spin orientations; i.e., the spin making an angle θ with the x -axis initially, finally makes an angle $-\theta$ with the x -axis. Thus, if one could constrain all spins to remain in the x - y plane, then the spin-spin interactions would not alter the size of the echo pulse at all. In fact, however, the spin-spin interactions force individual spins out of the x - y plane, and for those spins the reversal pulse of Hahn does not produce a complete reversal in our sense. This is the origin of the decrease in size of the echo pulse due to spin-spin interactions, and it is of no fundamental interest in our present discussion.

§ 3. The origin of randomness in statistical mechanics

Having realized that the fine-grained entropy S contains meaningful and measurable information about the ensemble of systems under study, we now investigate what must be done to allow S to increase with time. The constancy of S in the usual theory, equation (1.4), follows directly from the fact that the time development of the density matrix U is given by a similarity transformation, and this latter fact follows directly from the Liouville equation (1.1). *As long as the ensemble can be described validly by a Hamiltonian formalism, the fine-grained entropy fails to increase with time.* Since the spin-echo experiment forces us to retain the fine-grained definition of entropy, we must necessarily give up the completely causal, Hamiltonian description of the time-development of the ensemble, in order to obtain an increasing entropy.

Let us return to the spin-echo experiment. The size of the spin-echo pulse is in fact less than the size of initial pulse, the more so the longer the time T in the experiment. The spins fail to return to a fully aligned configuration. Under the conditions which we have outlined in section 2,* the relaxation time characteristic for this failure to re-align is the relaxation time for interchange of energy between the system of spins and the lattice vibrations. If the time T of the

* In the actual spin-echo experiment, spin-spin interactions also produce failure to align. However, for reasons given at the end of section 2, we consider this effect accidental, whereas the effect of the spin-lattice interaction is basic.

experiment is much longer than this "spin-lattice" relaxation time, the spin system has reached full thermal equilibrium, and no spin-echo pulse can be observed; the spin system has "forgotten" its initial state. Thus, *the approach to true equilibrium is governed by interactions between the system and the outside world, not by interactions within the system itself*. The latter interactions lead only to pseudo-equilibrium, not to true equilibrium.

A particularly striking case is obtained by imagining a system initially in a pure quantum state, ψ_0 . That is, the density matrix U_0 at time $t=0$ is a projection operator onto ψ_0 , and the entropy $S=0$. If there are no interactions with outside world, the wave function ψ_0 develops in time into a wave function $\psi(t)$, and the density matrix at time t is the projection operator onto $\psi(t)$. Since this is still one pure state, the entropy is still zero. There is simply no mechanism in this picture by which the system can go from an initial pure state to a final statistical distribution of states.

The failure of S to increase with time is due to the fact that we have overidealized an "isolated" system. Every system in statistical mechanics must be thought of as enclosed within walls of some kind. Every wall has an outside as well as an inside, and the outside surface of the wall is subject to collisions with molecules outside the system proper. We now proceed to point out that there is an important philosophical distinction between molecules inside the system and molecules outside the walls. Molecules inside the system are in principle accounted for by equations of motion plus initial conditions, i.e., they can be described causally. Molecules outside the system are *in principle* not amenable to a causal description, and *must of necessity* be described in stochastic terms. The momentum and energy transferred between outside molecules and the system proper then acts as a source of true randomness influencing the dynamical behaviour of the system inside the walls. We maintain that this is the origin of randomness and increasing entropy in statistical mechanics.

To see that the outside molecules *must* be treated statistically, let us imagine the opposite for a moment. In order to make a causal description of the motion of the outside molecules, we would have to include them in the Hamiltonian and specify their initial coordinates and momenta. That is, we would have to count these molecules formally as part of a larger system under study. This larger system, however, also has a wall, and there are outside molecules beyond *that* wall. Hence, we have arrived at the following dilemma:

Either we make a causal, Hamiltonian description of the whole Universe, or else we must allow for an essential element of randomness in the description of the motion of the limited system under study.

The random element here is not due to accidental shortcomings of the observer, but rather to the fact that the observer restricts his observations to a finite part of the Universe. Whereas human observers are not in fact restricted to coarse-grained experiments they are surely incapable of observing the whole Universe at

once. The whole measurement process depends on a dichotomy between observer and system observed. A man trying to observe the whole Universe simultaneously would be completely confounded by logical as well as by practical difficulties. He could not use any measuring apparatus (since that is part of the Universe), and he would have to observe, among other things, the workings of his own brain. Thus, there is a sound philosophical basis for the assertion that we are *in principle* limited to observing finite parts of the Universe, and are therefore forced to introduce a random element (due to the remainder of the Universe interacting with our limited system) into the basic equations of motion of the system. Conversely, a Laplace demon who observes the whole Universe at once would describe the world in terms quite different from statistical mechanics; he would not need to introduce any random element into his description.

Once this point of view is accepted, the conventional discussions of the ergodic problem become irrelevant, since they start from the assumption of perfectly reflecting, stationary walls, and Hamiltonian equations of motion. These assumptions lead inevitably to constant fine-grained entropy S , and hence to quasi-equilibrium rather than true equilibrium. The ergodic problem has been discussed from the present point of view by Lebowitz and co-workers.^{5), 6), 7), 8)} The present work provides a philosophical foundation for the model adopted in these papers.

It is a corollary of this point of view that the number of particles of which the system is composed need not be large. In principle, one single molecule inside the box is enough. As a result of the random impacts of outside molecules on the walls of the box, the motion of the one inside molecule is not determined causally by a Hamiltonian equation of motion, but contains a random, statistical element which changes an initial pure state into a statistical distribution of states, and eventually into the equilibrium, Maxwell-Boltzmann distribution. The large number of molecules in actual systems is a great help in carrying out calculations in statistical mechanics, but it is not an essential aspect of statistical mechanics. *Statistical mechanics is not the mechanics of large, complicated systems; rather it is the mechanics of limited, not completely isolated systems.*

We believe that the above arguments dispose of the usual objections against introducing an explicit source of randomness directly into the basic equations of statistical mechanics. A mechanics of limited systems cannot be entirely causal, even if the mechanics of the whole Universe is causal (which latter of course we do not know). It remains to deal with the objection that the system inside a calorimeter reaches internal thermal equilibrium, at its own temperature, long before it reaches thermal equilibrium with the world outside the calorimeter. To discuss this objection, let us return to the Gedanken-experiment of section 2.

Let us make a crude estimate of the time required for the system of our Gedanken-experiment to approach true equilibrium. We introduce the variable x' as follows: With perfect mirror walls, for a particle with initial velocity v , we define

$$x' = vt. \quad (3.1)$$

The relation between x' and the actual position x is then as follows:

$$\begin{aligned} \text{For } -a < x' < a, \quad x &= x'. \\ \text{For } a < x' < 3a, \quad x &= 2a - x'. \\ \text{For } 3a < x' < 5a, \quad x &= x' - 4a. \\ \text{For } 5a < x' < 7a, \quad x &= 6a - x'. \\ &\dots\dots\dots \end{aligned} \quad (3.2)$$

It should be noted that the mapping $x' \rightarrow x$ is a many-to-one mapping.

Now let us allow for the thermal motion of the two walls by assuming that each wall has a mass M and a randomly varying velocity V ; an elastic impact between a gas particle of mass m and initial velocity v results in reflection of the particle with an altered speed. The change in speed (ignoring the change in direction) is given by

$$\delta v = \frac{2MV + 2mv}{M + m}. \quad (3.3)$$

If $M \gg m$, and $\frac{1}{2}MV^2 \sim \frac{1}{2}mv^2 \sim \frac{1}{2}kT$, (3.3) can be approximated by

$$\delta v \cong 2V. \quad (3.4)$$

Since V is a random variable, so is the speed change δv . Let us denote the speed change at the first impact (at $x' = a$) by δv_1 , at the second impact ($x' = 3a$) by δv_2 , etc. Then the variable x' , from which x can be deduced according to (3.2), is given by an equation more complicated than (3.1), namely:

$$\begin{aligned} x' = vt + \delta v_1 \left(t - \frac{a}{v} \right) + \delta v_2 \left(t - \frac{a}{v} - \frac{2a}{v + \delta v_1} \right) \\ + \delta v_3 \left(t - \frac{a}{v} - \frac{2a}{v + \delta v_1} - \frac{2a}{v + \delta v_1 + \delta v_2} \right) \\ + \dots\dots\dots \end{aligned} \quad (3.5)$$

The series in (3.5) breaks off after k terms, where k is the number of wall impacts the particle has suffered before time t .

Since we need only a crude estimate of the relaxation time, we replace this complicated expression by a much simpler one:

$$x' \cong vt + \delta v_1 \left(t - \frac{a}{v} \right) + \delta v_2 \left(t - \frac{3a}{v} \right) + \delta v_3 \left(t - \frac{5a}{v} \right) + \dots\dots\dots \quad (3.6)$$

This is a reasonable first approximation based on the idea that the quantities δv_1 , δv_2 , etc. in the denominators of (3.5) all have zero average values, and thus, on the average, $v + \delta v_1 + \delta v_2$, for example, can be replaced by v .

The mean value of x' from equation (3.6) is given by (3.1). We are interested in the fluctuation of x' around this mean value. From (3.6) we get

$$\overline{(x' - vt)^2} \cong \overline{(\delta v_1)^2} \left(t - \frac{a}{v}\right)^2 + \overline{(\delta v_2)^2} \left(t - \frac{3a}{v}\right)^2 + \overline{(\delta v_3)^2} \left(t - \frac{5a}{v}\right)^2 + \dots \quad (3.7)$$

The cross terms vanish on averaging since the speed changes at different wall impacts are statistically independent. We now use $\overline{(\delta v_i)^2} = 4\bar{V}^2$ and approximate the series by an integral to get

$$\overline{(x' - vt)^2} \cong \frac{2vt^3}{3a} \bar{V}^2. \quad (3.8)$$

This quantity must be compared with $(2a)^2$, for once the uncertainty in x' exceeds $2a$ appreciably, the memory of the initial distribution-in- x has been effectively lost, without hope of recovery even by a reversal experiment. Introducing the mean number of wall impacts by :

$$r = vt/2a \quad (3.9)$$

the condition for the attainment of equilibrium becomes :

$$\frac{\overline{(x' - vt)^2}}{4a^2} \cong \frac{vt^3}{6a^3} \bar{V}^2 = \frac{4r^3}{3} \frac{\bar{V}^2}{v^3} \gg 1. \quad (3.10)$$

Solving for the mean number of impacts required, we get, upon replacing v^2 by its thermal average value,

$$r \gg \sqrt[3]{M/m} \cdot \text{Number of wall impacts for loss of memory.} \quad (3.11)$$

The corresponding relaxation time is rT_0 , where T_0 is given by (2.2). That is, r is the ratio of the relaxation time for true loss of memory of the initial distribution, to the relaxation time for the apparent loss of memory associated with pseudo-equilibrium. It should be noted that the mass of the wall, M , enters into (3.11). Thus the estimate depends on properties of the wall, and cannot be made without considering the wall itself as an essential element.

Let us now consider another relaxation time: If the average kinetic energy of the wall,

$$\frac{1}{2} M \bar{V}^2 = \frac{1}{2} k T_w \quad (3.12)$$

differs somewhat from the average kinetic energy of the gas particles,

$$\frac{1}{2} m \bar{v}^2 = \frac{1}{2} k T_g \quad (3.13)$$

then the wall impacts lead not only to loss of memory of the initial distribution, i.e., attainment of the distribution function (2.3) with $T = T_g$, but eventually the temperature of the gas must approach the temperature of the wall (the latter is maintained constant by collisions with outside molecules). The change in energy of a gas particle in a wall collision is given by

$$\frac{1}{2}m(v')^2 - \frac{1}{2}mv^2 = 4mM/(M+m)^2 \cdot [\frac{1}{2}MV^2 - \frac{1}{2}mv^2 - \frac{1}{2}(M-m)vV].$$

Averaging over the random velocities V of the wall, the last term on the right drops out. With the usual approximation $M \gg m$, we get

$$\overline{\frac{1}{2}m(v')^2 - \frac{1}{2}mv^2} \cong (4m/M) (\frac{1}{2}kT_w - \frac{1}{2}mv^2). \quad (3.14)$$

It follows from this that the fractional change in the effective temperature of the gas, produced by each wall collision, is of order m/M . Thus the number of wall collisions required to bring the gas temperature close to the temperature of the wall is of order:

$$r' \gg M/m \cdot \text{Number of wall impacts for effective energy transfer between gas and wall.} \quad (3.15)$$

The difference between (3.11) and (3.15) becomes significant if the mass ratio M/m is large. For example, consider a box of dimensions $a \sim 1$ cm, with gas particles of mean speed of order 10^5 cm/sec, so that the "coarse-grained relaxation time" T_0 is of the order of 10^{-5} sec. Assuming for the sake of illustration that $M/m \sim 10^9$, then the relaxation time for loss of memory of the initial distribution is of order $rT_0 \sim 10^{-2}$ sec, whereas the relaxation time for full temperature equilibrium between wall and gas is of order $r'T_0 \sim 10^4$ sec, i.e., several hours. In other words, *the wall collisions produce loss of memory of the initial distribution much more quickly than they produce full thermal equilibrium with the wall.*

A "coarse-grained" measurement of the distribution function $U(x, v)$ at a time $t \sim 10^{-4}$ sec, say, would give no hint of the retention of memory; thus, from the coarse-grained point of view, internal equilibrium has been attained at such time. In fact, however, only quasi-equilibrium has been attained, as can be demonstrated directly by carrying out a reversal experiment.

On the other hand, at time $t \sim 1$ sec, say, the memory of the initial distribution has been completely lost, and no experiment, no matter how fine-grained, can distinguish this system from one which started out from full equilibrium (distribution function (2.3)) right at $t=0$. This loss of memory is *not* connected with observation of the system during the intervening time interval. The loss of memory is an objective, physical phenomenon which takes place irrespective of the presence of observers.

In spite of this complete loss of memory of the initial distribution, the *energy* transfer between gas and wall can be ignored, and the gas is effectively in an adiabatic enclosure. This, then, answers the objections concerning the calorimeter: the relaxation time for loss of memory is very much less than the relaxation time for effective interchange of energy, and thus the effect of the outside molecules in producing loss of memory must be allowed for in discussions of the ergodic problem, even though the effect of the wall in producing thermal equilibrium with the outside world can be ignored completely. The general feeling that the outside molecules cannot be of fundamental importance is due simply to a confusion of

these two quite different relaxation times.

It should be noted, in closing, that in very many practical cases the relaxation time for complete loss of the initial distribution is longer than the observed relaxation time for the approach to "equilibrium": after all, usually one does not do a spin-echo experiment, and hence usually no distinction is made between equilibrium and quasi-equilibrium. Since the relaxation time for attainment of quasi-equilibrium is shorter than the relaxation time for attainment of true equilibrium, the former is measured by most experiments. This, however, is of no concern to us here in this discussion of the basic principles, important as it may be from a more practical point of view.

Although it may perhaps be possible to replace, formally, the actual effect of the wall by a specially adjusted, ad hoc coarse-graining procedure,¹⁰⁾ this procedure must necessarily depend on the detailed properties of the wall, and must differ essentially from the "ordinary" coarse-graining—for example, in leading to a much longer relaxation time. We feel that such a re-interpretation of the meaning of coarse-graining would be artificial and unsatisfactory. Once it is admitted that the thermal motion of the wall is essential for the attainment of true equilibrium, then this motion should be taken into account as such, not disguised as a coarse-grained measuring process.

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Equation of State of High Temperature Plasma

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It is generally accepted that the equation of state of the Debye and Hückel theory, originally developed for strong electrolytes basing on the classical statistical mechanics, is applicable to the high temperature plasmas in thermal equilibrium. However, if we were to apply the classical statistical mechanics to the fully ionized plasmas, the partition function would diverge because of the short range attraction between a pair of positive and negative charges, and there is a doubt whether the contribution of this infinite attraction may not overcome the contribution of the Debye-Hückel term.

In this paper the equation of state of high temperature plasmas is investigated in consideration of the quantum mechanics; and it is shown that the Debye-Hückel approximation surely applies to the plasmas of low density of the order $10^{15} \sim 10^{17}$ or so at high temperatures where $\lambda_e \lesssim a_0/Z$, where Z is the charge of a nucleus, λ_e the de Broglie wave length of an electron and a_0 the Bohr radius. This result is obtained by reducing the problem to that of a suitable classical gas and confirming that the contribution of the watermelon terms—which is considered as the leading correction to the Debye-Hückel approximation—is negligible compared with that of the ring terms considered in the Debye-Hückel approximation.

§ 1. Introduction

At a first sight, the high temperature plasmas are considered to be dealt with the classical statistical mechanics, if their density is so low and their temperature is so high that they may be considered as fully ionized. However, the presence of the infinite attraction between positive and negative charges introduces difficulties when investigated by the classical statistical mechanics because the classical Boltzmann factor is infinite for the state where a positive charge coalesces to a negative charge, leading to the divergence of the partition function. It has been frequently suggested that the difficulties are to be removed by consideration of the quantum mechanics, without any practical calculations.¹⁾ The purpose of this paper is to carry out some investigations along this line. That is, we will investigate the system on the basis of the quantum mechanics and show that the corrections to the Debye-Hückel approximation is negligible for fully ionized plasmas at temperatures $\lambda_e \lesssim a_0/Z$.

The analysis is made as follows. We will start with the Slater sum, approximate it in the form of an exponential involving the sum of effective potentials of

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two particles, and then apply to it the virial expansion of the classical statistical mechanics. Taking the effective potentials suitably, we evaluate the free energy in the watermelon approximation.^{2,3)} The result is that the correction to the free energy due to watermelon terms is negligible compared with the term of rings,^{2,3,4)} corresponding to the Debye-Hückel approximation.

This investigation supplements the analysis of Abe⁵⁾ who has shown that the contribution of the watermelon terms is small compared with that of the ring terms for electron gases in the uniform positive background at high temperatures. In his work, Abe has treated the problem by the classical statistical mechanics without any difficulty because divergence of the classical partition function has not occurred there.

§ 2. Slater sum

The system investigated in this paper is an aggregate of N particles of $N_n = N/(Z+1)$ nuclei and $N_e = NZ/(Z+1)$ electrons interacting by the Coulomb potential in volume V . The region of temperature $kT = 1/\beta$ and density $\rho = N/V$ to be investigated is assumed to be such that the probability that a nucleus and electrons are in the bound state is small and also the effect of the quantum statistics is negligible. This effect will be discussed in § 6.

Here we remark on the notations which will be used in the following. We will discriminate the particles, nuclei and electrons, by assigning an integer i ($=1, 2, \dots, N$) to each of them where the i -th particle is a nucleus or an electron according as

$$1 \leq i \leq N/(Z+1) \text{ or } N/(Z+1) < i \leq N.$$

Further, we will use the notations ν, ν' etc., denoting the species of a particle: they are equal either to n indicating a nucleus or to e indicating an electron. That is, ν_i , denoting the species of the i -th particle, is

$$\nu_i = \begin{cases} n, & 1 \leq i \leq N/(Z+1) \\ e, & N/(Z+1) < i \leq N. \end{cases} \quad (2.1)$$

The potential energy of the system is

$$\phi^{(N)}(\mathbf{r}^N) = \sum_{i>j} v_{\nu_i \nu_j}(r_{ij}), \quad (2.2)$$

$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$, with

$$v_{\nu \nu'}(r) = -e_\nu e_{\nu'} / r, \quad (2.2')$$

where $e_n = Ze$ and $Z_n = Z$ and $Z_e = -1$, $-e$ being the electronic charge. The Hamiltonian is

$$H = - \sum_{i=1}^N \frac{\hbar^2}{2m_{\nu_i}} \frac{\partial^2}{\partial \mathbf{r}_i^2} + \sum_{N \geq i > j \geq 1} v_{\nu_i \nu_j}(r_{ij}), \quad (2.3)$$

where $m_n = M$ is the nuclear mass and $m_e = m$ is the electronic mass. The partition function in the quantum theory is⁶⁾

$$Z \equiv e^{-\beta A} = \lambda_n^{-3N_n} \lambda_e^{-3N_e} \int d\mathbf{r}^N S^{(N)}(\mathbf{r}^N), \quad (2.4)$$

where the Slater sum $S^{(N)}(\mathbf{r}^N)$ is given by

$$S^{(N)}(\mathbf{r}^N) = \lambda_n^{3N_n} \lambda_e^{3N_e} \sum_i \phi_i^{(N)}(\mathbf{r}^N) e^{-\beta E_i} \phi_i^{(N)*}(\mathbf{r}^N) \quad (2.5)$$

in terms of the eigenfunctions $\phi_i^{(N)}(\mathbf{r}^N)$ and the eigenvalues E_i of (2.3):

$$(H - E_i) \phi_i^{(N)}(\mathbf{r}^N) = 0.$$

Here, $\lambda_v = h/\sqrt{2\pi m_v kT}$ is the de Broglie wave length.

The partition function could be evaluated in the form of a power series in density for low density systems of short range interaction by means of the Ursell expansion as Uhlenbeck and Beth⁷⁾ did. For systems with long range interaction, we need some devices. We have developed such a procedure for the classical cases as to sum up some graphs systematically.^{2), 3), 8)} Now, it is convenient to introduce an approximation by which we can directly apply it to our problem. For this purpose, it will be natural to apply the procedure of Mayer⁹⁾ to divide the potential of average force into component ones, rather than the Ursell development. Now, define the potential of average force, $W^{(N)}(\mathbf{r}^N)$, by

$$S^{(N)}(\mathbf{r}^N) = \exp\{-\beta W^{(N)}(\mathbf{r}^N)\}. \quad (2.6)$$

Then introduce the component potentials $w_{v_{i_1} \dots v_{i_n}}^{(n)}(\mathbf{r}^n)$ by

$$W_{v_i v_j}^{(2)}(\mathbf{r}_i, \mathbf{r}_j) = w_{v_i v_j}^{(2)}(\mathbf{r}_i, \mathbf{r}_j) \quad (2.7'')$$

$$W_{v_i v_j v_k}^{(3)}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) = w_{v_i v_j}^{(2)}(\mathbf{r}_i, \mathbf{r}_j) + w_{v_i v_k}^{(2)}(\mathbf{r}_i, \mathbf{r}_k) + w_{v_j v_k}^{(2)}(\mathbf{r}_j, \mathbf{r}_k) \\ + w_{v_i v_j v_k}^{(3)}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) \quad (2.7''')$$

.....

$$W_{v_1 \dots v_N}^{(N)}(\mathbf{r}_1, \dots, \mathbf{r}_N) = W^{(N)}(\mathbf{r}^N) = \sum_{i > j} w_{v_i v_j}^{(2)}(\mathbf{r}_i, \mathbf{r}_j) + \sum w_{v_i v_j v_k}^{(3)}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) \\ + \dots + w_{v_1 \dots v_N}^{(N)}(\mathbf{r}_1, \dots, \mathbf{r}_N). \quad (2.7^N)$$

Then the expansion of the partition function in powers of quantum parameter \hbar^2 by Kirkwood⁶⁾ shows that $W^{(N)}(\mathbf{r}^N)$ is different from $\Phi^{(N)}(\mathbf{r}^N)$ only when the gradient of $\Phi^{(N)}(\mathbf{r}^N)/kT$ is appreciable compared with the inverse of the de Broglie wave lengths, $1/\lambda_n$ and $1/\lambda_e$. That is, $W^{(N)}(\mathbf{r}^N)$ may be considered to be equal to $\Phi^{(N)}(\mathbf{r}^N)$ for the configurations for which N particles are far apart from each other; for such a configuration $w_{v\nu'}^{(2)}(\mathbf{r}, \mathbf{r}') = v_{\nu\nu'}(|\mathbf{r} - \mathbf{r}'|)$ and $w^{(3)} = w^{(4)} = \dots = 0$. When two particles come in the range where the gradient of $v_{\nu\nu'}(r)/kT$ is appreciable compared with $1/\lambda_{\nu\nu'}$, $w_{v\nu'}^{(2)}(\mathbf{r}, \mathbf{r}')$ is different from $v_{\nu\nu'}(|\mathbf{r} - \mathbf{r}'|)$, where

$$\lambda_{\nu\nu'} = h/\sqrt{2\mu_{\nu\nu'} kT}, \quad \mu_{\nu\nu'} = m_\nu m_{\nu'} / (m_\nu + m_{\nu'}).$$

When three particles come into this range, $w_{vv',v''}^{(3)}(\mathbf{r}, \mathbf{r}', \mathbf{r}'')$ will have some value, and so on. However, our system is of low density; and there will be no need to consider the effect of the possibility that more than two particles approach each other within the distance λ_v at the same time. We will drop $w^{(3)}, w^{(4)}, \dots$ and adopt the approximation:

$$W^{(N)}(\mathbf{r}^N) = \sum_{i>j} w_{vi}v_j^{(2)}(r_{ij}) \quad (2.8)$$

or

$$S^{(N)}(\mathbf{r}^N) = \exp \left\{ -\beta \sum_{i>j} w_{vi}v_j^{(2)}(r_{ij}) \right\}. \quad (2.8')$$

Then the partition function (2.4) with (2.8') is reduced just to the form usually investigated in the classical theory of fluids.

Our problems are first to evaluate $w_{vv'}^{(2)}(\mathbf{r}, \mathbf{r}')$ and next to apply the methods developed for classical cases.

The formula to give $w_{vv'}^{(2)}(\mathbf{r}, \mathbf{r}')$ is

$$S_{vv'}^{(2)}(\mathbf{r}, \mathbf{r}') = \exp \{ -\beta w_{vv'}^{(2)}(\mathbf{r}, \mathbf{r}') \} = S_{vv'}^{(2)}(|\mathbf{r} - \mathbf{r}'|) \quad (2.9)$$

$$S_{vv'}^{(2)}(r) = \exp \{ -\beta w_{vv'}^{(2)}(r) \} = \pi^{-3/2} \lambda_{vv'}^3 \sum_i \psi_i(\mathbf{r}) e^{-\beta \varepsilon_i} \psi_i^*(\mathbf{r}), \quad (2.10)$$

$$\left\{ -\frac{\hbar^2}{2\mu_{vv'}} \frac{\partial^2}{\partial \mathbf{r}^2} + v_{vv'}(r) \right\} \psi_i(\mathbf{r}) = \varepsilon_i \psi_i(\mathbf{r}) \quad (2.11)$$

where $\mu_{vv'} = m_v m_{v'} / (m_v + m_{v'})$. As the eigenfunctions and eigenvalues are known for the Coulomb potential, the calculation of $w_{vv'}^{(2)}(r)$ is possible in principle. (Some result in this direction is given in Appendix I). The analysis of Uhlenbeck and Beth⁷⁾ on $S_{vv'}^{(2)}(r)$ and the above discussions based on the work of Kirkwood⁶⁾ show that $w_{vv'}^{(2)}(r)$ can be identified with $v_{vv'}(r)$ in the region where the gradient of $v_{vv'}(r)/kT$ is small compared with the inverse of the de Broglie wave length $1/\lambda_{vv'} = \sqrt{2\mu_{vv'}kT}/\hbar$. We expect that $w_{vv'}^{(2)}(r)$ is quite different from $v_{vv'}(r)$ in the region where the gradient of $v_{vv'}(r)/kT$ is not small compared with $1/\lambda_{vv'}$.

We notice at this stage that $w_{ne}^{(2)}(r)$ cannot have a singular behavior such as $v_{ne}(r)$ at short distance and so the contribution from the region of small r must not be so large. In this way the difficulty can be removed by replacing $v_{vv'}(r)$ in the classical case by $w_{vv'}^{(2)}(r)$ in the quantum case. In the following, we will apply the watermelon approximation^{2,3)} to our problem and show that the correction to the ring approximation⁴⁾ is surely small. For our purpose, a rough estimation of $w_{vv'}^{(2)}(r)$ will be sufficient, which we will try in the following section.

§ 3. Pseudopotential

We could not decide the detailed form of an obstacle by the observation of the scattering of waves with wavelength longer than that of the diameter of the obstacle. For instance, we may consider a nucleus at the origin either as a square well potential of diameter a and depth $-3\hbar^2/m\alpha^2$, or as the potential $-4\pi\hbar^2 a/m \cdot \delta(\mathbf{r})$

or $-\hbar^2/ma \cdot \delta(r-a)$ for the scattering of a slow neutron with wavelength longer than a .¹⁰⁾

By its definition, in the evaluation of $w_{vv'}^{(2)}(r)$, we have only to consider correctly the states with $\beta\epsilon_i \lesssim 1$, that is, the states with wavelength longer than the de Broglie wavelength. That is, our $w_{vv'}^{(2)}(r)$ is to be obtained even if we replace $v_{vv'}(r)$ in (2.8) by a suitable $v'_{vv'}(r)$ which is different from $v_{vv'}(r)$ only in the region of small r . Now, if we have had a pseudopotential $v'_{vv'}(r)/kT$ which is different from $v_{vv'}(r)/kT$ only in the region where $r < a_{vv'}$ and the gradient of which is small compared with $1/\lambda_{vv'}$ in this range of r , then this $v'_{vv'}(r)$ is also identifiable to $w_{vv'}^{(2)}(r)$.*

Then the $v'_{vv'}(r)$ identifiable to $w_{vv'}^{(2)}(r)$ is the pseudopotential of $v_{vv'}(r)$, which is equal to $v_{vv'}(r)$ for $r > a_{vv'}$ ($a_{vv'} = \alpha\lambda_{vv'}$, α is a suitable numerical factor of order $0.1 \sim 0.3$), and which is a slowly varying function of r near the origin. If we allow $v'_{vv'}(r)$ to be different from $v_{vv'}(r)$ for $r < a_{vv'}$, we can take, for instance,

$$v'_{vv'}(r) = \begin{cases} v'_{vv'}(0) = \frac{3}{a_{vv'}^3} \int_0^{a_{vv'}} r^2 dr v_{vv'}(r), & r < a_{vv'}, \\ v_{vv'}(r), & r > a_{vv'}, \end{cases} \quad (3.1a)$$

or

$$v'_{vv'}(r) = \begin{cases} v_{vv'}(a_{vv'}) + \left\{ -v_{vv'}(a_{vv'}) + \frac{12}{a_{vv'}^3} \int_0^{a_{vv'}} r^2 dr v_{vv'}(r) \right\} \left(1 - \frac{r}{a_{vv'}} \right), & r < a_{vv'}, \\ v_{vv'}(r), & r > a_{vv'}. \end{cases} \quad (3.1b)$$

For the Coulomb potential:

$$v_{vv'}(r) = e_v e_{v'}/r, \quad (3.2)$$

they are, respectively,

$$v'_{vv'}(r) = \begin{cases} \frac{3}{2} e_v e_{v'}/a_{vv'}, & r < a_{vv'} \\ e_v e_{v'}/r, & r > a_{vv'} \end{cases} \quad (3.3a)$$

and

$$v'_{vv'}(r) = \begin{cases} 3 e_v e_{v'}/a_{vv'} (1 - 2r/3a_{vv'}), & r < a_{vv'}, \\ e_v e_{v'}/r, & r > a_{vv'}; \end{cases} \quad (3.3b)$$

these are to be compared with the $w_{vv'}^{(2)}(r)$ obtained for the high temperature case by solving the two-body problem, (2.9)–(2.11) (Appendix I, (A.20), (A.22)):

$$w_{vv'}^{(2)}(r) = 2\sqrt{\pi} \frac{e_v e_{v'}}{\lambda_{vv'}} \left(1 - \frac{2r}{\sqrt{\pi} \lambda_{vv'}} \right), \quad r < \lambda_{vv'} < a_0/Z. \quad (3.4)$$

* This is never the necessary condition for $w_{vv'}^{(2)}(r)$.

The order of magnitude is correct, so we may use any of (3.3a) or (3.3b) for an approximation to $w_{vv'}^{(2)}(r)$ in rough evaluation of the effect of this short-range part of $w_{vv'}^{(2)}(r)$. In the next section, we will use (3.3a) as an approximation to $w_{vv'}^{(2)}(r)$ for the free energy and the equation of state.

§ 4. Free energy and equation of state

The problem of calculating the free energy of a high temperature plasma in consideration of the quantum effect has been reduced to a calculation of the partition function of a classical gas with the interaction

$$W^{(N)}(\mathbf{r}^N) = \sum_{i>j} w_{v_i v_j}^{(2)}(r_{ij}), \quad (4.1)$$

$$w_{vv'}^{(2)}(r) = \begin{cases} v'_{vv'}(r), & r < a_{vv'} \\ v_{vv'}(r), & r > a_{vv'} \end{cases}$$

It is well known that for the problems in which $w_{vv'}^{(2)}(r)$ is Coulombic at large distance, in each term the ordinary virial expansion diverges, and physically reasonable results can be attained only by summing suitable contributions. A method of dealing with such a problem is given in author's paper³⁾ for the case of multi-component systems. The results for the problem, in which the long-range part of the potential is given by $e_v e_{v'} \phi(r)$, $\phi(r) = 1/r$, have been given by Eq. (70) with (66), (62) and the first equation of (67) in that paper in the ring approximation and Eq. (71) with (66), (62) and (67) in the watermelon approximation. By substituting the explicit form of the long-range part of the potential (62) in that paper, we have for the ring approximation

$$\frac{A}{VkT} = \sum_v \rho_v \ln \frac{\rho_v \lambda_v^3}{e} + \frac{A'}{VkT}, \quad (4.2)$$

$$-\left(\frac{A'}{VkT}\right)_R = -\frac{\kappa^3}{12\pi} - \frac{1}{4} \sum_v \sum_{v'} \rho_v \rho_{v'} \int d\mathbf{r} \{f_{1vv'}(r)\}^2. \quad (4.3)$$

Here the subscript R stands for ring;

$$\kappa^2 \equiv 4\pi \sum_v \rho_v e_v^2 / kT; \quad (4.4)$$

and $f_{1vv'}(r)$ is given by

$$\{1 + f_{1vv'}(r)\} \exp \{-f_{1vv'}(r)\} = \exp \{-\beta(w_{vv'}^{(2)}(r) - e_v e_{v'} / r)\}, \quad (4.5)$$

that is

$$f_{1vv'}(r) = 0 \text{ for } r > a_{vv'} \quad (4.5')$$

and $f_{1vv'}(r)$ for $r < a_{vv'}$ is to be determined by (4.5) with $w_{vv'}^{(2)}(r) = v'_{vv'}(r)$; \sum_v means that $\sum_v A_v = A_n + A_e$. The formula for the watermelon approximation is simplified to*

* The first terms of the right-hand side of Eq. (64) and Eq. (73) in ref. 3) were erroneously printed as $\kappa^3/96\pi$, which should read $5\kappa^3/96\pi$.

$$-\left(\frac{A'}{VkT}\right)_{WM} = \frac{\kappa^3}{12\pi} + \left(\frac{5\kappa^3}{96\pi} - \frac{\kappa^3}{12\pi}\right) + J^{(1)} + J^{(3)} \quad (4.6)$$

$$J^{(1)} = 1/2 \cdot \sum_{\nu} \sum_{\nu'} \rho_{\nu} \rho_{\nu'} \int_{a_{\nu\nu'}}^{\infty} 4\pi r^2 dr \left\{ \exp\left(-\frac{e_{\nu} e_{\nu'}}{kTr}\right) - 1 \right\} \quad (4.6')$$

$$J^{(3)} = 1/2 \cdot \sum_{\nu} \sum_{\nu'} \rho_{\nu} \rho_{\nu'} \int_0^{a_{\nu\nu'}} 4\pi r^2 dr \left\{ \exp\left(-\frac{v'_{\nu\nu'}(r)}{kT} - \frac{e_{\nu} e_{\nu'} (e^{-\kappa r} - 1)}{kTr}\right) - 1 \right\} \quad (4.6'')$$

by eliminating the $f_{1\nu\nu'}(r)$ from the formulae in the previous paper.³⁾

The purpose of this and next sections is to evaluate the right-hand side of Eq. (4.6). Evaluation of the contribution of the long-range part of the watermelons, the third term of (4.6), is to be made by expanding the exponential; for details, see Appendix IIa. The result obtained by dropping $O(ka_{\nu\nu'})$ and $O(e_{\nu} e_{\nu'} \kappa / kT)$ compared with 1 is

$$J^{(1)} = -\left(\frac{5\kappa^3}{96\pi} - \frac{\kappa^3}{12\pi}\right) + \frac{1}{2} \sum_{\nu} \sum_{\nu'} \rho_{\nu} \rho_{\nu'} \{J_{\nu\nu'}^{(1,1)} + J_{\nu\nu'}^{(1,2)} + J_{\nu\nu'}^{(1,3)}\} \quad (4.7)$$

$$J_{\nu\nu'}^{(1,1)} = 2\pi e_{\nu} e_{\nu'} a_{\nu\nu'}^2 / kT \quad (4.8)$$

$$J_{\nu\nu'}^{(1,2)} = -2\pi e_{\nu}^2 e_{\nu'}^2 a_{\nu\nu'} / (kT)^2 \quad (4.9)$$

$$J_{\nu\nu'}^{(1,3)} = \frac{2\pi e_{\nu}^3 e_{\nu'}^3}{(kT)^3} \left\{ 2 \ln \gamma + \ln 3 - \frac{11}{6} + \ln \frac{\kappa |e_{\nu} e_{\nu'}|}{kT} - K(t'_{\nu\nu'}) \right\}, \quad (4.10)$$

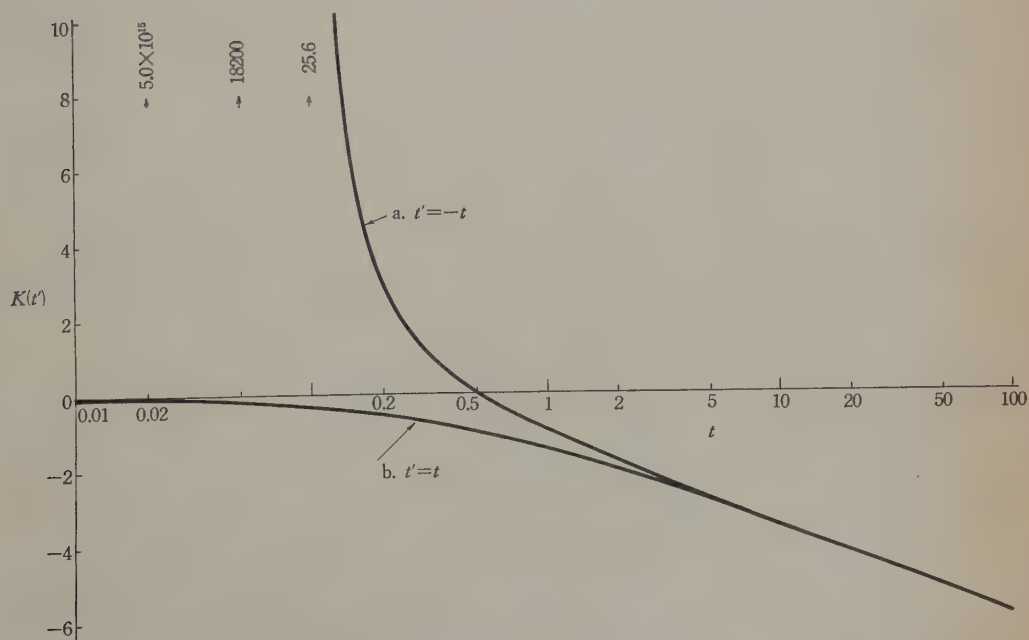


Fig. 1. $K(t') - t'$. a. $t' < 0$, b. $t' > 0$.

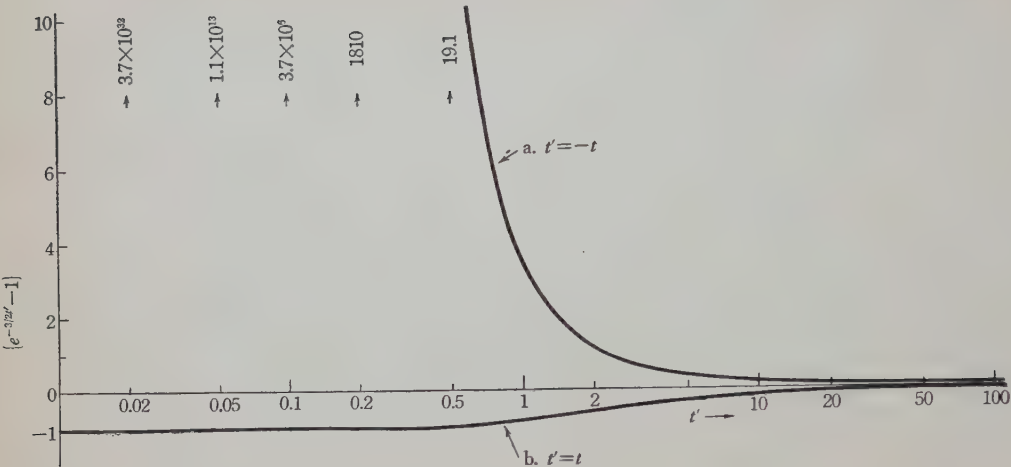


Fig. 2. $\{\exp(-3/2t') - 1\} - t'$. a. $t' < 0$, b. $t' > 0$.

where

$$K(t') = E_s(-1/t') + e^{-1/t'}(t' - t'^2 + 2t'^3) - 3t' + 3t'^2 - 2t'^3 \tag{4.11}$$

and

$$t'_{vv'} = kTa_{vv'}/e_v e_{v'} \tag{4.12}$$

The graph of $K(t')$ is given in Fig. 1. $\ln \gamma = 0.5772$ is Euler's number.

Evaluation of the short-range part of the watermelons, the last term of (4.6), is made for the pseudopotential (3.3a), in Appendix IIb: the result is

$$J^{(s)} = \frac{1}{2} \sum_v \sum_{v'} \rho_v \rho_{v'} J^{(s)}_{vv'} \tag{4.13}$$

$$J^{(s)}_{vv'} = \frac{4\pi a_{vv'}^3}{3} \left\{ \exp\left(-\frac{3}{2t'_{vv'}}\right) - 1 \right\} \tag{4.13'}$$

where $O(\kappa a_{vv'})$ compared with 1 has been dropped. The graph $\{\exp(-3/2t') - 1\}$ is given in Fig. 2. (4.6) then is integrated as

$$-\left(\frac{A'}{VkT}\right)_{WM} = \frac{\kappa^3}{12\pi} + J^{(l,1)} + J^{(l,2)} + J^{(l,3)} + J^{(s)} \tag{4.14}$$

with

$$J^{(l,n)} = \frac{1}{2} \sum_v \sum_{v'} \rho_v \rho_{v'} J^{(l,n)}_{vv'}, \quad n = 1, 2, 3, \tag{4.15}$$

where $J^{(l,1)}_{vv'}$, $J^{(l,2)}_{vv'}$, $J^{(l,3)}_{vv'}$ and $J^{(s)}$ are given by (4.8)–(4.13').

The equation of state is given by

$$\frac{p}{kT} = \rho - \frac{A'}{VkT} + \sum_v \rho_v \frac{\partial}{\partial \rho_v} \frac{A'}{VkT} \tag{4.16}$$

Substitution of $(A'/VkT)_{WM}$ given by (4.14), gives

$$\frac{p}{kT} = \rho - \frac{\kappa^3}{24\pi} - \frac{1}{2} \sum_{\nu} \sum_{\nu'} \rho_{\nu} \rho_{\nu'} \left(1 + \frac{\kappa}{2} \frac{\partial}{\partial \kappa} \right) \{ J_{\nu\nu'}^{(l,1)} + J_{\nu\nu'}^{(l,2)} + J_{\nu\nu'}^{(l,3)} + J_{\nu\nu'}^{(s)} \} \quad (4.17)$$

$$= \rho + \frac{\kappa^3}{24\pi} + \left(\frac{A'}{VkT} \right)_{WM} - \frac{\kappa}{2} \sum_{\nu} \sum_{\nu'} \rho_{\nu} \rho_{\nu'} \frac{\pi e_{\nu}^3 e_{\nu'}^3}{3(kT)^3} \quad (4.17')$$

For details, see Appendix IIc.

§ 5. Numerical computations for hydrogen and iron plasmas

The evaluation of the magnitude of the contribution from each term of (4.14) is given for hydrogen plasma of densities 10^{15} and 10^{17}cm^{-3} and temperatures 10^6 and 10^8 °K and for iron plasma of densities 10^{15} and 10^{17}cm^{-3} at temperature 10^8 °K. The results are given in Tables I and II respectively for hydrogen and iron plasmas, where the temperature and density dependence of each term is also given, in order that we can evaluate the magnitude of these terms at different temperatures and densities. The dependences of the last two columns are somewhat complex. The

Table I. Hydrogen plasma.

ρ	kT	κ	λ_{ne}	t'_{ne}	Rings	Watermelons			
			$=\lambda_{ee}/\sqrt{2}$		$\kappa^3/12\pi$	$J(l,1)$	$J(l,2)$	$J(l,3)$	$J(s)**$
		$\rho^{1/2}T^{-1/2}$	$T^{1/2}$	$T^{1/2}$	$\rho^{3/2}T^{-3/2}$	ρ^2T^{-2}	$\rho^2T^{-5/2}$	$\rho^2T^{-3}f_1(T)$	$\rho^2T^{-3/2}f_2(T)$
cm^{-3}	$^{\circ}\text{K}$	cm^{-1}	cm	—	cm^{-3}	cm^{-3}	cm^{-3}	cm^{-3}	
10^{15}	10^8	4.58×10^2	1.32×10^{-9}	-19.8	2.55×10^6	cm^{-3} 0.00* (0(1))	cm^{-3} -2.47 $\times 10^{-1}$	cm^{-3} -5.3 $\times 10^{-3}$	0.11 (-50.4)
10^{15}	10^6	4.58×10^3	1.32×10^{-8}	-1.98	2.55×10^9	0.00 (0(10 ⁴))	-2.47 $\times 10^4$	-1.94 $\times 10^3$	2.1 $\times 10^4$ (-1.1 $\times 10^4$)
10^{17}	10^8	4.58×10^3	1.32×10^{-9}	-19.8	2.55×10^9	0.00 (0(10 ⁴))	-2.47 $\times 10^3$	-5.3 $\times 10$	1.1 $\times 10^3$ (-5.04 $\times 10^5$)
10^{17}	10^6	4.58×10^4	1.32×10^{-8}	-1.98	2.55×10^{12}	0.00 (0(10 ⁸))	-2.47 $\times 10^8$	-1.94 $\times 10^7$	2.1 $\times 10^8$ (-1.1 $\times 10^8$)

* 0.00 because we are adopting the approximation $a_{\nu\nu'} = a_{\nu\nu'}$; $a_{ne} = a_{ee}/\sqrt{2}$.

** If there is a hard core of radius a_{ee} between electrons, the value is replaced by that in bracket.

Table II. Iron plasma.

ρ	kT	κ	λ_{ne}	t'_{ne}	Rings	Watermelons			
			$=\lambda_{ee}/\sqrt{2}$		$\kappa^3/12\pi$	$J^{(l,1)}$	$J^{(l,2)}$	$J^{(l,3)}$	$J^{(s)*}$
10^{15}	10^8	2.34×10^3	1.32×10^{-9}	-0.5	3.38×10^8	7.96	-13.3	-1.49 $\times 10^4$	9.33 (-199)
10^{17}	10^8	2.34×10^4	1.32×10^{-9}	-0.5	3.38×10^{11}	7.96×10^4	-1.33 $\times 10^5$	-1.16 $\times 10^8$	9.33 $\times 10^4$ (-2.0 $\times 10^6$)

* If there is a hard core of radius a_{ee} between electrons, the value is replaced by that in the bracket.

temperature dependence $f_1(T)$ is that in the curly bracket of (4.10), which is mainly governed by the behavior of $K(t'_{ne})$ which is shown in Fig. 1a. The dependence of $f_2(T)$ is that in the curly bracket of Eq. (4.13') which is shown in Fig. 2a. These graphs show that they may be taken to be of order 1~5 when

$\lambda_e \lesssim a_0/Z$ while when $\lambda_e > a_0/Z$ these factors increase as $\exp(r/\sqrt{kT})$ for $\nu=n$ and $\nu'=e$. But, even with very large values of $f_1(T)$ and $f_2(T)$, we find that the watermelon term in our approximation contributes very little compared with the ring term, $\kappa^3/12$, down to the temperature $\lambda_e = 10a_0/Z$, $T = 5 \times 10^7 \text{K}$ for hydrogen plasma and $T = 2 \times 10^{10} \text{K}$ for iron plasma or lower temperatures for the densities of order 10^{15} or 10^{17}cm^{-3} . However, our calculation based on an arbitrarily chosen pseudopotential is not sufficient to claim that the Debye-Hückel approximation is sufficient in these regions. A more careful evaluation of $w_{\nu\nu'}^{(2)}(r)$ and a more careful treatment of the bound states seem necessary,¹¹⁾ because in the temperature region for $\lambda_e > a_0/Z$ the contribution of bound states to $w_{\nu\nu'}^{(2)}(r)$ overcomes that of the scattering states, as is seen by comparing (A.16) with (A.17) or (A.18) in Appendix I.

The result shows that the contribution of the watermelon terms as a correction to the ring approximation is surely small in the temperature and density region where plasmas are considered to be fully ionized.

Here, it is noticed that the first term of (4.2), the ideal gas term*, is of the order of ρ and so the contribution of the ring is also negligible and we may say that as far as the free energy is concerned the ideal gas approximation is sufficiently good for high temperature plasmas. It can be seen from Eq. (4.17) that as to the pressure the situation is the same.

The evaluations in Tables I and II have been made by using $v'_{\nu\nu'}(r)$ given by (3.3a) with $a_{\nu\nu'} = \lambda_{\nu\nu'}/4$. Someone may consider this choice of $v'_{\nu\nu'}(r)$ might be essential. To see that it is not the case, we can easily evaluate them by taking $v'_{\nu\nu'}(r)$ given by (3.3a) with $a_{\nu\nu'} = \lambda_{\nu\nu'}/8$, then we find that the order of magnitude does not change for the respective term even though we have overestimate the attraction between a positive and a negative charge. This concludes that the result is fairly firm.

§ 6. Effect of quantum statistics

The systems under consideration are of so small density that the probability that more than two identical particles approach within the distance of the order $\lambda_{\nu\nu'}$ at the same time can be negligible. In such a case, we will be able to adopt the approximation

$$W^{(N)}(\mathbf{r}^N) = \sum_{i>j} w_{\nu_i\nu_j}^{(2)}(r_{ij}), \quad (6.1)$$

even if we take into account the quantum statistics, if some modifications are made about the definition of $w_{\nu\nu'}^{(2)}(r)$.

When the quantum statistics is considered, we have to group the particles ac-

* The numerical evaluation of this ideal gas term of the free energy is not given in Tables I and II, because it depends on what isotope is concerned.

cording to the spin of the particles as well as the species of them. Then we may consider that the particles with different spins can be distinguishable. Between a pair of distinguishable particles we have no effect of quantum statistics and $w_{\nu\nu}^{(2)}(r)$ for a pair of different kind of particles and for a pair of the same kind of particles with different spins are taken to be defined by (2.10)–(2.11). While for a pair of the same kind of particles with the same spin, we have to introduce some modification in the definition of $w_{\nu\nu}^{(2)}(r)$; It is

$$S_{\nu\nu}^{(2)}(r) = \exp\{-\beta w_{\nu\nu}^{(2)}(r)\} \\ = \frac{1}{2} \sum_s e^{-\beta \varepsilon_s} |\phi_s(\mathbf{r}) \mp \phi_s(-\mathbf{r})|^2. \quad (6.2)$$

The upper sign is for the fermions and the lower for the bosons. This results in an attraction between bosons of the same spin and a repulsion between fermions of the same spin as it is well known.

Now, let us estimate the order of magnitude of this additional interaction by that of the free particles. For the free particles of the same spin,¹²⁾

$$-\beta w_{\nu\nu}^{(2)}(r) = \ln\{1 \mp e^{-4\pi r^2/\lambda_{\nu\nu}^2}\}. \quad (6.3)$$

This is a short-range interaction of the range $\lambda_{\nu\nu}$ and will affect the magnitude of the watermelon terms.

In the above, we have been neglecting λ_{nn} as very small, and here again we may safely neglect the additional interaction between a pair of nuclei of the same spin. For electrons of the same spin, this additional interaction is a repulsion of the range λ_{ee} . The main effect of this repulsion appears in $J^{(s)}$ among the watermelon terms. Its order of magnitude is read by comparing the value of $J^{(s)}$ in the bracket in Tables I and II. The change of $w_{ee}^{(2)}(r)$ between electrons does not very much affect the resultant magnitude of each term. Thus, we may say that the result of the preceding sections holds even when we take into account the quantum statistics.

§ 7. Conclusion

Fully ionized plasmas at high temperatures are investigated on the basis of quantum theory. By reducing the quantum problem to a suitable classical one, the correction to the Debye-Hückel approximation is evaluated for the free energy and the equation of state. The correction due to watermelon terms—which is considered as the leading correction and which should have been infinite if quantum effects had not been taken into account—was shown to be very small compared with the Debye-Hückel term, which, in turn, is negligible compared with the ideal gas term in the region of temperatures and densities we are interested in.

Thus, we conclude that as far as the free energy and the equation of state for $\lambda_e \lesssim a_0/Z$ are concerned, the plasmas can be taken to be ideal.

The situation is expected to be quite different for the transport coefficients. In this case, the characters of collisions must be fully taken account of in order

to obtain the correct behaviors. The investigations of this problem by the similar method will be the subject of the following work.

Investigation of the equation of state at the somewhat lower temperatures, where the bound states play the essential role, has been retained for future.

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Appendix I. Slater sum of two bodies

Ia. Nucleus-Electron

We deal with the two-body problem of a nucleus of charge Ze and mass M and an electron of charge $-e$ and mass m . The two-body problem is

$$\left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{r}^2} - \frac{Ze^2}{r} \right\} \psi_i(\mathbf{r}) = \varepsilon_i \psi_i(\mathbf{r}), \quad (\text{A} \cdot 1)$$

by neglecting the difference of $\mu_{ne} = mM/(m+M)$ and m . The eigenfunctions and eigenvalues of the bound states are¹³⁾

$$\psi(\mathbf{r}) = R_{nl}(r) Y_{lm}^*(\vartheta, \varphi) \quad (\text{A} \cdot 2)$$

with $m = -l, \dots, l-1, l$ and $l = 0, 1, 2, \dots, n-1$. Here

$$Y_{lm}(\vartheta, \varphi) = 1/\sqrt{2\pi} \cdot P_{lm}(\vartheta) e^{im\varphi} \quad (\text{A} \cdot 3)$$

$$R_{nl}(r) = \frac{1}{(2l+1)!} \sqrt{\frac{(n+l)!}{(n-l-1)! 2n}} \left(\frac{2Z}{na_0} \right)^{3/2} e^{-Zr/na_0} \left(\frac{2Zr}{na_0} \right)^l \\ \times F(-n+l+1, 2l+2; 2Zr/na_0) \quad (\text{A} \cdot 4)$$

$$\varepsilon_n = -\frac{Z^2 e^2}{2n^2 a_0}, \quad a_0 \equiv \frac{\hbar^2}{me^2}, \quad (\text{A} \cdot 5)$$

and those of the continuous states are

$$\psi(\mathbf{r}) = R_{kl}(r) Y_{lm}(\vartheta, \varphi) \quad (\text{A} \cdot 6)$$

$$R_{kl}(r) = \left(\frac{2\pi Z/ka_0}{1 - e^{-2\pi Z/ka_0}} \right)^{1/2} \prod_{s=1}^l \left[s^2 + \left(\frac{Z}{ka_0} \right)^2 \right]^{1/2} \frac{1}{(2l+1)!} (2kr)^l e^{-ikr} \\ \times F(iZ/ka_0 + l + 1, 2l + 2; 2ikr) \quad (\text{A} \cdot 7)$$

$$\varepsilon_k = \frac{k^2 e^2}{2a_0} = \frac{\hbar^2 k^2}{2m}. \quad (\text{A} \cdot 8)$$

The level density of these states is given by the asymptotic form of the solution

$$R_{kl}(r) \sim \frac{1}{kr} \cos \left(kr + \frac{Z}{ka_0} \ln 2kr - \frac{\pi}{2} (l+1) + \sigma_l \right) \quad (\text{A} \cdot 9)$$

$$\sigma_l = \arg \Gamma(l+1 + iZ/ka_0) \quad (\text{A} \cdot 10)$$

where

$$F(\alpha, \beta; x) = 1 + \frac{\alpha}{\beta} x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)2!} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)3!} x^3 + \dots$$

Noticing that, because of the addition theorem for Legendre functions,¹⁴⁾

$$\sum_{m=-l}^l Y_{lm}^*(\vartheta, \varphi) Y_{lm}(\vartheta, \varphi) = \frac{2l+1}{4\pi},$$

we have for the Slater sum of two particles

$$S_{\text{ne}}^{(2)}(r) = S_{\text{disc}}^{(2)}(r) + S_{\text{cont}}^{(2)}(r), \quad (\text{A} \cdot 11)$$

$$S_{\text{disc}}^{(2)}(r) = \frac{\lambda_e^3}{4\pi} \sum_{l=0}^{\infty} (2l+1) \sum_{n=l+1}^{\infty} e^{\beta Z^2 e^2 / 2n^2 a_0} |R_{nl}(r)|^2 \quad (\text{A} \cdot 12)$$

$$S_{\text{cont}}^{(2)}(r) = \frac{\lambda_e^3}{4\pi} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} dk \frac{dn}{dk} e^{-\beta \tilde{\hbar}^2 k^2 / 2m} |N R_{kl}(r)|^2. \quad (\text{A} \cdot 13)$$

Here the second sum is to be calculated first by confining the r in the range R , then the level density dn/dk is

$$\frac{dn}{dk} = \frac{1}{\pi} \left(R - \frac{2}{k^2 a_0} \ln 2kR + \frac{Z}{k^2 a_0} + \frac{d\sigma_l}{dk} \right) \quad (\text{A} \cdot 14)$$

$$\frac{d\sigma_l}{dk} = -\frac{Z}{2k^2 a_0} \left\{ \frac{\Gamma'(l+1+iZ/ka_0)}{\Gamma(l+1+iZ/ka_0)} + \frac{\Gamma'(l+1-iZ/ka_0)}{\Gamma(l+1-iZ/ka_0)} \right\} \quad (\text{A} \cdot 14')$$

and the normalization constant N is equal to

$$N = \sqrt{2k^2/R} \quad (\text{A} \cdot 15)$$

for small l . To see the behavior near the origin, we get the expansion at $r=0$ up to the first order of r , which is

$$S_{\text{disc}}^{(2)}(r) = \frac{\lambda_e^3 Z^3}{\pi a_0^3} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^3} e^{\beta Z^2 e^2 / 2n^2 a_0} \right\} \left(1 - \frac{2Zr}{a_0} \right) + O(r^2), \quad (\text{A} \cdot 16)$$

$$S_{\text{cont}}^{(2)}(r) = \frac{\lambda_e^3}{2\pi^2} \int_0^{\infty} k^2 dk e^{-\beta \tilde{\hbar}^2 k^2 / 2m} \cdot \frac{2\pi Z / ka_0}{1 - e^{-2\pi Z / ka_0}} \cdot \left(1 - \frac{2Zr}{a_0} \right) + O(r^2). \quad (\text{A} \cdot 17)$$

$S_{\text{cont}}^{(2)}(r)$ developed in powers of β is

$$S_{\text{cont}}^{(2)}(r) = \left(1 + \frac{\lambda_e Z}{a_0} + O\left(\frac{\lambda_e Z}{a_0}\right)^2 \right) \left(1 - \frac{2Zr}{a_0} \right) + O(r^2). \quad (\text{A} \cdot 18)$$

At a very high temperature, $\lambda_e < a_0/Z$, we may approximate $S_{\text{ne}}^{(2)}(r)$ by $S_{\text{cont}}^{(2)}(r)$ and put

$$S_{\text{ne}}^{(2)}(r) \doteq e^{Z\lambda_e/a_0 - 2Zr/a_0} = e^{\beta(2\pi Z e^2/\lambda_e)(1-2r/\lambda_e)} \quad (\text{A} \cdot 19)$$

and then

$$w_{ne}^{(2)}(r) \doteq -\frac{2\pi Ze^2}{\lambda_e} \left(1 - \frac{2r}{\lambda_e}\right). \quad (\text{A} \cdot 20)$$

Note that $\lambda_e = \sqrt{\pi} \lambda_{ne}$.

Ib. Electron-Electron

For a pair of particles of the same kind, the eigenstates are all continuous. The Slater sum of two electrons with different spins can be calculated in analogy with the preceding paragraphs. The result is

$$S_{ee}^{(2)}(r) = \left(1 - \frac{\sqrt{\pi} \lambda_{ee}}{a_0}\right) \left(1 + \frac{2r}{a_0}\right) + O(r^2) \quad (\text{A} \cdot 21)$$

$$w_{ee}^{(2)}(r) \doteq \frac{2\pi e^2}{\sqrt{\pi} \lambda_{ee}} \left(1 - \frac{2r}{\sqrt{\pi} \lambda_{ee}}\right) \quad (\text{A} \cdot 22)$$

which is $S_{\text{cont.}}^{(2)}(r)$ if $Z = -1$ and $\lambda_{ee} = \sqrt{2} \lambda_{ne} = \sqrt{2} \lambda_e / \sqrt{\pi}$. For a pair of the same spin Eq. (4.13) must be modified with consideration of the Fermi statistics and is obtained as

$$S_{ee}^{(2)}(r) = \frac{2\pi^{3/2} \lambda_{ee}^3}{4\pi} \sum_{l=\text{odd}} (2l+1) \int_0^\infty dk \frac{dn}{dk} |NR_{kl}(r)|^2 \quad (\text{A} \cdot 23)$$

$$= O(r^2). \quad (\text{A} \cdot 23')$$

From this we see that $S_{ee}^{(2)}(0) = 0$, that is $w_{ee}^{(2)}(0) = \infty$ for a pair of the same spin.

Ic. Nucleus-Nucleus

At this stage, how to derive the $w_{nn}^{(2)}(r)$ for a pair of nuclei will be obvious. For the nuclei, λ_{nn} is very small compared with the preceding two cases. Here we will not consider it in detail, assuming a_{nn} to be zero.

Appendix II. Evaluation of the watermelon terms

IIa. Long-range part

Here the contribution of the long-range part of watermelons, the third term of the right-hand side of (4.6):

$$J^{(l)} = \frac{1}{2} \sum_{\nu} \sum_{\nu'} \rho_{\nu} \rho_{\nu'} I_{\nu\nu'}^{(l)} \quad (\text{B} \cdot 1)$$

$$I_{\nu\nu'}^{(l)} \equiv \int_{a_{\nu\nu'}}^{\infty} 4\pi r^2 dr \left\{ \exp\left(-\frac{e_{\nu} e_{\nu'}}{kT} \frac{e^{-\kappa r}}{r}\right) - 1 \right\} \quad (\text{B} \cdot 1')$$

is undertaken. In calculating the summand, we expand the exponential in the integrand:

$$I_{\nu\nu'}^{(l)} = I_{\nu\nu'}^{(l,1)} + I_{\nu\nu'}^{(l,2)} + I_{\nu\nu'}^{(l,3)} \quad (\text{B} \cdot 2)$$

$$I_{\nu\nu'}^{(l,1)} = \int_{a_{\nu\nu'}}^{\infty} 4\pi r^2 dr \left(-\frac{e_{\nu} e_{\nu'}}{kT} \frac{e^{-\kappa r}}{r} \right) \quad (\text{B} \cdot 3)$$

$$I_{\nu\nu'}^{(l,2)} = \int_{\alpha_{\nu\nu'}}^{\infty} 4\pi r^2 dr \frac{1}{2!} \left(-\frac{e_{\nu}e_{\nu'}}{kT} \frac{e^{-\kappa r}}{r} \right)^2 \quad (\text{B}\cdot 4)$$

$$I_{\nu\nu'}^{(l,h)} = \int_{\alpha_{\nu\nu'}}^{\infty} 4\pi r^2 dr \sum_{m=3}^{\infty} \frac{1}{m!} \left(-\frac{e_{\nu}e_{\nu'}}{kT} \frac{e^{-\kappa r}}{r} \right)^m \quad (\text{B}\cdot 5)$$

Integrations of (B·3) and (B·4) are easily performed and we have

$$I_{\nu\nu'}^{(l,1)} = -4\pi \frac{e_{\nu}e_{\nu'}}{kT} \frac{1 + \kappa \alpha_{\nu\nu'}}{\kappa^2} e^{-\kappa \alpha_{\nu\nu'}} \quad (\text{B}\cdot 6)$$

$$= -4\pi \frac{e_{\nu}e_{\nu'}}{kT\kappa^2} \left\{ 1 - \frac{1}{2} \kappa^3 \alpha_{\nu\nu'}^2 + \frac{1}{2} \cdot \frac{2}{3} \kappa^3 \alpha_{\nu\nu'}^3 - \frac{1}{3!} \cdot \frac{3}{4} \kappa^4 \alpha_{\nu\nu'}^4 + \dots \right\}, \quad (\text{B}\cdot 6')$$

$$I_{\nu\nu'}^{(l,2)} = \pi \left(\frac{e_{\nu}e_{\nu'}}{kT} \right)^2 \frac{1}{\kappa} e^{-2\kappa \alpha_{\nu\nu'}} \quad (\text{B}\cdot 7)$$

$$= \pi \left(\frac{e_{\nu}e_{\nu'}}{kT} \right)^2 \frac{1}{\kappa} \left\{ 1 - 2\kappa \alpha_{\nu\nu'} + \frac{4}{2!} \kappa^2 \alpha_{\nu\nu'}^2 - \frac{8}{3!} \kappa^3 \alpha_{\nu\nu'}^3 + \dots \right\}. \quad (\text{B}\cdot 7')$$

Note that, when the contribution to the $J^{(l)}$ is calculated, the first term of (B·6') drops because of the neutrality of our system ($\sum_{\nu} \rho_{\nu} e_{\nu} = 0$) and the first term of (B·7') is equal to $-(5\kappa^3/96 - \kappa^3/12)$, which just cancels the second term of (4·6).

Integration of (B·5) is fairly cumbersome. It was performed by Abe⁵⁾ for the case of $e_{\nu}e_{\nu'} > 0$ and $\alpha_{\nu\nu'} = 0$. For our case where $\alpha_{\nu\nu'} \neq 0$ and $e_{\nu}e_{\nu'}$ may be negative, we have to resort to another approach, in a more elementary but less transparent way than Abe's; that is, we apply the partial integration until (B·5) is written in the form which includes no integrations other than integral exponential

$$E_l(-z) = \int_0^z dx \frac{1}{x} (e^{-x} - 1) + \ln \gamma + \ln |z|, \quad (\text{B}\cdot 8)$$

where $\ln \gamma = 0.5772$ is Euler's number. In order to simplify the description somewhat, we will here use $|e_{\nu}e_{\nu'}|/kT$ as the unit of length and then introduce the dimensionless parameters

$$\eta_{\nu\nu'} \equiv \frac{|e_{\nu}e_{\nu'}|}{kT} \kappa, \quad t_{\nu\nu'} \equiv \frac{kT}{|e_{\nu}e_{\nu'}|} \alpha_{\nu\nu'} \quad (\text{B}\cdot 9)$$

and write $t = r \cdot kT/|e_{\nu}e_{\nu'}|$; η_{ee} is the parameter λ in Abe's paper. Then (B·5) is

$$I_{\nu\nu'}^{(l,h)} = \left(\frac{|e_{\nu}e_{\nu'}|}{kT} \right)^3 \cdot 4\pi \int_{t_{\nu\nu'}}^{\infty} t^2 dt \sum_{m=3}^{\infty} \frac{(-\epsilon_{\nu\nu'})^m}{m!} \frac{e^{-m\eta_{\nu\nu'}/t}}{t^m} \quad (\text{B}\cdot 10)$$

where

$$\epsilon_{\nu\nu'} \equiv e_{\nu}e_{\nu'}/|e_{\nu}e_{\nu'}|. \quad (\text{B}\cdot 11)$$

After the partial integrations, it reduces to

$$\begin{aligned}
I_{\nu\nu'}^{(l,h)} &= \left(\frac{|e_\nu e_{\nu'}|}{kT} \right)^3 4\pi \sum_{s=3}^{\infty} \eta_{\nu\nu'}^{s-3} \left\{ -\frac{\epsilon_{\nu\nu'}^s s^{s-3}}{s!(s-3)!} \int_{s\eta_{\nu\nu'} t'_{\nu\nu'}}^{\infty} dx \frac{e^{-x}}{x} \right. \\
&\quad \left. - (-1)^s \sum_{n=s+1}^{\infty} \frac{(-\epsilon_{\nu\nu'})^n n^{s-3}}{n!(n-3)(n-4)\cdots(n-s)} \frac{e^{-n\eta_{\nu\nu'} t'_{\nu\nu'}}}{t'_{\nu\nu'}^{n-s}} \right\} \\
&= \left(\frac{e_\nu e_{\nu'}}{kT} \right)^3 4\pi \sum_{s=3}^{\infty} \eta'_{\nu\nu'}{}^{s-3} \left\{ -\frac{s^{s-3}}{s!(s-3)!} \int_{s\eta'_{\nu\nu'} t'_{\nu\nu'}}^{\infty} dx \frac{e^{-x}}{x} \right. \\
&\quad \left. - (-1)^s \sum_{n=s+1}^{\infty} \frac{(-1)^n n^{s-3}}{n!(n-3)(n-4)\cdots(n-s)} \frac{e^{-n\eta'_{\nu\nu'} t'_{\nu\nu'}}}{t'_{\nu\nu'}{}^{n-s}} \right\} \quad (\text{B}\cdot 12)
\end{aligned}$$

where

$$\eta'_{\nu\nu'} = e_\nu e_{\nu'} \kappa / kT, \quad t'_{\nu\nu'} = kT a_{\nu\nu'} / e_\nu e_{\nu'}. \quad (\text{B}\cdot 13)$$

In the temperature and density region in which we are interested $\eta'_{\nu\nu'}$ is $10^{-6} \sim 10^{-10}$, we drop the higher powers of $\eta'_{\nu\nu'}$ and retain only $s=3$:

$$\begin{aligned}
I_{\nu\nu'}^{(l,h)} &= \left(\frac{e_\nu e_{\nu'}}{kT} \right)^3 \cdot 4\pi \left\{ -\frac{1}{3!} \int_{3\eta'_{\nu\nu'} t'_{\nu\nu'}}^{\infty} dx \frac{e^{-x}}{x} + \sum_{n=4}^{\infty} \frac{(-1)^n}{n!(n-3)} \frac{e^{-n\eta'_{\nu\nu'} t'_{\nu\nu'}}}{t'_{\nu\nu'}{}^{n-3}} \right\} \\
&\quad \times (1 + O(\eta'_{\nu\nu'})). \quad (\text{B}\cdot 14)
\end{aligned}$$

Next, expanding in powers of $\eta'_{\nu\nu'} t'_{\nu\nu'} = \kappa a_{\nu\nu'}$, we have

$$\begin{aligned}
I_{\nu\nu'}^{(l,h)} &= \frac{4\pi}{6} \left(\frac{e_\nu e_{\nu'}}{kT} \right)^3 \left\{ 2\ln\gamma + \ln 3 - \frac{11}{6} + \ln \frac{\kappa |e_\nu e_{\nu'}|}{kT} - K(t'_{\nu\nu'}) \right\} \\
&\quad \times (1 + O(\kappa a_{\nu\nu'}) + O(\eta_{\nu\nu'})), \quad (\text{B}\cdot 15)
\end{aligned}$$

$$K(t') = E_4(-1/t') + e^{-1/t'} (t' - t'^2 + 2t'^3) - 3t' + 3t'^2 - 2t'^3. \quad (\text{B}\cdot 16)$$

The behavior of $K(t')$ can be calculated by using the expansion of $E_4(-x)$ at small x and the asymptotic expansion at large x .¹⁵⁾ It is shown that $K(t') \rightarrow 0$ when $t' \rightarrow 0$ if $t' > 0$, by which we can confirm that our result is surely identical to that of Abe,⁵⁾ if our result is applied to the electron gas in the positive background. The curve of $K(t')$ has been given in Fig. 2.

Substituting (B·6'), (B·7') and (B·15) in (B·1) with (B·2), we obtain

$$J^{(n)} = -\left(\frac{5\kappa^3}{96\pi} - \frac{\kappa^3}{12\pi} \right) + \frac{1}{2} \sum_{\nu} \sum_{\nu'} \rho_{\nu} \rho_{\nu'} \left\{ J_{\nu\nu'}^{(l,1)} + J_{\nu\nu'}^{(l,2)} + J_{\nu\nu'}^{(l,3)} \right\} \quad (\text{B}\cdot 17)$$

$$J_{\nu\nu'}^{(l,1)} = \frac{2\pi e_\nu e_{\nu'} a_{\nu\nu'}^2}{kT} (1 + O(\kappa a_{\nu\nu'})) \quad (\text{B}\cdot 18)$$

$$J_{\nu\nu'}^{(l,2)} = -\frac{2\pi e_\nu^2 e_{\nu'}^2 a_{\nu\nu'}^2}{(kT)^2} (1 + O(\kappa a_{\nu\nu'})) \quad (\text{B}\cdot 19)$$

$$J_{\nu\nu'}^{(l,3)} = \frac{2\pi e_\nu^3 e_{\nu'}^3}{3(kT)^3} \left\{ 2 \ln r + \ln 3 - \frac{11}{6} + \ln \frac{\kappa |e_\nu e_{\nu'}|}{kT} - K(t'_{\nu\nu'}) \right\} \\ \times (1 + O(\kappa a_{\nu\nu'}) + O(\gamma'_{\nu\nu'})) \quad (\text{B} \cdot 20)$$

with $t'_{\nu\nu'} = kT a_{\nu\nu'} / e_\nu e_{\nu'}$.

IIb. Short-range part

Adopting (3.3a) as the short-range part of the effective potential, we calculate

$$J^{(s)} = \frac{1}{2} \sum_\nu \sum_{\nu'} \rho_\nu \rho_{\nu'} \int_0^{a_{\nu\nu'}} 4\pi r^2 dr \left\{ \exp \left(-\frac{3}{2t'_{\nu\nu'}} - \frac{e_\nu e_{\nu'} (e^{-\kappa r} - 1)}{kTr} \right) - 1 \right\}. \quad (\text{B} \cdot 21)$$

Noticing that $\kappa r < \kappa a_{\nu\nu'} \ll 1$ for $r < a_{\nu\nu'}$ and

$$-\frac{e_\nu e_{\nu'} (e^{-\kappa r} - 1)}{kTr} = -\frac{1}{t'_{\nu\nu'}} \cdot \frac{a_{\nu\nu'}}{r} (e^{-\kappa r} - 1) \sim -\frac{1}{t'_{\nu\nu'}} \cdot \kappa a_{\nu\nu'} \text{ for } r \leq a_{\nu\nu'},$$

which is negligible compared with $-3/2t'_{\nu\nu'}$, we can approximate this by

$$J^{(s)} = \frac{1}{2} \sum_\nu \sum_{\nu'} \rho_\nu \rho_{\nu'} \cdot \frac{4\pi a_{\nu\nu'}^3}{3} \left\{ \exp \left[-\frac{3}{2t'_{\nu\nu'}} (1 + O(\kappa a_{\nu\nu'})) \right] - 1 \right\}. \quad (\text{B} \cdot 22)$$

IIc. Pressure

The last derivatives of (4.17) are calculated as follows:

$$\frac{\kappa}{2} \frac{\partial}{\partial \kappa} J_{\nu\nu'}^{(l,1)} = \frac{\kappa}{2} \cdot \frac{2\pi e_\nu e_{\nu'}}{kT\kappa^3} \left\{ e^{-\kappa a_{\nu\nu'}} (1 + \kappa a_{\nu\nu'} + 2\kappa^2 a_{\nu\nu'}^2) - 1 \right\} \\ = -\frac{2\pi e_\nu e_{\nu'} \kappa a_{\nu\nu'}^3}{3kT} (1 + O(\kappa a_{\nu\nu'})) = J_{\nu\nu'}^{(l,1)} \times O(\kappa a_{\nu\nu'}), \quad (\text{B} \cdot 23)$$

$$\frac{\kappa}{2} \frac{\partial}{\partial \kappa} J_{\nu\nu'}^{(l,2)} = -\frac{\kappa}{2} \cdot \frac{\pi e_\nu^2 e_{\nu'}^2}{(kT)^2 \kappa^2} \left\{ e^{-2\kappa a_{\nu\nu'}} (1 + 2\kappa a_{\nu\nu'}) - 1 \right\} \\ = \frac{\pi e_\nu^2 e_{\nu'}^2 \kappa a_{\nu\nu'}^2}{(kT)^2} (1 + O(\kappa a_{\nu\nu'})) = J_{\nu\nu'}^{(l,2)} \times O(\kappa a_{\nu\nu'}), \quad (\text{B} \cdot 24)$$

$$\frac{\kappa}{2} \frac{\partial}{\partial \kappa} J_{\nu\nu'}^{(l,3)} = \frac{\pi e_\nu^3 e_{\nu'}^3}{3(kT)^3} \quad (\text{B} \cdot 25)$$

$$\frac{\kappa}{2} \frac{\partial}{\partial \kappa} J_{\nu\nu'}^{(s)} = \frac{\kappa}{2} \cdot \frac{4\pi a_{\nu\nu'}^3}{3} \cdot \frac{e_\nu e_{\nu'}}{kT} e^{-3/2t'_{\nu\nu'} + e_\nu e_{\nu'} \kappa/kT} = J_{\nu\nu'}^{(s)} \times O(\kappa a_{\nu\nu'}). \quad (\text{B} \cdot 26)$$

When we neglect the $O(\kappa a_{\nu\nu'})$, we have only to retain $(\kappa/2) \cdot (\partial/\partial \kappa) J_{\nu\nu'}^{(l,3)}$.

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On the Theory of Superconductivity

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Bardeen, Cooper and Schrieffer's theory of superconductivity is discussed by developing an exact treatment of strong coupling approximation. With respect to the thermal properties, their results are verified: The energy of the ground state and the magnitude of the energy gap are found to be identical with the exact answers in the strong coupling limit. However, the argument for the magnetic properties such as the Meissner effect seems to leave some questions. We have discussed the Meissner effect by means of the gauge invariant method presented in the previous paper. In order to maintain the invariance exactly the case of strong coupling between electrons is investigated. Then the "interaction current" plays an essential role and is shown to diverge with the size of the system. In view of the qualitative resemblance of the BCS variational solution to that of the strong coupling approximation, it may be inferred that the above conclusion seems to hold also in the weak coupling case. This will suggest the importance of the interaction current and the requirement of the exact gauge invariance contrary to Anderson's arguments.

§ 1. Introduction

In recent years a great amount of light has been thrown on the mysteries of superconducting phenomena. One of them was presented by Bardeen, Cooper and Schrieffer¹⁾. They have revealed the fact that a system of interacting Fermi particles through attractive two-body forces has quite different properties from those of a system with Fermi distribution, however weak the coupling may be. They have started with the discussion after reducing the Hamiltonian of the system into a form which contains only the interaction between those particles found in a narrow shell of momentum space around the Fermi surface. A similar result has been obtained by Bogolyubov who has considered the electron phonon system in its original form²⁾. These peculiar properties give the origin of superfluidity of the system. Moreover, Bardeen, Cooper and Schrieffer have claimed that they have explained the magnetic property, the so-called Meissner effect, by means of their model.

However, in these discussions, there are some questions to be examined. In the BCS theory, we must first of all be careful about the influence of reducing the Hamiltonian. Any thorough investigation of the effect of neglected terms may be a hard problem. Furthermore, they have applied a variational method making use of the trial function with the total number of electrons unspecified. It may give reasonable result for the ground state energy, but might be entirely misleading for

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the excited states, since the constant terms with respect to the total number have not been treated carefully. The first problem will be outside the scope of this paper. We will give here an alternative method to treat the BCS Hamiltonian, and develop an exact theory of strong coupling limit which may be useful for checking the BCS results that are derived irrespective of the coupling strength. Then it will be found that their answers are quite reliable in this limit and that the energy gap will exist also in the weak coupling limit. The third question is the discussion about the Meissner effect. The idealized Hamiltonian which they used is not gauge invariant because of the momentum dependent cut-off on the effective interaction between electrons. As has been argued especially by Schafroth and Buckingham³⁾, a proof of gauge invariance lies at the core of the problem of superconductivity, since the violation of this invariance does lead very easily to the Meissner effect.

Anderson⁴⁾ has shown by introducing the collective excitations of the system that the BCS theory is approximately gauge invariant in the weak coupling case, but the examination of the equation of continuity in the zero momentum limit alone is not sufficient for this problem. It will ensure the fact that the momentum dependence of the effective interaction between electrons gives small effects on the longitudinal component of the current, but it cannot tell anything about the transverse components. Many other authors, e.g., Blatt, Matsubara, Yosida and Rickayzen⁵⁾ discussed this problem along the same line. Although their results may essentially be correct, there seem to remain some points that are not so clear. One of them is the same as for Anderson's arguments, that is, the non gauge-invariance of the Hamiltonian they used. The second point is the validity of random phase approximation. We know the applicability of this approximation in the case of high density electron gas but in any other cases we have no criterion to believe the validity of this approximation. The third point is the expression of the paramagnetic current operator. They have taken into account a part of this operator which can be expressed in terms of Bose operators they introduced, and neglected the remaining terms. Since it is not so easy to clarify these points exactly, we would rather like to treat a gauge invariant model which seems at first sight leading to the Meissner effect.

In the previous paper⁶⁾ we have considered whether the gauge invariance and a simple energy gap model given phenomenologically can lead to the Meissner effect. The result was negative; we found a peculiar singularity in the expression of electric current instead of the London diamagnetism. In this calculation the ground state was considered to be the Fermi distribution. One may naively suspect that this simple choice of wave function would lead to the singular results, and that if we took a more realistic wave function we would obtain a reasonable answer. It is the second purpose of this paper to examine this point. We will reconsider the BCS theory in its original form from the gauge invariant standpoint, modifying the magnetic interaction of the electron system according to the method presented in the previous paper. We have not taken into account the collective excitation of the

system, since it involves a term which does not conserve the total electron number to which our procedure is not applicable. In order to obtain the gauge invariant results, it is further necessary to take into account the effect of electron-electron correlations in a gauge invariant way. For instance, the perturbation treatment of the correlations secures the invariance in each order of the perturbation expansion. The perturbation theory is, however, not applicable to the present problem since it is not able to give the energy gap necessary to explain the thermal property of the system. Moreover, it gives the interaction currents which vanish when the size of the system becomes large. The perturbation series is in fact shown to be divergent in this case. On the other hand, the variational solution used by BCS is not gauge invariant in nature. It is not the exact solution of their model, but that of the further reduced Hamiltonian, as shown by Valatin⁷. This further reduced system evidently violates the gauge invariance since the reduced Hamiltonian can create or annihilate two electrons at the same time, breaking the conservation law of the total electric charge. Therefore one cannot obtain the gauge invariant result by means of the BCS variational solution even if the formulation is modified so as to satisfy this invariance.

Instead of these solutions, one may make use of the strong coupling approximation⁸, where the invariance is again maintained in each order of the inverse power series of the coupling constant of electron correlations. In the lowest order, the interaction current which is proportional to the coupling constant is most effective and the term due to the kinetic energy may be neglected. Although the strong coupling approximation is not reliable quantitatively, it will give some qualitative insight into the problem. The electron system will have a distribution in the momentum space which is diffused around the Fermi surface even in the weak coupling case. Such a diffused distribution gives characteristic properties of the system which are quite different from those of the Fermi distribution. The strong coupling solution is nothing but the uniform distribution of the electrons in a narrow shell of the momentum space around the Fermi surface. It is only the extreme case of the diffused distribution, so we may expect that it will inform us of some qualitative aspects of the BCS model irrespective of the coupling strength.

In Section 2, the BCS model is solved in the case of the strong coupling. At first the simple strong coupling approximation is performed, the results of which are verified afterwards by means of a method developed by Tomonaga⁹, which is valid under a less stringent condition than the simple strong coupling approximation. Thus the energies of the ground state and the first excited states are obtained in the form of the inverse power series with respect to the coupling constant. They give the finite energy gap. In Section 3, the magnetic interaction of the BCS model is introduced in the gauge invariant way according to the procedure given in the previous paper⁶. The current is calculated in the lowest order approximation of the strong coupling and a peculiar divergence is found. In Section 4, the limit of validity of our result is considered. Investigating the qualitative resemblance

of the BCS variational solution to that of the strong coupling approximation, it may be inferred that the divergence found in Section 3 would not be characteristic of the strong coupling case. It is probable that the interaction current associated with the requirement of the gauge invariance seems to be rather effective also in the weak coupling case. In Appendix 1, detailed properties of second excited states of the strong coupling limit are listed. In Appendix 2, a part of the detailed calculations for the diamagnetic current is given. And in Appendix 3, the various intermediate states which have nonvanishing matrix elements in the expression for the paramagnetic current are listed.

§ 2. Strong coupling approximation of BCS theory

Let us begin with the strong coupling approximation of the BCS theory⁸⁾. In the BCS theory the Hamiltonian is given by

$$H = \sum_{\mathbf{k}s} \omega_{\mathbf{k}} n_{\mathbf{k}s} - V \sum_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}}^* b_{\mathbf{k}'} \quad (2.1)$$

where $\omega_{\mathbf{k}}$ is the Bloch energy and $n_{\mathbf{k}s}$ is the single-particle number operator defined by

$$n_{\mathbf{k}s} = c_{\mathbf{k}s}^* c_{\mathbf{k}s}$$

in terms of electron creation and annihilation operators $c_{\mathbf{k}s}^*$ and $c_{\mathbf{k}s}$. V is the average coupling constant which is inversely proportional to the size of the system and $b_{\mathbf{k}}$ means the annihilation operator of an electron pair with opposite momenta and spins and is defined by

$$b_{\mathbf{k}} = c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}. \quad (2.2)$$

The summation of the second term in (2.1) is restricted to the region

$$\omega_f - \omega < \omega_{\mathbf{k}} < \omega_f + \omega \quad (2.3)$$

with ω_f , the Fermi energy of an electron, and ω , the average phonon frequency.

It is not so easy to obtain a reasonable answer to the problem (2.1) since the electron pair which is annihilated or created by $b_{\mathbf{k}}$ or $b_{\mathbf{k}}^*$ does not obey any definite statistics, that is, it has the main character of a Bose particle but in some respects behaves like a Fermi particle. This may be shown by the following commutation relations,

$$\begin{aligned} [b_{\mathbf{k}}, b_{\mathbf{k}'}^*]_- &= (1 - n_{\mathbf{k}\uparrow} - n_{-\mathbf{k}\downarrow}) \delta_{\mathbf{k}\mathbf{k}'}, \\ [b_{\mathbf{k}}, b_{\mathbf{k}'}]_- &= 0, \\ [b_{\mathbf{k}}, b_{\mathbf{k}'}]_+ &= 2b_{\mathbf{k}} b_{\mathbf{k}'} (1 - \delta_{\mathbf{k}\mathbf{k}'}). \end{aligned} \quad (2.4)$$

These relations can be simplified since the interaction term of the Hamiltonian (2.1) does not give the excitation which breaks up the electron pair. If the system is composed of electrons all in pairs from the start, each state will satisfy the relation

$$b_{\mathbf{k}}^* b_{\mathbf{k}} \Psi = n_{\mathbf{k}\uparrow} n_{-\mathbf{k}\downarrow} \Psi = \frac{1}{2} (n_{\mathbf{k}\uparrow} + n_{-\mathbf{k}\downarrow}) \Psi. \quad (2.5)$$

On account of this, the relations (2.4) can be written as

$$\begin{aligned} [b_{\mathbf{k}}, b_{\mathbf{k}'}]_- &= [b_{\mathbf{k}}, b_{\mathbf{k}'}^*]_- = 0, \quad \text{if } \mathbf{k} \neq \mathbf{k}', \\ [b_{\mathbf{k}}, b_{\mathbf{k}}]_+ &= 0, \quad [b_{\mathbf{k}}, b_{\mathbf{k}}^*]_+ = 1. \end{aligned} \quad (2.6)$$

Now it is easy to obtain the explicit representation of $b_{\mathbf{k}}$ operators which satisfies Eqs. (2.6). It is represented by a matrix

$$b_{\mathbf{k}} = (\sigma_x + i\sigma_y)_{\mathbf{k}}/2 = m_{x\mathbf{k}} + im_{y\mathbf{k}}, \quad (2.7)$$

where $\sigma_{\mathbf{k}}$ is the Pauli spin matrix and $\mathbf{m}_{\mathbf{k}}$ is the "angular momentum" operator. In this way, we have assigned one "spin" to each state of the electron pair. The spin points downwards when the state is occupied by the pair, otherwise it points upwards. The right-hand side of (2.7) is nothing but the operator which turns the direction of the spin \mathbf{k} upwards. If the electron pair is a usual Fermi particle, some sign function is necessary in (2.7). The absence of this is the only difference between our system and the Fermion system.¹⁰⁾

By means of (2.7), one obtains

$$b_{\mathbf{k}}^* b_{\mathbf{k}} = (1/2) - m_{z\mathbf{k}}. \quad (2.8)$$

$$b_{\mathbf{k}'}^* b_{\mathbf{k}} + b_{\mathbf{k}}^* b_{\mathbf{k}'} = 2(m_{x\mathbf{k}'} m_{x\mathbf{k}} + m_{y\mathbf{k}'} m_{y\mathbf{k}}); \quad \text{if } \mathbf{k} \neq \mathbf{k}'. \quad (2.9)$$

The Hamiltonian (2.1) now takes the form

$$H = E_f + N(0)\omega^2 - \sum_{\mathbf{k}} 2\varepsilon_{\mathbf{k}} m_{z\mathbf{k}} + (V - 2\omega_f) M_z - V(\mathbf{M}^2 - M_z^2) \quad (2.10)$$

where E_f is the Fermi energy of the system, $N(0)$ the density of the Bloch states of one spin per unit energy at the Fermi surface, $\varepsilon_{\mathbf{k}}$ the Bloch energy measured relative to the Fermi surface defined by $\varepsilon_{\mathbf{k}} = \omega_{\mathbf{k}} - \omega_f$ and $\mathbf{M} = \sum_{\mathbf{k}} \mathbf{m}_{\mathbf{k}}$ is the total angular momentum of our spin system. M_z is evidently a constant of motion which is connected with the total number of the electron pairs. If one takes the number of pair states twice as much as the number of electron pairs, the value of M_z can be shown to vanish by means of (2.8). In this case the Hamiltonian may be further simplified and is given by

$$H = E_f + N(0)\omega^2 - \sum_{\mathbf{k}} 2\varepsilon_{\mathbf{k}} m_{z\mathbf{k}} - V\mathbf{M}^2. \quad (2.11)$$

First we shall consider the problem in a classical way, treating the spin as a classical quantity, a vector with magnitude one half. It is necessary to inquire what arrangement of the spin directions gives the minimum value for the Hamiltonian (2.11). To this end, we write

$$m_{x\mathbf{k}} = (1/2) \sin \theta_{\mathbf{k}} \cos \varphi_{\mathbf{k}} \quad (2.12)$$

$$m_{y\mathbf{k}} = (1/2) \sin \theta_{\mathbf{k}} \sin \varphi_{\mathbf{k}} \quad (2.13)$$

$$m_{z\mathbf{k}} = (1/2) \cos \theta_{\mathbf{k}} \quad (2.14)$$

and substitute these expressions into (2.11). $\theta_{\mathbf{k}}$'s and $\varphi_{\mathbf{k}}$'s are determined by the variation under the condition $\sum_{\mathbf{k}} \cos \theta_{\mathbf{k}} = 0$. Introducing the undetermined multiplier λ , one may use a quantity

$$\begin{aligned} \mathcal{H} &= H - \lambda \sum_{\mathbf{k}} \cos \theta_{\mathbf{k}} \\ &= E_f + N(0) \omega^2 - \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} + \lambda) \cos \theta_{\mathbf{k}} - V/4 \cdot \sum_{\mathbf{k}, \mathbf{k}'} \sin \theta_{\mathbf{k}} \sin \theta_{\mathbf{k}'} \cos (\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}'}). \end{aligned} \quad (2.15)$$

The angles $\theta_{\mathbf{k}}$, $\varphi_{\mathbf{k}}$ are now determined by the equations

$$0 = \frac{\partial \mathcal{H}}{\partial \theta_{\mathbf{k}}} = (\varepsilon_{\mathbf{k}} + \lambda) \sin \theta_{\mathbf{k}} - \frac{V}{2} \cos \theta_{\mathbf{k}} \sum_{\mathbf{k}'} \sin \theta_{\mathbf{k}'} \cos (\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}'}) \quad (2.16)$$

$$0 = \frac{\partial \mathcal{H}}{\partial \varphi_{\mathbf{k}}} = \frac{V}{2} \sin \theta_{\mathbf{k}} \sum_{\mathbf{k}'} \sin \theta_{\mathbf{k}'} \sin (\varphi_{\mathbf{k}} - \varphi_{\mathbf{k}'}). \quad (2.17)$$

There are three types of solutions for (2.17), that is, $\sin \theta_{\mathbf{k}} = 0$, $\varphi_{\mathbf{k}} = \varphi^0$ and $\varphi_{\mathbf{k}} = \varphi^0 + \pi$ where φ^0 is a constant independent of \mathbf{k} . The first solution $\sin \theta_{\mathbf{k}} = 0$ will not give the energy minimum since the direction of spin \mathbf{k} is the same as in the case of the Fermi distribution. The second solution $\varphi_{\mathbf{k}} = \varphi^0$ gives a smaller value for (2.15) than what is given by the third solution. Thus it becomes apparent that all the spins are arranged in a plane which contains the z axis. The angle $\theta_{\mathbf{k}}$ between the direction of the spin \mathbf{k} and that of the z axis is given by

$$\cos \theta_{\mathbf{k}} = \varepsilon_{\mathbf{k}} / \sqrt{\varepsilon_{\mathbf{k}}^2 + \varepsilon_0^2}, \quad (2.18)$$

by means of (2.16). Here, ε_0 denotes the quantity

$$\varepsilon_0 = \omega / \sinh(1/VN(0)). \quad (2.19)$$

The distribution of the directional cosines of the spins is now shown to be described in terms of BCS' distribution function $h_{\mathbf{k}}$. Using (2.18), the Hamiltonian (2.15) takes the value

$$\mathcal{H} = E_f - 2N(0) \omega^2 / [\exp(2/VN(0)) - 1], \quad (2.20)$$

which is nothing but the value given by BCS' variational method.

Now let us return to quantum theory and study the strong coupling approximation. The Schrödinger equation of our system may be written in the form

$$(H_0 + H_{\text{int}}) \Phi = W \Phi, \quad (2.21)$$

with

$$H_0 = -V \mathbf{M}^2, \quad (2.22)$$

$$H_{\text{int}} = - \sum_{\mathbf{k}} 2\varepsilon_{\mathbf{k}} m_{z\mathbf{k}},$$

$$W = E - E_f - N(0) \omega^2, \quad (2.23)$$

where E is the energy of the system.

It is quite easy to obtain the eigenvalues and eigenfunctions of H_0 . If one defines the function $\phi(J, K)$ by the equations

$$\mathbf{M}^2 \phi(J, K) = J(J+1) \phi(J, K), \quad (2.24)$$

$$M_z \phi(J, K) = K \phi(J, K), \quad (2.25)$$

the eigenfunction of H_0 is given by $\phi(J, 0)$ which belongs to the eigenvalue $-VJ(J+1)$.

In the case of the ground state, J takes the largest value $N(0)\omega$ which is taken when all spins are arranged upwards at the same time. We shall denote this value by M from now on.

$$M \equiv N(0)\omega. \quad (2.26)$$

The ground state energy E_M and the eigenfunction of H_0 is, thus, given by

$$E_M = -VM(M+1), \quad (2.27)$$

$$\phi(M, 0) = \frac{1}{\sqrt{(2M)!}} Y^M |0\rangle, \quad (2.28)$$

where $Y = \sum_{\mathbf{k}} (m_{x\mathbf{k}} - im_{y\mathbf{k}})$ and $|0\rangle$ means the vacuum state which is represented by

$$|0\rangle = \phi(M, M) = \prod_{\mathbf{k}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\mathbf{k}}. \quad (2.29)$$

By means of (2.7), $\phi(M, 0)$ may be expressed in an alternative form

$$\phi(M, 0) = \frac{1}{\sqrt{(2M)!}} \left(\sum_{\mathbf{k}} b_{\mathbf{k}}^* \right)^M |0\rangle, \quad (2.30)$$

that means a kind of uniform distribution in which every configuration is mixed up with equal probability. This is an extreme case contrary to the Fermi distribution.

The first excited states of H_0 are degenerate in $2M-1$ fold, whose eigenfunctions take the form

$$\phi_i(M-1, 0) = \frac{1}{\sqrt{(2M-2)!}} Y^{M-1} \phi_i(M-1, M-1), \quad i=1, \dots, 2M-1, \quad (2.31)$$

where

$$\phi_i(M-1, M-1) = \sum_{\mathbf{k}} \varphi_i(\mathbf{k}) b_{\mathbf{k}}^* |0\rangle. \quad (2.32)$$

These are orthogonal to $\phi(M, M-1)$ which leads to the condition

$$\sum_{\mathbf{k}} \varphi_i(\mathbf{k}) = 0. \quad (2.33)$$

Thus the energy and the eigenfunction of the first excited states take the forms

$$E_{M-1} = -V(M-1)M, \quad (2.34)$$

$$\phi_i(M-1, 0) = \frac{1}{\sqrt{(2M-2)!}} Y^{M-1} \sum_{\mathbf{k}} \varphi_i(\mathbf{k}) b_{\mathbf{k}}^* |0\rangle, \quad i=1, \dots, 2M-1. \quad (2.35)$$

The orthonormalization condition among these states leads to

$$\delta_{ij} = \sum_{\mathbf{k}=1}^{2M} \varphi_i^*(\mathbf{k}) \varphi_j(\mathbf{k}), \quad (2.36)$$

and the completeness condition for the subspace with $J=M-1$ to

$$\sum_{i=1}^{2M-1} \varphi_i^*(\mathbf{k}) \varphi_i(\mathbf{k}') = \delta_{\mathbf{k}\mathbf{k}'} - (1/2M). \quad (2.37)$$

We shall remark here the existence of the finite energy gap between the ground and the first excited states. By means of (2.27) and (2.34) the energy difference is shown to be

$$E_{M-1} - E_M = 2VM = 2VN(0)\omega \quad (2.38)$$

which does not vanish even when the normalization volume becomes infinite.

Similar discussions can be applied to the second excited states, the results of which will be summarized in Appendix 1. Now we will consider the effects of the H_{int} term for the energy values of the ground and the first excited states. For this purpose one may use the perturbation formulas which are given by

$$\begin{aligned} W &= E_J + \langle H_{\text{int}} U \rangle_J, \\ U &= \sum_{n=0}^{\infty} U^{(n)}, \\ U^{(n+1)} &= \frac{1}{a} H_{\text{int}} U^{(n)} - \sum_{m=0}^n \langle H_{\text{int}} U^{(m)} \rangle_J \frac{1}{a} U^{(n-m)}, \\ U^{(0)} &= 1, \quad \frac{1}{a} = \frac{1-P}{E_J - H_0}, \end{aligned} \quad (2.39)$$

where $\langle \dots \rangle_J$ means the expectation value in the unperturbed state $\psi(J, 0)$ and P is the projection operator to $\psi(J, 0)^{(1)}$. By means of these formulas, the ground state energy will be obtained in the form of the inverse power series of the coupling constant V as follows:

$$W_G = E_M + \langle H_{\text{int}} U \rangle_M = E_M + \sum_{n=1}^{\infty} A^{(n)} W_G. \quad (2.40)$$

The first order correction of the energy is given by

$$A^{(1)} W_G = \langle \psi(M, 0) | H_{\text{int}} | \psi(M, 0) \rangle, \quad (2.41)$$

by means of (2.39). This quantity vanishes since the expectation value of $m_{z\mathbf{k}}$ in the ground state does not depend on \mathbf{k} due to the uniform distribution of the system.

In order to find the second order correction

$$A^{(2)} W_G = \langle \psi(M, 0) | H_{\text{int}} \frac{1}{a} H_{\text{int}} | \psi(M, 0) \rangle, \quad (2.42)$$

one may use the relation

$$H_{\text{int}}\psi(M, 0) = \sqrt{\frac{2M}{2M-1}} \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_i^*(\mathbf{k}) \psi_i(M-1, 0), \quad (2.43)$$

which is obtained from (2.23), (2.28), (2.35) and (2.37). Then the right-hand side of (2.42) can be easily calculated and gives the result

$$\Delta^{(2)} W_G = -\frac{1}{3x} N(0) \omega^2 - \frac{1}{6x} \omega, \quad (2.44)$$

in which the terms proportional to M^{-1} are neglected and a simplified notation x is introduced in place of $VN(0)$ which is larger than unity in the case of strong coupling, i. e.,

$$x \equiv VN(0).$$

Following these procedures, calculation is performed up to fifth order, employing the various properties of the second excited states summarized in Appendix 1. The third and the fifth order corrections vanish again. Finally the ground state energy is found to be

$$\begin{aligned} E_G &= E_f + N(0) \omega^2 + W_G \\ &= E_f - \left[x - 1 + \frac{1}{3x} - \frac{1}{45x^3} + O\left(\frac{1}{x^5}\right) \right] N(0) \omega^2 - \left[x + \frac{1}{6x} - \frac{1}{24x^3} + O\left(\frac{1}{x^5}\right) \right] \omega. \end{aligned} \quad (2.45)$$

We shall compare this result with that obtained by BCS (2.20). When the latter is expanded in the inverse power series of x , it can be easily seen that their result just coincides with the right-hand side of Eq. (2.45) except the last term. This last term is not proportional to the total number of electrons but is essential in the argument of the excitation of the system. Since BCS has discarded this term entirely, their discussion about the energy gap of the system is not so reliable as the argument of the ground state energy.

Now, we will consider the next problem, the energy of the first excited states. It is rather complicated on account of the degeneracy of unperturbed states. It will be found that the first order energy correction vanishes by making use of the relation

$$\begin{aligned} H_{\text{int}}\psi_j(M-1, 0) &= \sqrt{\frac{2M}{2M-1}} \sum_{\mathbf{k}} \varphi_j(\mathbf{k}) \varepsilon_{\mathbf{k}} \psi(M, 0) \\ &+ 2 \sqrt{\frac{2M-2}{2M-3}} \sum_{\mathbf{k}\mathbf{k}'} \varepsilon_{\mathbf{k}} \varphi_j(\mathbf{k}') \varphi_i^*(\mathbf{k}, \mathbf{k}') \psi_i(M-2, 0), \end{aligned} \quad (2.46)$$

which is derived from (2.35) and (A1.3). Therefore the degeneracy of the states remains unchanged. Second order energy correction $\Delta^{(2)} W_{\text{ex}}$ is given by

$$\Delta^{(2)} W_{\text{ex}} \delta_{ij} = (\psi_i(M-1, 0) | H_{\text{int}} \frac{1}{a} H_{\text{int}} | \psi_j(M-1, 0)), \quad (2.47)$$

which is to be calculated making use of (2.46) and (A1.5) and gives the result

$$\begin{aligned} A^{(2)} W_{ex} \delta_{ij} = & -\frac{1}{(2M-3)V} \cdot \frac{2M}{3} \omega^2 \delta_{ij} - \frac{1}{(M-1)(2M-3)V} \sum_{\mathbf{k}, \mathbf{k}'} \varepsilon_{\mathbf{k}} \varepsilon_{\mathbf{k}'} \varphi_i^*(\mathbf{k}) \varphi_j(\mathbf{k}') \\ & + \frac{2M}{(M-1)(2M-3)V} \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}^2 \varphi_i^*(\mathbf{k}) \varphi_j(\mathbf{k}). \end{aligned} \quad (2.48)$$

Multiplying $\varphi_i(\mathbf{k})$ on both sides of (2.48), summing up with respect to the index i and noticing (2.37), one obtains the relations

$$\sum_{\mathbf{k}'} \{ (z - 2M\varepsilon_{\mathbf{k}}^2) \delta_{\mathbf{k}\mathbf{k}'} + \varepsilon_{\mathbf{k}} \varepsilon_{\mathbf{k}'} + \varepsilon_{\mathbf{k}'}^2 \} \varphi_j(\mathbf{k}') = 0, \quad \mathbf{k} = 1, \dots, 2M, \quad (2.49)$$

where z denotes the quantity

$$z = (M-1)(2M-3)V A^{(2)} W_{ex} + \frac{2}{3}(M-1)M\omega^2. \quad (2.50)$$

The relations (2.49) and (2.33) compose the system of $(2M+1)$ coupled homogeneous linear equations which determines the values of $2M$ unknowns, $\varphi_j(\mathbf{k})$ and $A^{(2)} W_{ex}$. In order to give a set of significant solutions, the rank of a matrix formed by the coefficients of the equations must be smaller than $2M$. The condition for this takes the form

$$z \left(\sum_{\mathbf{k}} \frac{1}{z - 2M\varepsilon_{\mathbf{k}}^2} \right)^2 \prod_{\mathbf{k}} (z - 2M\varepsilon_{\mathbf{k}}^2) = 0. \quad (2.51)$$

The left-hand side of (2.51) is a polynomial of order $2M-1$ in z . We shall show that $2M-1$ roots of (2.51) are all real, by counting the number and degeneracy of real roots.

In order to simplify the discussions we will introduce a set of notations E_i instead of ε_i as follows. Arrange a set of numbers $\varepsilon_1^2, \dots, \varepsilon_{2M}^2$ according to their magnitude and give notation $E_0^2 (=0), E_1^2, \dots, E_n^2$ to each magnitude of them. If the number of $\varepsilon_{\mathbf{k}}^2$'s which have the magnitude E_i^2 is N_i , one evidently obtains

$$\sum_{i=0}^n N_i = 2M.$$

Moreover, N_i is not smaller than two, since $\varepsilon_{\mathbf{k}} = \varepsilon_{-\mathbf{k}}$ and $\mathbf{k} \neq 0$. When a dimensionless quantity Y is introduced by

$$z = 2M\omega^2 Y, \quad (2.52)$$

the values of Y which satisfy Eq. (2.51) are found to be

$$\begin{aligned} Y=0, & \quad N_0-1 \text{ fold degenerate roots,} \\ Y=(E_{\mathbf{k}}/\omega)^2, & \quad N_{\mathbf{k}}-2 \text{ fold degenerate roots, } \mathbf{k}=1, \dots, n, \\ Y=Y_p, & \quad \text{double roots, } p=1, \dots, n, \end{aligned} \quad (2.53)$$

where Y_p is defined by the equation

$$\sum_{i=0}^n \frac{N_i}{Y_p - (E_i/\omega)^2} = 0. \quad (2.54)$$

In this way $2M-1$ real roots of (2.51) are obtained and thus it is verified that Eq. (2.51) has only real roots. The degeneracy of the first excited states are removed in part by the second order perturbation; the largest value of Y is unity and the smallest zero. Since the third order correction vanishes again, the energies of the first excited states now take the values

$$\begin{aligned} E_{\text{ex}} &= E_f + N(0)\omega^2 + W_{\text{ex}} \\ &= E_f - \left[x - 1 + \frac{1}{3x} + O\left(\frac{1}{x^3}\right) \right] N(0)\omega^2 + \left[x + \frac{2Y-1}{2x} + O\left(\frac{1}{x^3}\right) \right] \omega. \end{aligned} \quad (2.55)$$

The so-called "energy gap" of the system is the energy difference between the ground state and the lowest energy state among the first excited states whose Y takes the value zero. Thus the energy gap is given by

$$E.G. = (E_{\text{ex}})_{Y=0} - E_G = \left[2x - \frac{1}{3x} + O\left(\frac{1}{x^3}\right) \right] \omega \quad (2.56)$$

which is again identical with the value given by BCS in the strong coupling limit. This identification seems to be accidental, since BCS' treatment is not accurate in this order as stated before.

We have so far discussed only the case of pair excitations but it is possible to consider the excitations which break up some electron pairs in the strong coupling approximation. Since the pair states which are occupied by an electron of the broken pair lose the correlations with other states, one may still assign spins to other electron pair states and make discussions similar to those developed above. For instance, if there are two electrons one of which is in a state $-\mathbf{k}_1 \downarrow$, and the other in a state $\mathbf{k}_2 \uparrow$, ($\mathbf{k}_1 \neq \mathbf{k}_2$), the Hamiltonian (2.1) takes a form

$$H = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} + \sum_{\mathbf{k}}' 2\omega_{\mathbf{k}} b_{\mathbf{k}}^* b_{\mathbf{k}} - V \sum_{\mathbf{k}\mathbf{k}'}' b_{\mathbf{k}}^* b_{\mathbf{k}'}, \quad (2.57)$$

in which the summation does not extend over the states \mathbf{k}_1 and \mathbf{k}_2 . We may again introduce the operator $\mathbf{m}_{\mathbf{k}}$ defined by (2.7), and the total angular momentum \mathbf{M}' of the spin system by

$$\mathbf{M}' = \sum_{\mathbf{k}}' \mathbf{m}_{\mathbf{k}}. \quad (2.58)$$

The Hamiltonian (2.57) will then be written as

$$H = E_f + N(0)\omega^2 - \sum_{\mathbf{k}}' 2\varepsilon_{\mathbf{k}} m_{z\mathbf{k}} - V\mathbf{M}'^2, \quad (2.59)$$

with the condition $M_z' = 0$. The ground state solution of (2.59) in the strong coupling limit is found to be

$$\phi = \frac{1}{\sqrt{(2M-2)!}} Y_2^{M-1} C_{\mathbf{k}_1\downarrow}^* C_{\mathbf{k}_2\uparrow}^* |0\rangle, \quad (2.60)$$

with $Y_2 = \sum_{\mathbf{k}} b_{\mathbf{k}}^*$, giving the energy

$$E = E_f - (x-1)N(0)\omega^2 + x\omega, \quad (2.61)$$

which turns out to be identical in the lowest order approximation of strong coupling with the result of pair excitation (2.55). Similar discussions can be made for various types of excitations in which many electron pairs are broken up. Thus we have obtained the energy gap for the case of single electron excitations.

There may be a criticism about the validity of the strong coupling approximation. That is, in order to secure the convergence of the perturbation series in (2.39) we must first satisfy the condition

$$\frac{1}{(2VM)^2} \sum_i |\langle \psi_i(M-1, 0) | H_{int} | \psi(M, 0) \rangle|^2 \ll 1. \quad (2.62)$$

The left-hand side of (2.62) can be calculated by making use of (2.43) and (2.37), giving rise to the condition $M/6x^2 \ll 1$, or

$$x \gg \sqrt{M} \quad (2.63)$$

which is never satisfied however strong the coupling may be, when the size of the system becomes infinite. In spite of this fact, the expression for the ground state energy (2.45) may give a reasonable result if the coupling is so strong that the size independent condition $x \gg 1$ is satisfied. This is because (2.45) is a simple power series in x^{-1} . Actually we can confirm this anticipation because we can derive the same result by means of an approximation method developed by Tomonaga which is valid for $x \gg 1^9$. Thus the energy eigenvalue given by (2.45) is correct even though the condition (2.62) is not satisfied. As for the eigenfunctions, however, the convergence of the perturbation series is very bad. It is a power series of \sqrt{M}/x but not of x^{-1} , and one must carry out the perturbation calculation up to the (M/x^2) -th order in order to obtain a sensible eigenfunction. We shall quote here the lecture of Tomonaga that gives another method to solve (2.21) in the strong coupling case. It will be shown that this method is valid under the size independent condition $x^{-1} \ll 1$.

We shall consider only the pair excitations; single particle excitations may be treated in a similar manner. Then the Schrödinger function in general takes the form

$$\varphi = \sum_{n=0}^M \frac{1}{\sqrt{n!} (2M-2n)!} Y^{M-n} \sum_{\mathbf{k}_1 \dots \mathbf{k}_n} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n) b_{\mathbf{k}_1}^* \dots b_{\mathbf{k}_n}^* |0\rangle \quad (2.64)$$

which can be inferred from the forms (2.28), (2.35) and (A1.3). Here φ_n 's are symmetric functions which vanish when any two variables take a common value and satisfy a set of subsidiary conditions

$$\sum_{\mathbf{k}} \varphi_n(\mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_{n-1}) = 0, \quad \mathbf{k}_1, \dots, \mathbf{k}_{n-1} = 1, \dots, 2M. \quad (2.65)$$

This is a generalization of relations (2.33) and (A1.3) which are imposed from orthogonality conditions. Eq. (2.65) can be verified rather easily when one notices the facts that (2.65) does lead to the required orthogonality conditions and it is composed of a set of linearly independent relations as many as the orthogonality conditions. Since the latter conditions are linear with respect to φ_n , Eq. (2.65) is thus shown to be necessary and sufficient.

The equations for φ_n -functions can be derived from (2.21) in a general way. It is easy to obtain the following relation

$$\begin{aligned} & \sum_{\mathbf{k}, \mathbf{k}_1, \dots, \mathbf{k}_n} \varepsilon_{\mathbf{k}} m_{z\mathbf{k}} Y^{M-n} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n) b_{\mathbf{k}}^* \dots b_{\mathbf{k}_n}^* |0\rangle \\ &= -Y^{M-n} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_n} (\varepsilon_{\mathbf{k}_1} + \dots + \varepsilon_{\mathbf{k}_n}) \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n) b_{\mathbf{k}_1}^* \dots b_{\mathbf{k}_n}^* |0\rangle \\ & - \frac{M-n}{n+1} Y^{M-n-1} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{n+1}} \{ \varepsilon_1 \varphi_n(\mathbf{k}_2 \dots \mathbf{k}_{n+1}) \\ & + \varepsilon_2 \varphi_n(\mathbf{k}_1 \mathbf{k}_3 \dots \mathbf{k}_{n+1}) + \dots + \varepsilon_{n+1} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n) \} b_{\mathbf{k}_1}^* \dots b_{\mathbf{k}_{n+1}}^* |0\rangle. \quad (2.66) \end{aligned}$$

The coefficient of the first term on the right-hand side of (2.66) may be written as

$$\begin{aligned} & (\varepsilon_{\mathbf{k}_1} + \varepsilon_{\mathbf{k}_2} + \dots + \varepsilon_{\mathbf{k}_n}) \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n) = \left[(\varepsilon_{\mathbf{k}_1} + \dots + \varepsilon_{\mathbf{k}_n}) \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n) \right. \\ & \left. - \frac{1}{2(M-n+1)} \sum_{i=1}^n \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_{i-1} \mathbf{k} \mathbf{k}_{i+1} \dots \mathbf{k}_n) \right] \\ & + \frac{1}{2(M-n+1)} \sum_{i=1}^n \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_{i-1} \mathbf{k} \mathbf{k}_{i+1} \dots \mathbf{k}_n). \quad (2.67) \end{aligned}$$

The first term gives an amplitude of a state whose magnitude of \mathbf{M} is $M-n$, whereas the second term is associated with a state whose magnitude of \mathbf{M} is $M-n+1$. On the other hand the coefficient of the second term of (2.66) takes the form

$$\begin{aligned} & \varepsilon_1 \varphi_n(\mathbf{k}_2 \dots \mathbf{k}_{n+1}) + \varepsilon_2 \varphi_n(\mathbf{k}_1 \mathbf{k}_3 \dots \mathbf{k}_{n+1}) + \dots + \varepsilon_{n+1} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n) \\ &= \left[\varepsilon_1 \varphi_n(\mathbf{k}_2 \dots \mathbf{k}_{n+1}) + \varepsilon_2 \varphi_n(\mathbf{k}_1 \mathbf{k}_3 \dots \mathbf{k}_{n+1}) + \dots + \varepsilon_{n+1} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n) \right. \\ & + \frac{1}{M-n} \sum_{i=1}^{n+1} (\varepsilon_1 + \dots + \varepsilon_{n+1} - \varepsilon_i) \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_{i-1} \mathbf{k}_{i+1} \dots \mathbf{k}_{n+1}) \\ & - \frac{1}{2(M-n)(2M-2n+1)} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n+1} \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_n(\mathbf{k} \mathbf{k}_1 \dots \mathbf{k}_{i-1} \mathbf{k}_{i+1} \dots \mathbf{k}_{j-1} \mathbf{k}_{j+1} \dots \mathbf{k}_{n+1}) \Big] \\ & - \frac{1}{M-n} \sum_{i=1}^{n+1} \left[(\varepsilon_1 + \dots + \varepsilon_{n+1} - \varepsilon_i) \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_{i-1} \mathbf{k}_{i+1} \dots \mathbf{k}_{n+1}) \right. \\ & - \frac{1}{2(M-n+1)} \sum_{\substack{j=1 \\ (j \neq i)}}^{n+1} \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_{j-1} \mathbf{k} \mathbf{k}_{j+1} \dots \mathbf{k}_{i-1} \mathbf{k}_{i+1} \dots \mathbf{k}_{n+1}) \Big] \\ & - \frac{1}{2(2M-2n+1)(M-n+1)} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n+1} \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_{j-1} \mathbf{k} \mathbf{k}_{j+1} \dots \mathbf{k}_{i-1} \mathbf{k}_{i+1} \dots \mathbf{k}_{n+1}). \quad (2.68) \end{aligned}$$

The first term corresponds to a state with magnitude of $M=M-n-1$, the second term to a state with $M-n$ and the third to a state with $M-n+1$.

Substituting (2.67) and (2.68) into (2.66), one obtains

$$\begin{aligned} & \sum_{\mathbf{k}\mathbf{k}_1\cdots\mathbf{k}_n} \varepsilon_{\mathbf{k}} m_{z\mathbf{k}} Y^{M-n} \varphi_n(\mathbf{k}_1\cdots\mathbf{k}_n) b_{\mathbf{k}_1}^* \cdots b_{\mathbf{k}_n}^* |0\rangle \\ &= -\frac{M-n}{n+1} Y^{M-n-1} \sum_{\mathbf{k}_1\cdots\mathbf{k}_{n+1}} (\text{The first term of (2.68)}) b_{\mathbf{k}_1}^* \cdots b_{\mathbf{k}_{n+1}}^* |0\rangle \\ & \quad - \frac{n Y^{M-n+1}}{2(2M-2n+1)} \sum_{\mathbf{k}\mathbf{k}_1\cdots\mathbf{k}_{n-1}} \varepsilon_{\mathbf{k}} \varphi_n(\mathbf{k}\mathbf{k}_1\cdots\mathbf{k}_{n-1}) b_{\mathbf{k}_1}^* \cdots b_{\mathbf{k}_{n-1}}^* |0\rangle. \end{aligned} \quad (2.69)$$

For $n=M$ the above relation must be slightly changed, which is not considered here, however, since it makes no difference for the following discussions. If we insert the expression (2.64) into (2.21) and compare on the both sides of (2.21) the coefficients of each configuration with a definite M^2 value, we will obtain the following relations among the φ_n -functions,

$$\begin{aligned} & (E_{M-n} - W) \varphi_n(\mathbf{k}_1\cdots\mathbf{k}_n) + \sqrt{\frac{2(M-n)}{(n+1)(2M-2n-1)}} \sum_{i=1}^{n+1} \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_{n+1}(\mathbf{k}_1\cdots\mathbf{k}_{i-1}\mathbf{k}\mathbf{k}_i\cdots\mathbf{k}_n) \\ & + \sqrt{\frac{2M-2n+2}{n(2M-2n+1)}} \{ \varepsilon_{\mathbf{k}_1} \varphi_{n-1}(\mathbf{k}_2\cdots\mathbf{k}_n) + \cdots + \varepsilon_{\mathbf{k}_n} \varphi_{n-1}(\mathbf{k}_1\cdots\mathbf{k}_{n-1}) \} \\ & + \frac{2}{\sqrt{n(2M-2n+1)(2M-2n+2)}} \left\{ \sum_{i=1}^n (\varepsilon_{\mathbf{k}_1} + \cdots + \varepsilon_{\mathbf{k}_n} - \varepsilon_{\mathbf{k}_i}) \varphi_{n-1}(\mathbf{k}_1\cdots\mathbf{k}_{i-1}\mathbf{k}\mathbf{k}_{i+1}\cdots\mathbf{k}_n) \right\} \\ & - \frac{1}{2(2M-2n+3)} \sum_{\substack{i,j=1 \\ (i \neq j)}}^n \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_{n-1}(\mathbf{k}_1\cdots\mathbf{k}_{i-1}\mathbf{k}\mathbf{k}_{i+1}\cdots\mathbf{k}_{j-1}\mathbf{k}\mathbf{k}_{j+1}\cdots\mathbf{k}_n) \} = 0, \end{aligned} \quad (2.70)$$

for $n=0, 1, \dots, M-1$.

These equations can be solved approximately if we may assume that only φ_n 's with $n \ll M$ play a dominant role. In this case we may neglect the ratio n/M in (2.70) and these equations will be reduced into simpler forms

$$\begin{aligned} & \frac{1}{\sqrt{n}} [\varepsilon_{\mathbf{k}_1} \varphi_{n-1}(\mathbf{k}_2\cdots\mathbf{k}_n) + \cdots + \varepsilon_{\mathbf{k}_n} \varphi_{n-1}(\mathbf{k}_1\cdots\mathbf{k}_{n-1})] + (2nMV + E_M - W) \varphi_n(\mathbf{k}_1\cdots\mathbf{k}_n) \\ & + \frac{1}{\sqrt{n+1}} \left[\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_{n+1}(\mathbf{k}\mathbf{k}_1\cdots\mathbf{k}_n) + \cdots + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \varphi_{n+1}(\mathbf{k}_1\cdots\mathbf{k}_n\mathbf{k}) \right] = 0. \end{aligned} \quad (2.71)$$

One is able to obtain the solution of these equations in the form

$$\varphi_n = C_n \{ \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n - 1/N \cdot (\varepsilon_2^2 \varepsilon_3 \cdots \varepsilon_n + \varepsilon_2 \varepsilon_3^2 \cdots \varepsilon_n + \cdots) \}, \quad (2.72)$$

in which the second and the following terms are considered in order to satisfy the requirements (2.65). However, here again these terms may be neglected since they are nearly n/M times smaller than the first term. Substituting the first terms of (2.72) into (2.71) one obtains the relations among C_n

$$\sqrt{n} C_{n-1} + (2nMV + E_M - W) C_n + \sqrt{n+1} \frac{2}{3} M \omega^2 C_{n+1} = 0. \quad (2.73)$$

Obviously (2.73) have a solution

$$C_n = \frac{C}{\sqrt{n!}} \left(-\frac{1}{2MV} \right)^n,$$

$$W = E_M - \omega^2/3V, \quad (2.74)$$

where C is an arbitrary factor determined by the normalization condition. One obtains the energy of the system from (2.23) and (2.74)

$$E = E_f + N(0)\omega^2 + E_M - \omega^2/3V$$

$$= E_f - [x - 1 + 1/3x]N(0)\omega^2 - x\omega \quad (2.75)$$

which has the value essentially agreeing with the result of the simple strong coupling approximation (2.45). In order to find C we calculate the normalization integral

$$1 = (\Psi\Psi) = \sum_{n=0}^M \sum_{\mathbf{k}_1 \dots \mathbf{k}_n} \varphi_n^*(\mathbf{k}_1 \dots \mathbf{k}_n) \varphi_n(\mathbf{k}_1 \dots \mathbf{k}_n)$$

$$= C^2 \sum_{n=0}^M \frac{1}{n!} \left(\frac{1}{2MV} \right)^{2n} (\sum \epsilon_{\mathbf{k}}^2)^n = C^2 \exp \left(\frac{\omega^2}{6MV^2} \right).$$

Therefore

$$C = \exp \left(-\frac{\omega^2}{12MV^2} \right) = \exp \left(-\frac{M}{12x^2} \right) \quad (2.76)$$

and the approximate eigenfunction is

$$\varphi_n(\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_n) = \exp \left(-\frac{M}{12x^2} \right) \cdot \frac{1}{\sqrt{n!}} (-1)^n \left(\frac{M}{6x^2} \right)^{n/2} \cdot \left(\frac{2}{3} M \omega^2 \right)^{-n/2} \epsilon_1 \epsilon_2 \dots \epsilon_n.$$

$$(2.77)$$

One sees that C becomes very small when the size of the system becomes large and, as will be seen in (2.78) below, higher configurations become more and more important in such a case. This result just corresponds to the situation discussed below (2.63).

The probability with which the system exists in a state φ_n is given by

$$P_n = \frac{1}{n!} \left(\frac{M}{6x^2} \right)^n \exp \left(-\frac{M}{6x^2} \right), \quad (2.78)$$

then the mean value of n is given by

$$\bar{n} = \sum_{n=0}^M n P_n \simeq M/6x^2. \quad (2.79)$$

In order that our approximation be valid it is necessary and sufficient that only φ_n 's with $n \ll M$ play a dominant role. Now we find from (2.78) that the system has the Poisson distribution with respect to these configurations. Therefore the above assumption will be valid if

$$\bar{n} \ll M, \quad (2.80)$$

which gives

$$x \gg 1. \quad (2.81)$$

If this condition is satisfied the eigenfunction has a negligibly small value for $n \approx M-1$, so that the boundary condition at $n=M-1$ is automatically satisfied. Thus our approximation is shown to be valid in the strong coupling case under the condition (2.81) which is independent of the size of the system. Thus it is known that the two treatments, the simple strong coupling approximation in Section 2 and the more advanced one discussed here, give the same answer with respect to the energy eigenvalue in spite of the fact that the former requires the more stringent condition (2.63). From (2.79) we can infer that, if we carry through the higher order perturbation calculation up to the order of M/x^2 , we would obtain the result identical with the result obtained here. Further our result (2.77) shows that the perturbation series for the eigenfunction will be a power series of \sqrt{M}/x but not of x^{-1} .

Our description of the system of electron pairs in terms of spins associated with each pair state may not give any mathematical advantage in the case of weak coupling, but provides us with some valuable physical pictures about what is going on in such a system. In the strong coupling limit treated above, the spin-spin interactions are predominant just as in ferromagnetism. The third term in Eq. (2.11) is interpreted as the interaction between spins and the external magnetic field which varies from spin to spin. It seems at first sight that the latter interaction may be predominant in the weak coupling limit, but this is not the case, since, however weak the coupling may be, the spin-spin interactions close to the Fermi surface always surpass this interaction at $\varepsilon_k \sim 0$. Thus we may get a finite energy gap also in the weak coupling case.

§ 3. The Meissner effect in the strong coupling limit

We shall now consider the magnetic interaction of the system in a gauge invariant way. Since the interaction energy between electrons (2.1) depends on the velocity of electrons, the magnetic interaction is to be modified according to the procedure given in our previous paper⁹⁾. It is true that there are some ambiguities in this procedure associated with the choice of the path for the line integral of vector potential. However, it seems most natural to choose a straight path for this, not only because of simplicity but also of the fact that it will correspond to taking the magnetic interaction in the "well-ordered" form¹³⁾.

Now, the Hamiltonian (2.1) may be written in the form

$$H = \frac{1}{2m} \int \nabla \varphi^*(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) d\mathbf{x} - \frac{V}{4} \int \varphi^*(\mathbf{x}) V(\mathbf{x}-\mathbf{y}) \sigma_y \varphi^*(\mathbf{y}) d\mathbf{x} d\mathbf{y} \\ \cdot \int \varphi(\mathbf{u}) V(\mathbf{u}-\mathbf{v}) \sigma_y \varphi(\mathbf{v}) d\mathbf{u} d\mathbf{v}. \quad (3.1)$$

where m is the electron mass, $\varphi(\mathbf{x})$ the electron wave field, σ_y the y -component of Pauli's spin matrix and $V(\mathbf{x})$ is a function defined by

$$V(\mathbf{x}) = 1/\Omega \cdot \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (3.2)$$

Ω is the normalization volume and the summation is extended only over those \mathbf{k} 's which satisfy the condition (2.3). Consequently, the electric current is found according to I to be given by

$$j_\mu(\mathbf{x}) = j_\mu^0(\mathbf{x}) + j_\mu^p(\mathbf{x}) + j_\mu^d(\mathbf{x}), \quad (3.3)$$

where $j_\mu^0(\mathbf{x})$ is the ordinary current which is given from the kinetic energy term in (3.1) and the last two terms, called the "interaction current", take the forms

$$j_\mu^p(\mathbf{x}) = -(ieV/4) \int d\mathbf{z} d\mathbf{z}' d\mathbf{y} d\mathbf{y}' V(\mathbf{z}-\mathbf{z}') V(\mathbf{y}-\mathbf{y}') \varphi^*(\mathbf{z}) \sigma_y \varphi^*(\mathbf{z}') \varphi(\mathbf{y}) \sigma_y \varphi(\mathbf{y}') \\ \cdot \left[\int_y^{z'} \delta(\mathbf{x}-\mathbf{s}) ds_\mu + \int_{y'}^z \delta(\mathbf{x}-\mathbf{s}) ds_\mu \right], \quad (3.4)$$

$$j_\mu^d(\mathbf{x}) = -(e^2V/4c) \int d\mathbf{z} d\mathbf{z}' d\mathbf{y} d\mathbf{y}' V(\mathbf{z}-\mathbf{z}') V(\mathbf{y}-\mathbf{y}') \varphi^*(\mathbf{z}) \sigma_y \varphi^*(\mathbf{z}') \varphi(\mathbf{y}) \sigma_y \varphi(\mathbf{y}') \\ \cdot \left[\int_y^{z'} \mathbf{A} ds + \int_{y'}^z \mathbf{A} ds \right] \left[\int_y^{z'} \delta(\mathbf{x}-\mathbf{s}) ds_\mu + \int_{y'}^z \delta(\mathbf{x}-\mathbf{s}) ds_\mu \right] \quad (3.5)$$

in which \mathbf{A} is a vector potential and only the terms up to second order in e are retained. Correspondingly, the Hamiltonian becomes

$$H(\mathbf{A}) = H_0(\mathbf{A}) - (1/c) \int d\mathbf{x} (\mathbf{j}^p(\mathbf{x}) \mathbf{A}(\mathbf{x})), \quad (3.6)$$

where $H_0(\mathbf{A})$ means the Hamiltonian which contains only the ordinary magnetic interaction.

It may be of interest to consider the magnitude of these currents in the strong coupling limit of electron-electron interaction on account of the reasons stated in Introduction. It must be noticed that here again the question arises whether the simple strong coupling approximation is sensible. As was mentioned in Section 2 the convergence of the perturbation series for the eigenfunction is very bad; so it seems, at first sight, that the simple use of the zero order eigenfunction cannot give any sensible result. We can show, however, that in spite of this fact, the conclusion reached by the simple consideration needs not be modified by the more precise consideration. Therefore we will first calculate the electric current by making use of the lowest order eigenfunction, and then will return to the discussion about the above mentioned question. In the lowest order approximation of strong coupling the expectation value of the electric current will be proportional to the coupling constant V . Therefore it is not necessary to take into account the ordinary current $j_\mu^0(\mathbf{x})$ which does not depend on V . In order to obtain the result up to the e^2

term one has only to take the expectation value of (3.5) in the ground state and after applying the first order perturbation method by means of the last term of (3.6) the contribution of (3.4) will be estimated.

If we choose a straight path for the line integrals in (3.5), the Fourier transform of the diamagnetic current $j_\mu^a(\mathbf{x})$ takes the form

$$\begin{aligned} j_\mu^a(\mathbf{k}) &= \frac{1}{\sqrt{\Omega}} \int e^{-i\mathbf{k}\cdot\mathbf{x}} j_\mu^a(\mathbf{x}) d\mathbf{x} \\ &= \frac{e^2 V}{2c\Omega^3} \sum_{\lambda, \mathbf{p}, \mathbf{p}', \mathbf{q}, \mathbf{q}', \mathbf{k}'} \{c_{\mathbf{p}\uparrow}^* c_{\mathbf{p}'\downarrow}^* - c_{\mathbf{p}\downarrow}^* c_{\mathbf{p}'\uparrow}^*\} \{c_{\mathbf{q}\uparrow} c_{\mathbf{q}'\downarrow} - c_{\mathbf{q}\downarrow} c_{\mathbf{q}'\uparrow}\} A_\lambda(\mathbf{k}') \int_0^1 dt dt' \\ &\quad \cdot \int d\mathbf{z} d\mathbf{z}' d\mathbf{y} d\mathbf{y}' V(\mathbf{z}-\mathbf{z}') V(\mathbf{y}-\mathbf{y}') \exp[-i\mathbf{p}\mathbf{z} - i\mathbf{p}'\mathbf{z}' + i\mathbf{q}\mathbf{y} + i\mathbf{q}'\mathbf{y}' \\ &\quad - i\mathbf{k}\{\mathbf{y} + (\mathbf{z}' - \mathbf{y})t'\}] (z_\mu' - y_\mu) \\ &\quad \cdot [\exp(i\mathbf{k}'\{\mathbf{y} + (\mathbf{z}' - \mathbf{y})t\}) \cdot (z_\lambda' - y_\lambda) + \exp(i\mathbf{k}'\{\mathbf{y}' + (\mathbf{z} - \mathbf{y}')t\}) \cdot (z_\lambda - y_\lambda')]. \end{aligned} \quad (3.7)$$

In order to calculate the expectation value of (3.7) in the ground state $\phi(M, 0)$, one may make use of a relation

$$\begin{aligned} &(\phi(M, 0) | \{c_{\mathbf{p}\uparrow}^* c_{\mathbf{p}'\downarrow}^* - c_{\mathbf{p}\downarrow}^* c_{\mathbf{p}'\uparrow}^*\} \{c_{\mathbf{q}\uparrow} c_{\mathbf{q}'\downarrow} - c_{\mathbf{q}\downarrow} c_{\mathbf{q}'\uparrow}\} | \phi(M, 0)) \\ &= -(\delta_{\mathbf{q}}^{\mathbf{p}} \delta_{\mathbf{q}'}^{\mathbf{p}'} + \delta_{\mathbf{q}}^{\mathbf{p}'} \delta_{\mathbf{q}'}^{\mathbf{p}}) [2I(\mathbf{p}) I(\mathbf{p}') + I(\mathbf{p}) E(\mathbf{p}') + E(\mathbf{p}) I(\mathbf{p}') + \frac{1}{2} E(\mathbf{p}) E(\mathbf{p}')] \\ &\quad - \delta_{\mathbf{p}}^{\mathbf{p}'} \delta_{\mathbf{q}'}^{\mathbf{q}} E(\mathbf{p}) E(\mathbf{q}), \end{aligned} \quad (3.8)$$

which can be obtained by means of (2.28). We have neglected here the terms proportional to M^{-1} and introduced the following notations: $\delta_{\mathbf{q}}^{\mathbf{p}}$ is the Kronecker δ -symbol which is equal to unity if \mathbf{p} is identical with \mathbf{q} and vanishes otherwise, and

$$I(\mathbf{p}) = \begin{cases} 1, & \text{if } \omega_{\mathbf{p}} < \omega_f - \omega \\ 0, & \text{otherwise,} \end{cases} \quad (3.9)$$

$$E(\mathbf{p}) = \begin{cases} 1, & \text{if } \omega_f - \omega \leq \omega_{\mathbf{p}} \leq \omega_f + \omega, \\ 0, & \text{otherwise.} \end{cases} \quad (3.10)$$

It will be found that the first term on the right-hand side of (3.8) gives only a negligible contribution to the current, since this term gives essentially the same correlation of electrons which will also be taken into account in the perturbation theory. This consequence is expected from the fact that in the perturbation approximation the correlation energy of the system takes a value independent of the total electron number. The last term of (3.8) is therefore most effective in this case. It has quite different characters from other terms of the perturbation series. We may interpret the role of this term as follows, making use of an idea of "pseudo-particle" introduced by Bogolyubov²⁾ and Valatin⁷⁾. When a pair of pseudo-

particles are annihilated by $\varphi^*(z')$ and $\varphi(y')$ in (3.5) and then created by $\varphi^*(z)$ and $\varphi(y)$, the matrix element of (3.5) will take the form of the second term of (3.8), that is, this term is given only by the scattering of a pair of pseudo-particles. We will further remark here the fact that the appearance of these terms is a common feature of the strong coupling approximation and the variational method used by BCS.

According to the above remarks the expectation value of (3.7) now takes the form

$$\begin{aligned} \langle j_\mu^a(\mathbf{k}) \rangle &= (\psi(M, 0) | j_\mu^a(\mathbf{k}) | \psi(M, 0)) \\ &= -\frac{e^2 V}{2c\Omega^2} \sum_\lambda A_\lambda(\mathbf{k}) \int d^3 R R_\lambda R_\mu \int_0^1 dt dt' e^{i(t-t')\mathbf{k}\mathbf{R}} \\ &\quad \cdot \left\{ \left(\sum_{\mathbf{p}} E(\mathbf{p}) \right)^2 + \left(\sum_{\mathbf{p}} E(\mathbf{p}) E\left(\mathbf{p} + \mathbf{k} \frac{2-t-t'}{2}\right) \right) \left(\sum_{\mathbf{q}} E(\mathbf{q}) E\left(\mathbf{q} + \mathbf{k} \frac{t+t'}{2}\right) \right) \right\} \\ &\quad + \frac{e^2 V}{8c\Omega^2} \sum_{\lambda, \mathbf{p}, \mathbf{p}', \mathbf{q}} E(\mathbf{p}) E(\mathbf{p} + \mathbf{k} + \mathbf{q}) E(\mathbf{p}') E(\mathbf{q} - \mathbf{p}') A_\lambda(\mathbf{k}) \int_0^1 dt dt' \\ &\quad \cdot \int d\mathbf{R} d\mathbf{r} e^{i\mathbf{q}\mathbf{r} + i\mathbf{k}\mathbf{R}(t-t') + i(\mathbf{k}\mathbf{r}/2)(t+t')} r_\lambda r_\mu, \end{aligned} \quad (3.11)$$

where we have neglected a term which is antisymmetric with respect to the interchange of suffices λ and μ , since the final result must have evidently a symmetric form.

After some examinations the second term on the right-hand side of (3.11) will be found to be negligibly small; the argument for this will be given in detail in Appendix 2.

Introducing a pair of variables x and y defined by

$$x = t - t', \quad y = (t + t')/2, \quad (3.12)$$

the first term in (3.11) may be written as

$$\begin{aligned} \langle j_\mu^a(\mathbf{k}) \rangle &= -\frac{e^2 V}{c\Omega^2} \sum_\lambda A_\lambda(\mathbf{k}) \int d^3 R R_\lambda R_\mu \int_0^1 dx \int_{x/2}^{1-x/2} dy \cos x \mathbf{k}\mathbf{R} \\ &\quad \cdot \left\{ \left(\sum_{\mathbf{p}} E(\mathbf{p}) \right)^2 + \left(\sum_{\mathbf{p}} E(\mathbf{p}) E(\mathbf{p} + \mathbf{k}y) \right) \left(\sum_{\mathbf{q}} E(\mathbf{q}) E(\mathbf{q} + \mathbf{k}(1-y)) \right) \right\}. \end{aligned} \quad (3.13)$$

One is able to obtain an explicit form for the quantities in the parentheses on the right-hand side of (3.13); when the magnitude of \mathbf{k} is smaller than $2m\omega/k_f$ (k_f = the Fermi momentum) the following relation may be derived;

$$\sum_{\mathbf{p}} E(\mathbf{p}) E(\mathbf{p} + \mathbf{k}y) = \frac{\Omega}{(2\pi)^3} \left\{ F - \pi a^2 k y + \frac{\pi}{6} k^3 y^3 \right\}, \quad (3.14)$$

where F and a^2 are constants defined by

$$F = \int d\mathbf{p} E(\mathbf{p}), \quad (3.15)$$

$$a^2 = 4m\omega_f. \quad (3.16)$$

Making use of the relation (3.14) and integrating (3.13) by parts with respect to x , one finds

$$\langle j_\mu^a(\mathbf{k}) \rangle = -\frac{e^2 V}{(2\pi)^6 c} \sum_\lambda A_\lambda(\mathbf{k}) \int d^3 R R_\lambda R_\mu \int_0^1 dx \frac{\sin x \mathbf{k} \mathbf{R}}{\mathbf{k} \mathbf{R}} \sum_{n=0}^6 \alpha_n x^n \quad (3.17)$$

where

$$\begin{aligned} \alpha_0 &= 2F^2 - \pi F a^2 k + \frac{\pi F k^3}{6}, & \alpha_4 &= \frac{\pi^2 k^4}{48} \left(a^2 - \frac{k^2}{4} \right), \\ \alpha_1 &= -\frac{1}{4} \pi F k^3 + \frac{\pi^2 k^2}{2} \left(a^4 - \frac{a^2 k^2}{6} \right), & \alpha_5 &= \frac{\pi^2 k^6}{384}, \\ \alpha_2 &= \frac{1}{8} \pi F k^3 + \frac{\pi^2 k^2 a^2}{4} \left(\frac{k^2}{2} - a^2 \right), & \alpha_6 &= -\frac{\pi^2 k^6}{2304}, \\ \alpha_3 &= \frac{\pi^2 k^4}{12} \left(\frac{k^2}{24} - a^2 \right), \end{aligned} \quad (3.18)$$

Now we shall consider the integration over $d^3 R$. When one makes use of the polar coordinate system with the axis parallel to the direction of vector \mathbf{k} , $R_\lambda R_\mu$ may be written effectively as

$$R_\lambda R_\mu = R^2 \left\{ \frac{1}{2} \sin^2 \theta \left(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2} \right) + \cos^2 \theta \frac{k_\lambda k_\mu}{k^2} \right\}, \quad (3.19)$$

on account of the symmetric form of the integrand, where R the magnitude and θ is the polar angle of vector \mathbf{R} . Although (3.17) will diverge when $d^3 R$ integration is performed, this divergence is necessary in order to compensate the infinitesimal quantity V which is proportional to $1/\Omega$. Here one has to employ some artificial technique to avoid the ambiguity of treating an infinite quantity. For this purpose, we shall first cut off the contribution from large R values to the integration by means of a damping factor $\exp(-pR)$, and after obtaining the final result we shall remove the effect of this "cut off" technique taking the limit $p \rightarrow 0$. Then, introducing a variable

$$u = \cos \theta \quad (3.20)$$

one finds the relation

$$\begin{aligned} \int d^3 R R_\lambda R_\mu \frac{\sin x \mathbf{k} \mathbf{R}}{\mathbf{k} \mathbf{R}} &= \frac{2\pi}{k} \int_{-1}^1 \frac{du}{u} \left\{ \frac{1}{2} (1-u^2) \left(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2} \right) + u^2 \frac{k_\lambda k_\mu}{k^2} \right\} \\ &\quad \times \int_0^\infty e^{-pR} R^3 \sin(xkuR) dR \end{aligned}$$

$$\begin{aligned}
 &= 48\pi p x \int_{-1}^1 du \left\{ \frac{1}{2} (1-u^2) \left(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2} \right) + u^2 \frac{k_\lambda k_\mu}{k^2} \right\} \frac{p^2 - x^2 k^2 u^2}{\{p^2 + x^2 k^2 u^2\}^4} \\
 &= 48\pi p x \left[\frac{1}{2} \left(\frac{1}{3p^2 \xi^2} + \frac{1}{2p^4 \xi} + \frac{1}{2p^5 x k} \tan^{-1} \frac{xk}{p} \right) \left(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2} \right) + \frac{2}{3\xi^3} \frac{k_\lambda k_\mu}{k^2} \right],
 \end{aligned} \tag{3.21}$$

where ξ is defined by

$$\xi = p^2 + x^2 k^2. \tag{3.22}$$

Substituting the equation (3.21) into (3.17) and performing the x -integration, one now gets the explicit form for the diamagnetic part of the electric current as follows

$$\begin{aligned}
 \langle j_\mu^d(\mathbf{k}) \rangle = & -\frac{3e^2 V}{8\pi^5 c p^3} \sum_\lambda A_\lambda(\mathbf{k}) \left[\left(\frac{A(k)}{2k^2} + \frac{B(k)}{2pk} \tan^{-1} \frac{k}{p} \right) \left(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2} \right) \right. \\
 & \left. + \frac{\alpha_0}{3k^2} \frac{k_\lambda k_\mu}{k^2} \right]
 \end{aligned} \tag{3.23}$$

where the terms which vanish as $p \rightarrow 0$ are neglected and the following notations are introduced

$$\begin{aligned}
 A(k) &= \frac{\alpha_0}{3} + \sum_{n=1}^6 \frac{\alpha_n}{n+1} \\
 &= \frac{2}{3} F^2 - \frac{1}{3} \pi F a^2 k + \frac{\pi^2 a^4}{6} k^2 - \frac{1}{36} \pi F k^3 - \frac{\pi^2 a^2}{60} k^4 + \frac{\pi^2}{5040} k^6, \\
 B(k) &= \sum_{n=0}^6 \frac{\alpha_n}{n+1} \\
 &= 2F^2 - \pi F a^2 k + \frac{\pi^2 a^4}{6} k^2 + \frac{1}{12} \pi F k^3 - \frac{\pi^2 a^2}{60} k^4 + \frac{\pi^2}{5040} k^6.
 \end{aligned} \tag{3.24}$$

Now we shall consider the magnitude of the paramagnetic current. To the lowest order in \mathbf{A} and $1/V$, the contribution of paramagnetic part is given by

$$\begin{aligned}
 \langle j_\mu^p(\mathbf{k}) \rangle = & -\frac{1}{c} \sum_\lambda \sum_i' \{ (\psi(M, 0) | j_\mu^p(\mathbf{k}) | \Phi_i) (\Phi_i | j_\lambda^p(-\mathbf{k}) | \psi(M, 0)) \\
 & + (\psi(M, 0) | j_\lambda^p(-\mathbf{k}) | \Phi_i) (\Phi_i | j_\mu^p(\mathbf{k}) | \psi(M, 0)) \} A_\lambda(\mathbf{k}) / (E_0 - E_i)
 \end{aligned} \tag{3.25}$$

in which the summation is extended over the complete set of eigenstates Φ_i of the unperturbed Hamiltonian (2.22) except the ground state $\psi(M, 0)$; E_i is the energy of the state Φ_i , and $j_\mu^p(\mathbf{k})$ is the Fourier transform of the paramagnetic current $j_\mu^p(\mathbf{x})$, (3.4), which takes the form

$$\begin{aligned}
 j_\mu^p(\mathbf{k}) &= \frac{1}{\sqrt{\Omega}} \int e^{-i\mathbf{k}\cdot\mathbf{x}} j_\mu^p(\mathbf{x}) d\mathbf{x} \\
 &= \frac{ieV}{2\Omega^{5/2}} \sum_{\mathbf{p}\mathbf{p}'\mathbf{q}\mathbf{q}'} (c_{\mathbf{p}\uparrow}^* c_{\mathbf{p}'\downarrow}^* - c_{\mathbf{p}\downarrow}^* c_{\mathbf{p}'\uparrow}^*) (c_{\mathbf{q}\uparrow} c_{\mathbf{q}'\downarrow} - c_{\mathbf{q}\downarrow} c_{\mathbf{q}'\uparrow}) \int_0^1 dt \int dz dz' dy dy'
 \end{aligned}$$

$$\cdot V(\mathbf{z}-\mathbf{z}') V(\mathbf{y}-\mathbf{y}') \exp[-i\mathbf{p}\mathbf{z}-i\mathbf{p}'\mathbf{z}'+i\mathbf{q}\mathbf{y}+i\mathbf{q}'\mathbf{y}'-i\mathbf{k}\{\mathbf{y}+(\mathbf{z}'-\mathbf{y})t\}] \quad (3.26)$$

$$\cdot (z_{\mu}'-y_{\mu}).$$

We have here taken a straight path for the line integrals in (3.4).

There are fourteen types of intermediate states which have nonvanishing matrix elements in the expression (3.25). The explicit forms of these states are listed in Appendix 3. Examining the contributions of each type of intermediate states, it will be found that almost all types give only negligible contributions to the current which vanish with the size of the system. Finite current is obtained through three types of intermediate states; one is the state $\phi_4(\mathbf{p}_2's, \mathbf{p}_1's)$ in which an electron with momentum $\hbar\mathbf{p}_2'(I(\mathbf{p}_2'))$ and spin s is annihilated and a single electron with momentum $\hbar\mathbf{p}_1'(E(\mathbf{p}_1'))$ and spin s is excited,¹ other electrons being in the ground state; another type of intermediate states is $\phi_7(\mathbf{p}_1's, \mathbf{p}_2'-s)$ where one electron pair is broken, one of the two electrons making single electron excitation with momentum $\hbar\mathbf{p}_2', (E(\mathbf{p}_2'))$ and spin $-s$ and the other electron going into the state with momentum $\hbar\mathbf{p}_1'$ and spin s which lies outside the narrow shell around the Fermi surface in the momentum space. Such states will be denoted by the symbol $O(\mathbf{p}_1')$, where

$$O(\mathbf{p}_1') = \begin{cases} 1 & \text{if } \omega_f + \omega < \omega_{\mathbf{p}_1'}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.27)$$

The third type of intermediate states is the state $\phi_{12}(-\mathbf{p}_1'\downarrow, \mathbf{p}_2'\uparrow)$ given by breaking off of an electron pair again, where each electron of the broken pair makes a single electron excitation with momenta $\hbar\mathbf{p}_1'$, and $\hbar\mathbf{p}_2', (E(\mathbf{p}_1'))$ and $E(\mathbf{p}_2'))$, respectively. The wave functions of these states are found to be (see Appendix 3)

$$\phi_4(\mathbf{p}_2's, \mathbf{p}_1's) = \frac{\sqrt{2}}{\sqrt{(2M)!}} c_{\mathbf{p}_2's} Y_1^M c_{\mathbf{p}_1's}^* |0\rangle I(\mathbf{p}_2') E(\mathbf{p}_1'), \quad (3.28)$$

$$\phi_7(\mathbf{p}_1's, \mathbf{p}_2'-s) = \frac{\sqrt{M}}{\sqrt{(2M-1)!}} c_{\mathbf{p}_1's}^* c_{\mathbf{p}_2'-s}^* Y_1^{M-1} |0\rangle O(\mathbf{p}_1') E(\mathbf{p}_2') \quad (3.29)$$

$$\phi_{12}(-\mathbf{p}_1'\downarrow, \mathbf{p}_2'\uparrow) = \frac{1}{\sqrt{(2M-2)!}} Y_2^{M-1} c_{-\mathbf{p}_1'\downarrow}^* c_{\mathbf{p}_2'\uparrow}^* |0\rangle E(\mathbf{p}_1') E(\mathbf{p}_2') (1 - \delta_{\mathbf{p}_2'}^{\mathbf{p}_1'}) \quad (3.30)$$

where Y_n is an operator defined by

$$Y_n = \sum_{\mathbf{k}} b_{\mathbf{k}}^*, \quad (3.31)$$

the summation being extended over the electron pair states except those n states that are occupied by singly excited electrons. The excitation energies of ϕ_4 and ϕ_7 are both VM and that of ϕ_{12} is $2VM$.

The criterion of finding these effective intermediate states is again given by a similar discussion applied for the case of diamagnetic current. Finite contributions

for the current are given only through the correlation of electrons similar to the scattering of pseudo-particles, or in other words, only through the matrix elements of (3.26) which contain the factor $\delta_{-p'}^p$ or $\delta_{-q'}^q$, as in (3.8). The matrix elements of (3.26) between the ground state $\phi(M, 0)$ and each one of the three excited states listed above (3.28–30) contain terms other than those which contain a factor $\delta_{-p'}^p$, etc. These terms give again only negligible contributions vanishing with the size of the system. Neglecting such terms the matrix elements of the operator in (3.26)

$$Q = (c_{p\uparrow}^* c_{p'\downarrow}^* - c_{p\downarrow}^* c_{p'\uparrow}^*) (c_{q\uparrow} c_{q'\downarrow} - c_{q\downarrow} c_{q'\uparrow}) \quad (3.32)$$

will be found to be

$$(\phi_4(p_2's, p_1's) | Q | \phi(M, 0)) \simeq -\frac{1}{\sqrt{2}} \delta_{-p'}^p \{ \delta_{p_2'}^{q'} \delta_{-q}^{p_1'} + \delta_{-q'}^{p_1'} \delta_{p_2'}^q \} E(p_1') I(p_2') E(p), \quad (3.33)$$

$$(\phi_7(p_1's, p_2'-s) | Q | \phi(M, 0)) \simeq \mp \frac{1}{\sqrt{2}} \delta_{-q'}^q \{ \delta_p^{p_1'} \delta_{p_2'}^{p'} + \delta_{p'}^{p_1'} \delta_{p_2'}^p \} O(p_1') E(p_2') E(q), \quad (3.34)$$

where the upper or lower sign is taken according as the spin direction s is \uparrow or \downarrow , and

$$(\phi_{12}(-p_1'\downarrow, p_2'\uparrow) | Q | \phi(M, 0)) \simeq \frac{1}{2} [\delta_{-q'}^q (\delta_p^{p_2'} \delta_{-p'}^{p_1'} + \delta_{p'}^{p_2'} \delta_{-p}^{p_1'}) E(q) - \delta_{-p'}^p (\delta_{-q'}^{p_2'} \delta_q^{p_1'} + \delta_{-q}^{p_2'} \delta_q^{p_1'}) E(p)] E(p_1') E(p_2'). \quad (3.35)$$

Making use of these relations the matrix elements of the current (3.26) may be written as

$$(\phi_4(p_2's, p_1's) | j_\mu^p(k) | \phi(M, 0)) = -\frac{ieVF}{2\sqrt{2}(2\pi)^3\Omega^{1/2}} E(p_1') I(p_2') \delta_{p_2'}^{k+p_1'} \int_0^1 dt \int d^3R R_\mu e^{-ikRt}, \quad (3.36)$$

The integration with respect to dt can be performed easily.

Introducing the polar coordinate system and the polar angle of vector \mathbf{R} as in the case of the diamagnetic current, R_μ may be put effectively as

$$R_\mu = R \cos \theta k_\mu / k, \quad (3.37)$$

on account of the symmetric form of the integrand. Now one finds

$$\begin{aligned} \int d^3R \int_0^1 dt R_\mu e^{-ikRt} &= 2\pi \frac{k_\mu}{ik^2} \int_0^\infty e^{-pR} R^2 dR \int_0^\pi \sin \theta d\theta \{ 1 - e^{-ikR \cos \theta} \} \\ &= 4\pi \frac{k_\mu}{ik^2} \int_0^\infty e^{-pR} R^2 dR \left\{ 1 - \frac{\sin kR}{kR} \right\} = 8\pi \frac{k_\mu}{ik^2} \left\{ \frac{1}{p^3} - \frac{p}{(k^2 + p^2)^2} \right\}, \end{aligned} \quad (3.38)$$

where the damping factor $\exp(-pR)$ is again introduced. The second term in the parentheses of (3.38) can be discarded in the limit $p \rightarrow 0$. By means of the relation (3.38), (3.36) turns out to be

$$(\phi_4(\mathbf{p}_2' s, \mathbf{p}_1' s) | j_\mu^p(\mathbf{k}) | \psi(M, 0)) = -\frac{eVF}{2\sqrt{2}\pi^2\Omega^{1/2}} E(\mathbf{p}_1') I(\mathbf{p}_2') \delta_{\mathbf{p}_2' \mathbf{p}_1'}^{\mathbf{k}+\mathbf{p}_1'} \frac{k_\mu}{k^2 p^3}. \quad (3.39)$$

Similarly we find

$$(\phi_7(\mathbf{p}_1' s, \mathbf{p}_2' -s) | j_\mu^p(\mathbf{k}) | \psi(M, 0)) = \pm \frac{eVF}{2\sqrt{2}\pi^2\Omega^{1/2}} O(\mathbf{p}_1') E(\mathbf{p}_2') \delta_{\mathbf{p}_2' \mathbf{p}_1'}^{\mathbf{k}-\mathbf{p}_1'} \frac{k_\mu}{k^2 p^3}, \quad (3.40)$$

with the sign determined before, and

$$(\phi_{12}(-\mathbf{p}_1' \downarrow, \mathbf{p}_2' \uparrow) | j_\mu^p(\mathbf{k}) | \psi(M, 0)) = -\frac{eVF}{\pi^2\Omega^{1/2}} E(\mathbf{p}_1') E(\mathbf{p}_2') \delta_{\mathbf{p}_1' \mathbf{p}_2'}^{\mathbf{k}+\mathbf{p}_2'} \frac{k_\mu}{k^2 p^3}, \quad (3.41)$$

where we have assumed that $\mathbf{k} \neq 0$.

On the other hand one is able to derive the relation

$$M = \Omega F / 2(2\pi)^3. \quad (3.42)$$

When these various relations are put in Eq. (3.25), the paramagnetic current is now found to be

$$\begin{aligned} \langle j_\mu^p(\mathbf{k}) \rangle = & \frac{e^2 VF}{\pi^4 c \Omega p^3 k^2} \left\{ \int d\mathbf{p} E(\mathbf{p}) I(\mathbf{k}+\mathbf{p}) + \int d\mathbf{p} O(\mathbf{p}) E(\mathbf{k}+\mathbf{p}) \right. \\ & \left. + 2 \int d\mathbf{p} E(\mathbf{p}) E(\mathbf{k}+\mathbf{p}) \right\} \sum_\lambda A_\lambda(\mathbf{k}) \frac{k_\lambda k_\mu}{k^2}. \end{aligned} \quad (3.43)$$

The first two integrations in the parentheses of (3.43) may be easily performed and give the results:

$$\int d\mathbf{p} E(\mathbf{p}) I(\mathbf{k}+\mathbf{p}) = \pi \left[2m k (\omega_f - \omega) - \frac{1}{12} k^3 \right], \quad (3.44)$$

$$\int d\mathbf{p} O(\mathbf{p}) E(\mathbf{k}+\mathbf{p}) = \pi \left[2m k (\omega_f + \omega) - \frac{1}{12} k^3 \right], \quad (3.45)$$

where the magnitude of \mathbf{k} is taken to be smaller than $2m\omega/k_f$. By means of Eqs. (3.14), (3.44) and (3.35) it may be seen that the quantity in the parentheses of (3.43) gives the value α_0/F . Further, we can express the factor Ω on the right-hand side of (3.43) in terms of p , making use of the relation

$$\Omega = 4\pi \int_0^\infty e^{-pR} R^2 dR = \frac{8\pi}{p^3}. \quad (3.46)$$

Thus one finally obtains

$$\langle j_{\mu}^p(\mathbf{k}) \rangle = -\frac{e^2 V \alpha_0}{8\pi^5 c p^3 k^2} \sum_{\lambda} A_{\lambda}(\mathbf{k}) \frac{k_{\lambda} k_{\mu}}{k^2}. \quad (3.47)$$

Adding up both expressions (3.23) and (3.47), we now find the total electric current in the lowest order approximation of strong coupling. The non-gauge invariant terms cancel each other. By means of (3.46) and a relation

$$\Omega = 2\pi^2 N(0)/mk_f, \quad (3.48)$$

which may be easily verified, the total current will be given by

$$\langle j_{\mu}(\mathbf{k}) \rangle = -\frac{3e^2 x}{2^6 \pi^4 c m k_f} \left(\frac{A(k)}{k^2} + \frac{\pi B(k)}{2pk} \right) \sum_{\lambda} \left((\delta_{\lambda\mu} - \frac{k_{\lambda} k_{\mu}}{k^2}) A_{\lambda}(\mathbf{k}) \right), \quad (3.49)$$

in the limit $p \rightarrow 0$, in terms of a notation x which was introduced in Section 2, ($x \equiv VN(0)$). This expression diverges in the limit $p \rightarrow 0$; in other words, the current density of the system diverges as the size of the system becomes infinite. Moreover, this expression has a peculiar singularity in the limit $k \rightarrow 0$. The degree of this singularity is higher than that of a simple energy gap model discussed in the previous paper⁰. Now we will return to the problem left at the beginning of this section, i. e., the applicability of simple strong coupling approximation to calculate the electric current. Since, as is now known, the diamagnetic current (3.7) plays an essential role, one may only consider the expectation value of (3.7) now using the full eigenfunction (2.64) in its approximate form (2.77).

Making use of the notation (2.24), (2.25) and also of the approximation (2.77), the eigenfunction (2.64) may be written as

$$\phi = \sum_n \exp\left(-\frac{M}{12x^2}\right) \frac{1}{\sqrt{n!}} \left(\frac{M}{6x^2}\right)^{n/2} \phi(M-n, 0). \quad (3.50)$$

Our task is now to calculate $\langle j_{\mu}^a(\mathbf{k}) \rangle$ with this eigenfunction. In doing this one has to calculate quantities like

$$(\phi(M-n, 0) | \{c_{p\uparrow}^* c_{p'\downarrow}^* - c_{p\downarrow}^* c_{p'\uparrow}^*\} \{c_{q\uparrow} c_{q'\downarrow} - c_{q\downarrow} c_{q'\uparrow}\} | \phi(M-n', 0)). \quad (3.51)$$

This can be shown to vanish when the difference between n and n' exceeds two. Moreover, by a consideration similar to that which leads to (3.8) and (3.11) and also to (3.33), (3.34) and (3.35) we notice that here again terms corresponding to the scattering of pseudo-particles are important. Then we find that for $n=n'$ and $n \ll M$, (3.51) is essentially equal to (3.8), the difference being a small quantity of the order n/M . This is due to the fact that in $\phi(M-n, 0)$ only a very small fraction, i. e., n/M , of the particles are in states different from those of $\phi(M, 0)$. Further, for $n' \neq n$, (3.51) has a negligibly small value, because of the orthogonality condition (2.65) (cf. also Appendix 3). This does not lead to a finite amount even after summed up with respect to the states $\phi(M-n, 0)$ and $\phi(M-n', 0)$, since (3.51) vanishes when the difference between n and n' exceeds two.

Thus in our approximation we may simply put

$$\langle j_{\mu}^a(\mathbf{k}) \rangle = (\psi(M, 0) | j_{\mu}^a(\mathbf{k}) | \psi(M, 0)) \exp\left(-\frac{M}{6x^2}\right) \sum_n \frac{1}{n!} \left(\frac{M}{6x^2}\right)^n,$$

which shows that the effect of the higher configurations is just compensated by the normalization factor, giving rise to

$$\langle j_{\mu}^a(\mathbf{k}) \rangle = \langle \psi(M, 0) | j_{\mu}^a(\mathbf{k}) | \psi(M, 0) \rangle. \quad (3.52)$$

In this way our simple calculation of the electric current can be justified in spite of the breakdown of the condition (2.63).

§ 4. Discussion

It is now clear that BCS' theory of superconductivity may not account for the Meissner effect at least in the strong coupling limit, although, on the other hand, it was successful in explaining the thermal properties of a superconducting system. We have remarked two peculiarities in the expression for the electric current density, one of them is the divergence and the other is the singularity in the limit $k \rightarrow 0$.

It has been found that among the various correlations of electrons the most essential correlations are given by the "scattering term of pseudo-particles" such that, in the case of the diamagnetic current (3.5), a pseudo-particle is annihilated at \mathbf{y}' and created at \mathbf{y} , while the other pseudo-particle is annihilated at \mathbf{z}' and created at \mathbf{z} . We may assume the spatial points \mathbf{y} and \mathbf{y}' to coincide with good approximation, since the function $V(\mathbf{y}-\mathbf{y}')$ will decrease rapidly as the distance between the two points increases. Similarly \mathbf{z} and \mathbf{z}' are taken to be identical. We may therefore interpret the electron-electron interaction taken by BCS as a type of quasi-local interactions between pseudo-particles. This quasi-local interaction is of a very long range. The correlation arising from this interaction does not decrease, however far the two spatial points \mathbf{z} and \mathbf{y} may be separated. That is, a pseudo-particle interacts with every other pseudo-particle with the constant strength irrespective of the relative distance. These long range correlations may give rise to an infra-red divergence in various properties of the system. Indeed, the energy of the system has this type of divergence since it is proportional to M^2 , and the electric current is now found to be divergent. Instead of employing these long range interactions, we have introduced an infinitesimal coupling constant V in order to compensate the divergence. But it is to be expected that a compensation of an infinite quantity by an infinitesimal one would be a delicate procedure. It may be successful in one case but in another case it might not be so. Actually the infinitesimal V has given a reasonable result for the energy, but failed to give a finite answer for the current.

Now we shall consider the variational solution used by BCS. It has a character which is qualitatively not different from that of strong coupling solution, as

stated before. The BCS solution gives a diffused type of electron distribution around the Fermi surface. Consequently, the expectation value of the diamagnetic current (3.5) by the BCS solution, for instance, will be given essentially by the correlation of pseudo-particle scattering type and take the form similar to (3.13), the main alternation being involved in a slight change of the constants α_n , etc. It seems therefore that the infra-red divergence obtained above will appear also in the case of BCS solution. We may thus conclude that this feature will be inherent of the interaction current of the BCS model, irrespective of the coupling strength. Although the interaction current is proportional to the coupling constant which is small in the weak coupling limit, the matrix element of the current will probably diverge and we cannot discard the effect of the interaction current even in the weak coupling limit.

The second peculiarity in the expression for the electric current is the singularity in the limit $k \rightarrow 0$. This means physically that the total electric current diverges. It has been found in the previous paper that such a singularity has been obtained in the simple energy gap model, too. The appearance of such an unphysical result seems to be a common feature of rather simple energy gap models, irrespective of the origin of the energy gap, that is, whether it is given phenomenologically in a simple model or it is obtained as the result of simple correlations between electrons. At least it may be certain that the energy gap and the gauge invariance do not necessarily account for the Meissner effect. The detailed mechanism of correlations would play an essential role in this phenomenon. In these cases, it is valuable to find some methods to obtain an approximate value for the penetration depth of the applied magnetic field. One of such methods was given by Blatt et al.¹³⁾ which makes use of a kind of variational procedure. As a common feature of the variational technique their approach may not be so clear physically, but practically we shall be able to obtain a reliable result.

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Appendix 1. The properties of second excited states of H_0

We shall here summarize the results concerning the second excited states of H_0 obtained by the method similar to that in Section 2 for the first excited states.

$$\text{energy:} \quad E_{M-2} = -V(M-2)(M-1), \quad (\text{A1.1})$$

$$\text{degree of degeneracy:} \quad M(2M-3), \quad (\text{A1.2})$$

eigenfunctions :

$$\left\{ \begin{aligned} \phi_i(M-2, 0) &= \frac{1}{\sqrt{(2M-4)!}} Y^{M-2} \sum_{\mathbf{k}\mathbf{k}'} \phi_i(\mathbf{k}, \mathbf{k}') b_{\mathbf{k}}^* b_{\mathbf{k}'}^* |0\rangle, \\ &\quad i=1, 2, \dots, M(2M-3), \\ \phi_i(\mathbf{k}, \mathbf{k}') &= \phi_i(\mathbf{k}', \mathbf{k}), \quad \phi_i(\mathbf{k}, \mathbf{k}) = 0 \\ \sum_{\mathbf{k}'} \phi_i(\mathbf{k}, \mathbf{k}') &= 0, \quad \mathbf{k}=1, \dots, 2M, \end{aligned} \right. \quad (\text{A1}\cdot 3)$$

conditions of orthonormalization :

$$2 \sum_{\mathbf{k}\mathbf{k}'} \phi_i^*(\mathbf{k}, \mathbf{k}') \phi_j(\mathbf{k}, \mathbf{k}') = \delta_{ij}, \quad (\text{A1}\cdot 4)$$

completeness conditions :

$$\begin{aligned} \sum_i \phi_i^*(\mathbf{p}, \mathbf{q}) \phi_i(\mathbf{p}', \mathbf{q}') &= (1 - \delta_{\mathbf{p}\mathbf{q}}) (1 - \delta_{\mathbf{p}'\mathbf{q}'} \\ &\quad \left[\frac{1}{4} (\delta_{\mathbf{p}\mathbf{p}'} \delta_{\mathbf{q}\mathbf{q}'} + \delta_{\mathbf{p}\mathbf{q}'} \delta_{\mathbf{q}\mathbf{p}'}) + \frac{1}{2(2M-1)(2M-2)} \right. \\ &\quad \left. - \frac{1}{8(M-1)} (\delta_{\mathbf{p}\mathbf{p}'} + \delta_{\mathbf{p}\mathbf{q}'} + \delta_{\mathbf{q}\mathbf{p}'} + \delta_{\mathbf{q}\mathbf{q}'}) \right]. \end{aligned} \quad (\text{A1}\cdot 5)$$

Appendix 2. Proof of the fact that the second term on the right-hand side of (3.11) vanishes

We will verify the assertion that the second term on the right-hand side of (3.11) vanishes on putting the vector \mathbf{k} in the E factors to be zero. This will have no considerable effect on the result of our investigation since these E factors are all definite quantities free from any singularities. At least we must have the exact result for the lowest order term of the Laurent expansion series in powers of k , even with the above approximation.

The quantity to be examined now takes the form

$$\begin{aligned} &\frac{e^2 V}{8c\Omega^3} \sum_{\mathbf{p}\mathbf{p}'\mathbf{q}\lambda} E(\mathbf{p}) E(\mathbf{p}') E(\mathbf{q}+\mathbf{p}) E(\mathbf{q}+\mathbf{p}') A_\lambda(\mathbf{k}) \int_0^1 dt dt' \int d\mathbf{R} d\mathbf{r} \\ &\quad \times e^{i\mathbf{q}\mathbf{r} + i\mathbf{k}\mathbf{R}(t-t') + i(\mathbf{k}\mathbf{r}/2)(t+t')} r_\lambda r_\mu. \end{aligned} \quad (\text{A2}\cdot 1)$$

Integrating (A2.1) with respect to dt and dt' , one has

$$(\text{A2}\cdot 1) = \frac{-e^2 V}{4\pi^2 c (2\pi)^9} \int d^3 q F(q)^2 \sum_\lambda A_\lambda(\mathbf{k}) \int d^3 r \frac{r_\lambda r_\mu}{\mathbf{k}\mathbf{r}} e^{i\mathbf{q}\mathbf{r}} (e^{i\mathbf{k}\mathbf{r}} - 1) \int d\mathbf{R} \frac{e^{i\mathbf{k}\mathbf{R}} - 1}{\mathbf{k}\mathbf{R}}, \quad (\text{A2}\cdot 2)$$

where the simplified notation $F(q)$ is introduced which is defined by

$$\sum_{\mathbf{p}} E(\mathbf{p}) E(\mathbf{p}+\mathbf{q}) = \frac{\Omega}{(2\pi)^3} F(q), \quad (\text{A2}\cdot 3)$$

whose explicit form with small q can be easily derived by means of (3.14). The $d\mathbf{R}$ integral in (A2.2) is performed immediately by making use of a damping factor $\exp(-pR)$ and gives the result

$$\int d\mathbf{R} e^{-pR} \frac{e^{i\mathbf{k}\mathbf{R}} - 1}{\mathbf{k}\mathbf{R}} = \frac{4\pi i}{p} \left\{ \frac{1}{p^2 + k^2} + \frac{1}{kp} \tan^{-1} \frac{k}{p} \right\}. \quad (\text{A2.4})$$

As this integral diverges in the second order of $1/p$, it is now clear that we must consider only the divergent contributions of the remaining two integrals. $r_\lambda r_\mu$ may be replaced by an equivalent form similar to (3.19). Then the remaining integrals in (A2.2) are calculated as follows:

$$\begin{aligned} & \int d\mathbf{q} F(q)^2 \int d\mathbf{r} e^{-pr} \frac{r_\lambda r_\mu}{\mathbf{k}\mathbf{r}} e^{i\mathbf{q}\mathbf{r}} (e^{i\mathbf{k}\mathbf{r}} - 1) \\ &= \frac{8\pi^2 i}{k} \int_0^\infty dq q F(q)^2 \int_{-1}^1 \frac{dx}{x} \left[\frac{1}{2} (1-x^2) \left(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2} \right) + x^2 \frac{k_\lambda k_\mu}{k^2} \right] \\ & \quad \times \int_0^\infty e^{-pr} r^2 \sin qr \sin kr x dr \\ &= \frac{16\pi^2 i p}{k} \int_0^\infty dq q F(q)^2 \int_{-1}^1 \frac{dx}{x} \left[\frac{1}{2} (1-x^2) \left(\delta_{\lambda\mu} - \frac{k_\lambda k_\mu}{k^2} \right) + x^2 \frac{k_\lambda k_\mu}{k^2} \right] \\ & \quad \times \left\{ \frac{4p^2}{\{p^2 + (q-kx)^2\}^3} - \frac{3}{\{p^2 + (q-kx)^2\}^2} \right\}, \quad (\text{A2.5}) \end{aligned}$$

in which the integration over x has to be taken in the sense of the principal value. Now we will examine the "non-gauge invariant" term which is proportional to $k_\lambda k_\mu / k^2$. Integration over x gives

$$\begin{aligned} & \int_{-1}^1 dx \left[\frac{4p^2 x}{\{p^2 + (q-kx)^2\}^3} - \frac{3x}{\{p^2 + (q-kx)^2\}^2} \right] \\ &= \frac{p^2 + (q-k)(q-3k)}{2k^2 \{p^2 + (q-k)^2\}^2} - \frac{p^2 + q^2 + 3k^2 + 4qk}{2k^2 \{p^2 + (q+k)^2\}^2}. \quad (\text{A2.6}) \end{aligned}$$

The second term on the right-hand side of (A2.6) does not give any divergent contribution to (A2.5), since it has no singularities as a function of q . Therefore this term may be discarded. The effect of the first term is estimated as

$$\begin{aligned} & \frac{16\pi^2 i}{k} p \frac{k_\lambda k_\mu}{k^2} \int_0^\infty dq q F(q)^2 \frac{p^2 + (q-k)(q-3k)}{2k^2 \{p^2 + (q-k)^2\}^2} \\ & \cong \frac{8\pi^2 i p^3 k_\lambda k_\mu}{k^4} F(k)^2 \int_{-\infty}^\infty \frac{dq}{\{p^2 + (q-k)^2\}^2} = \frac{4\pi^2 i k_\lambda k_\mu}{k^4} F(k)^2 \quad (\text{A2.7}) \end{aligned}$$

which is also a finite magnitude and may be neglected. It is now found that (A2.1) has no non-gauge invariant contribution.

The gauge invariant term in (A2.5) can be calculated in a similar way. Integrating with respect to x one finds

$$\begin{aligned} & \int_{-1}^1 \frac{dx}{x} \left\{ \frac{4p^2}{\{p^2 + (q-kx)^2\}^3} - \frac{3}{\{p^2 + (q-kx)^2\}^2} \right\} \\ &= -\frac{1}{\eta^3} \left(3qp - \frac{q^3}{p} \right) \left\{ \tan^{-1} \frac{q-k}{p} - \tan^{-1} \frac{q+k}{p} \right\} - \frac{p^2 - 3q^2}{2\eta^3} \log \frac{p^2 + (q-k)^2}{p^2 + (q+k)^2} \\ &+ \frac{1}{2\eta^3} \left\{ \frac{p^2 - 7q^2 + 4qk}{p^2 + (q-k)^2} - \frac{p^2 - 7q^2 - 4qk}{p^2 + (q+k)^2} \right\} + \frac{1}{\eta} \left\{ \frac{p^2 - q^2 + qk}{\{p^2 + (q-k)^2\}^2} - \frac{p^2 - q^2 - qk}{\{p^2 + (q+k)^2\}^2} \right\}, \end{aligned} \quad (\text{A2.8})$$

where η is defined by

$$\eta = p^2 + q^2. \quad (\text{A2.9})$$

If p is put to zero, (A2.8) has a pair of singular points as a function of q . One of them is $q=k$, the other $q=0$. When we perform the integration over q in (A2.5) by means of (A2.8), it will be found that the former singularity at $q=k$ gives only a finite contribution just as in the case of (A2.7). Therefore the effect of this singular point may be neglected. The contribution from the latter singularity at $q=0$ to the integral (A2.5) is estimated as follows,

$$\begin{aligned} & \frac{8\pi^2 i}{k} p \int_0^{q_0} dq q F(q)^2 \left[\frac{\pi}{\eta^3} \left(3qp - \frac{q^3}{p} \right) - \frac{p^2 - 3q^2}{\eta^3} \log \frac{|k-q|}{k+q} \right. \\ & \quad \left. + \frac{q}{2\eta^3} \left(\frac{4k-7q}{(q-k)^2} + \frac{7q+4k}{(q+k)^2} \right) \right] \\ & \cong \frac{8\pi^2 i}{k} p F^2 \int_0^{q_0} dq q^2 \left[\frac{\pi}{\eta^3} \left(3p - \frac{q^2}{p} \right) + 2 \frac{(p^2 - 3q^2)}{k\eta^3} + \frac{4}{k\eta^2} \right], \end{aligned} \quad (\text{A2.10})$$

in which q_0 is an arbitrary positive number. The last two terms in the parentheses may be easily found to give only a finite contribution in the limit $p \rightarrow 0$, which is to be discarded. The first term is calculated by an elementary method and one may find that the diverging contributions in the limit $p \rightarrow 0$ cancel exactly.

Thus it is verified that (A2.1) gives no contribution at all.

Appendix 3. The list of the intermediate states which have nonvanishing matrix elements in the expression (3.25)

There are fourteen types of intermediate states which have nonvanishing matrix elements in the expression (3.25). We summarize here the wave functions of each state, making use of the notations defined in (3.9), (3.10) and (3.27).

$$\phi_1(\mathbf{p}_1 s, \mathbf{p}_2 s) = \frac{1}{\sqrt{(2M)!}} c_{\mathbf{p}_1 s}^* c_{\mathbf{p}_2 s} Y^M |0\rangle O(\mathbf{p}_1) I(\mathbf{p}_2)$$

$$\phi_{2t}(\mathbf{p}_1 s, \mathbf{p}_2 s) = \frac{1}{\sqrt{(2M-2)!}} c_{\mathbf{p}_1 s}^* c_{\mathbf{p}_2 s} Y^{M-1} \sum_{\mathbf{k}} \varphi_t(\mathbf{k}) b_{\mathbf{k}}^* |0\rangle O(\mathbf{p}_1) I(\mathbf{p}_2)$$

$$\begin{aligned} \phi_3(\mathbf{p}_1 s, \mathbf{p}_2 - s; \mathbf{k}_1 s', \mathbf{k}_2 s'') &= \frac{1}{\sqrt{(2M-2)!}} c_{\mathbf{p}_1 s}^* c_{\mathbf{p}_2 - s} Y_2^{M-1} c_{\mathbf{k}_1 s'}^* c_{\mathbf{k}_2 s''}^* |0\rangle \\ &\times O(\mathbf{p}_1) I(\mathbf{p}_2) E(\mathbf{k}_1) E(\mathbf{k}_2) \end{aligned}$$

$$\phi_4(\mathbf{p}_1 s, \mathbf{p}_2 s) = \frac{\sqrt{2}}{\sqrt{(2M)!}} c_{\mathbf{p}_1 s} Y_1^M c_{\mathbf{p}_2 s}^* |0\rangle I(\mathbf{p}_1) E(\mathbf{p}_2)$$

$$\phi_{5t}(\mathbf{p}_1 s, \mathbf{p}_2 s) = \frac{\sqrt{2}}{\sqrt{(2M-2)!}} c_{\mathbf{p}_1 s} Y_1^{M-1} \sum_{\mathbf{k}}' \varphi_t(\mathbf{k}) b_{\mathbf{k}}^* c_{\mathbf{p}_2 s}^* |0\rangle I(\mathbf{p}_1) E(\mathbf{p}_2)$$

$$\begin{aligned} \phi_6(\mathbf{p}_1 s, \mathbf{p}_2 s, \mathbf{k}_1 s', \mathbf{k}_2 - s') &= \frac{\sqrt{2}}{\sqrt{(2M-2)!}} c_{\mathbf{p}_1 s} Y_3^{M-1} c_{\mathbf{p}_2 s}^* c_{\mathbf{k}_1 s'}^* c_{\mathbf{k}_2 - s'}^* |0\rangle \\ &\times I(\mathbf{p}_1) E(\mathbf{p}_2) E(\mathbf{k}_1) E(\mathbf{k}_2) \end{aligned}$$

$$\phi_7(\mathbf{p}_1 s, \mathbf{p}_2 - s) = \frac{\sqrt{M}}{\sqrt{(2M-1)!}} c_{\mathbf{p}_1 s}^* Y_1^{M-1} c_{\mathbf{p}_2 - s}^* |0\rangle O(\mathbf{p}_1) E(\mathbf{p}_2)$$

$$\phi_{8t}(\mathbf{p}_1 s, \mathbf{p}_2 - s) = \sqrt{\frac{M-1}{(2M-3)!}} c_{\mathbf{p}_1 s}^* Y_1^{M-2} \sum_{\mathbf{k}}' \varphi_t(\mathbf{k}) b_{\mathbf{k}}^* c_{\mathbf{p}_2 - s}^* |0\rangle O(\mathbf{p}_1) E(\mathbf{p}_2)$$

$$\begin{aligned} \phi_9(\mathbf{p}_1 s, \mathbf{p}_2 - s, \mathbf{k}_1 s', \mathbf{k}_2 - s') &= \sqrt{\frac{M-1}{(2M-3)!}} c_{\mathbf{p}_1 s}^* Y_3^{M-2} c_{\mathbf{p}_2 - s}^* c_{\mathbf{k}_1 s'}^* c_{\mathbf{k}_2 - s'}^* |0\rangle \\ &\times O(\mathbf{p}_1) E(\mathbf{p}_2) E(\mathbf{k}_1) E(\mathbf{k}_2) \end{aligned}$$

$$\phi_{10t} = \frac{1}{\sqrt{(2M-2)!}} Y^{M-1} \sum_{\mathbf{k}} \varphi_t(\mathbf{k}) b_{\mathbf{k}}^* |0\rangle$$

$$\phi_{11t} = \frac{1}{\sqrt{(2M-4)!}} Y^{M-2} \sum_{\mathbf{k}\mathbf{k}'} \varphi_t(\mathbf{k}, \mathbf{k}') b_{\mathbf{k}}^* b_{\mathbf{k}'}^* |0\rangle$$

$$\phi_{12}(\mathbf{p}_1 s, \mathbf{p}_2 - s) = \frac{1}{\sqrt{(2M-2)!}} Y_2^{M-1} c_{\mathbf{p}_1 s}^* c_{\mathbf{p}_2 - s}^* |0\rangle E(\mathbf{p}_1) E(\mathbf{p}_2)$$

$$\phi_{13t}(\mathbf{p}_1 s, \mathbf{p}_2 - s) = \frac{1}{\sqrt{(2M-4)!}} Y_2^{M-2} \sum_{\mathbf{k}}' \varphi_t(\mathbf{k}) b_{\mathbf{k}}^* c_{\mathbf{p}_1 s}^* c_{\mathbf{p}_2 - s}^* |0\rangle E(\mathbf{p}_1) E(\mathbf{p}_2)$$

$$\begin{aligned} \phi_{14}(\mathbf{p}_1 s, \mathbf{p}_2 - s, \mathbf{k}_1 s', \mathbf{k}_2 - s') &= \frac{1}{\sqrt{(2M-4)!}} Y_4^{M-2} c_{\mathbf{p}_1 s}^* c_{\mathbf{p}_2 - s}^* c_{\mathbf{k}_1 s'}^* c_{\mathbf{k}_2 - s'}^* |0\rangle \\ &\times E(\mathbf{p}_1) E(\mathbf{p}_2) E(\mathbf{k}_1) E(\mathbf{k}_2) \end{aligned}$$

In these expressions each c^* factor which is put between Y_n and $|0\rangle$ must be interpreted to give a single particle excitation. The summation over \mathbf{k} with prime means to be extended over the electron pair states except those which are occupied by singly excited electrons. Y_n is then defined by

$$Y_n = \sum_{\mathbf{k}}' b_{\mathbf{k}}^*.$$

The intermediate states which give rise to the required divergence for (3.25) are such ones as are obtained by the correlation of pseudo-particle scattering type with the ground state. In other words, they are the states which are obtained from the ground state through one or two pair excitations. We can immediately see from the wave functions listed above that they are $\phi_{2i}, \phi_4, \phi_{5i}, \phi_7, \phi_{8i}, \phi_{10i}, \phi_{11i}, \phi_{12}$ and ϕ_{13i} . However, the contributions from $\phi_{2i}, \phi_{5i}, \phi_{8i}, \phi_{10i}, \phi_{11i}$ and ϕ_{13i} are so greatly reduced on account of the conditions (2.33) and (A1.3) that they may be neglected. Thus, it is necessary only to investigate the effects of the remaining three states ϕ_4, ϕ_7 and ϕ_{12} as was done in Section 3.

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On Levinson's Theorem in the Theory of Multi-Channel Scattering

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Using a Lee-type model, we have investigated Levinson's theorem in the case of multi-channel scatterings. In this case, contrary to the case of potential scatterings or Dyson's model, it seems that Levinson's theorem becomes ambiguous in interpreting the physical meaning of the result obtained.

§ 1. Introduction

Recently M. Ida¹⁾ showed, using Dyson's model,²⁾ that Levinson's theorem³⁾ which is well known in quantum mechanics does not generally hold in the quantized field theory and that this feature is related to the existence of the redundant solutions of Low's scattering equation.

Although the result acquired by Ida may be said to be an extension of the theorem to that in field theory, it lacks physical reality in dealing with the scattering phenomena of elementary particles, because we have only one channel for N - θ scattering in Dyson's model. Ida proved that for Dyson's model Levinson's theorem is modified as

$$\delta(\infty) - \delta(\mu) = (m_0 - m)\pi, \quad (1)$$

with μ the mass of θ -particle, m_0 and m the number of bare V -fields and that of N - θ bound states.¹⁾ It is of much interest whether such a physical interpretation is also possible for Levinson's theorem extended to the case of multi-channel scatterings. In the multi-channel case the phase shift is generally a complex quantity and this is an enough reason to expect that the equality (1) should be modified.

The purpose of this paper is to investigate in what form Levinson's theorem is reproduced in the case where many channels exist.

§ 2. Multi-channel scatterings and Levinson's theorem

For simplicity, let us consider a Lee-type model with two scattering channels, in which one kind of N -particle, two kinds of V -particles and two kinds of θ -particles are mutually interacting.

The Hamiltonian for this model is given by*

* It must be noted that in the present paper, for the sake of simplicity, only the magnitude of momentum is taken into consideration. It will be seen, however, that this simplification does not affect the main feature of the conclusion of our argument.

$$H = H_0 + H_1,$$

$$H_0 = \sum_{\lambda=1}^2 |v_\lambda\rangle M_\lambda \langle v_\lambda| + \sum_{\alpha=1}^2 \int_{m_\alpha}^{\infty} d\omega |\varphi_\alpha(\omega)\rangle \omega \langle \varphi_\alpha(\omega)|, \quad (2)$$

$$H_1 = \sum_{\lambda=1}^2 \sum_{\alpha=1}^2 g_{\alpha\lambda} \int_{m_\alpha}^{\infty} d\omega f_\alpha(\omega) [|v_\lambda\rangle \langle \varphi_\alpha(\omega)| + |\varphi_\alpha(\omega)\rangle \langle v_\lambda|],$$

where

M_λ : bare mass of V -particle,

$$\omega = M_0 + \sqrt{\mu_\alpha^2 + k^2}, \quad m_\alpha = M_0 + \mu_\alpha,$$

M_0 : mass of N -particle,

μ_α : mass of θ -particle, ($\mu_1 < \mu_2$),

$g_{\alpha\lambda}$: unrenormalized coupling constant,

$|v_\lambda\rangle$: state of a bare V -particle,

$|\varphi_\alpha(\omega)\rangle$: state of a pair N - θ ,

and the cut-off factor $f_\alpha(\omega)$ is related to the usual one, $v_\alpha(k)$, in the momentum representation as

$$f_\alpha(\omega) = \sqrt{k_\alpha} v_\alpha(k_\alpha) / 2\pi, \quad k_\alpha = \sqrt{(\omega - M_0)^2 - \mu_\alpha^2}. \quad (3)$$

We first make an eigenvalue equation for the N - θ bound state. From the Schrödinger equation

$$(H - E)|E\rangle = 0,$$

we obtain

$$\begin{aligned} (M_\lambda - E) \langle v_\lambda | E \rangle + \sum_{\alpha} g_{\alpha\lambda} \int_{m_\alpha}^{\infty} d\omega f_\alpha(\omega) \langle \varphi_\alpha(\omega) | E \rangle &= 0, \\ (\omega - E) \langle \varphi_\alpha(\omega) | E \rangle + \sum_{\lambda} g_{\alpha\lambda} f_\alpha(\omega) \langle v_\lambda | E \rangle &= 0, \end{aligned} \quad (4)$$

thereby

$$\sum_{\lambda=1}^2 A_{\nu\lambda}(E) \langle v_\lambda | E \rangle = 0, \quad (5)$$

where

$$A_{\nu\lambda}(E) \equiv (M_\nu - E) \delta_{\nu\lambda} - \sum_{\alpha} g_{\alpha\nu} g_{\alpha\lambda} \int_{m_\alpha}^{\infty} d\omega \frac{f_\alpha^2(\omega)}{\omega - E}. \quad (6)$$

Accordingly, the energy eigenvalues E_n for the N - θ bound states are given by the roots of the equation

$$D(E) = \det |A_{\nu\lambda}(E)| = 0. \quad (7)$$

Next the T -matrix for N - θ scattering, $N+\theta_\beta \rightarrow N+\theta_\alpha$, is calculated:

$$\begin{aligned} T_{\alpha\beta}(\omega) &= \langle \varphi_\alpha(\omega) | H_1 | \varphi_\beta^+(\omega) \rangle = - \langle \varphi_\alpha(\omega) | H_1 (H - \omega^+)^{-1} H_1 | \varphi_\beta(\omega) \rangle \\ &= -f_\alpha(\omega) f_\beta(\omega) \sum_{\nu, \lambda} g_{\alpha\nu} g_{\beta\lambda} \langle v_\nu | (H - \omega^+)^{-1} | v_\lambda \rangle, \end{aligned} \quad (8)$$

$$\omega^+ \equiv \omega + i\epsilon$$

where $|\varphi_\alpha^+(\omega)\rangle$ stands for an out-going solution of the Schrödinger equation. The expression

$$S_{\nu\lambda}'(\omega^+) \equiv \langle v_\nu | (H - \omega^+)^{-1} | v_\lambda \rangle,$$

which is regarded as a "propagator" of a clothed V -particle, can be evaluated in the following manner. We set

$$\begin{aligned} (H - \omega^+)^{-1} | v_\lambda \rangle &\equiv |\chi_\lambda(\omega^+)\rangle, \\ |v_\lambda\rangle &= (H - \omega^+) |\chi_\lambda(\omega^+)\rangle, \end{aligned}$$

and operate $\langle v_\nu |$ and $\langle \varphi_\alpha(\omega) |$ to the latter from the left respectively, then we have

$$\begin{aligned} \delta_{\nu\lambda} &= (M_\nu - \omega^+) \langle v_\nu | \chi_\lambda(\omega^+) \rangle + \sum_{\alpha} g_{\alpha\nu} \int_{m_\alpha}^{\infty} d\omega' f_\alpha(\omega') \langle \varphi_\alpha(\omega') | \chi_\lambda(\omega^+) \rangle, \\ 0 &= (\omega' - \omega^+) \langle \varphi_\alpha(\omega') | \chi_\lambda(\omega^+) \rangle + \sum_{\mu} g_{\alpha\mu} f_\alpha(\omega') \langle v_\mu | \chi_\lambda(\omega^+) \rangle, \end{aligned}$$

accordingly

$$\delta_{\nu\lambda} = \sum_{\mu} A_{\nu\mu}(\omega^+) \langle v_\mu | \chi_\lambda(\omega^+) \rangle = \sum_{\mu} A_{\nu\mu}(\omega^+) S_{\mu\lambda}'(\omega^+),$$

which shows that S' is the inverse matrix of A , namely,

$$S_{\nu\lambda}'(\omega^+) = \widetilde{A}_{\nu\lambda}(\omega^+) / D(\omega^+), \quad (9)$$

where $\widetilde{A}_{\nu\lambda}(\omega^+)$ is the cofactor of $A_{\nu\lambda}(\omega^+)$. $A_{\nu\lambda}$ and D have the same forms as those defined by (6) and (7) respectively. Substituting (9) into (8), we get the T -matrix for the elastic scattering ($\alpha = \beta$):

$$T_{\alpha\alpha}(\omega^+) = -f_\alpha^2(\omega) Q_\alpha(\omega^+) / D(\omega^+), \quad (10)$$

$$Q_\alpha(\omega^+) = \sum_{\nu, \lambda} g_{\alpha\nu} g_{\alpha\lambda} \widetilde{A}_{\nu\lambda}(\omega^+). \quad (11)$$

The T -matrix is also represented by the phase shift $\delta_\alpha(\omega)$ as

$$T_{\alpha\alpha}(\omega^+) = -\frac{1}{\pi} e^{i\delta_\alpha(\omega)} \sin \delta_\alpha(\omega). \quad (12)$$

Here it should be remarked that the phase shift $\delta_\alpha(\omega)$ is generally complex.

We are now in a position to investigate the relationship between the phase shift and the number of bound states. Define the function $F_\alpha(z)$ by

$$F_\alpha(z) = z Q_\alpha(z) / D(z), \quad (13)$$

where

$$\begin{aligned}
 D(z) &= (M_1 - z)(M_2 - z) - (M_1 - z) \sum_{\alpha} g_{\alpha 2}^2 J_{\alpha}(z) \\
 &\quad - (M_2 - z) \sum_{\alpha} g_{\alpha 1}^2 J_{\alpha}(z) + \kappa^2 J_1(z) J_2(z), \\
 Q_{\alpha}(z) &= g_{\alpha 1}^2 (M_2 - z) + g_{\alpha 2}^2 (M_1 - z) - \kappa^2 J_{\beta}(z), \quad (\alpha \neq \beta), \\
 \kappa^2 &\equiv (g_{11} g_{22} - g_{12} g_{21})^2, \\
 J_{\alpha}(z) &\equiv \int_{m_{\alpha}}^{\infty} d\omega \frac{f_{\alpha}^2(\omega)}{\omega - z}.
 \end{aligned}$$

$F_{\alpha}(z)$ is a function defined on the z -plane with a branch-cut from $z = m_1$ to $z = +\infty$ on the real axis, and we can show that it has the following properties:

- i) It is regular except for a finite number of poles (=zeros of $D(z)$).
- ii) It has a finite number of zeros (=zeros of $Q_{\alpha}(z)$ and $z=0$).
- iii) $\lim_{\substack{z \rightarrow \omega^+ \\ |z| \rightarrow \infty}} F_{\alpha}(z) = \omega e^{i\delta_{\alpha}(\omega)} \sin \delta_{\alpha}(\omega) / \pi f_{\alpha}^2(\omega)$, $\omega \geq m_{\alpha}$.
- iv) $\lim_{|z| \rightarrow \infty} z F'_{\alpha}(z) / F_{\alpha}(z) = 0$.

In addition, since it holds that for real ω

$$\begin{aligned}
 \lim_{\omega \rightarrow -\infty} Q_{\alpha}(\omega) &= +\infty, \\
 Q_{\alpha}'(\omega) &< 0, \quad \text{for } \omega < m_{\beta} (\alpha \neq \beta),
 \end{aligned}$$

we see for $\omega \leq m_{\beta}$ that

$$\begin{aligned}
 Q_{\alpha}(\omega) &\text{ has one zero, if } Q_{\alpha}(m_{\beta}) \leq 0, \\
 Q_{\alpha}(\omega) &\text{ has no zero, if } Q_{\alpha}(m_{\beta}) > 0.
 \end{aligned} \tag{14}$$

$Q_{\alpha}(\omega)$ has no zero in the range $\omega > m_{\beta}$ except for the special case of $\kappa^2 = 0$, because in this range $J_{\alpha}(\omega^+)$ has a non-vanishing imaginary part. Furthermore, we can show that there is no complex zero. (See Appendix.) From these facts we see that $Q_2(z)$ cannot have any zero in the interval $m_1 < \omega < m_2$ while $Q_1(z)$ may have one.

All the zeros of $D(z)$ on the real axis lie in the range $\omega < m_2$, and they give the energies of bound states. (See Appendix.) If $D(z)$ has one zero in the interval $m_1 < \omega < m_2$, it necessarily coincides with that of $Q_1(z)$. This can be seen as follows: If

$$D(\omega) = (M_1 - \omega)(M_2 - \omega) + [(M_1 - \omega)g_{22}^2 + (M_2 - \omega)g_{21}^2]J_2(\omega) + Q_1(\omega)J_1(\omega) = 0$$

for some value of ω in that interval, the imaginary part of this equality is

$$\text{Im } D(\omega) = Q_1(\omega) \cdot \pi f_1^2(\omega) = 0,$$

showing that $Q_1(\omega)$ is also zero. Accordingly, this zero of $D(z)$ is not a pole of $F_1(z)$.⁴⁾

Now the integration of the function $F'_{\alpha}(z)/F_{\alpha}(z)$ along the path indicated in Fig. 1 gives, due to Cauchy's theorem,

$$\frac{1}{2i} \oint_C dz \frac{F'_\alpha(z)}{F_\alpha(z)} = (\nu_\alpha - \bar{\nu})\pi, \quad (15)$$

where ν_α and $\bar{\nu}$ are the numbers of zeros of $zQ_\alpha(z)$ and $D(z)$, respectively, in the range $-\infty < z < m_1$ of the real axis. On the other hand, the contribution from the large circle vanishes owing to the property iv), and we have only to perform the integration along the path just above and under the branch cut.

The case $\alpha=1$. We can distinguish the following three cases:

1) There is one zero (denoted ω_0) of $Q_1(z)$ but no zero of $D(z)$ in the interval $m_1 < z < m_2$.

2) $Q_1(z)$ and $D(z)$ have one common zero in that interval.

3) There is no zero of $Q_1(z)$ in that interval.

In the case 1) we have

$$\begin{aligned} \frac{1}{2i} \oint_C dz \frac{F'_1(z)}{F_1(z)} &= \frac{1}{2i} \int_{m_1}^{\infty} d\omega \left[\frac{F'_1(\omega^+)}{F_1(\omega^+)} - \frac{F'_1(\omega^-)}{F_1(\omega^-)} \right] = \text{Re}[\delta_1(\infty) - \delta_1(m_1)] \\ &\quad + \lim_{\eta \rightarrow 0} \text{Im}[\{\log \sin \delta_1(\infty) - \log \sin \delta_1(\omega_0 + \eta)\} \\ &\quad + \{\log \sin \delta_1(\omega_0 - \eta) - \log \sin \delta_1(m_1)\}] - \pi \\ &= \text{Re}[\delta_1(\infty) - \delta_1(m_1)] + \arg \frac{\sin \delta_1(\infty)}{\sin \delta_1(m_1)} - \pi, \end{aligned}$$

where we have used the fact that $\delta_1(\omega)$ is real if $m_1 < \omega < m_2$ and that $\arg \sin \delta_1(\omega_0 + \eta) = \arg \sin \delta_1(m_2)$. In the case 2) and 3) the term $-\pi$ does not appear. In any case we can write

$$(N_1 - \bar{N})\pi = \text{Re}[\delta_1(\infty) - \delta_1(m_1)] + \arg \frac{\sin \delta_1(\infty)}{\sin \delta_1(m_2)}, \quad (16)$$

where N_1 is the number of zeros of $zQ_1(z)$ and \bar{N} the number of the bound states.

The case $\alpha=2$. In this case the integral interval is divided into a physical and unphysical regions ($m_1 < z < m_2$). For the latter the observed values of the phase shift are not given. We assume, therefore, that the values of the phase shift in the unphysical region are given by "analytic continuation". This enables us to make use of the property iii) of $F_2(z)$ in the interval $m_1 < z < m_2$. Now $Q_2(z)$ has no zero in the interval $m_1 < z < m_2$, while the zero of $D(z)$, if exists in

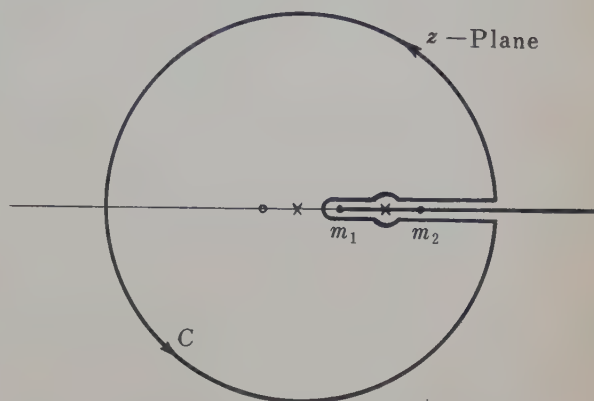


Fig. 1. The zero (O) of $zQ_\alpha(z)$ and the zeros (X) of $D(z)$.

this interval, survives. We have then two distinguished cases:

1) $D(z)$ has one zero (denoted ω_0) in the interval $m_1 < z < m_2$.

$$(N_2 - \bar{N})\pi = \text{Re}[\delta_2(\infty) - \delta_2(m_1)] + \lim_{\eta \rightarrow 0} \arg \left[\frac{\sin \delta_2(\infty)}{\sin \delta_2(\omega_0 + \eta)} + \frac{\sin \delta_2(\omega_0 - \eta)}{\sin \delta_2(m_1)} \right]. \quad (17)$$

($\text{Re } \delta_2$ has no pole at ω_0)

2) $D(z)$ has no zero in that interval.

$$(N_2 - \bar{N})\pi = \text{Re}[\delta_2(\infty) - \delta_2(m_1)] + \arg \frac{\sin \delta_2(\infty)}{\sin \delta_2(m_1)}. \quad (18)$$

Here N_2 and \bar{N} stand for the number of zeros of $zQ_2(z)$ and that of the bound states.

The above mentioned “*analytic continuation*” is not possible in the strict sense, because we may have only a finite number of observed values of the phase shift. Nevertheless, we can expect that an approximate continuation would be possible taking advantage of the facts that $\delta_2(\omega)$ must have branch points at m_1 and m_2 , and that $F_2(z)$ has poles at the energies of the bound states. (It would be worth noting that a simple extrapolation from the low energy region is never valid because $\omega = m_2$ is a branch point.)

§ 3 Conclusion

We could see, in the two-channel case, how phase shifts are related to the number of bound states. The number N_α ($\alpha=1, 2$) which appear in (16) or (17) and (18) is the number of zeros of $zQ_\alpha(z)$ and it is seen from (14) to be 2 or 1, which is equal to or smaller by one than the number of the bare V -fields. Thus we can not generally interpret N_α as the number of the bare V -fields. The situation will become much more complicated in the general case when we have many bare V -fields, and we are compelled to reach a negative conclusion that Levinson's theorem has nothing to do with determining the number of discrete eigenvalues of the free Hamiltonian. Besides, in the scatterings of the heavier θ -particles we meet with the difficulty of *analytic continuation* of the phase shifts.

Calculating the K -matrix and the elastic or inelastic scattering cross sections, we also see that the phase shifts and the cross sections have the possibility of showing the so-called “*cusplike behavior*”⁵⁾ at the threshold of the other channel. Appearance of these features is due to the integral $\text{Re } J_\alpha(\omega^+)$ in (13). Discussions on other problems, say, the effective range expansion in multi-channel scatterings, etc., are omitted, being beyond the range of our purpose.

In conclusion the authors would like to express their sincere thanks to Prof. R. Utiyama for his continual encouragement and suggestions. They are also indebted to Messrs. K. Yamamoto and R. Sugano for many helpful discussions.

Appendix

We show that $Q_\alpha(z)$ has no complex zero. Set $z=x+iy$, then the imaginary part of $Q_\alpha(x+iy)$ is

$$y \left[-g_{\alpha 1}^2 - g_{\alpha 2}^2 - \kappa^2 \int_{m_\beta}^{\infty} d\omega \frac{f_\beta^2(\omega)}{(\omega-x)^2 + y^2} \right]. \quad (\text{A} \cdot 1)$$

If y does not vanish, this is different from zero. (Q. E. D.)

Next, it is shown that $D(\omega^+)$ has no zero in the range $\omega > m_2$. For this purpose we rewrite $D(\omega^+)$ as

$$\begin{aligned} D(\omega^+) &= \kappa^2 \left[J_1(\omega^+) - \frac{1}{\kappa^2} g_{21}^2(M_2 - \omega) - \frac{1}{\kappa^2} g_{22}^2(M_1 - \omega) \right] \\ &\quad \times \left[J_2(\omega^+) - \frac{1}{\kappa^2} g_{11}^2(M_2 - \omega) - \frac{1}{\kappa^2} g_{12}^2(M_1 - \omega) \right] \\ &\quad - \frac{1}{\kappa^2} [g_{21}g_{11}(M_2 - \omega) - g_{22}g_{12}(M_1 - \omega)]^2 \\ &\equiv A(\omega^+)B(\omega^+) - C^2(\omega) = 0 \end{aligned} \quad (\text{A} \cdot 2)$$

and divide A and B into real and imaginary parts:

$$\left. \begin{aligned} A &= a + ia', & a' &= \kappa\pi f_1^2(\omega) > 0 \\ B &= b + ib', & b' &= \kappa\pi f_2^2(\omega) > 0 \end{aligned} \right\} \quad \text{for } \omega > m_2. \quad (\text{A} \cdot 3)$$

Then we have

$$\begin{aligned} \text{Re } D(\omega^+) &= ab - a'b' - c^2 = 0, \\ \text{Im } D(\omega^+) &= ab' + a'b = 0. \end{aligned} \quad (\text{A} \cdot 4)$$

The latter gives that a and b have the sign opposite to each other, while the former gives $ab > 0$. This contradiction indicates that $D(\omega^+)$ has no zero in the range $\omega > m_2$.

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Equilibrium Deformation of Ra^{226}

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The Hamiltonian of a single particle moving in a pear-shaped potential has been discussed. In order to investigate the effect of spin-orbit coupling in a field of octupole deformation a simpler radial form has been suggested. A Nilsson's Representation is chosen and eigenvalues are calculated using second order perturbation. It has been shown that in order to explain the occurrence of the 253 Kev (1-) level in Ra^{226} , the octupole deformation parameter a_3 comes out to be 0.02.

§ 1. Introduction

The systematic occurrence of relatively lowlying odd parity rotational states¹⁾ of even nuclei in the neighbourhood of Radium, suggests an odd parity vibration, most likely the octupole vibration. This, in its turn, raises the interesting question that whether an egg-shaped nucleus is more stable than the more familiar spheroidal one. It has already been shown²⁾ that, adding up the individual particle energies in an anisotropic harmonic oscillator potential, the spheroidal shape becomes more stable than a spherical shape.

Lee and Inglis³⁾ have calculated the energies of the nucleon moving in an average potential, whose surfaces run parallel to that of a pear-shaped nucleus. They have come to the conclusion that although a pear-shape is not energetically stable, the difference of energy is not much between a prolate spheroid and a pear-shaped nucleus. But there are several approximations in their calculations, and they have neglected the spin orbit coupling term, which will considerably affect the energies of individual nucleons.

§ 2. Mathematical formulation

We have devised a formalism in which the effect of spin-orbit interaction on the energies of the nucleons moving in a pear-shaped potential can be easily calculated.

The equipotential surface is given by the equation

$$r' = \sqrt{\frac{2V}{M\omega_0^2}} [1 + a_1 Y_{10}(\theta' \phi') + a_2 Y_{20}(\theta' \phi') + a_3 Y_{30}(\theta' \phi')]^{-1/2} \quad (1)$$

with usual notations.

To fix the center of mass of this surface one gets

$$a_1 \sim \frac{9\sqrt{3}}{2\sqrt{35\pi}} a_2 a_3 \quad (2)$$

while for volume preservation,

$$\omega_0^2 \sim \omega_0^{02} \left[1 + \frac{5}{16\pi} (a_2^2 + a_3^2) \right], \quad (3)$$

neglecting higher order terms in a_1 , a_2 and a_3 . In (3) ω_0^0 corresponds to the angular frequency in a spherical nucleus. Using (2) and (3), we can write (1) as

$$V(r'\theta') = \frac{1}{2} M \omega_0^{02} r'^2 \left\{ 1 + \frac{5}{16\pi} (a_2^2 + a_3^2) \right\} \cdot \\ \left[1 + \frac{9\sqrt{3}}{2\sqrt{35\pi}} a_2 a_3 Y_{10}(\theta'\phi') + a_2 Y_{20}(\theta'\phi') + a_3 Y_{30}(\theta'\phi') \right]. \quad (4)$$

The total Hamiltonian of the particle is given by

$$H_{\text{pear}} = -\frac{\hbar^2}{2M} \nabla'^2 + V(r'\theta') + C\mathbf{l} \cdot \mathbf{s} + D\mathbf{l}^2 \quad (5)$$

where C and D are explained in reference 2).

When $a_3=0$, (5) reduces to the familiar Nilsson Hamiltonian for spheroidal nucleus. Thus from (4) and (5) we get after co-ordinate transformation

$$H_{\text{pear}} - H_{\text{spheroid}} = -\frac{\hbar\omega_0^0}{2} \cdot \frac{5}{32\pi} a_3^2 (\nabla^2 - r^2) + \frac{\hbar\omega_0^0}{2} \cdot \frac{5}{32\pi} a_3^2 a_2 r^2 Y_{20}(\theta\phi) \\ - \chi \hbar\omega_0^0 \frac{4}{3} \sqrt{\frac{\pi}{5}} \eta r^2 \left[\frac{9\sqrt{3}}{2\sqrt{35\pi}} a_3 Y_{10}(\theta\phi) + \frac{a_3}{a_2} Y_{30}(\theta\phi) \right] \quad (6)$$

where in order to compare with Nilsson's²⁾ Hamiltonian we have put for convenience

$$\frac{1}{2\chi} a_2 \left[1 + \frac{5}{32\pi} a_3^2 \right] = -\frac{4}{3} \sqrt{\frac{\pi}{5}} \eta \quad (7)$$

and $a_2 \sim -2\delta$ with $\chi=0.05$ as suggested by Nilsson.

Since the Z -component of the total angular momentum of the particle j_z commutes with (5), the eigen-states can be specified by

$$|j_z\rangle = \sum_N a_N |Nj_z\rangle \quad (8)$$

where $|Nj_z\rangle$ are the Nilsson wave function in a spheroidal potential. The extra energy due to the pear-shaped deformation can be found from the equation

$$a_N' E_2 = \sum_j a_N \langle Nj_z | H_{\text{pear}} - H_{\text{spheroid}} | N'j_z \rangle \quad (9)$$

by exact diagonalization. Whereas with use of perturbation calculations, this extra energy will be given by

$$\begin{aligned} \frac{\Delta E}{\hbar\omega_0^0} = \sum_i & \left[\left(N_i + \frac{3}{2} \right) \frac{5}{32\pi} a_3^2 + \chi \langle N_i j_z | \frac{5}{64\pi} a_3^2 a_2 r^2 Y_{20}(\theta\phi) | N_i j_z \rangle \right] \\ & + \chi \sum_{N_i' N_i''} \frac{|\langle N_i j_z | K \left[\frac{9\sqrt{3}}{2\sqrt{35}\pi} a_3 r^2 Y_{10}(\theta\phi) + \frac{a_3}{a_2} r^2 Y_{30}(\theta\phi) \right] | N_i' j_z \rangle|^2}{E_{N_i' j_z} - E_{N_i'' j_z}} \quad (10) \\ & (N_i' = N_i \pm 1) \end{aligned}$$

where

$$K \simeq -\frac{4}{3} \sqrt{\frac{\pi}{5}} \gamma$$

and $E_{N_i j_z}$ are the corresponding Nilsson's eigen-values. In Equation (10) summation over i denotes addition of the single particle energies for a particular configuration.

§ 3. Numerical calculations

With use of the above formulation, the difference in energy between a pear-shaped and a spheroidal nucleus for Ra^{226} is calculated. The equilibrium quadrupole deformation for Ra^{226} , using Nilsson's eigen values, is $a_2 = -0.4191$ which corresponds to $\gamma = +4$.

The energy for a single nucleon moving in such a pear-shaped potential is plotted against a_3 in Fig. 1. It is obvious that the introduction of the pear-shaped deformation markedly affects the single particle energy levels.

We have calculated the expectation values of the second and third terms in (6) using first and second order perturbation treatment. In order to check the accuracy of our perturbation calculation we have also calculated the same for $j_z = 11/2$ and $9/2$, by exact diagonalization of (9). The

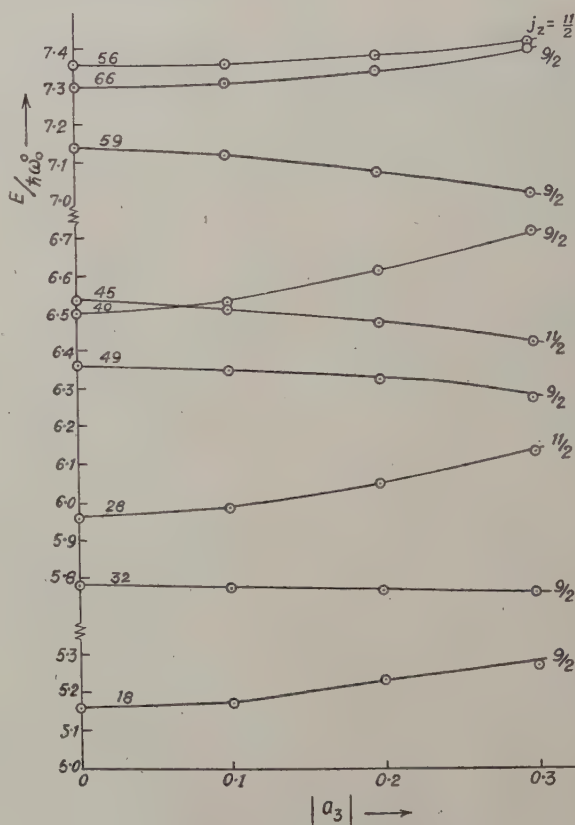


Fig. 1. Single particle energy levels in a pear-shaped potential.

results are tabulated below for $a_3=0.1$, in units of $\chi\hbar\omega_0^0$.

	$j_z=11/2$	By exact diagonalization	From perturbation calculation
Nilsson's Orbit No.	56	+0.036	+0.035
	45	-0.524	-0.553
	28	+0.497	+0.528

It can be seen that the difference between the two calculations is very small. For $j_z \leq 7/2$, the matrix that is to be handled is of very high order which makes it rather difficult to solve by exact diagonalization.

Assuming a configuration as given by Nilsson for $\gamma=+4$ for Ra^{226} , the total difference in energy, ΔE , is calculated by trace-technique. The sum of the traces of the matrices for different j_z corresponding to all the seventy-five orbits of Nilsson have been found. The contribution to this energy from the particles in unfilled levels in Ra^{226} , as calculated by perturbation treatment, is then subtracted. The required energy difference comes out to be

$$\Delta E/\hbar\omega_0^0 = 50.4759 a_3^2 + 45.2103 a_3^3 \quad (11)$$

where the first term comes from volume preservation and the second term from the pear-shaped deformation.

Thus since both terms in (11) are positive for any value of a_3 , the difference in energy is always positive. In order to excite the 253 Kev (1-) level in Ra^{226} , (11) gives $a_3 \sim 0.02$.

§ 4. Discussion

Thus we have found that introduction of the spin-orbit interaction, although greatly affects the single particle energy levels, does not make any further improvement on Inglis³⁾ results (in case of Ra^{226}). But similar calculations for other regions (e.g., Sm^{162}) are necessary before one can draw any definite conclusions. Our calculations reveal that in the Ra^{226} region the 1- level should be looked upon as octupole vibration about a spheroidal equilibrium shape. It will be interesting to see whether the altered wave function can explain the moment of inertia of the odd parity bands, which remain unexplained until now¹⁾.

Recently Suekane⁴⁾ has calculated the octupole rigidity of some hypothetical nuclei, and found that, although C_2 (quadrupole rigidity) are very large at the closed shells of magic numbers, C_3 (the octupole rigidity) are not always large here. Hence he has concluded that the nuclei in the neighbourhood of these closed shells, the lowlying levels due to octupole type collective oscillations may be observed. But unfortunately he has not carried out any calculations in Ra-region, where this type of oscillation is more pronounced.

It may be mentioned that our estimation of a_3 (~ 0.02) agrees remarkably with that of Fröman⁵⁾ from his considerations on odd parity α -transition of even-even nuclei in this region.

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Superexchange Interaction

—The Four Electron Model—

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The superexchange interaction of the four-electron model is investigated, including various excited configurations. The only assumption used is that the overlap between atomic orbitals is small. It is shown that the superexchange interaction of the four-electron model can be represented by a spin Hamiltonian $2J\mathbf{s}_A\mathbf{s}_B$, even when the non-orthogonality of wave functions is taken into account, and even when various mechanisms of superexchange interaction are included simultaneously in the effective interaction J . It is also shown that single-configuration approximation like that used in the previous paper (I) gives a result which is very close to the correct one in the four-electron model. We argue, therefore, that as long as we look for the difference in energy between the parallel and the ordered antiparallel states, the essential features are involved in single-configuration approximation even when it is applied to real crystals as in (I).

§ 1. Introduction

Since Anderson¹⁾ discussed the origin of superexchange interaction, various models which lead to spin dependent energy of ionic crystals have been proposed.²⁾ However, these discussions are limited to qualitative understanding of the interaction and not suited for application to real crystals. On the other hand, Yamashita and the author^{3)**} proposed a method which can apply to real crystals in evaluating a numerical value of the spin dependent part of the crystal energy.***

In this approach we have made three assumptions: (1) Overlap between ions is small. (2) The ground state of an antiferromagnet is close to the ordered antiparallel spin state. (3) Instead of configuration interaction a single-configuration approximation is adopted. We have seen that the first assumption is well satisfied in MnO. We see, however, that the second and the third assumptions are fundamental in our approach, and they cannot be removed without serious difficulties. In order to express the ground state of an antiferromagnet as a linear combination of various states of different spin configurations it is necessary to know the transi-

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** We shall refer to this paper as (I) in the following.

*** The numerical results of MnO will appear in Progress of Theoretical Physics Vol. 23, No. 1 (1960).

tion matrix between these states. But as is well known, it is very difficult to obtain it in the case of N -body problem owing to non-orthogonality of wave functions. Moreover, the configuration interaction in N -body problem gives rise to a divergence difficulty when carried out to higher order processes.

In the case of four-electron model, however, we can both carry out the configuration interaction and calculate the energy of the singlet state as well as the ordered antiparallel state. Thus it may serve to give some idea for the approximation that we adopted in the N -body problem to compare the energy of the state of single-configuration approximation with that of configuration mixing and to have some relation among the energies of the triplet, the ordered antiparallel and the singlet states in the case of the four-electron model.

In § 2 we see that in the ground configuration (the so-called totally ionic state) the energy difference between the triplet and the antiparallel state is the same as that between the antiparallel and the singlet state.

In § 3 we carry out the configuration mixing and see that the energies of the parallel, the ordered antiparallel and the singlet states are separated by an equal amount from each other. Therefore the spin interaction of the four-electron model can be represented by an equivalent spin Hamiltonian, $2J\mathbf{s}_A\mathbf{s}_C$, J being the magnitude of the energy separation.* In § 4 we obtain the energy difference between the parallel and the ordered antiparallel states in the single-configuration approximation used in (I) and see that it is very close to the above J .

§ 2. Totally ionic states

Consider three nuclei, A , B and C , in a row with distance a . Let $\varphi_s(\mathbf{r})$ be an atomic orbital of s type belonging to the atom A , $\varphi_p(\mathbf{r})$ be that of p type belonging to B , and $\varphi'_s(\mathbf{r})$ be that of s type belonging to C . These are real orbitals. We take into account the following two overlap integrals and assume that the overlap between φ_s and φ'_s is negligibly small:

$$\int \varphi_p(\mathbf{r})\varphi_s(\mathbf{r})d\mathbf{v} = - \int \varphi_p(\mathbf{r})\varphi'_s(\mathbf{r})d\mathbf{v} = S. \quad (1)$$

We assume that in the ground state atoms A and C are occupied by a single electron and atom B is filled with two electrons of opposite spin. The total Hamiltonian of the system is

$$H = \Sigma_i H_1(i) + \Sigma_{i \neq j} e^2 / 2r_{ij}, \quad (2)$$

where

$$H_1(i) = p_i^2 / 2m - e^2 \Sigma_g Z_g / r_{ig}. \quad (3)$$

* This was also shown by F. Keffer and T. Oguchi in the case of the Anderson model, using a spin operator method. They have shown that Anderson's method to treat a four-electron model can be extended even when the non-orthogonality of wave functions is taken into account. The author wishes to express his thanks to them for sending a preprint of their investigation before publication.

Here i and j specify electrons and g a nucleus, and r_{ig} means the distance between the i -th electron and the g -th nucleus whose atomic number is denoted by Z_g . On the other hand, r_{ij} is the distance between the i -th and j -th electrons.

Let us consider a parallel spin state ϕ_n and two antiparallel spin states ϕ_1 and ϕ_2 , given by

$$\phi_n = (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_s' \alpha \varphi_p' \beta|, \quad (4)$$

$$\phi_1 = (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_p' \beta \varphi_s' \beta|, \quad (5)$$

$$\phi_2 = (4!)^{-1/2} |\varphi_s' \alpha \varphi_p \alpha \varphi_p' \beta \varphi_s \beta|. \quad (6)$$

The average energies of these states are given by⁴⁾

$$\begin{aligned} E = & \sum_i (i|i) + \frac{1}{2} \sum_{ij} [(ijij) - (ijji)] \\ & - \sum_{ij} P_{ij} \{ (j|i) + \sum_m [(jmim) - (jmmi)] \} \\ & + \frac{1}{2} \sum_{ijmn} P_{im} P_{jn} [(mni j) - (mnji)], \end{aligned} \quad (7)$$

$$\text{where} \quad (i|j) = \int \varphi_i(1) H_1(1) \varphi_j(1) dx_1, \quad (8)$$

$$(ijmn) = \int \varphi_i(1) \varphi_j(2) (e^2/r_{12}) \varphi_m(1) \varphi_n(2) dx_1 dx_2, \quad (9)$$

$$P_{ij} = \delta_{ij} - (1+S)^{-1}_{ij}, \quad (10)$$

$$S_{ij} = \int \varphi_i(1) \varphi_j(1) dx_1 - \delta_{ij}. \quad (11)$$

Here i, j, m and n specify atomic orbitals including spin direction. The summations over i, j, m and n should be taken over all orbitals which are contained in the wave function in question. As in (I), we expand the average energy in powers of S ,* assuming a rapid convergence. Then up to the second order terms we obtain

$$E = \sum_i (i|i) + \sum_{i>j} [(ijij) - (ijji)] + \sum_{i>j} B(i, j), \quad (12)$$

$$\begin{aligned} \text{where} \quad B(i, j) = & -2S_{ij} [(j|i) + \sum_{m \neq i} (jmim)] \\ & + S_{ij}^2 [(i|i) + (j|j) + \sum_{m \neq i} (imim) + \sum_{m \neq j} (jmjm) - (ijji)]. \end{aligned} \quad (13)$$

Therefore, up to the second order of S the average energy of the states ϕ_n , ϕ_1 and ϕ_2 are expressed as follows:

$$\begin{aligned} E_n = & (\phi_n H \phi_n) / (\phi_n \phi_n) = e_0 + B(p\alpha, s\alpha) + B(p\alpha, s'\alpha) - (p\alpha s\alpha s\alpha p\alpha) \\ & - (p\alpha s'\alpha s'\alpha p\alpha), \end{aligned} \quad (14)$$

$$\begin{aligned} E_1 = & (\phi_1 H \phi_1) / (\phi_1 \phi_1) = e_0 + B(p\alpha, s\alpha) + B(p\beta, s'\beta) - (p\alpha s\alpha s\alpha p\alpha) \\ & - (p\beta s'\beta s'\beta p\beta), \end{aligned} \quad (15)$$

* We define the order of magnitude of matrix elements as follows: $(p|s)$, $(ps'ss')$, $(psss)$ and $(pspp)$ are first order quantities in S and $(pspp)$ is a second order quantity.

$$E_2 \equiv (\psi_2 H \psi_2) / (\psi_2 \psi_2) = e_0 + B(p\alpha, s'\alpha) + B(p\beta, s\beta) - (p\alpha s'\alpha s'\alpha p\alpha) - (p\beta s\beta s\beta p\beta), \quad (16)$$

where

$$e_0 = 2(s|s) + 2(p|p) + 4(psps) + (ss'ss') + (pppp). \quad (17)$$

It can be easily seen that these three expressions have the same value. As was shown in (I), the difference in energy between ψ_{11} and ψ_1 (or ψ_2) arises from a fourth order part of the expansion of (7). It was already calculated in (I) and was given in Eq. (15) of that article.

Next, let us calculate the energy difference between ψ_1 and the singlet state. The singlet state and the triplet state with $M_s=0$ are given, respectively, by

$$\psi_s = 2^{-1/2}(\psi_1 + \psi_2), \quad (18)$$

$$\psi_t = 2^{-1/2}(\psi_1 - \psi_2). \quad (19)$$

By using the definition of E_1 , we have the average energies of these states

$$E_s = (\psi_s H \psi_s) / (\psi_s \psi_s) = E_1 - [E_1(\psi_1 \psi_2) - (\psi_1 H \psi_2)], \quad (20)$$

$$E_t = (\psi_t H \psi_t) / (\psi_t \psi_t) = E_1 + [E_1(\psi_1 \psi_2) - (\psi_1 H \psi_2)]. \quad (21)$$

From (20) and (21) we see

$$E_t - E_s = E_1 - E_s = E_1(\psi_1 \psi_2) - (\psi_1 H \psi_2). \quad (22)$$

Non-diagonal matrix elements such as $(\psi_1 H \psi_2)$ can be obtained straightforwardly,⁵⁾ as long as the four-electron problem is concerned. We obtain:

$$(\psi_1 \psi_2) = S^4 \quad (23)$$

$$\text{and} \quad (\psi_1 H \psi_2) = 4S^3(p|s) - 2S^2[2(pss'p) - (pssp)]. \quad (24)$$

Substituting these expressions into (22), we see that it is the same as (15) of (I), as should be the case. Thus, we have seen that the interaction between spins of the atoms A and C can be represented by a spin Hamiltonian $2J\mathbf{s}_A\mathbf{s}_C$, where J is given by (22).

§ 3. Configuration interaction

Next, we mix various excited states into the ground state, which we classify into two categories: (1) One or two electrons transfer from an atom to another or two other atoms. (2) An electron of an atom is excited to a higher orbital belonging to the same atom. In the former states we assume that φ_s or φ_s' accommodates the transferring electron and in the latter we consider only an s type orbital (denoted by φ_c) belonging to the atom B as the excited orbital. In the following we shall show a list of all excited states which are compatible with $M_s=0$.

$$\begin{aligned}
 \phi_3 &= (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_s \beta \varphi_s' \beta| \\
 \phi_4 &= (4!)^{-1/2} |\varphi_s \alpha \varphi_s' \alpha \varphi_p \beta \varphi_s' \beta| \\
 \phi_5 &= (4!)^{-1/2} |\varphi_s \alpha \varphi_s' \alpha \varphi_p \beta \varphi_s \beta| \\
 \phi_6 &= (4!)^{-1/2} |\varphi_s' \alpha \varphi_p \alpha \varphi_s \beta \varphi_s' \beta| \\
 \phi_7 &= (4!)^{-1/2} |\varphi_s \alpha \varphi_s' \alpha \varphi_s \beta \varphi_s' \beta| \\
 \phi_8 &= (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_p \beta \varphi_s \beta| \\
 \phi_9 &= (4!)^{-1/2} |\varphi_s' \alpha \varphi_p \alpha \varphi_p \beta \varphi_s' \beta| \\
 \phi_a &= (4!)^{-1/2} |\varphi_s \alpha \varphi_c \alpha \varphi_p \beta \varphi_s' \beta| \\
 \phi_b &= (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_c \beta \varphi_s' \beta| \\
 \phi_c &= (4!)^{-1/2} |\varphi_s' \alpha \varphi_c \alpha \varphi_p \beta \varphi_s \beta| \\
 \phi_d &= (4!)^{-1/2} |\varphi_s' \alpha \varphi_p \alpha \varphi_c \beta \varphi_s \beta| \\
 \phi_e &= (4!)^{-1/2} |\varphi_s \alpha \varphi_s' \alpha \varphi_p \beta \varphi_c \beta| \\
 \phi_f &= (4!)^{-1/2} |\varphi_p \alpha \varphi_c \alpha \varphi_s \beta \varphi_s' \beta|
 \end{aligned} \tag{25}$$

We calculate the energy of the system which is now expressed as a linear combination of these states by the perturbation method. As we shall see later, the excited states are divided into two classes: (1) the matrix element with the ground state is of order S , and (2) is of order S^2 . The second order perturbed energy from the latter states is of order S^4 , while it is of order S^2 from the former states. We carry the perturbation process to the third order for the former states and carry it to the second order for the latter, in order to obtain a consistent approximation which involves all terms up to the fourth order of S .

We first calculate the energy of the ordered antiparallel state. For its ground state we take ϕ_1 (Eq. (5)). States ϕ_3 and ϕ_4 have first order matrix elements with ϕ_1 and $\phi_7, \phi_8, \phi_9, \phi_a, \phi_b, \phi_e$ and ϕ_f have the second order ones. Then we assume the ordered antiparallel state is expressed as

$$\phi_{ap} = \phi_1 + \sum_i \lambda_i \phi_i + \sum_j \mu_j \phi_j. \tag{26}$$

Here the summations over i and j are to be taken over $i=3, 4$ and $j=7, 8, 9, a, b, e$ and f , respectively. We shall later see that λ 's are of order S and μ 's are of order S^2 . Then the following expression for the energy of the state, ϕ_{ap} , is correct up to S^4 terms.

$$\begin{aligned}
 E_{ap} &= (\phi_{ap} H \phi_{ap}) / (\phi_{ap} \phi_{ap}) \\
 &= E_1 + (1/S_1) \sum_i [2\lambda_i (H_{1i} - E_1 S_{1i}) + \lambda_i^2 (H_i - E_1 S_i)] \\
 &\quad + 2 \sum_{i>j} \lambda_i \lambda_j (H_{ij} - E_1 S_{ij}) \\
 &\quad - [\sum_i (2\lambda_i S_{1i} + \lambda_i^2 S_i)] \sum_i [2\lambda_i (H_{1i} - E_1 S_{1i}) + \lambda_i^2 (H_i - E_1 S_i)] \\
 &\quad + \sum_j \{ 2\mu_j [H_{1j} - E_1 S_{1j} + \sum_i \lambda_i (H_{ij} - E_1 S_{ij})] + \mu_j^2 (H_j - E_1 S_j) \}.
 \end{aligned} \tag{27}$$

Here the meaning of the notations is :

$$H_{ij} = (\psi_i H \psi_j), \quad S_{ij} = (\psi_i \psi_j),$$

$$H_i = H_{ii}, \quad S_i = S_{ii}, \quad E_i = H_i / S_i.$$

The right-hand side of (27) is separated into three parts: S^0 , S^2 and S^4 parts. We determine λ_i so as to minimize the S^2 parts and insert the value of λ_i so determined into the S^4 parts. The error caused in this process is higher than S^4 . The S^2 part is minimum with respect to λ_i when

$$\lambda_3 = -\lambda_4 = \lambda \equiv -A/B, \quad (28)$$

where

$$A = (p|s) + (psss) + (ppps) + (ps'ss') - S[(p|p) + 2(psp) + (pppp)], \quad (29)$$

$$B = (s|s) - (p|p) - (spsp) + (ssss) + (ss'ss') - (pppp). \quad (30)$$

Then the first line of (27) becomes

$$-2\Delta E - 2J_2 + 2J_2', \quad (31)$$

where

$$\Delta E = A^2/B \quad (32)$$

$$J_2 = (2\lambda S + \lambda^2)[B(p, s) - (pssp)] + S^2\Delta E, \quad (33)$$

$$J_2' = 2\lambda[-2S(pssp) + 3S^2(ppps) + S^2(psss) - S^3(pppp) - S^3(psp)]$$

$$+ \lambda^2[-2S(psss) + 2S(ppps) + S^2(ssss) - S^2(pppp)]. \quad (34)$$

The meaning of these terms will be given later.

The second line of (27) is written as

$$2\lambda_3\lambda_4(H_{34} - E_1S_{34}) = -2\lambda^2(H_{34} - E_1S_{34}). \quad (35)$$

This term is due to the third order process of perturbation, that is, $\psi_1 \rightarrow \psi_3 \rightarrow \psi_4 \rightarrow \psi_1$ and is of the fourth order of S , because $H_{34} - E_1S_{34}$ is of the second order. The third line of (27) becomes simply $4\Delta E(2\lambda S + \lambda^2)$, which is due to the normalization condition. The minimum value of the last line of (27) with respect to μ 's is easily obtained as

$$-\Sigma_j[(H_{1j} - E_1S_{1j}) + \Sigma_i\lambda_i(H_{ij} - E_1S_{ij})]^2/(E_j - E_1). \quad (36)$$

These terms are the energy gains which result from mixing configurations $\psi_7, \psi_8, \psi_9, \psi_a, \psi_b, \psi_c$ and ψ_f . Here, $H_{1j} - E_1S_{1j}$ is the direct transition matrix from ψ_1 to ψ_j and $\lambda_i(H_{ij} - E_1S_{ij})$ comes from second order process, first to ψ_i then to ψ_j . This is the only effect which occurs when various mechanisms are included simultaneously.

Next we calculate the energy of the triplet state. We take for the ground state (4) rather than (19), although the result does not depend on the choice. Then all the excited states which are connected with ψ_n are given by

$$\begin{aligned}
\psi_{t3} &= (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_s' \alpha \varphi_s \beta| \\
\psi_{t4} &= (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_s' \alpha \varphi_s' \beta| \\
\psi_{t6} &= (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_c \alpha \varphi_s' \beta| \\
\psi_{tf} &= (4!)^{-1/2} |\varphi_s' \alpha \varphi_p \alpha \varphi_c \alpha \varphi_s \beta|.
\end{aligned} \tag{37}$$

We assume the triplet state is given as follows :

$$\psi_t(\text{C. I.}) = \psi_{t1} + \lambda_3 \psi_{t3} + \lambda_4 \psi_{t4} + \mu_e \psi_{t6} + \mu_f \psi_{tf}. \tag{38}$$

The average energy of the state (38) is given also by (27), where matrix elements are those between triplet states. The term which corresponds to (31) is

$$-24E + 2J_2' + 2J_3, \tag{39}$$

where

$$\begin{aligned}
J_3 &= 2\lambda[2S(ppss') + 3S^2(ppps) - S^2(pss's) + S^3(pppp) + S^3(psp's)] \\
&\quad + \lambda^2[2S(ppps) - 2S(ps'ss') + S^2(ss'ss') - S^2(pppp)].
\end{aligned} \tag{40}$$

The term which corresponds to (35) is absent, because $H_{34} - E_1 S_{34}$ vanishes in this case. The term corresponding to the third line of (27) becomes $44E(2\lambda S + \lambda^2)$ as before.

Then the difference in energy between the parallel and the antiparallel states is

$$E_t - E_{ap} = J_1 + 2J_2 + 2J_3 + J_4 + J_5 + J_6 + J_7, \tag{41}$$

where

$$J_1 = E_{t1} - E_1, \tag{42}$$

$$J_4 = 2\lambda^2(H_{34} - E_1 S_{34}), \tag{43}$$

$$J_5 = [H_{17} - E_1 S_{17} + 2\lambda(H_{37} - E_1 S_{37})]^2 / (E_7 - E_1) \tag{44}$$

$$J_6 = [(H_{18} - E_1 S_{18}) + \lambda(H_{38} - E_1 S_{38})]^2 / (E_8 - E_1) \tag{45}$$

$$J_7 = 2[H_{1a} - E_1 S_{1a} - \lambda(H_{4a} - E_1 S_{4a})]^2 / (E_a - E_1) \tag{46}$$

and J_2 and J_3 are defined by (33) and (40), respectively. Here $J_1 = E_{t1} - E_1$ is given by (22) or by (15) of (I). (We see E_t in (22) is the energy of the triplet state of $M_s = 0$, while E_{t1} is that of $M_s = 1$; they should be equal to each other.) This term is due to overlap S between the atoms A and B . In order to interpret the $2J_2 + 2J_3$ terms, we will look for the changes of overlap and exchange terms of (14) and (15) when $N(\varphi_p + i\varphi_s)$ is substituted in it instead of $\varphi_p\beta$, where N is a normalization constant. The change of $B(p\alpha, s\alpha)$ is seen to be J_2' .^{*} This term occurs both in (14) and in (15). On the other hand, the change of $B(p\alpha, s'\alpha)$

* The $\varphi_p\beta$ function is contained in $B(p\alpha, s\alpha)$ as is seen from (13).

is J_3 in the parallel case, (14), while in the antiparallel case, (15), $-J_2$ is the change of $B(p\beta, s'\beta) - (p\beta s'\beta s'\beta p\beta)$. Thus the energy difference $J_2 + J_3$ results from the transfer of the $\varphi_{p\beta}$ electron to the φ_s orbital. Including the effect of transfer to the $\varphi_{s'}$ orbital, we obtain the $2(J_2 + J_3)$ term. We may interpret the $2J_2$ term to correspond to the usual Anderson term, while $2J_3$ term has not been pointed out previously. As stated above, it originates from the overlap energy $B(p\alpha, s'\alpha)$ and does not arise when orthogonality between wave functions is assumed. The J_4 term arises from the fact that the third order perturbation process $\phi_1 \rightarrow \phi_3 \rightarrow \phi_4 \rightarrow \phi_1$ is present only in the antiparallel case. So, this is the Anderson-Hasegawa term, which is positive only when $H_{34} - E_1 S_{34}$ is positive. By direct calculation of matrix elements,

$$H_{34} - E_1 S_{34} = 2S(ppps) - S^2(pppp) + (pps's') + 2\lambda SB$$

is shown. The first three terms may be supposed to be small because of cancellation. Then the sign of the Anderson-Hasegawa term is determined by that of λS , which may be positive in the usual case. The above four terms arise from mixing ϕ_3 and ϕ_4 , in which one electron of atom B transfers to atoms A and C , respectively. The J_5 term arises from the fact that the configuration ϕ_7 , in which two electrons transfer from atom B to atoms A and C simultaneously, is present only in the antiparallel case and is always positive. We may regard J_2 to J_5 as arising from *transfer* mechanism, because they are due to configurations in which one or two electrons transfer from atom B to atoms A and C . The J_6 term is due to configurations ϕ_8 and ϕ_9 , which are present only in the antiparallel case and is always positive. In a real crystal these configurations appear in the usual band model (in MnO these are such configurations as $[\text{Mn}^+\text{O}^{--}\text{Mn}^{+++}]$). The last term, J_7 , corresponds to the Slater mechanism. We note the excited states of the Slater mechanism, ϕ_a, ϕ_b, ϕ_e and ϕ_f are divided into two classes. One of them arises from one-electron transition process from the ground state, the other from two-electron transition process. For the antiparallel state the former is ϕ_a and ϕ_b and the latter is ϕ_e and ϕ_f . For the parallel state the latter is ϕ_{ie} and ϕ_{if} , whereas the former states have zero matrix elements with the ground state because of symmetry. Therefore, the effect of two-electron transition cancels out in the energy difference, while the effect of states of one-electron transition favours the energy of the antiparallel state. This is the J_7 term.

Next, we calculate the energy of the singlet state. For the ground state we take ϕ_s , (18). For the excited states all states from ϕ_3 to ϕ_f are necessary.

$$\phi_s(\text{C. I.}) = \phi_s + \sum_{i=3\dots 6} \lambda_i \phi_i + \sum_{j=7\dots f} \lambda_j \phi_j. \quad (47)$$

Following a similar procedure to the antiparallel case we have calculated the energy of the state, ϕ_s (C.I.). Then it can be shown that the difference in energy between the antiparallel and the singlet states obtained in this way, is just equal to that between the triplet and the antiparallel states, (41).

§ 4. Single configuration

Although the treatment of configuration mixing developed in the previous section is quite general, except for the assumption that the overlap between atomic orbitals is small and the energy can be expanded in powers of S , the extension of configuration mixing to the N body-problem cannot be carried out without divergence difficulty. To avoid this difficulty we have developed a method³⁾ to include various excited configurations in the frame of single-configuration approximation. Although we cannot say how good this approximation is in the N -body problem, we can compare the approximation with the exact approach in the case of a four-electron model. This may serve to get some idea of the approximation used in the N body-problem.

As in (I) we assume the following wave function for the parallel, and the antiparallel states, respectively.

$$\phi_t = (4!)^{-1/2} |\varphi_s \alpha \varphi_p \alpha \varphi_s' \alpha \phi_p \beta| \quad (48)$$

$$\phi_{ap} = (4!)^{-1/2} |\phi_s \alpha \phi_{p1} \alpha \phi_{p2} \beta \phi_s' \beta| \quad (49)$$

$$\phi_p = (1 + 2\lambda_3 S + \lambda_3^2 - 2\lambda_4 S + \lambda_4^2)^{-1/2} (\varphi_p + \lambda_3 \varphi_s + \lambda_4 \varphi_s'), \quad (50)$$

$$\phi_{p1} = (1 - 2\lambda_4 S + \lambda_4^2 + \mu_a^2 + 2\lambda_4 \mu_a T)^{-1/2} (\varphi_p + \lambda_4 \varphi_s' + \mu_a \varphi_c), \quad (51)$$

$$\phi_{p2} = (1 + 2\lambda_3 S + \lambda_3^2 + \mu_b^2 + 2\lambda_3 \mu_b T)^{-1/2} (\varphi_p + \lambda_3 \varphi_s + \mu_b \varphi_c), \quad (52)$$

$$\phi_s = (1 + \mu_s^2) (\varphi_s + \mu_s \varphi_s'), \quad (53)$$

$$\phi_s' = (1 + \mu_s'^2) (\varphi_s' + \mu_s' \varphi_s), \quad (54)$$

$$T = \int \varphi_s \varphi_c dv. \quad (55)$$

First, we calculate the energy of the triplet state, ϕ_t , which can be expanded as

$$\phi_t \propto \phi_{t1} + \lambda_3 \phi_{t3} + \mu_4 \phi_{t4}. \quad (56)$$

Compared with (38), this function does not involve ϕ_{te} and ϕ_{tf} , because, as stated previously, these states are connected with ϕ_{t1} by two-electron transition process. Thus the energy of ϕ_t is given by

$$E_{t1} - 2\Delta E + 2J_2' + 2J_3 + 4\Delta E(2\lambda S + \lambda^2). \quad (57)$$

Next, we calculate the energy of ϕ_{ap} . We have two methods to obtain it. First, since ϕ_{ap} is a single Slater determinant its energy is given by (7) where φ 's are replaced by ϕ 's. We can expand it in terms of λ 's and μ 's just as in (27). By a straightforward calculation we see that it is just the same as (27) except that μ_7 is replaced by $\lambda_3 \lambda_4$ and the terms which contain μ_e and μ_f are absent. This can be most easily seen by virtue of the second method, in which ϕ_{ap} is expanded as

$$\phi_{ap} \propto \phi_1 + \lambda_3 \phi_3 + \lambda_4 \phi_4 + \lambda_3 \lambda_4 \phi_7 + \mu_3 \phi_8 + \mu_3 \phi_9 + \mu_a \phi_a + \mu_b \phi_b, \quad (58)$$

where higher configurations which are not given in (25) are omitted. In (58) ψ_7 appears with the coefficient $\lambda_3\lambda_4$ and ψ_e and ψ_f are absent. Then the difference in energy between ψ_i and ψ_{ap} is given by

$$J_1 + 2J_2 + 2J_3 + J_4 + J_5' + J_6 + J_7, \quad (59)$$

where
$$J_5' = -[2\lambda_3\lambda_4(\bar{H}_{17} + 2\lambda_3\bar{H}_{37}) + (\lambda_3\lambda_4)^2(E_7 - E_1)]. \quad (60)$$

We notice that the Slater term of this approximation is equal to that of configuration interaction, since in this approximation configurations, ψ_e and ψ_f , which are those of two-electron excitation, are absent in the both states, the effect of these configurations being cancelled out in the energy difference. The J_5' term of (59) results because the configuration ψ_7 , which is a state of two-electron excitation, appears under the coefficient of $\lambda_3\lambda_4$ in the single-configuration approximation, while in the configuration interaction it appears with the coefficient μ_7 , which can be varied independently to give a minimum of energy.

§ 5. Discussions

We see from the results of the previous sections that in the frame of configuration interaction the difference in energy between the triplet and the antiparallel states is equal to that between the antiparallel and the singlet states, thus the spin interaction between atoms A and C can be represented by an equivalent spin Hamiltonian, $2J\mathbf{s}_A\mathbf{s}_C$, where J is given by (41). We have seen that this is true even when the non-orthogonality of wave functions is taken into account, and even when various mechanisms of superexchange interaction are included simultaneously in the effective interaction J .

We have also seen that the distinction among various mechanisms is rather clear, even when various mechanisms are considered simultaneously. We emphasize in particular the distinction between the Slater mechanism (J_7) and the mechanism of two electron transition (J_5). As long as we are concerned with the energy difference between the parallel and the antiparallel states, the essential feature of the former is taken into account by single-configuration approximation, while the proper account of the latter can be taken only by the configuration interaction.

We may regard each term of the interaction (41) as resulting from four distinct origins. (1) J_1 . This term is due to overlap S between ions. (2) J_2 to J_5 . We may regard these terms as arising from *transfer* mechanism. The Anderson and the Anderson-Hasegawa mechanisms as well as the mechanism of simultaneous transfer are included in it. In single-configuration approximation they are those terms which involve λ 's which represent the degree of transfer. (3) J_6 . This term arises from the usual band model. (4) J_7 . This is due to the Slater mechanism, in which the deformation of atom B is taken into consideration.

In § 4 we saw that the difference in energy between the triplet and the antiparallel states of single-configuration approximation is equal to the correct one

(41) except that J_5 of (41) is replaced by J_5' . We have seen³⁾ by a rough numerical calculation that J_5' is nearly equal to J_5 . Thus we can say that when we calculate the difference in energy between the triplet and the antiparallel states in the frame of single-configuration approximation we can obtain a result which is very close to the correct one. We notice the above statement is validated by the following circumstance: Although we cannot take proper account of the states of two-electron excitation in single-configuration approximation (that is, ϕ_7 appears with coefficient of $\lambda_3\lambda_4$ instead of μ_7 , and ϕ_e and ϕ_f are not included in ϕ_{ap}), they have no large effects on the difference between the energies of the parallel and antiparallel states. Thus the conclusion of our investigation is as follows: (1) As long as we look for the difference in energy between the parallel and the ordered antiparallel states, the essential features are involved in single-configuration approximation even when it is applied to real crystals. (2) The correlation effect, to account for which we must invoke the configuration interaction, plays its essential role only in such problems as looking for the ground spin configuration of antiferromagnets or as determining coefficients λ and μ when overlaps between ions are large and the expansion in S of energy is invalidated.

Acknowledgements

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Internal Structure and Evolution of Very Massive Stars

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The internal structure of very massive stars with 46.8 solar masses is investigated in their early stage of evolution. Starting from an initial homogeneous model, their evolution is followed to the stage where the hydrogen in the core diminished to nine per cent in weight. The inhomogeneous models consist of three regions, a hydrogen-rich envelope, a hydrogen-consuming core, and a semi-convective intermediate zone. In the last zone, the mean molecular weight varies continuously and its distribution is determined by the condition of convective neutrality, where the gradient of mean molecular weight resulting from the convective mixing between the envelope and the core is taken into account. It is shown that the stars stay in the H - R diagram nearly at the same position as the main sequence stars until the hydrogen in the core is mostly exhausted and their life is about 3×10^6 years.

§ 1. Introduction

Recently, the internal structure and evolution of very massive stars heavier than ten solar masses are investigated by several authors,¹⁾⁻⁵⁾ and it is found that the convective unstable region grows (in terms of the mass fraction contained in it) as the evolutionary phases proceed in contrast to the less massive stars in which the convective core shrinks as the hydrogen contents of the core decrease.³⁾⁻⁵⁾ This new phenomenon of growing convective instability, which is characteristic of very massive stars, has been considered in detail by Schwarzschild and Härm³⁾ and Sakashita et al.^{4)*} independently. As the mass of stars chosen in Paper I is 15.6 solar masses, the effect of convective instability is small so that the newly appeared convective zone is approximated by a radiative one.

On the other hand, in the case of Schwarzschild and Härm,³⁾ the range of stellar mass is so wide that the effect of this zone becomes more serious. They treated this as a semi-convective one in which convective motion joins the hydrogen-rich envelope and hydrogen-consuming convective core and the variation of mean molecular weight due to this convective mixing is determined so as to maintain the convective neutrality. The condition for the convective neutrality used by them is that the adiabatic gradient is equal to the radiative one. In the semi-convective region, however, there exists a gradient of mean molecular weight resulting from

* This paper will be referred to as "Paper I" in the following.

the convective mixing, and this gradient in turn suppresses the upward and downward driving force giving rise to the convective motion if the mean molecular weight of the material element remains unchanged during its motion. Therefore, the condition to maintain convective neutrality should be modified by taking account of the effect of the gradient of mean molecular weight.

It is the purpose of this paper to investigate the structure of very massive stars consisting of three regions, the hydrogen-rich envelope, hydrogen-consuming core and semi-convective intermediate zone in which the distribution of mean molecular weight is determined so as to satisfy the condition of the convective neutrality that the driving force for the convective motion vanishes everywhere.

§ 2. Construction of models

First, the initial homogeneous model is constructed by the usual method using non-dimensional variables introduced by Schwarzschild as described in Paper I. Chemical composition and opacity are taken in accordance with Paper I, i.e., $X=0.9$, $Y=0.08$, $Z=0.02$, and opacity is due to electron scattering only. With a fixed parameter $A=144.2$ of envelope solutions and with the chemical composition adopted above, the mass of stars is determined from Equation (8) in Paper I as $46.8M_{\odot}$. Using this value of the fixed parameter, the basic equation (7) in Paper I is integrated inwards for appropriate values of the remaining parameter of envelope, C , by the numerical method.

As for the convective core solutions including radiation pressure, the results of Henrich's⁽⁶⁾ calculation are used. Fitting conditions at the interface between radiative envelope and convective core are the same as in Paper I, that is, the continuity of U , V and β , or U , V and $(n+1)$. In this way, the mathematical characteristics are completely determined for the initial homogeneous model.

Secondly, inhomogeneous models immediately following the homogeneous one must be constructed. In order to construct them the physical processes, which occur in succession slightly after the homogeneous stage and determine the variation of the chemical composition, are to be considered. Hydrogen-burning nuclear reactions at the center of star reduce the hydrogen content, and the mixing by convective motion in the core will be fully effective. Thus the hydrogen content of the core will decrease homogeneously, meanwhile the mass of a convective unstable region grows gradually.

The depletion of hydrogen in the core reduces the temperature gradient since the main opacity is due to electron scattering, and so this will tend to make the outer boundary of the core to be in radiative equilibrium. However, there appears a new convective instability at the innermost region of the envelope. If the convective motion due to this instability is effective enough, hydrogen in the envelope and helium in the core will be mixed by this motion. The gradient of mean molecular weight resulting from mixing will in turn suppress the convective motion

and this will cease when the convective neutrality condition is satisfied.

On the other hand, if the convection is not effective, there will appear a radiative inner zone and a convective outer zone between the core and the envelope. The appearance and the growth of such a purely radiative intermediate zone will depend on the details of the mixing processes such as the local variation in the rates of mass and energy transports.

In this paper we shall investigate the former case. This will be considered as a limiting case where the purely radiative zone does not appear. Then, successive depletion of hydrogen in the core and successive growth of convective instability in the envelope give rise to the convective motion between the core and the envelope, and the variation of mean molecular weight in this region is self-adjusted to maintain the convective neutrality. The condition of convective neutrality should be thus modified to take account of the effect of the gradient of mean molecular weight arising from mixing so as to generate continually the driving force to maintain the convective motion. In this case the condition of

Table 1. Mathematical and physical characteristics of models.

Model	0 (Homogeneous)	1	2
$\log C$	-3.6185	-3.6160	-3.6145
U_1	1.486	1.476	1.470
V_1	4.626	4.624	4.623
$(n+1)_1$	3.347	3.351	3.352
$\log x_1$	-0.374	-0.375	-0.375
$\log q_1$	-0.193	-0.192	-0.192
$\log p_1$	0.616	0.618	0.619
$\log t_1$	-0.571	-0.570	-0.570
β_1	0.819	0.818	0.817
U_f		2.205	2.756
V_f		2.100	0.580
$(n+1)_f$		2.620	1.523
$\log x_f$		-0.553	-0.848
$\log q_f$		-0.525	-1.267
$\log p_f$		1.181	1.534
$\log t_f$		-0.383	-0.217
β_f		0.721	0.429
X_c	0.900	0.730	0.098
β_c	0.735	0.687	0.426
$\log T_c$	7.645	7.654	7.716
$\log L/L_\odot$	5.368	5.371	5.372
$\log R/R_\odot$	0.850	0.852	0.852
$\log T_e$	4.679	4.679	4.679
life (years)	0	1.5×10^6	3.3×10^6

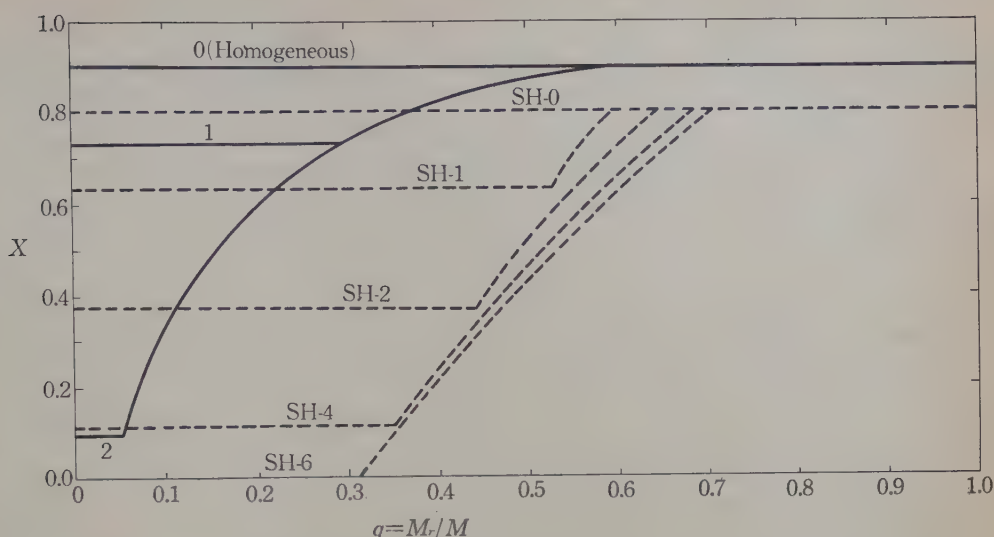


Fig. 1 The variation of the hydrogen content from the center to the surface. Dotted lines show the results obtained by Schwarzschild and Härm for stars with $M=28.2 M_{\odot}$.

convective neutrality that the driving force for the convection motion just vanishes is given by

$$\frac{1}{(n+1)_{\text{rad}}} = \frac{1}{(n+1)_{\text{ad}}} + \frac{\beta}{4-3\beta} \frac{d \log \mu}{d \log P}, \quad (1)$$

where $(n+1)_{\text{rad}}$ and $(n+1)_{\text{ad}}$ are defined by Equation (13) in Paper I. On the other hand, under the condition that the last term in Equation (1) is dropped as considered by Schwarzschild and Härm,³⁾ finite driving force is active in the negative direction so that no convective mixing will occur between the core and the envelope.

According to the above consideration, we can construct the models of very massive stars in their early phases. The model consists of three regions, a hydrogen rich envelope, a hydrogen consuming core and a semi-convective zone between them in which the distribution of mean molecular weight is determined by Equation (1), or more explicitly, by the following differential equation,

$$\frac{d \log \mu}{d \log x} = \left\{ \frac{d \log t}{d \log x} - \left(\frac{8-6\beta}{32-24\beta-3\beta^2} \right) \cdot \frac{d \log p}{d \log x} \right\} \frac{(4-3\beta)}{\beta}. \quad (1')$$

Equation (1') for the semi-convective intermediate zone should be added to the basic equation (7) in Paper I, and be simultaneously satisfied. The inhomogeneous models are then constructed as follows. The envelope solutions are obtained for slightly different values of the free parameter C . From the point where the envelope solution reaches the condition of convective instability, the semi-convective zone is integrated inwards and this is fitted to the convective core solution according to

the conditions of continuity of U , V and β . By this fitting procedure, the mathematical characteristics of models are completely determined for the definite value of parameter C .

To transform the non-dimensional mathematical quantities to the physical ones, it is necessary to equate the rate of energy production to the stellar luminosity. The energy production in the convective core is given by Equation (17) in Paper I in which we adopt

$$\varepsilon_0 = 6 \times 10^{-101} X_H X_{CN}, \quad s = 14.17,$$

according to Burbidge et al.⁷⁾

Likewise, all the physical quantities of stars are obtained for the given mass.

§ 3. Results and discussions

The mathematical and physical characteristics of models are shown in Table I, and the distribution of hydrogen in stars is plotted in Fig. 1. We see that the very massive stars stay nearly at the same position in the H - R diagram even hydrogen in the core has been mostly exhausted, the effective temperature retaining the same value and luminosity increasing only slightly through the whole stages in consideration.

Though the data of the H - R diagram for very massive stars to be compared with the theoretical results have never been obtained, it may be possible to check the theoretical results, if it is allowed to extrapolate the H - R diagram of less massive stars to heavier ones. For this purpose, the investigation of the models in later stage, i.e., the gravitational contraction stage, and the life of staying on this stage might be desired.

As for the less massive stars, the internal structure in early phases of their evolution was investigated in the same way for the star of 15.6 solar masses. In this case it was found that the solution which consists of the radiative homogenous envelope, the semi-convective zone and the convective core did not exist. Therefore, there will be a critical mass between 15.6 and 46.8 solar masses, above which such a solution exists. Below the critical mass, the semi-convective zone might not develop as evolution proceeds and some part of the intermediate zone might be in a state of radiative equilibrium. For more detailed investigations about the formation of the semi-convective zone, it may be desired to study the physical mixing processes occurring in a star, for instance, the rate of mass transport by convective motion.

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On the Possibility of the Two-Fermion Interaction

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As the general source for non-leptonic weak interactions, an elementary two-fermion (A - n) interaction is introduced and investigated. The calculated asymmetry factors and decay rates for Λ and Σ decays show that this interaction, with some corrections due to the strong pion-baryon interactions, does not seem sufficient to describe these decay processes. Hence we cannot ascribe the main features of weak interactions to the two-fermion interaction only.

§ 1. Introduction

It is remarkable that in weak interactions such quantities as space inversion parity, charge conjugation parity, strangeness, isotopic spin, etc., are not conserved, though they are known to be very good quantum numbers in strong interactions.

There have been several attempts to explain some of these violation characters on a general footing. As far as the non-conservation of space parity and charge conjugation parity in neutrino processes is concerned, the left-handed two-component neutrino theory¹⁾ seems to provide a very good explanation. As for the non-leptonic processes, the $V-A$ chiral-current interactions (the Fermi interactions of current-current type between four fermions) are suggested by Feynman-Gell-Mann²⁾ and Sudarshan-Marshak³⁾.

At present, however, our knowledge on the strangeness (S) and the isotopic spin (I) quantum numbers, which might be related to some inner degrees of freedom other than those which are concerned with the Lorentz space, remains more or less at a phenomenological stage. For example, the strangeness selection rule $|\Delta S|=1$ and the iso-selection rule $|\Delta I|=1/2$ are usually put as *ad hoc* assumptions on baryon-meson weak interactions.

But among many proposed physical interpretations for the strangeness, Sakata's composite particle model⁴⁾ gives us a very simple and natural way of understanding. Supposing that all the strongly interacting particles should be composed of a limited number of fundamental particles, i.e., nucleon and Λ -hyperon, as well as their antiparticles, the strangeness quantum number can be regarded as the particle number of constituent Λ 's appearing in the composite particle. Furthermore, we can say an interaction is weak or strong according to whether it causes the kind-changing transition of a fundamental particle or not. The last property corresponds to Nishijima-Gell-Mann's rule.

On the other hand, we have no general explanation for the iso-selection rule

$|\Delta I|=1/2$. Of course it is not certain whether this form of iso-selection rule should be the actual one. Indeed, using the chiral-current interactions, which were compatible assumptions with Sakata's model, Okubo, Marshak and Sudarshan⁶⁾ showed that the experimental data for Λ -decay could be reproduced in quite a different way from the interpretation based on the rule $|\Delta I|=1/2$. With all the interesting feature, unfortunately, the chiral-current interaction hypothesis does not seem to be good enough to account for $K^0 \rightarrow 2\pi$ decay rates.⁶⁾

Thus, we take here the $|\Delta I|=1/2$ rule for an acceptable one, and try to find its origin in a simpler interaction, rather than in *ad hoc* application of the rule to every possible weak process. An interesting suggestion in this line was given by Wentzel.⁷⁾ He introduced the idea of "spurion", a spurious 'particle' with isotopic spin $1/2$, which is emitted or absorbed in weak processes without carrying any energy, momentum and charge. This, however, was in itself a formal device.

In this note we assume that all the non-leptonic weak interactions should be considered as combined effects of the virtual two-fermion ($\Lambda \rightarrow n$) transition and the strong pion-baryon interactions. The two-fermion interaction is rather unfamiliar to us, but it is not so curious from the standpoint of the composite model for elementary particles, and further it gives a physical ground equivalent to the spurion concept.* In actual calculations we deal with the non-leptonic hyperon decays. Our results obtained are, however, not in good agreement with the experimental data. This fact might be considered that two-fermion interaction is not sufficient for being accepted as the sole source of weak interactions.

§ 2. Two-fermion interaction

According to the composite particle picture, we may guess that $|\Delta I|=1/2$ should also be realized by the transition of Λ to nucleon. As the simplest formulation we can introduce the elementary two-fermion interaction giving rise to the virtual transition $\Lambda \rightarrow n$ (neutron), whose Hamiltonian is written (in obvious notations):

$$H_w = \int d\mathbf{x} g (\bar{n} [1 + \epsilon \gamma_5] \Lambda) + \text{H. C.} \quad (1)$$

The coupling constant g has the dimension of mass.

Hamiltonian (1) may be regarded as the coupling of Λ and n with spurion which is not written down explicitly, and thus the rule $|\Delta I|=1/2$ is guaranteed. The possible momentum dependence of the coefficients g and ϵ in the elementary interaction (1) is disregarded. The finite value of ϵ can be considered as the universal source of non-leptonic parity violations, and in the actual calculations we impose the so-called one-to-one law, $|\epsilon|=1$.

*We have already pointed out another possible model which functions as a spurion.⁸⁾ It consists of the scalar interaction of scalar or pseudoscalar K^0 meson with the virtual lepton (especially muon) loop in the vacuum. This mechanism was first considered by Schwinger.⁹⁾

The interaction of this type is first suggested by Okun' in his article on composite model.¹⁰⁾ Later, Sawyer¹¹⁾ considered two-fermion interaction $\Lambda \rightarrow n$ together with $\Sigma^+ \rightarrow p$ and $\Sigma^0 \rightarrow n$ by basing on a different standpoint.

As it is too complicated to treat the interaction (1) straightforwardly in the framework of the composite model, we confine ourselves to the investigation of the possibility that all the non-leptonic weak interactions should be reduced to (1) with some added corrections due to strong interactions:

$$H_{N\pi} = \int dx iG(\bar{N}(\boldsymbol{\tau} \cdot \boldsymbol{\pi})\gamma_5 N) \quad (2)$$

$$H_{\Lambda\pi} = \int dx G'((\bar{\Sigma} \cdot \boldsymbol{\pi}) \begin{bmatrix} i\gamma_5 \\ 1 \end{bmatrix} \Lambda) + \text{H. C.} \quad (3)$$

$$H_{\Sigma\pi} = \int dx iG'(\bar{\Sigma}\gamma_5 \times \Sigma) \cdot \boldsymbol{\pi}. \quad (4)$$

In these strong interactions we take the Σ -hyperons and the pions for the usual particles, and this will correspond to some approximation from the view-point of the composite particle model.

No other leptonless weak interactions like the Yukawa interactions of $(\Lambda N\pi)$ - or $(\Sigma N\pi)$ -type, or four-fermion Fermi couplings (chiral-current interactions) are introduced in what follows.

§ 3. Λ - and Σ -decays

The recent experimental data on Λ - and Σ -hyperon decays are summarized as follows:¹²⁾

Decay mode	Partial decay rate Γ (10^{10} sec^{-1})	Branching ratio	Asymmetry factor α
$\Lambda \rightarrow p + \pi^-$	0.24 ± 0.02	$\{ 0.63 \pm 0.03$	$\alpha_{\Lambda^-} = 0.85^{+0.15}_{-0.21}$
$\Lambda \rightarrow n + \pi^0$	0.14 ± 0.02	$\{ 0.37 \pm 0.03$	$\alpha_{\Lambda^0} ?$
$\Sigma^+ \rightarrow p + \pi^0$	0.51 ± 0.07	$\{ 0.47 \pm 0.04$	$ \alpha_{\Sigma^0} \geq 0.70 \pm 0.30$
$\Sigma^+ \rightarrow n + \pi^+$	0.57 ± 0.07	$\{ 0.53 \pm 0.04$	$ \alpha_{\Sigma^+} \leq 0.03 \pm 0.11$
$\Sigma^- \rightarrow n + \pi^-$	0.58 ± 0.05	—	$ \alpha_{\Sigma^-} \sim \alpha_{\Sigma^+} $

The kinematical analysis with the $|A| = 1/2$ rule shows that the branching ratio of Λ -decay modes is exactly 2:1. But the other quantities, such as asymmetry factors of Λ and Σ , and the branching ratio of the Σ -decay, cannot be predicted only by this rule.

Now the assumptions made in the previous section allow us to calculate these quantities. The lowest order perturbation corresponds to the diagrams given in Figs. 1-(O-a), -(O-b), but to this order the $\Sigma^+ \rightarrow p + \pi^0$ decay does not occur. We must, therefore, take into account the higher order effects of virtual pion interactions. (Figs. 1-(I), ..., (IV).)

We neglect the contributions of diagrams (III) and (IV), for they turn out to be relatively small. On the other hand, diagrams (I) and (II) are reduced to the similar ones as diagrams (O-a) and (O-b) with the substitution of the effective two-fermion interactions $\Lambda \rightarrow n$, $\Sigma^+ \rightarrow p$ and $\Sigma^0 \rightarrow n$ in place of (1). These effective two-fermion interactions have momentum dependences, which we have disregarded in the elementary interaction (1), and which bring a non-zero asymmetry factor for $\Sigma^+ \rightarrow p + \pi^0$ mode. (cf. ref.¹¹⁾)

The perturbational calculation including up to the second order pion corrections gives the following matrix elements :

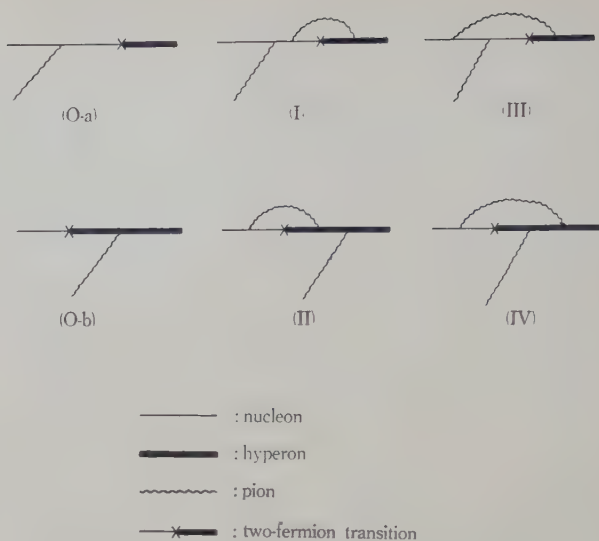


Fig. 1 Feynman diagrams containing the two-fermion transition $\Lambda \rightarrow n$.

$$M(\Lambda \rightarrow p + \pi^-) = \frac{-i\sqrt{2} Gg}{m_{\Lambda}^2 - m^2} \left\{ \langle p | (i\mathbf{k}) \begin{bmatrix} \varepsilon + \gamma_5 \\ \varepsilon + \gamma_5 \end{bmatrix} | \Lambda \rangle \right. \\ \left. + \frac{G'^2}{(4\pi)^2} \langle p | (i\mathbf{k}) \begin{bmatrix} \frac{1}{\eta_1} B_1 \varepsilon - \frac{1}{\eta_2} A_1 \gamma_5 \\ -\frac{1}{\eta_4} B_1 \varepsilon + \frac{1}{\eta_3} A_1 \gamma_5 \end{bmatrix} | \Lambda \rangle \right\}, \quad (5)$$

$$M(\Lambda \rightarrow n + \pi^0) = \frac{iGg}{m_{\Lambda}^2 - m^2} \left\{ \langle n | \quad \quad | \Lambda \rangle + \frac{G'^2}{(4\pi)^2} \langle n | \quad \quad | \Lambda \rangle \right\}, \quad (6)$$

$$M(\Sigma^+ \rightarrow p + \pi^0) = \frac{-i\sqrt{2} G'g}{m_{\Sigma}^2 - m^2} \\ \times \frac{G^2}{(4\pi)^2} \langle p | (i\mathbf{k}) \begin{bmatrix} \left(B_3 + \frac{G''}{G} B_1 \right) \varepsilon + \left(A_2 - \frac{G''}{G} A_1 \right) \gamma_5 \\ -i \left(A_3 + \frac{G''}{G} A_1 \right) - i \left(B_2 - \frac{G''}{G} B_1 \right) \varepsilon \gamma_5 \end{bmatrix} | \Sigma^+ \rangle, \quad (7)$$

$$M(\Sigma^+ \rightarrow n + \pi^+) = \frac{-iG'g}{m_{\Sigma}^2 - m^2} \left\{ \langle n | (i\mathbf{k}) \begin{bmatrix} \eta_1 \varepsilon - \eta_2 \gamma_5 \\ -i\eta_3 + i\eta_4 \varepsilon \gamma_5 \end{bmatrix} | \Sigma^+ \rangle \right\}$$

$$+ \frac{G^2}{(4\pi)^2} \langle n | (i\mathbf{k}) \left[\begin{array}{l} \left(2B_3 + \frac{G''}{G} B_1 \right) \varepsilon + \left(2A_2 - \frac{G''}{G} A_1 \right) \gamma_5 \\ -i \left(2A_3 + \frac{G''}{G} A_1 \right) - i \left(2B_2 - \frac{G''}{G} B_1 \right) \varepsilon \gamma_5 \end{array} \right] | \Sigma^+ \rangle \right\}, \quad (8)$$

$$M(\Sigma^- \rightarrow n + \pi^-) = \frac{-iG'g}{m_\Sigma^2 - m^2} \left\{ \langle n | (i\mathbf{k}) \left[\begin{array}{l} \gamma_1 \varepsilon - \gamma_2 \gamma_5 \\ -i\gamma_3 + i\gamma_4 \varepsilon \gamma_5 \end{array} \right] | \Sigma^- \rangle \right. \\ \left. + \frac{G''}{(4\pi)^2} \langle n | (i\mathbf{k}) \left[\begin{array}{l} -\frac{G''}{G} B_1 \varepsilon + \frac{G''}{G} A_1 \gamma_5 \\ i \frac{G''}{G} A_1 - i \frac{G''}{G} B_1 \varepsilon \gamma_5 \end{array} \right] | \Sigma^- \rangle \right\}, \quad (9)$$

where

$$i\mathbf{k} = i\mathbf{k}_\mu \gamma_\mu \quad (10)$$

(k_μ : energy-momentum 4-vector of the outgoing pion),

$$\eta_1 = \frac{m_\Sigma + m}{m_A + m} = 1.04, \quad \eta_2 = \frac{m_\Sigma - m}{m_A - m} = 1.43, \quad (11)$$

$$\eta_3 = \frac{m_\Sigma + m}{m_A - m} = 12.17, \quad \eta_4 = \frac{m_\Sigma - m}{m_A + m} = 0.12,$$

(m, m_A, m_Σ : masses of nucleon, A , and Σ)

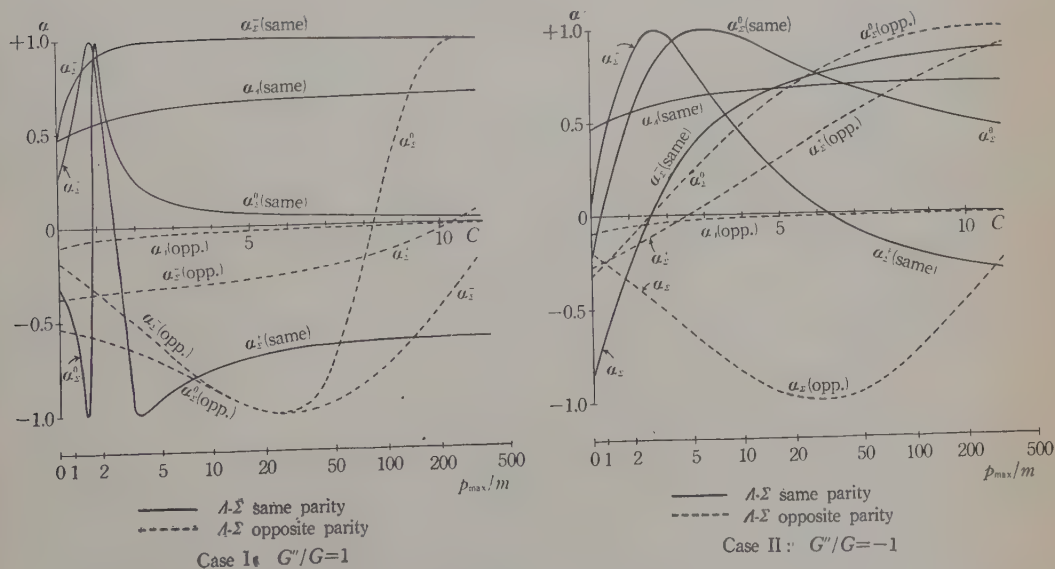


Fig. 2. Calculated asymmetry factors for A and Σ decays. (α : asymmetry factor)

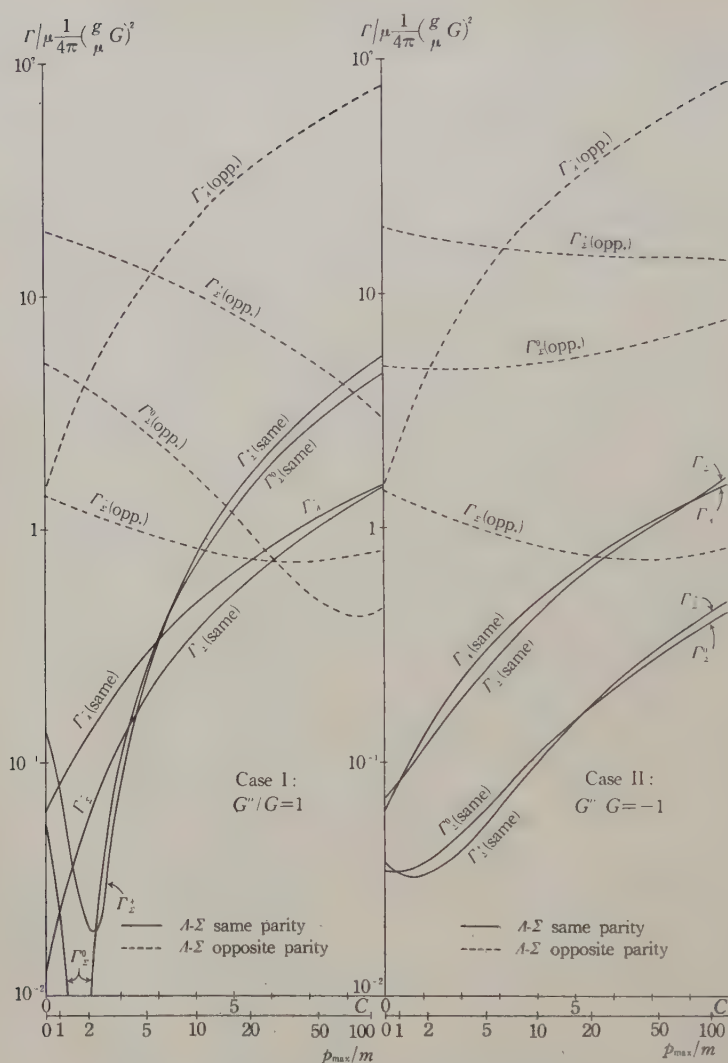


Fig. 3 Calculated decay rates for Λ and Σ decays.
(Γ : partial decay rate; μ : pion mass)

and the coefficients A_i and B_i ($i=1, 2, 3$) of the mesonic correction terms are given by

$$\begin{aligned} A_1 &= -C + 0.14, & B_1 &= C + 1.31, \\ A_2 &= -C + 0.48, & B_2 &= C - 4.42, \\ A_3 &= -C + 16.33, & B_3 &= C - 3.08. \end{aligned} \quad (12)$$

Here C is the cut-off parameter related to the maximum virtual pion momentum p_{max} :

$$C=2\left[\ln\left\{\frac{p_{\max}}{m}+\left(\frac{p_{\max}^2}{m^2}+1\right)^{1/2}\right\}-\frac{p_{\max}}{m}\left(\frac{p_{\max}^2}{m^2}+1\right)^{-1/2}\right]. \quad (13)$$

The coupling constants for strong interactions are assumed to be $G^2/4\pi=G'^2/4\pi=G''^2/4\pi\approx 4\pi$, and the upper or lower line in each matrix element corresponds to the case of the same or the opposite $\Lambda-\Sigma$ relative parity, respectively.

Figs. 2 and 3 show the calculated asymmetry factors and decay rates as the functions of the cut-off parameter C . We put $\varepsilon=1$ in accordance with the positive experimental value of α_+ . Within the natural cut-off momentum region, we can hardly find any satisfactory interpretation of the experimental values from these graphs.

Though we have made several additional assumptions in the course of calculations, the resulting features seem to indicate that the two-fermion interaction (1) is not sufficient as the sole primary interaction coexisting with the strong interactions (2)–(4). If the four-fermion weak interaction exists beside (1), the conclusion will be modified.

§ 4. Discussions

Takeda and Kato¹³⁾ proposed the global weak interaction and investigated the effect of the pion cloud around the baryon core. The latter gives rise to some effective two-fermion interaction like (1). Applying the static meson theory, they pointed out the possibility of explaining the experimental asymmetry and branching ratios. On the contrary, our calculation is based on the relativistic perturbation theory and contains the full contribution of the pair-producing effects. The static approximation for our case gives no asymmetry at all.

It has been shown in the preceding section that we cannot ascribe the main features of weak interactions to the two-fermion interaction of $\Lambda-n$ type. But this does not mean the exclusion of the existence of this interaction. Furthermore, the straightforward application of the composite theory might bring the circumstances better. At any rate we must not pass by the possible two-fermion interaction, when we investigate the types of the actual interactions.

Finally we make a remark on the lepton process. If the two-fermion interaction should exist in the lepton family, the expected form would be the $\mu\rightarrow e$ transition :

$$H_{\mu-e}=\int d\mathbf{x} g_{\mu-e}(\bar{\mu}[1+\varepsilon_{\mu-e}\gamma_5]e)+\text{H.C.} \quad (14)$$

This interaction, however, would not reveal itself in a real process, since the matrix elements for $\mu\rightarrow e+\gamma$ decay, which is the only detectable process induced by (14), totally cancel out, provided that coefficients $g_{\mu-e}$ and $\varepsilon_{\mu-e}$ in (14) have no momentum dependence and that leptons have no anomalous magnetic moments.

In concluding the paper, the author wishes to express his sincere thanks to

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Note added in proof : Recently Oneda and Sakita (B. Sakita, private communication) investigated A and K decays with the A - n two-fermion interaction as well as four-fermion chiral-current interactions, and found a plausible explanation for decay processes. In their considerations the two-fermion interaction might be regarded as a phenomenological one representing the $|\Delta I|=1/2$ part of the chiral-current interactions.

Semi-Phenomenological Interpretation of the Optical Model in Nuclear Reactions from the Point of View of Fluctuation-Dissipation Theorem

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The behaviors of the nuclear optical model for the elastic scattering of neutrons at low energies are investigated on the theoretical basis of the fluctuation-dissipation theorem. The theory starts from a Schrödinger equation with the optical potential and the fluctuating source function. The source function, which is a representative of motions of the compound nucleus, is subject to the fluctuation-dissipation theorem, in which the imaginary part of the optical potential is proportional to the correlation function of the fluctuating source function. From this it is found that the strength function characterizing nuclear reactions is represented by the Fourier transform of the correlation function of the fluctuating wave function, and that the average rate of energy dissipation of the compound nucleus is proportional to the strength function and temperature of the compound nucleus.

§ 1. Introduction

Since the success of the optical model,¹⁾ it is found that the one-particle approximation is fairly good even in the problems of nuclear reactions at low energies. This model can describe the giant resonances in the cross sections for the elastic scattering of neutrons at low energies by large nuclei. The optical model consists of the one-particle approximation and the implicit use of the notion of irreversible processes. Concerning the latter, the important contributions have recently been given by Bloch²⁾ and Hayakawa et al.³⁾ to the theoretical basis for the optical model. In the present paper the behaviors of the optical model will be investigated by a semi-phenomenological equation of irreversible processes in the one-particle approximation. In a forthcoming paper, this equation will be derived from the fundamental many-particle equation, together with the formation of the optical potential.

Following Hayakawa et al., the phenomena should be considered in the following way. The time-intervals of the incident wave packets are so shorter than the average periods of compound nuclei that the incident neutrons regard the nuclear oscillations as a random fluctuation. Hence a neutron in the compound nucleus suffers the probability dissipation appearing in the Lorentzian type of relaxation, which is to be observed as the "giant resonance". The width of the giant resonance estimated in their theories is in agreement with the experimental value

and is considerably smaller than the value obtained by Lane, Thomas and Wigner.⁴⁾ It is evident that the phenomena considered in the above papers are nothing but some kind of irreversible processes. This fact has already been suggested by Porter,⁵⁾ who has shown a Nyquist-like theorem for nuclear reactions.

In the present work, we shall start from the reasonable assumption that motions of a neutron in the compound nucleus are described by a one-particle Schrödinger equation with an optical potential and a fluctuating source function. Such an equation is analogous to the Langevin equation in the theory of Brownian motion. From this it is shown, without resort to detailed treatment of the many-particle equations, that the various quantities characterizing nuclear reactions are closely related to the correlation function of the fluctuating part of the wave function, and particularly, that random oscillations of the compound nucleus must be responsible for the imaginary part of the optical potential and the strength function can be written by the Fourier transform of the correlation function of the wave function and the temperature of the compound nucleus.

As is suggested above, each neutron in question may move like a Brownian particle in a compound nucleus which oscillates in a random fashion. Here we first outline the Brownian motion of a molecule in a solution or an electron in a resistor. Such a particle collides with surrounding molecules in the solution or surrounding lattices in the resistor in thermal equilibrium. Because of a numerous freedom of scatterers, the particle in question experiences a random motion, that is to say, a Brownian motion, and loses its energy. Let the equation of motion for this particle be written in the form

$$m\ddot{x} = F, \quad (1)$$

where m and x are the mass and the position coordinate of the particle, and F represents the total forces acting on it. Since the average energy dissipation is expressed by a damping force $-\beta\dot{x}$, the force F can be divided into the two parts as follows: $F = -\beta\dot{x} + f$, in which f is the pure random force with the vanishing average value. Thus one gets the equation

$$m\ddot{x} + \beta\dot{x} = f. \quad (2)$$

Equation (1) is a dynamical one, while (2) becomes a semi-phenomenological equation if f is characterized by a statistical law. For a statistical f , Equation (2) is nothing but a Langevin equation in the theory of Brownian motion. As is well known, the random force f is characterized by

$$\left. \begin{aligned} \langle f(t) \rangle &= 0, \\ \langle f(t)f(t') \rangle &= 2\beta\kappa T \delta(t-t'), \end{aligned} \right\} \quad (3)$$

where $\langle A \rangle$ stands for the average value of A over all possible values of f , that is, the ensemble average with respect to f . Here κ is the Boltzmann constant and

T the temperature of the surrounding heat bath. The second member of (3) is the fluctuation-dissipation theorem or the Nyquist theorem. This implies that the correlation function $\langle f(t)f(t') \rangle$ must be responsible for the friction constant β . Now we get the phenomenological equation

$$m\ddot{x}_0 + \beta\dot{x}_0 = 0 \quad (4)$$

for the mean position $x_0 = \langle x \rangle$.

In general, the statistical distribution law of the fluctuating force f becomes to be of the Gaussian type if f consists of many independent fluctuations. On the other hand, we are concerned with a Markoffian process if the relaxation of the surrounding heat bath is Lorentzian, that is, $\langle f(t)f(t') \rangle \propto \exp[-|t-t'|/\tau]$. In such a case, the correlation functions of various fluctuating quantities can be determined by $\langle f(t)f(t') \rangle$ only. When the relaxation time τ of f is sufficiently small, the correlation function of f can be expressed by $\delta(t-t')$ as (3) and the spectrum of f becomes white.

Now let us return to our problem of nuclear reactions. In contrast with the ordinary Brownian motion in a large heat bath with steady fluctuations and a definite temperature, the Brownian motion of neutron in the compound nucleus should be considered in a somewhat restricted sense. Fluctuations of the compound nucleus are caused by passing of the incident beam and abate by emitting an outgoing neutron. In other words, the compound nucleus heats up by interaction with the incident beam and cools down by emission of a neutron. However, we may treat the compound nucleus so as to be kept in thermal equilibrium with a definite temperature, since the time-length t_0 of the incident wave packet is considerably longer than the relaxation time of the compound nucleus. In most cases the relaxation time τ of f is much shorter than that of the wave function, τ_0 , so that the steady compound nucleus occurs if

$$t_0 \gg \tau_0 \quad \text{or} \quad \Delta \ll \Gamma, \quad (5)$$

where $\Delta \equiv (\hbar/t_0)$ and $\Gamma \equiv (\hbar/\tau_0)$ are the spread in energies of the incident wave packet and the width of the "giant resonance", respectively. On the other hand, the nuclear oscillations can be regarded as random if the average period T_0 of the compound nucleus is far longer than t_0 , that is,

$$T_0 \gg t_0 \quad \text{or} \quad D \ll \Delta, \quad (6)$$

where $D \equiv (\hbar/T_0)$ is the average distance among the fine-structure levels of the compound nucleus. Furthermore, most of the fine-structure levels at low energies have widths much narrower than the average distance D , so that the nuclear oscillations corresponding to the fine-structure levels are approximately independent of each other. Hence, if the interval Δ contains many fine-structure levels—this is expected for large nuclei, then it is concluded by the central limit theorem that the statistical distribution law of f becomes Gaussian. Consequently, the dis-

tribution law of the wave function or the reduced width of the compound nucleus is also Gaussian. This fact is confirmed from the experiments.⁶⁾ When all the above conditions are satisfied as expected in the actual nuclear reactions, the motion of a neutron in the compound nucleus can be treated as a Brownian motion which is just a steady Markoffian stochastic process with a Gaussian distribution law. Thus, in most cases, the neutron will dissipate its energy in a Lorentzian type of relaxation phenomena under action of random forces. The energy dissipation results in the probability dissipation for the amplitude of elastic scattering.

The phenomenological or semi-phenomenological equation for the amplitude $\chi(\mathbf{x}, t)$ may be derived from the fundamental many-particle equation through a similar procedure as derivation of (4) or (2) from (1). Such a procedure suggests that the amplitude $\chi(\mathbf{x}, t)$ may obey the semi-phenomenological equation

$$\left(i\hbar \frac{\partial}{\partial t} - \mathcal{H}\right)\chi = f, \quad (7)$$

where \mathcal{H} is the one-particle Hamiltonian containing the complex optical potential and f the fluctuating source function with the vanishing average value. (This will be justified together with the formation of the optical potential, in a forthcoming paper.) Besides the property

$$\langle f(\mathbf{x}, t) \rangle = 0, \quad (8)$$

where $\langle \dots \rangle$ means the average value over all possible f 's, the statistical character of f is uniquely determined by the correlation function $\langle f(\mathbf{x}, t)f^*(\mathbf{x}', t') \rangle$. The explicit form will be discussed in the next section. Denoting the average amplitude χ_0 by

$$\chi_0 \equiv \langle \chi \rangle, \quad (9)$$

then one readily gets the phenomenological equation

$$\left(i\hbar \frac{\partial}{\partial t} - \mathcal{H}\right)\chi_0 = 0. \quad (10)$$

Equation (10) to be compared with (4) describes the average motion of neutron in nuclear reactions, while Equation (7) corresponding to the Langevin equation (2) may prepare a semi-phenomenological method to discuss the effects of nuclear fluctuations on the average properties of nuclear reactions.

§ 2. Statistical characters of fluctuating source function

In this section we shall present the explicit form of the correlation function $\langle f(\mathbf{x}, t)f^*(\mathbf{x}', t') \rangle$. This is achieved by making direct use of the generalized Nyquist theorem.⁷⁾ First we introduce the notion of the "impedance".

Now we have the semi-phenomenological equation

$$\left(i\hbar \frac{\partial}{\partial t} - \mathcal{H}\right)\chi = f \quad (11)$$

for a neutron in nuclear reactions, where the one-particle Hamiltonian \mathcal{H} may be written as

$$\mathcal{H} \equiv -\frac{\hbar^2}{2m} \nabla^2 + \mathcal{V}. \quad (12)$$

Here \mathcal{V} is the so-called optical potential with a negative imaginary part. Denoting the fluctuating part of the wave function χ by ξ , that is,

$$\xi \equiv \chi - \langle \chi \rangle, \quad (13)$$

then ξ obeys the same equation

$$\left(i\hbar \frac{\partial}{\partial t} - \mathcal{H}\right)\xi = f \quad (14)$$

as (11) because of (8) or (10). ξ is a particular solution of (11) or (14) which vanishes as f tends to zero. For the Fourier transforms f_ω and ξ_ω defined by

$$\left. \begin{aligned} f_\omega(\mathbf{x}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} f(\mathbf{x}, t) dt, \\ \xi_\omega(\mathbf{x}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \xi(\mathbf{x}, t) dt, \end{aligned} \right\} \quad (15)$$

Equation (14) becomes

$$(\hbar\omega - \mathcal{H})\xi_\omega = f_\omega. \quad (16)$$

As is easily understood from (11) or (14), the average rate of energy dissipation in a volume V is determined by the equation

$$\frac{dW}{dt} = \int_V \left\langle f^* \frac{\partial \xi}{\partial t} + \frac{\partial \xi^*}{\partial t} f \right\rangle d^3\mathbf{x}. \quad (17)$$

Thus we can regard $\partial \xi / \partial t$ and f as a current density and a deriving force, respectively, in the thermodynamical sense. This fact permits us to define the "impedance"* of the compound nucleus by the ratio of f_ω to $-i\omega \xi_\omega$. Hence we rewrite Equation (16) in the form

$$-\mathcal{Z}_\omega(-i\omega \xi_\omega) = f_\omega, \quad (18)$$

where the "impedance" \mathcal{Z}_ω is defined by

$$\mathcal{Z}_\omega \equiv \frac{1}{i\omega} (\hbar\omega - \mathcal{H}). \quad (19)$$

* Strictly speaking, this should be the impedance *density*.

Here it is noted that $\text{Re } \mathcal{J}_\omega \geq 0$. One may use the "admittance" \mathcal{Y}_ω defined by the inverse of \mathcal{J}_ω , that is,

$$\mathcal{Y}_\omega \equiv \mathcal{J}_\omega^{-1} = \frac{i\omega}{\hbar\omega - \mathcal{H}}. \quad (20)$$

Taking (17) and (18) into account, the well-known fluctuation-dissipation theorem⁷⁾ gives us the formula

$$\langle | \int_V f(\mathbf{x}, t) d^3\mathbf{x} |^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega E(\omega, T) \iint_V 2 \langle \mathbf{x} | \text{Re } \mathcal{J}_\omega | \mathbf{x}' \rangle d^3\mathbf{x} d^3\mathbf{x}'. \quad (21)^*$$

Here $E(\omega, T)$ stands for the average energy of a neutron with the frequency ω in thermal equilibrium and becomes asymptotically κT for $\kappa T \gg \hbar\omega$. From (21) one can easily obtain the correlation function

$$\langle f_\omega(\mathbf{x}) f_{\omega'}^*(\mathbf{x}') \rangle = E(\omega, T) [2 \langle \mathbf{x} | \text{Re } \mathcal{J}_\omega | \mathbf{x}' \rangle] \delta(\omega - \omega'). \quad (22)$$

This determines the statistical characters of f together with[†] (8). In terms of the optical potential, the statistical characters of f are rewritten as

$$\left. \begin{aligned} \langle f_\omega(\mathbf{x}) \rangle &= 0, \\ \langle f_\omega(\mathbf{x}) f_{\omega'}^*(\mathbf{x}') \rangle &= \frac{1}{\omega} E(\omega, T) [-2 \langle \mathbf{x} | \text{Im } \mathcal{V} | \mathbf{x}' \rangle] \delta(\omega - \omega'). \end{aligned} \right\} \quad (23)^{**}$$

Hence it is concluded that the fluctuating source must be responsible for the imaginary part of the optical potential as expected in § 1. Equation (11) or (16) and the fluctuation-dissipation theorem (23) are just the starting point of our semi-phenomenological theory. $f(\mathbf{x}, t)$ may be expressed in the form

$$f(\mathbf{x}, t) = \frac{\delta H_I}{\delta \tilde{\xi}(\mathbf{x}, t)},$$

where H_I is the fluctuating interaction Hamiltonian between the neutron and the target nucleus.

Here it may be necessary to pay attention to the ω -dependence of the correlation function $\langle f_\omega f_{\omega'}^* \rangle$. The factor $\delta(\omega - \omega')$ in the right-hand side of (23) means that the correlation function $\langle f(\mathbf{x}, t) f^*(\mathbf{x}', t') \rangle$ is represented by a function of $(t - t')$ only with respect to time, that is to say, the nuclear fluctuations are steady. As is mentioned above in § 1, it is to be emphasized that such fluctuations occur only in the presence of the incident beam and vanish after the beam passes by the nucleus and a final neutron is emitted. This suggests that the right-hand

* $\text{Re } \mathcal{J}_\omega$ stands for the hermitian part of the operator \mathcal{J}_ω , namely, $\langle \mathbf{x} | \text{Re } \mathcal{J}_\omega | \mathbf{x}' \rangle^* = \langle \mathbf{x}' | \text{Re } \mathcal{J}_\omega | \mathbf{x} \rangle$. If the optical potential is local, we get $\langle \mathbf{x} | \text{Re } \mathcal{J}_\omega | \mathbf{x}' \rangle = \text{Re } \mathcal{J}_\omega(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}')$.

** $\text{Im } \mathcal{V}$ is the anti-hermitian part of the operator \mathcal{V} . For the local \mathcal{V} , $\langle \mathbf{x} | \text{Im } \mathcal{V} | \mathbf{x}' \rangle$ is proportional to $\delta(\mathbf{x} - \mathbf{x}')$. The non-locality of the optical potential is a reflection of the non-zero correlation length associated with the fluctuating source f .

side of (23) should contain a factor which cuts off the frequencies outside the interval $(\hbar/\tau \gtrsim |\omega| \gtrsim \hbar/t_0)$, τ and t_0 being the relaxation time of f and the time-length of the incident wave packet, respectively. It is natural that (22) or (23) expresses the vanishing fluctuations outside the compound nucleus at every instant.

§ 3. Fluctuations of wave function

Now we investigate fluctuations in the wave function of a neutron in nuclear reactions by means of the correlation function of the fluctuating wave function. The solution of (16) is written as

$$\left. \begin{aligned} \xi_\omega(\mathbf{x}) &= \int \langle \mathbf{x} | \frac{1}{\hbar\omega - \mathcal{H}} | \mathbf{x}' \rangle f_\omega(\mathbf{x}') d^3 \mathbf{x}' \\ \text{or} \quad \xi_\omega^*(\mathbf{x}) &= \int f_\omega^*(\mathbf{x}') \langle \mathbf{x}' | \frac{1}{\hbar\omega - \mathcal{H}^*} | \mathbf{x} \rangle d^3 \mathbf{x}', \end{aligned} \right\} \quad (24)$$

where $\langle \mathbf{x} | 1/(\hbar\omega - \mathcal{H}) | \mathbf{x}' \rangle$ is Green's function of Equation (16). From (22) and (24) one can immediately obtain the formula

$$\begin{aligned} \langle \xi_\omega(\mathbf{x}) \xi_\omega^*(\mathbf{x}') \rangle &= E(\omega, T) \langle \mathbf{x} | \frac{1}{\hbar\omega - \mathcal{H}} [2 \operatorname{Re} \mathcal{J}_\omega] \frac{1}{\hbar\omega - \mathcal{H}^*} | \mathbf{x}' \rangle \delta(\omega - \omega') \\ &= \frac{i}{\omega} E(\omega, T) \langle \mathbf{x} | \frac{1}{\hbar\omega - \mathcal{H}} - \frac{1}{\hbar\omega - \mathcal{H}^*} | \mathbf{x}' \rangle \delta(\omega - \omega'). \end{aligned} \quad (25)$$

Thus we get the correlation function

$$\langle \xi(\mathbf{x}, t) \xi^*(\mathbf{x}', t') \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \frac{i}{\omega} E(\omega, T) \langle \mathbf{x} | \frac{1}{\hbar\omega - \mathcal{H}} - \frac{1}{\hbar\omega - \mathcal{H}^*} | \mathbf{x}' \rangle. \quad (26)$$

In order to see the time-dependence of this integral, it is necessary to investigate the eigenvalue problem of the operator \mathcal{H} . The eigenvector $|n\rangle$ belonging to the eigenvalue E_n satisfies the equation

$$\mathcal{H}|n\rangle = E_n|n\rangle. \quad (27)$$

Since \mathcal{H} is not hermitian, the eigenvalue E_n must be a complex number with a negative imaginary part. Putting

$$E_n \equiv \varepsilon_n - i \frac{\Gamma_n}{2} \quad (\Gamma_n \geq 0), \quad (28)$$

ε_n is an energy level and Γ_n its width of the one-particle state in the compound nucleus. (The relaxation time τ_0 of the wave function in § 1 is nothing but the lifetime (\hbar/Γ_n) .) Thus we regard ε_n and Γ_n as the position and the width of the "giant resonance" in nuclear reactions, respectively. The vector sequence $\{|n\rangle\}$ has the property

$$\left. \begin{aligned} \langle \tilde{n} | m \rangle &= \delta_{nm}, \\ \sum_n |n\rangle \langle \tilde{n}| &= 1 \end{aligned} \right\} \quad (29)$$

for normalization and completeness, as is easily proved. Here we have used the time-reversed vector $|\tilde{n}\rangle$ of the vector $|n\rangle$.²⁾ By using the second member of (29), Green's function is written in the explicit representation

$$\text{or} \quad \left. \begin{aligned} \langle x | \frac{1}{\hbar\omega - \mathcal{H}} | x' \rangle &= \sum_n \frac{\langle x | n \rangle \langle \tilde{n} | x' \rangle}{\hbar\omega - E_n} \\ \langle x | \frac{1}{\hbar\omega - \mathcal{H}^*} | x' \rangle &= \sum_n \frac{\langle x | \tilde{n} \rangle \langle n | x' \rangle}{\hbar\omega - E_n^*} \end{aligned} \right\} \quad (30)$$

Hence, in the complex ω -plane, the function $\langle x | 1/(\hbar\omega - \mathcal{H}) | x' \rangle$ has poles in the lower half, while poles of $\langle x | 1/(\hbar\omega - \mathcal{H}^*) | x' \rangle$ occupy the upper half. From this and (26), it is concluded that the correlation function $\langle \hat{\xi}(x, t) \hat{\xi}^*(x', t') \rangle$ is represented in the following way:

$$\left. \begin{aligned} \langle \hat{\xi}(x, t) \hat{\xi}^*(x', t') \rangle &= \Phi_{\xi}^+(x, t; x', t') \quad \text{for } t > t', \\ &= \Phi_{\xi}^-(x, t; x', t') \quad \text{for } t < t', \end{aligned} \right\} \quad (31)$$

where Φ_{ξ}^+ and Φ_{ξ}^- are defined by the integrals

$$\left. \begin{aligned} \Phi_{\xi}^+(x, t; x', t') &\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \frac{1}{\omega} E(\omega, T) \langle x | \frac{i}{\hbar\omega - \mathcal{H}} | x' \rangle, \\ \Phi_{\xi}^-(x, t; x', t') &\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \frac{1}{\omega} E(\omega, T) \langle x | \frac{-i}{\hbar\omega - \mathcal{H}^*} | x' \rangle. \end{aligned} \right\} \quad (32)$$

Φ_{ξ}^+ vanishes for $t < t'$, while Φ_{ξ}^- is zero for $t > t'$. It is quite easy to prove the relation

$$\Phi_{\xi}^-(x, t; x', t') = \{\Phi_{\xi}^+(x', t'; x, t)\}^*.$$

From the Fourier transform of Φ_{ξ}^+ , we obtain the formula

$$\int_{-\infty}^{\infty} \Phi_{\xi}^+(x, t; x', t') e^{i\omega(t-t')} d(t-t') = \frac{1}{\omega} E(\omega, T) \langle x | \frac{i}{\hbar\omega - \mathcal{H}} | x' \rangle$$

or

$$\int_0^{\infty} \langle \hat{\xi}(x, t) \hat{\xi}^*(x', t') \rangle e^{i\omega(t-t')} d(t-t') = \frac{1}{\omega} E(\omega, T) \langle x | \frac{i}{\hbar\omega - \mathcal{H}} | x' \rangle.$$

This formula can be rewritten as

$$\langle x | \frac{1}{\hbar\omega - \mathcal{H}} | x' \rangle = \frac{-i\omega}{E(\omega, T)} \int_0^{\infty} \langle \hat{\xi}(x, t) \hat{\xi}^*(x', t') \rangle e^{i\omega(t-t')} d(t-t'). \quad (33)$$

In this formula Green's function $\langle \mathbf{x} | 1/(\hbar\omega - \mathcal{H}) | \mathbf{x}' \rangle$ of (16) is expressed as the Fourier transform of the correlation function of the fluctuating wave function. This fact is very interesting, for the observation of fluctuations in the wave function informs us about the important average property of nuclear reactions. Such a relation becomes more impressive by making use of the strength function.

Let us fix the coordinate \mathbf{x} at the point \mathbf{x}_0 on the nuclear surface. Since we can use there the reduced width^{2,4)} γ_n defined by

$$\gamma_n^2 \equiv \frac{\hbar^2 R_0}{2m} \langle \mathbf{x}_0 | n \rangle^2, \quad (34)$$

R_0 and m being the nuclear radius and the neutron mass, respectively, we obtain the formula

$$\text{Re} \langle \mathbf{x}_0 | \frac{i}{\hbar\omega - \mathcal{H}} | \mathbf{x}_0 \rangle = \frac{2\pi m}{\hbar^2 R_0} s(\hbar\omega), \quad (35)$$

where the relation $\langle \mathbf{x}_0 | n \rangle = \langle \tilde{n} | \mathbf{x}_0 \rangle$ and the assumption of real γ_n^2 have been used. Here $s(\hbar\omega)$ is the strength function defined by

$$\left. \begin{aligned} s(\hbar\omega) &\equiv \sum_n s_n(\hbar\omega), \\ s_n(\hbar\omega) &\equiv \frac{\gamma_n^2 \Gamma_n}{(2\pi) [(\hbar\omega - \varepsilon_n)^2 + \Gamma_n^2/4]} \end{aligned} \right\} \quad (36)$$

$s_n(\hbar\omega)$ is normalized by the condition

$$\int s_n(\hbar\omega) d(\hbar\omega) = \gamma_n^2.$$

Thus it is easy to obtain the formula

$$\text{Re} \int_0^\infty \langle \hat{\xi}(\mathbf{x}_0, t) \hat{\xi}^*(\mathbf{x}_0, t') \rangle e^{i\omega(t-t')} d(t-t') = \frac{2\pi m}{\hbar R_0} \cdot \frac{E(\omega, T)}{\hbar\omega} s(\hbar\omega). \quad (37)$$

This formula shows that observation of fluctuations in the wave function leads to the full knowledge of the strength function which plays an essential role in the average properties of nuclear reactions. Such a relationship can also be obtained from observation of the spectral intensity of the fluctuating wave function. The spectral intensity $G_{\xi}(\omega)$ of $\hat{\xi}$ is defined by

$$\int_{-\infty}^{\infty} G_{\xi}(\omega) d\omega \equiv \langle |\hat{\xi}(\mathbf{x}_0, t)|^2 \rangle = \Phi_{\xi}^+(\mathbf{x}_0, t; \mathbf{x}_0, t) + \Phi_{\xi}^-(\mathbf{x}_0, t; \mathbf{x}_0, t). \quad (38)$$

From the definition (26) and (32), we obtain the formula

$$\left. \begin{aligned} G_{\xi}(\omega) &= \frac{1}{\pi\omega} E(\omega, T) \text{Re} \langle \mathbf{x}_0 | \frac{i}{\hbar\omega - \mathcal{H}} | \mathbf{x}_0 \rangle, \\ &= \frac{2m}{\hbar R_0} \cdot \frac{E(\omega, T)}{\hbar\omega} s(\hbar\omega). \end{aligned} \right\} \quad (39)$$

That is, $G_{\xi}(\omega)$ informs us about $s(\hbar\omega)$.

Finally it is not unnecessary to derive a simple relation between the "admittance" and the fluctuation of the current $\dot{\xi} \equiv \partial \xi / \partial t$, for the admittance is defined as a ratio of f_{ω} to $\dot{\xi}_{\omega} = -i\omega \xi_{\omega}$. Defining the function $\Phi_{\xi}^{+}(\mathbf{x}, t; \mathbf{x}', t')$ by the integral

$$\Phi_{\xi}^{+}(\mathbf{x}, t; \mathbf{x}', t') \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} E(\omega, T) \langle \mathbf{x} | \mathcal{J}_{\omega} | \mathbf{x}' \rangle, \quad (40)$$

and $\Phi_{\xi}^{-}(\mathbf{x}, t; \mathbf{x}', t')$ by the relation

$$\Phi_{\xi}^{-}(\mathbf{x}, t; \mathbf{x}', t') \equiv \{\Phi_{\xi}^{+}(\mathbf{x}', t'; \mathbf{x}, t)\}^*,$$

respectively, we can put the correlation function of $\dot{\xi}$ in the following way,

$$\langle \dot{\xi}(\mathbf{x}, t) \dot{\xi}^*(\mathbf{x}', t') \rangle = \Phi_{\xi}^{\pm}(\mathbf{x}, t; \mathbf{x}', t') \quad \text{for } t \geq t', \quad (41)$$

as is easily proved. From (40) and (41) we immediately obtain the formula

$$\langle \mathbf{x} | \mathcal{J}_{\omega} | \mathbf{x}' \rangle = \frac{1}{E(\omega, T)} \int_0^{\infty} \langle \dot{\xi}(\mathbf{x}, t) \dot{\xi}^*(\mathbf{x}', t') \rangle e^{i\omega(t-t')} d(t-t'). \quad (42)$$

This is quite similar to the relationship between the admittance and the current-fluctuations in the theory of Brownian motions or noises.

On the nuclear surface the admittance becomes

$$\text{Re} \langle \mathbf{x}_0 | \mathcal{J}_{\omega} | \mathbf{x}_0 \rangle = \frac{\pi k^2}{\hbar R_0} s(\hbar\omega), \quad (43)$$

where we have used the relation $\hbar\omega = \hbar^2 k^2 / 2m$, k being the wave number of neutron. The spectral intensity $G_{\xi}(\omega)$ of $\dot{\xi}$ is connected with $G_{\xi}(\omega)$ as $G_{\dot{\xi}} = \omega^2 G_{\xi}$ and is expressed by

$$\left. \begin{aligned} G_{\dot{\xi}}(\omega) &= \frac{1}{\pi} E(\omega, T) \text{Re} \langle \mathbf{x}_0 | \mathcal{J}_{\omega} | \mathbf{x}_0 \rangle, \\ &= \frac{k^2}{\hbar R_0} E(\omega, T) s(\hbar\omega). \end{aligned} \right\} \quad (44)$$

§ 4. Surface admittance of compound nucleus and strength function of nuclear reaction

In the ordinary theory of nuclear reactions one makes use of the notion of the nuclear surface which introduces a sharp cut between the channel region and the nuclear region. Thus the wave function ξ of neutron obeys the equation

$$\begin{aligned} i\hbar \frac{\partial \xi}{\partial t} &= -\frac{\hbar^2}{2m} \nabla^2 \xi \quad \text{for } |\mathbf{x}| > R_0, \\ &= \left(-\frac{\hbar^2}{2m} \nabla^2 + \mathcal{V} \right) \xi \quad \text{for } |\mathbf{x}| < R_0, \end{aligned} \quad (45)$$

where R_0 is the radius of the compound nucleus. The wave function in the channel region must be connected with that in the nuclear region by the continuity condition for the logarithmic derivative of the wave function. Thus, for an observer in the channel region, the compound nucleus can be replaced by the value at the nuclear surface of the logarithmic derivative of the wave function in the compound nucleus. Since the logarithmic derivative is considered to be an "impedance" or an "admittance" of the compound nucleus, it is more convenient to formulate the problem of nuclear reactions in the terminology of circuit theory. The problem of nuclear reactions is equivalent to a transmission line loaded by an impedance, in which the transmission line and the load impedance correspond to the channel region and the compound nucleus, respectively. Such an equivalence can be achieved by the correspondence

$$I \equiv \frac{i\hbar}{\sqrt{2}} \frac{\partial \xi}{\partial t}, \quad V \equiv \frac{\hbar}{\sqrt{2}im} \nabla \xi. \quad (46)$$

The "current" I and the "voltage" V obey the equations

$$\left. \begin{aligned} RI + \nabla \cdot V &= 0, \\ C \frac{\partial V}{\partial t} + \nabla I &= 0 \end{aligned} \right\} \quad (47)$$

in the channel region, where $R \equiv (2/i\hbar)$ and $C \equiv m$. This is the same form as the equation for the Kelvin cable. Since this line has the distributed impedance $Z' \equiv R$ and the distributed admittance $Y' \equiv -i\omega C$, we get the propagation constant $\sqrt{Z'Y'} = ik$ and the characteristic impedance $Z_0 \equiv \sqrt{Z'/Y'} = (2/\hbar k)$ in the channel region. The surface impedance of the compound nucleus is defined by

$$Z(\omega) \equiv [V_\omega/I_\omega]_{R_0} = \frac{1}{im\omega} \left[\frac{d}{d|x|} \log \xi(x) \right]_{R_0}, \quad (48)$$

where V is the component of V normal to the nuclear surface. The inverse of $Z(\omega)$, that is,

$$Y(\omega) \equiv Z^{-1}(\omega), \quad (49)$$

is the surface admittance of the compound nucleus. Z or Y is determined by the wave function in the compound nucleus. If Z and Y are given, the various quantities of nuclear reactions can be calculated from them. The S -matrix is defined by the formula

$$S \equiv \frac{Z - Z_0}{Z + Z_0} e^{-2ikR_0} \quad (50)$$

as is well known in circuit theory. The cross section for elastic scattering is proportional to

$$|S-1|^2 = \left| \frac{-2Z_0}{Z+Z_0} e^{-2ikR_0} + (e^{-2ikR_0} - 1) \right|^2. \quad (51)$$

Thus we obtain the absorption probability $A(\omega)$ defined by

$$A(\omega) \equiv 1 - |S|^2 = \frac{4 \operatorname{Re} Z_0 Z}{|Z + Z_0|^2} = \frac{4 \operatorname{Re} Z_0 Y}{|1 + Z_0 Y|^2} \simeq 4 \operatorname{Re} Z_0 Y. \quad (52)$$

This is interpreted as the probability for formation of a compound nucleus by absorbing a neutron. At the same time, this is also the probability for decay of a compound nucleus by emitting a neutron, as is easily understood from the reciprocity theorem.

Now the compound nucleus can be replaced by the surface impedance Z concentrated at the nuclear surface, so that the fluctuating source may apparently be represented by the concentrated source, that is,

$$f(\mathbf{x}, t) = -\delta(|\mathbf{x}| - R_0) \frac{\hbar^2}{2m} \frac{\partial \hat{\xi}(\mathbf{x}, t)}{\partial |\mathbf{x}|}. \quad (53)$$

Substituting (53) into (17) and using (46), the average rate of energy dissipation is expressed by I and V as follows,

$$\frac{dW}{dt} = \oint \langle IV^* + I^* V \rangle_{R_0} R_0^2 d\Omega, \quad (54)$$

where Ω is the solid angle, hence the replacement (53) is reasonable.

If, for simplicity, we deal with only the S -waves, then the solution (24) at the nuclear surface gives us the surface admittance

$$Y(\omega) = \frac{\hbar^2}{2} \oint \langle \mathbf{x}_0 | \mathcal{J}_\omega | \mathbf{x}_0 \rangle R_0^2 d\Omega, \quad (55)$$

as is easily proved by (24), (46), (48), (49) and (53). We obtain the formula

$$\operatorname{Re} Y(\omega) = 2\pi^2 \hbar k^2 R_0 s(\hbar\omega) \quad (56)$$

from (43).

Substituting (42) into (55) and using (46), we get the formula

$$Y(\omega) = \frac{1}{E(\omega, T)} \int_0^\infty \left\{ \oint \langle I(\mathbf{x}_0, t) I^*(\mathbf{x}_0, t') \rangle R_0^2 d\Omega \right\} e^{i\omega(t-t')} d(t-t') \quad (57)$$

for the relation between the admittance and the current. This is naturally expected. Furthermore it is evident that the spectral intensity $G_I(\omega)$ of the fluctuating current I is determined by

$$G_I(\omega) = \frac{\hbar^2}{2} \oint G_i(\omega) R_0^2 d\Omega.$$

Hence we get

$$\left. \begin{aligned} G_I(\omega) &= \frac{1}{\pi} E(\omega, T) \operatorname{Re} Y(\omega), \\ &= 2\pi \hbar k^2 R_0 E(\omega, T) s(\hbar\omega). \end{aligned} \right\} \quad (58)$$

Both (57) and (58) show the relationship between the fluctuating quantities and the average properties in a way similar to those mentioned in the last section. Finally we write an interesting formula, for the average rate of energy emission from the compound nucleus,

$$\left. \begin{aligned} dP_{\omega} &= 2 \frac{Z_0}{|Z_0 + Z|^2} \cdot \frac{G_I(\omega)}{|Y|^2} d\omega, \\ &= \frac{1}{2\pi} E(\omega, T) A(\omega) d\omega, \\ &= E(\omega, T) (8\pi k R_0) s(\hbar\omega) d\omega. \end{aligned} \right\} \quad (59)$$

This is nothing but the formula* obtained by Porter⁵⁾.

§ 5. Conclusion

In the preceding sections, it has been shown that nearly all the phenomenological behaviors of nuclear reactions at low energies can be interpreted by fluctuations of the wave function or the source function of neutron. The theory starts from the semi-phenomenological equation (11) and the fluctuation-dissipation theorem (23). Throughout the present paper the optical potential \mathcal{V} is regarded as a given function (or operator). The construction of \mathcal{V} must be performed in the framework of the fundamental many-particle problem. The purpose of the present work is to emphasize the possibility of showing, without resort to detailed discussions about the many-particle problem, the fact that the optical model should describe the average properties of the fluctuating compound nucleus. Equation (11) is the most simplified form for description of fluctuations. Our next work is to justify Equation (11) and to construct the optical potential in the framework of the fundamental many-particle problem. This will be performed in a forthcoming paper.

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* Note that this is different from Porter's expression by 4π . However, this difference is quite trivial, for he used the reduced width defined by $\gamma_n^2 \equiv \frac{2\pi\hbar^2 R_0}{m} \langle x_0 | n \rangle^2$ in the place of (34).

Note added in proof: In the present paper, we have considered that the random motions of a neutron in question originates in the situation which a short wave packet ($T_0 \gg t_D$) regards nuclear oscillations as random oscillations. However, it may be plausible that the neutron beams produced by actual instruments would rather consist of a random mixture of long wave packets, whose central energies distribute over the same region as the energy spread of the above short wave packet. As is understood from van Hove and Toda's discussions, the similar asymptotic randomness can also be observed by a mixture beam. [L. van Hove; *Physica* **21** (1955), 517, *ibid.* **23** (1957), 441, M. Toda, *J. Phys. Soc. Japan* **13** (1958), 1266] At any rate, discussions and conclusions in the present paper are never altered.

Remarks on the Transformation Properties of the Dirac Equation

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The physical properties of dynamical variables in the extreme-relativistic representation will be discussed. Especially, it will be pointed out that in the choice of the proper "mean" position we must be careful, and that our results include those of Bose-Gamba-Sudarshan as special cases.

§ 1. Introduction and summary

The extreme-relativistic approximation, i.e. when $p \gg m$ (limit towards the Weyl equation), may be valuable for the discussion of high energy scattering phenomena. It is to be noted that the approach of Yennie-Ravenhall-Wilson¹⁾ and Brenner-Brown-Elton²⁾ to the scattering of fast electrons by nuclei is essentially the zeroth order approximation in such a theory. And it has also been pointed out by Bose-Gamba-Sudarshan³⁾ and the present author⁴⁾ that this approximation may be suggestive for the discussion of the chirality invariance in the theory of weak interactions. Hence the extreme-relativistic representation may be valuable in connection with such an approximation.

In the preceding paper, we discussed in a synthetical manner the transformation properties of the Dirac equation in the extreme-non-relativistic and extreme-relativistic representations.⁴⁾ Then, it was pointed out that there exist some ambiguities in the choice of the usual F-W-T (Foldy-Wouthuysen-Tani)⁵⁾ and of C-T (Cini-Touschek)⁶⁾ transformations respectively. It is not obvious *a priori* whether such different F-W-T (and/or C-T) transformations are equivalent to each other or not. Recently Pursey⁷⁾ has proved that within the validity of approximations the ambiguities in transformations give rise at most to unitary transformations connecting different approximate theories.

The purpose of this note is to investigate the physical significance of dynamical variables in the extreme-relativistic representation, which has been left untouched in the previous work.⁴⁾ Similar work has recently been carried out by Bose-Gamba-Sudarshan³⁾. As will be seen, the essential differences between their results and ours lie in the interpretation of operators in the new representation, and the former is contained in the latter as a special case. Specifically, in the extreme relativistic representation,* we shall first introduce a proper "mean" position

* Following Bose-Gamba-Sudarshan, we shall refer to the extreme-non-relativistic, Dirac- and extreme-relativistic representation simply as *C*-, *D*- and *E*-one, respectively.

operator, different from the usual one, such that its time derivative is the operator \mathbf{p}/E for the positive energy states and $-\mathbf{p}/E$ for the negative energy states, making a good correspondence to the conventional definition of the velocity of a particle. Then, we shall have a proper "mean" orbital angular momentum defined differently from the usual one in such a way that the spin appears related to the new position in a natural fashion. This is a constant of motion by itself. And a proper "mean" spin angular momentum is related simply to the helicity of the particle.

Throughout in this note we adopt the natural unit $\hbar=c=1$.

§ 2. Operators in D - and E -representation

In order to pass over to E -representation from D -representation, we may use the following unitary transformation,³⁾⁴⁾⁵⁾

$$U = \exp \left[-\frac{1}{2p} \beta (\boldsymbol{\alpha} \cdot \mathbf{p}) \tan^{-1} \frac{m}{p} \right]. \quad (1)$$

Then, the wave function ψ in D -representation corresponding to a state where the particle is located definitely at one point, passes over into the wave function $\psi' = U\psi$ apparently representing in E -representation a particle being spread over a finite region of dimensions, at least, of the order of $1/p$. Indeed, the position operator of the particle in E -representation is no longer the operator \mathbf{x} in D -representation, being rather a complicated one:

$$\mathbf{x}' = U\mathbf{x}U^{-1} = \mathbf{x} + \frac{im\beta\boldsymbol{\alpha}}{2Ep} - \frac{im(E+p)\beta(\boldsymbol{\alpha} \cdot \mathbf{p})\mathbf{p}}{2p^3E^2} - \frac{m^2(\boldsymbol{\sigma} \times \mathbf{p})}{2E(E+p)p^2}, \quad (2)$$

and the hamiltonian now reads

$$H' = UHU^{-1} = E \cdot \frac{(\boldsymbol{\alpha} \cdot \mathbf{p})}{p} \quad (3)$$

with

$$H = m\beta + (\boldsymbol{\alpha} \cdot \mathbf{p}). \quad (4)$$

Thus, in D -representation we want to find out the position operator \mathbf{X} which satisfies the relation

$$UXU^{-1} = \mathbf{X}' = \mathbf{x} \quad (5)$$

where \mathbf{x} may be the "mean" position in E -representation. We find

$$\mathbf{X} = U^{-1}\mathbf{x}'U = \mathbf{x} - \frac{im\beta\boldsymbol{\alpha}}{2Ep} + \frac{im(E+p)\beta(\boldsymbol{\alpha} \cdot \mathbf{p})\mathbf{p}}{2E^2p^3} - \frac{m^2(\boldsymbol{\sigma} \times \mathbf{p})}{2E(E+p)p^2} \quad (6)$$

It is well known that in D -representation the velocity operator is calculated as

$$\dot{\mathbf{x}} = i[\mathbf{x}, H]_- = \boldsymbol{\alpha}. \quad (7)$$

Then, in E -representation the velocity operator reads

$$\dot{\mathbf{x}}' = \boldsymbol{\alpha} - \frac{m\beta}{E\dot{p}} \mathbf{p} - \frac{m^2}{E(E+\dot{p})\dot{p}^2} (\boldsymbol{\alpha} \cdot \mathbf{p}) \mathbf{p}, \quad (8)$$

and its projection by $\hat{\mathbf{p}} \equiv \mathbf{p}/p$, which is a unit vector parallel with the momentum \mathbf{p} , is

$$(\dot{\mathbf{x}}' \cdot \hat{\mathbf{p}}) = \frac{1}{E} [(\boldsymbol{\alpha} \cdot \mathbf{p}) - m\beta] = -\gamma'_5 \cdot \frac{(\boldsymbol{\sigma} \cdot \mathbf{p})}{p} \quad (9)$$

with

$$\gamma'_5 = -\frac{(\boldsymbol{\alpha} \cdot \mathbf{p}) - m\beta}{E} \cdot \frac{(\boldsymbol{\sigma} \cdot \mathbf{p})}{p}. \quad (10)$$

It is to be noted that the momentum \mathbf{p} commutes always with the quantities in Eqs. (1) and (4). In order to interpret $\dot{\mathbf{X}}$ in Eq. (6), we calculate its time derivative

$$\dot{\dot{\mathbf{X}}} = \frac{E}{\dot{p}} \boldsymbol{\alpha} + \frac{m\beta}{E^2} \mathbf{p} - \frac{m^2(m^2 + 2\dot{p}^2 + E\dot{p})}{E^3(E+\dot{p})\dot{p}^3} \cdot (\boldsymbol{\alpha} \cdot \mathbf{p}) \mathbf{p}. \quad (11)$$

This is complicated, but its projection by $\hat{\mathbf{p}}$ simply reads

$$(\dot{\dot{\mathbf{X}}} \cdot \hat{\mathbf{p}}) = \frac{\dot{p}}{E} \cdot \frac{\beta m + (\boldsymbol{\alpha} \cdot \mathbf{p})}{E} \quad (12)$$

with similar properties as those in C -representation. In E -representation, the "mean" velocity operator may be

$$\dot{\mathbf{x}} = \frac{E}{\dot{p}} \boldsymbol{\alpha} - \frac{m^2(\boldsymbol{\alpha} \cdot \mathbf{p}) \mathbf{p}}{E\dot{p}^3}. \quad (13)$$

And its projection by $\hat{\mathbf{p}}$ simply reads

$$(\dot{\mathbf{x}} \cdot \hat{\mathbf{p}}) = \frac{\dot{p}}{E} \cdot \frac{(\boldsymbol{\alpha} \cdot \mathbf{p})}{p} \quad (14)$$

with the similar properties as those in C -representation, analogous to Eq. (12).

Now, in order to examine in detail the $\dot{\mathbf{x}}$ in Eq. (13), we calculate its time derivative

$$\ddot{\mathbf{x}} = -\frac{2E^2}{\dot{p}^3} (\boldsymbol{\sigma} \times \mathbf{p}) = 2E\gamma'_5 (\dot{\mathbf{x}} \times \mathbf{p}) / \dot{p} = -2i\dot{\mathbf{x}} + 2i\mathbf{p}. \quad (15)$$

From the first equality, we immediately see that $\ddot{\mathbf{x}}$ has no component along $\hat{\mathbf{p}}$, i.e. it satisfies

$$(\ddot{\mathbf{x}} \cdot \hat{\mathbf{p}}) = 0. \quad (16)$$

Following Dirac's procedure⁸⁾, we obtain by using the third equality of Eq. (15) and $\ddot{\mathbf{x}} = -2i\dot{\mathbf{x}}H'$

$$\dot{\mathbf{x}} = \frac{i}{2} \ddot{\mathbf{x}}_0 e^{-2iHt}/H' + \mathbf{p}/H'. \quad (17)$$

The $\ddot{\mathbf{x}}_0$ is a constant in time, being equal to the value of $\ddot{\mathbf{x}}$ at $t=0$. From Eq. (17), we see that the $\dot{\mathbf{x}}$ in E -representation consists of two parts, a constant \mathbf{p}/H' and an oscillatory part $(i/2)\ddot{\mathbf{x}}_0 e^{-2iHt}/H'$. This oscillates on the plane perpendicular to $\hat{\mathbf{p}}$, since $(\ddot{\mathbf{x}}_0 \cdot \hat{\mathbf{p}}) = 0$. Its frequency is high, being $2H'$, and is at least $2p$ when $p \gg m$. Moreover, it may be curious that the amplitude of this oscillation is E/p . Such a situation is quite different from that in D -representation where the amplitude of oscillation of the velocity α is isotropically unity.* The components of the $\dot{\mathbf{x}}$ in Eq. (13) do not commute with each another, and therefore we take up the x -component of this operator alone which reads

$$\dot{x} = \frac{E}{p} \alpha_x - \frac{m^2(\alpha \cdot \mathbf{p})}{Ep^3} p_x. \quad (13')$$

From Eq. (13') we see that its eigenvalue has the following properties:

- 1) For $p_x \rightarrow \infty$, p_y and p_z being fixed, it tends to ± 1 ;
- 2) For $p_x = p$ with $p_y = p_z = 0$, it is equal to $\pm p/E$;
- 3) For vanishing mass $m \rightarrow 0$, it tends to ± 1 ;
- 4) When $p_x \rightarrow 0$, p_y and/or p_z is not zero, it tends to $\pm E/p$.

Cases 1), 2) and 3) may be interpreted in a natural manner. As for the case 4), it seems to contradict the requirement of relativity, since the velocity of particle exceeds that of light. Such a difficulty suggests that the $X' = \mathbf{x}$ in E -representation is not the "mean" position operator in its proper sense. Thus, it may be mentioned that such E -position operator is a valid one only for the case where m^2/p^2 is negligibly small.

Next, we try to look for the proper "mean" position in E -representation. For this purpose, let us adopt the following position operator in this representation

$$\mathbf{x}_E = \mathbf{x} - \frac{1}{2p^2} (\boldsymbol{\sigma} \times \mathbf{p}), \quad (18)$$

from which has already been eliminated the oscillatory part of the $\dot{\mathbf{x}}$ given by Eq. (17), whose amplitude is E/p . This satisfies the usual commutation relation with the momentum \mathbf{p} . Indeed, its time derivative is given by

$$\dot{\mathbf{x}}_E = \frac{\mathbf{p}}{E} - \frac{(\boldsymbol{\alpha} \cdot \mathbf{p})}{p}, \quad (19)$$

* Let \mathbf{i} , \mathbf{j} and $\hat{\mathbf{p}}$ be mutually orthogonal unit vectors, then we get

$$(\dot{\mathbf{x}} \cdot \mathbf{i}) = \frac{E}{p} (\boldsymbol{\alpha} \cdot \mathbf{i}) \quad \text{and} \quad (\dot{\mathbf{x}} \cdot \mathbf{j}) = \frac{E}{p} (\boldsymbol{\alpha} \cdot \mathbf{j}).$$

As to the average value over a short lapse of time of these components, we have the following results,

$$\text{sp}[(\dot{\mathbf{x}} \cdot \hat{\mathbf{p}}) A_{\pm}'(\mathbf{p})] = \pm \frac{p}{E} \quad \text{and} \quad \text{sp}[(\dot{\mathbf{x}} \cdot \mathbf{i}) A_{\pm}'(\mathbf{p})] = \text{sp}[(\dot{\mathbf{x}} \cdot \mathbf{j}) A_{\pm}'(\mathbf{p})] = 0,$$

where the double sign denotes the positive or negative energy states respectively.

which is related to the momentum \mathbf{p} by a relation to be expected from the theory of relativity. At this stage, the proper position operator corresponding to the \mathbf{x}_E is given in D -representation by

$$\mathbf{X}_D = \mathbf{x} - \frac{1}{2p^2} (\boldsymbol{\sigma} \times \mathbf{p}) + \frac{im\beta(\boldsymbol{\alpha} \cdot \mathbf{p})\mathbf{p}}{2E^2 p^2}. \quad (20)$$

Hence, its time derivative reads

$$\dot{\mathbf{X}}_D = \frac{\mathbf{p}}{E} \cdot \frac{\beta m + (\boldsymbol{\alpha} \cdot \mathbf{p})}{E} \quad (21)$$

with the same properties as the above. And we always have

$$\ddot{\mathbf{X}}_D = \ddot{\mathbf{x}}_E = 0. \quad (22)$$

It is to be noted that such \mathbf{x}_E (and/or \mathbf{X}_D) is quite different from that of Bose-Gamba-Sundarshan.⁸⁾ It is likely that the very \mathbf{x}_E is the proper "mean" position operator of the spreading particle in E -representation. Eq. (18) (and/or Eq. (20)) shows that their components are not commutable with each other.* The commutator for the x and y component of \mathbf{x}_E and \mathbf{X}_D

$$[x_E, y_E]_- = \frac{1}{2ip^4} (\boldsymbol{\sigma} \cdot \mathbf{p}) p_z, \quad (23)$$

and

$$[X_D, Y_D]_- = \frac{1}{2ip^4} (\boldsymbol{\sigma} \cdot \mathbf{p}) p_z - \frac{m\beta(\boldsymbol{\alpha} \times \mathbf{p})_z}{2E^2 p^2}, \quad (24)$$

respectively. On the other hand, components of the proper "mean" velocity operator in E - (and/or D -) representation always commute with each other. Here we note that the internal structure of the spreading particle may be reflected in the proper "mean" position operator in E -representation.

Furthermore, the modification in the interpretation of operators involved in each representation has to be discussed also for the new orbital- and spin-angular momentum. As we may readily show, the proper "mean" orbital angular momentum reads

$$\mathbf{L}_E = (\mathbf{x}_E \times \mathbf{p}) = (\mathbf{x} \times \mathbf{p}) + \frac{1}{2} \boldsymbol{\sigma} - \frac{1}{2p^2} (\boldsymbol{\sigma} \cdot \mathbf{p}) \mathbf{p} \quad (25)$$

for E -representation, and

$$\mathbf{L}_D = (\mathbf{X}_D \times \mathbf{p}) = (\mathbf{x}_E \times \mathbf{p}) \quad (26)$$

for D -representation. It seems curious that there is no need to add the quantity $\frac{1}{2}\boldsymbol{\sigma}$ to the orbital angular momentum. A new spin angular momentum is

$$\boldsymbol{\sigma}_E = \frac{1}{p^2} (\boldsymbol{\sigma} \cdot \mathbf{p}) \mathbf{p} = \boldsymbol{\Sigma}_D \quad (27)$$

* Pryce already pointed out that there exists such a situation in the discussion of relativistic definitions of center of mass for systems of particles.⁹⁾

for each representation. In D -representation, except for the energy sign, our L_D and Σ_D coincide with those of Bose-Gamba-Sudarshan, in spite of the difference in the interpretation of operators. As was pointed out by Bose-Gamba-Sudarshan, it is of interest to note that the spin appears related to the proper “mean” position in a natural fashion, or in other words that L_E (and/or L_D) automatically includes the relativistic spin terms, and that σ_E (and/or Σ_D) is simply related to the longitudinal polarization of the particle. Moreover, it is to be noticed that L_E and σ_E (and/or L_D and Σ_D) are constants of motion, respectively.

Finally, if we wished to discuss the operator in E -representation

$$A_{\pm} = A'_{\pm} A A'_{\pm} \quad \text{with} \quad A'_{\pm} = \frac{1}{2} \left(1 \pm \frac{(\boldsymbol{\alpha} \cdot \mathbf{p})}{p} \right)$$

instead of a generic operator A , as Bose-Gamba-Sudarshan did, then we should immediately find out that our results coincide with theirs. For convenience, E - and D -operators are listed simply in Table I.

In conclusion, the author wishes to express his gratitude to professors Z. Koba and Y. Munakata for kind discussions. His acknowledgment is due to Prof. H. Yukawa for his continuous encouragement.

Table I Relevant E -operators in E - and D -representation in a proper sense.

E -representation	D -representation
“mean” position	
$\mathbf{x}_E = \mathbf{x} - \frac{1}{2p^2} (\boldsymbol{\sigma} \times \mathbf{p})$	$\mathbf{X}_D = \mathbf{x} - \frac{1}{2p^2} (\boldsymbol{\sigma} \times \mathbf{p}) + \frac{im\beta(\boldsymbol{\alpha} \cdot \mathbf{p})\mathbf{p}}{2E^2p^3}$
momentum	
\mathbf{p}	\mathbf{p}
“mean” velocity	
$\dot{\mathbf{x}}_E = \frac{\mathbf{p}}{E} \cdot \frac{(\boldsymbol{\alpha} \cdot \mathbf{p})}{p}$	$\dot{\mathbf{X}}_D = \frac{\mathbf{p}}{E} \cdot \frac{\beta\mathbf{m} + (\boldsymbol{\alpha} \cdot \mathbf{p})}{E}$
“mean” orbital angular momentum	
$(\mathbf{x}_E \times \mathbf{p}) = (\mathbf{x} \times \mathbf{p}) + \frac{1}{2} \boldsymbol{\sigma} - \frac{1}{2p^2} (\boldsymbol{\sigma} \cdot \mathbf{p}) \mathbf{p}$	$(\mathbf{X}_D \times \mathbf{p}) = (\mathbf{x}_E \times \mathbf{p})$
“mean” spin angular momentum	
$\boldsymbol{\sigma}_E = \frac{1}{p^2} (\boldsymbol{\sigma} \cdot \mathbf{p}) \mathbf{p}$	$\boldsymbol{\Sigma}_D = \boldsymbol{\sigma}_E$

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Electromagnetic Structure of the Nucleon. III—Static Limits and *S*-Wave Effects—

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The static limits of the 3π -state contribution (lowest order) to the electromagnetic structure of the nucleon are explicitly calculated. By comparing these with Bosco and Alfaro's results, the *S*-wave effect on the isoscalar part is investigated in detail, and it is shown that the *S*-wave effect is quite small (always less than 20%). Numerical evaluations of $\lim\langle r^2 \rangle_{1,2}^S$, etc., also are carried out, but the numerical values are too sensitive to cut-off methods to draw any quantitative conclusion.

§ 1. Introduction

In our previous works,¹⁾ we have investigated the electromagnetic structure of the nucleon meson-theoretically. Especially we calculated the 3π -state contribution to the isoscalar part in the lowest order ($\sim eg^0$) of relativistic perturbation theory. But the results obtained are very complicated, and so numerical estimations were carried out in very crude ways. We also presented expressions for the spectral functions in the “*no-loop approximation*” in which the closed loop is shrunk to one point but the open nucleon line is treated still relativistically. Though these expressions are much simpler than the original integrals, they are still too complicated to carry out the integrations analytically.

The charge distribution function (of the 3π -state lowest order contribution) is positive in the outer region and negative in the inner region, but in the no-loop approximation it becomes positive definite because the negative part then shrinks to a negative δ -function. The a. m. m. distribution function is positive in the inner region and also positive in the asymptotic region as will be verified in § 4, hence it may be positive definite.

Very recently, Bosco and Alfaro²⁾ have calculated the 3π -state contribution by the static theory, assuming an effective interaction Hamiltonian,

$$H_{\text{int}} = ie(\lambda/\mu^3) \varepsilon_{\lambda\mu\nu\rho} A_\lambda \frac{\partial\phi_1}{\partial x_\mu} \frac{\partial\phi_2}{\partial x_\nu} \frac{\partial\phi_3}{\partial x_\rho}. \quad (1.1)$$

Their results are both positive definite if one takes $\lambda > 0$.

In the present paper, we investigate the relations between our relativistic

results and their static ones. Namely, we calculate the static limits of the spectral functions in the no-loop approximation, as was done in I, § 4-1.*

For convenience of comparison, we should take the effective coupling constant λ in (1.1) as

$$\lambda = (16/\sqrt{\pi})f^3, \quad (1.2)$$

which is obtained by calculating the γ - 3π nucleon closed loop shown in Fig. 1. Then the differences between our static limits and their results manifestly correspond to S -wave contributions. We can thus evaluate the S -wave effects on the 3π -state contributions.

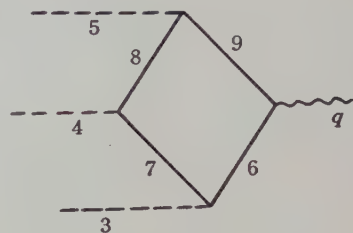


Fig. 1

In § 2, we shall calculate the static limits of our relativistic spectral functions in the no-loop approximation, and show that the S -wave effect vanishes for the charge spectral function and is less than 20% of the P -wave effect for the a. m. m. spectral function. In § 3, We shall summarize the numerical results of static limits of not only the isoscalar part but also the isovector part. In § 4, we shall investigate the behaviors near the thresholds of the spectral functions, which can be calculated also for the relativistic expressions by means of a new technique. Finally, we discuss our results.

The notations are the same as those of I throughout this paper, unless otherwise indicated.

§ 2. Static limits

The spectral functions in the no-loop approximation (see Fig. 2) of the 3π -state contributions given in II (6.26) are**

$$\begin{aligned} \alpha_1^S(m^2) &= \frac{3}{32\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{m^2}{M^2} \int \frac{\xi_0}{U_0^3} \theta(-V_0) dx^{(4)}, \\ \alpha_2^S(m^2) &= \frac{3}{8\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{1}{M^2} \int \frac{1}{U_0^3} [-M^2 \xi_0 + (-V_0)] \theta(-V_0) dx^{(4)}, \end{aligned} \quad (2.1)$$

with

$$\begin{aligned} U_0 &\equiv (x_1 + x_3)(x_2 + x_5) + (x_1 + x_3)x_4 + x_4(x_2 + x_5), \\ \xi_0 &\equiv (1/U_0)[x_1^2(x_2 + x_5) + x_2^2(x_1 + x_3) + (x_1 + x_2)^2 x_4], \\ V_0 &\equiv \xi_0 M^2 + (x_3 + x_4 + x_5)\mu^2 - (x_3 x_4 x_5 / U_0) m^2, \\ dx^{(4)} &\equiv \prod_{i=1}^5 dx_i \cdot \delta(1 - \sum_{i=1}^5 x_i), \quad x_i \geq 0. \end{aligned} \quad (2.2)$$

* But in the present case a new technique is necessary since the integrations of the relativistic results cannot be carried out analytically.

** We put $M' = M$.

The static limit is defined by the limit of $M \rightarrow \infty$. We can then easily show that the integrands in (2.1) are non-vanishing only for $x_1 = O(1/M)$ and $x_2 = O(1/M)$. Noticing this fact, we can expand (2.1) in powers of $1/M$. For $\alpha_1^S(m^2)$, the lowest order term of the $1/M$ -expansion vanishes, hence we must calculate the next order term, which is proportional to the $1/M$ -correction of $\alpha_1^S(m^2)$. These detailed calculations are presented in Appendix A.

Carrying out the integrations over x_1 and x_2 , we obtain the following expressions for static limits:

$$\begin{aligned} \lim \alpha_1^S(m^2) &= \frac{1}{2\pi^2} \left(\frac{f}{\mu} \right)^6 m^2 \int \frac{(-V_{st})^2}{U_{st}^{5/2}} \theta(-V_{st}) dx^{(2)}, \\ \lim \alpha_2^S(m^2) &= \frac{4}{\pi^3} \left(\frac{f}{\mu} \right)^6 M \int \frac{(-V_{st})^{3/2}}{U_{st}^{7/2}} \sqrt{x_4 + x_5} [(U_{st} - 2x_4 x_5) m^2 - U_{st} \mu^2] \\ &\quad \times \theta(-V_{st}) dx^{(2)}, \end{aligned} \quad (2.3)$$

$$1/M\text{-correction of } \alpha_1^S(m^2) = -(3m^2/4M^2) \lim \alpha_2^S(m^2) \quad (2.4)$$

with

$$\begin{aligned} U_{st} &\equiv x_3 x_5 + x_3 x_4 + x_4 x_5, \quad V_{st} \equiv (x_3 + x_4 + x_5) \mu^2 - (x_3 x_4 x_5 / U_{st}) m^2, \\ dx^{(2)} &\equiv dx_3 dx_4 dx_5 \delta(1 - x_3 - x_4 - x_5), \quad x_i \geq 0. \end{aligned} \quad (2.5)$$

We can carry out the integrations of (2.3) analytically. After rather tedious calculations, which are outlined in Appendix B, we obtain

$$\begin{aligned} \lim \alpha_1^S(m^2) &= \frac{f^6}{105\pi} \left(\frac{m}{\mu} - 3 \right)^3 \left[\left(\frac{m}{\mu} \right)^3 + 9 \left(\frac{m}{\mu} \right)^2 + 12 \frac{m}{\mu} - 3 - 3 \left(\frac{m}{\mu} \right)^{-1} \right], \\ \lim \alpha_2^S(m^2) &= \frac{f^6}{\pi^2} \frac{M}{\mu} \left[\frac{\kappa}{8} \left\{ \left(\frac{m}{\mu} \right)^4 + \left(\frac{m}{\mu} \right)^3 - 33 \left(\frac{m}{\mu} \right)^2 + 35 \frac{m}{\mu} + 53 \right. \right. \\ &\quad \left. \left. - 75 \left(\frac{m}{\mu} \right)^{-1} - 9 \left(\frac{m}{\mu} \right)^{-2} + 27 \left(\frac{m}{\mu} \right)^{-3} \right\} + 3 \left\{ \frac{m}{\mu} - 2 \left(\frac{m}{\mu} \right)^{-1} \right\} \sinh^{-1} \kappa \right] \end{aligned} \quad (2.6)$$

with

$$\kappa \equiv \sqrt{\{(m - \mu)/2\mu\}^2 - 1}. \quad (2.7)$$

Now, we denote the spectral functions calculated by the static theory with the Hamiltonian (1.1) (P -wave only) by $\alpha_{1,st}^S(m^2)$ and $\alpha_{2,st}^S(m^2)$. Then Bosco and Alfaro's results rewritten in our notation are*

* Unfortunately their calculation for the a.m.m. spectral function contains some mistakes. Their result should be reduced by a factor two.

Note added in proof: They informed us that these mistakes in their preprint were corrected in publication (in the Physical Review) according to the pointing-out of one of the authors (N.N.).

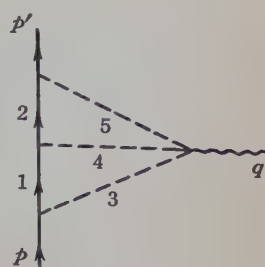


Fig. 2

$$\begin{aligned}
\alpha_{1,sl}^S(m^2) &= \lim \alpha_1^S(m^2), \\
\alpha_{2,sl}^S(m^2) &= \frac{f^6}{\pi^2} \frac{M}{\mu} \left[\frac{\kappa}{48} \left\{ 5 \left(\frac{m}{\mu} \right)^4 + 5 \left(\frac{m}{\mu} \right)^3 - 157 \left(\frac{m}{\mu} \right)^2 + 183 \frac{m}{\mu} \right. \right. \\
&\quad \left. \left. + 241 - 335 \left(\frac{m}{\mu} \right)^{-1} - 77 \left(\frac{m}{\mu} \right)^{-2} + 135 \left(\frac{m}{\mu} \right)^{-3} \right\} \right. \\
&\quad \left. + \left\{ \frac{3}{2} \frac{m}{\mu} - \left(\frac{m}{\mu} \right)^{-1} - \left(\frac{m}{\mu} \right)^{-3} \right\} \sinh^{-1} \kappa \right], \quad (2.8)
\end{aligned}$$

where the latter has been rewritten in a simpler form.

The differences between (2.6) and (2.8) correspond to the S -wave contributions. Therefore there is no S -wave effect for the charge spectral function, and the S -wave contribution for the a. m. m. spectral function is always less than 1/5 of the P -wave one (this will be explained in detail in the next section). The equality between $\alpha_{1,sl}^S(m^2)$ and $\lim \alpha_1^S(m^2)$ is verified also without carrying out the parametric integrations (see Appendix C).

The above results can be qualitatively understood in the following way. Since S -wave pions can appear only in a pair, the remaining pion must be P -wave in the 3π -state. Then the matrix element is proportional to the spin operator σ . Therefore this contribution to the charge density must vanish on account of its transformation property. Namely the S -wave contribution is present only for the a. m. m. distribution. Furthermore, the ϕ^2 -term, which is misleading in the scattering problem, does not contribute in the present case, because the matrix element involving it is proportional to the isospin operator τ and hence cannot contribute to the isoscalar part. This may be the reason why the S -wave effect is small.

§ 3. Numerical results

In this section we summarize the numerical results calculated by the static limits of not only the isoscalar part but also the isovector part (the lowest order term of the 2π -state contribution).

3-1. Numerical results in the spectral representations

The spectral functions of the static limits of the isovector part, which are easily obtained from I(3.3), are

$$\begin{aligned}
\lim \alpha_1^V(m^2) &= \frac{f^2}{\pi} \left\{ \frac{m}{\mu} - 2 \left(\frac{m}{\mu} \right)^{-1} \right\} \sqrt{\left(\frac{m}{2\mu} \right)^2 - 1}, \\
\lim \alpha_2^V(m^2) &= \frac{f^2}{4} \frac{M}{\mu} \left\{ \frac{m}{\mu} - 4 \left(\frac{m}{\mu} \right)^{-1} \right\}. \quad (3.1)
\end{aligned}$$

Fig. 3 shows $\lim Q^{S,V}$, $\lim \langle r^2 \rangle_1^{S,V}$, $\lim \mu^{S,V}$ and $\lim (\mu \langle r^2 \rangle_2)^{S,V}$ for various square cut-offs in the spectral functions. $\lim \langle r^2 \rangle_1^S$ is disappointingly small for the cut-off comparable to the nucleon mass.

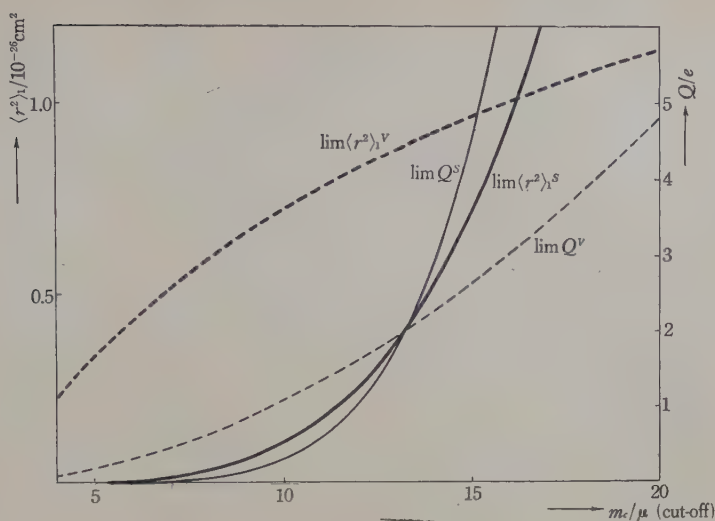


Fig. 3 (a)

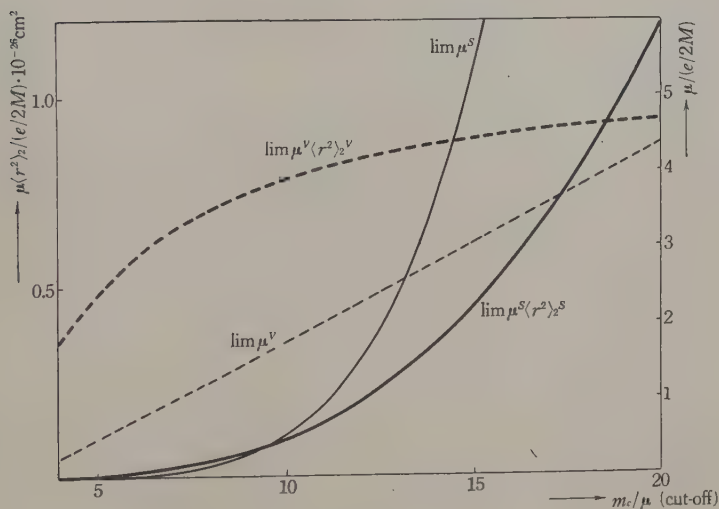


Fig. 3 (b)

3-2. Numerical results in the spatial representations

The distribution functions of the static limits of the isovector part are given in I(4.13) and I(4.12). As for the isoscalar part, $\lim \rho_1^S(r) = \rho_{1,st}^S(r)$ is given by Eq. (9) of Bosco and Alfaro, and $\lim \rho_2^S(r)$ can be calculated from (2.6) by means of I(2.7). Fig. 4 shows $\lim Q^{S,V}$, $\lim \langle r^2 \rangle_1^{S,V}$, $\lim \mu^{S,V}$ and $\lim (\mu \langle r^2 \rangle_2)^{S,V}$ for various square cut-offs in the distribution functions. $\lim \langle r^2 \rangle_1^S$ is very satisfactorily large. This strong discrepancy between $\lim \langle r^2 \rangle_1^{S,V}$'s calculated by the above two methods is due to the strong divergence of $\lim \alpha_1^S(m^2)$ for $m^2 \rightarrow \infty$.

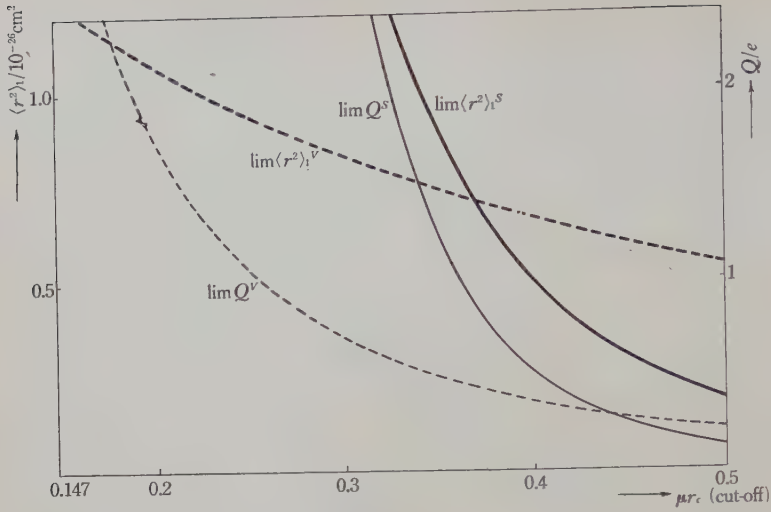


Fig. 4 (a)

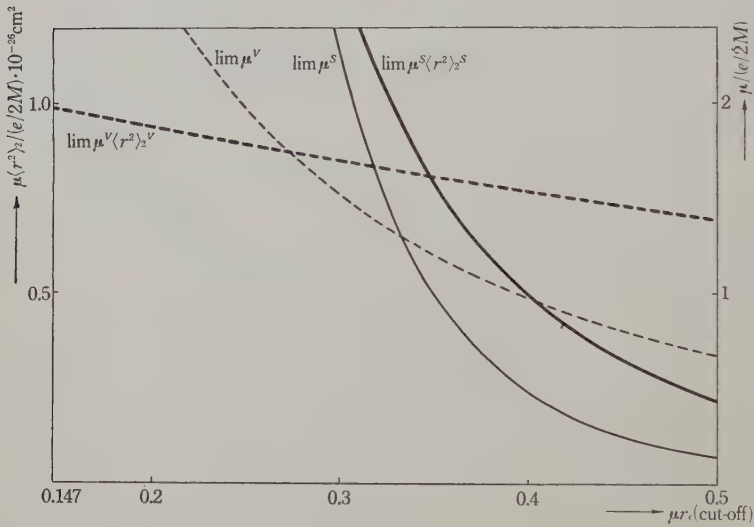


Fig. 4 (b)

3-3. S-wave effect

The S -wave contributions to $\alpha_2^V(m^2)$ and $\alpha_1^S(m^2)$ vanish as was stated in I, § 4, and in the previous section. The contributions to $\alpha_1^V(m^2)$ and $\alpha_2^S(m^2)$ do not vanish; Fig. 5 displays the ratios

$$\frac{\lim \alpha_1^V(m^2) - \alpha_{1,st}^V(m^2)}{\alpha_{1,st}^V(m^2)} = \frac{m^2 - 4\mu^2}{5m^2 - 8\mu^2}, \quad (3.2)$$
$$\frac{\lim \alpha_2^S(m^2) - \alpha_{2,st}^S(m^2)}{\alpha_{2,st}^S(m^2)}$$

where

$$\alpha_{1,et}^V(m^2) = \frac{f^2}{6\pi} \left\{ 5 \frac{m}{\mu} - 8 \left(\frac{m}{\mu} \right)^{-1} \right\} \sqrt{\left(\frac{m}{2\mu} \right)^2 - 1} \quad (3.3)$$

is easily obtained by using Bosco and Alfaro's technique.²⁾ Both of (3.2) vanish at the thresholds and monotonously increase up to $1/5$. The similarity between

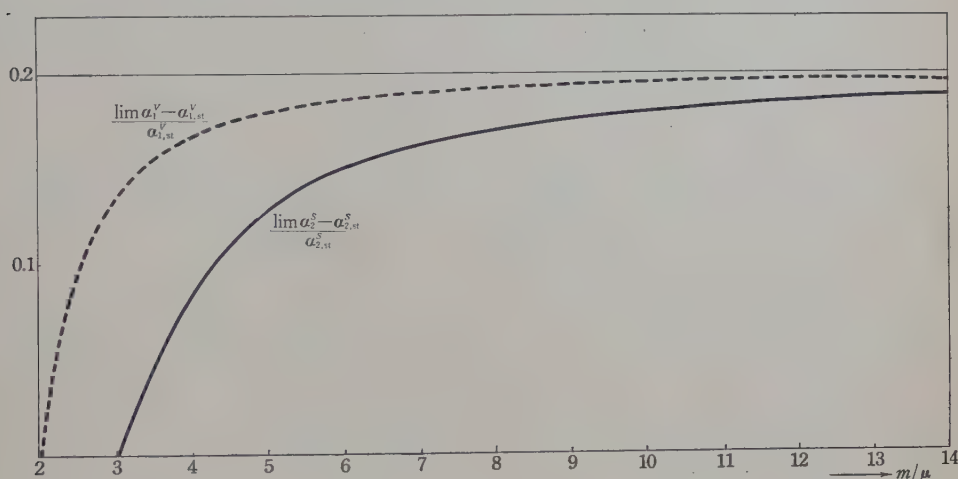


Fig. 5

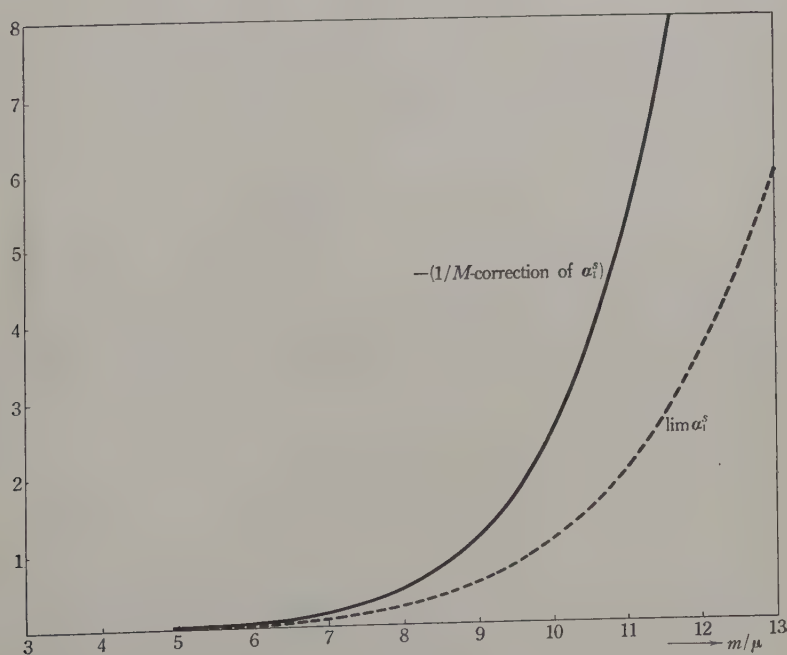


Fig. 6

these behaviors is rather remarkable.

3-4. Recoil effect

We have obtained the $1/M$ -correction of $\alpha_1^s(m^2)$ in (2.4) with (2.6). This is compared with $\alpha_{1,st}^s(m^2)$ in Fig. 6. We thus find that the recoil effect will be rather large as in the isovector part. The $1/M$ -expansion, however, is very bad as was stressed in I, § 4-3. So the relativistic evaluation is desirable, but it has so far been carried out only in a very crude way (see I § 5-1 and II § 5). In Table I, we compare Q^s and $\langle r^2 \rangle_1^s$ (for square cut-offs in the distribution functions) calculated by the relativistic and the static theory.

Table I

	unit	$r_0=1/M$		$r_0=2/M$		$r_0=1/2.5\mu$
		relativistic	static	relativistic	static	static
Q^s	e	6.6 ± 4	251	1.1 ± 0.5	3.62	0.526
$\langle r^2 \rangle_1^s$	10^{-26} cm^2	1.6 ± 0.6	32	0.65 ± 0.3	1.83	0.484

§ 4. Behaviors near the threshold

The behaviors of $\lim \alpha_{1,2}^s(m^2)$ and $\alpha_{1,2,st}^s(m^2)$ near the threshold $m=3\mu$ are easily calculated from (2.6) and (2.8); namely we have

$$\begin{aligned}\lim \alpha_1^s(m^2) &= \alpha_{1,st}^s(m^2) = \frac{4f^6}{3\pi} \varepsilon^3 \left[1 + \frac{2}{3} \varepsilon + \dots \right], \\ \lim \alpha_2^s(m^2) &= \frac{128}{45} \frac{f^6 M}{\pi^2 \mu} \varepsilon^{5/2} \left[1 + \frac{19}{24} \varepsilon + \dots \right], \\ \alpha_{2,st}^s(m^2) &= \frac{128}{45} \frac{f^6 M}{\pi^2 \mu} \varepsilon^{5/2} \left[1 + \frac{109}{168} \varepsilon + \dots \right],\end{aligned}\tag{4.1}$$

where $m=3\mu+\varepsilon\mu$, ($\varepsilon>0$).

The relativistic spectral functions in the no-loop approximation, (2.1), can also be calculated analytically only near the threshold. The method employed is explained in Appendix D. The results obtained are

$$\begin{aligned}\alpha_1^s(m^2) &= \frac{1}{64\pi^2} \cdot \frac{1}{3^{3/2}} \left(\frac{g^2}{4\pi} \right)^3 \frac{\mu^2}{M^2} \varepsilon^5 + \dots, \\ \alpha_2^s(m^2) &= \frac{1}{4\pi^3 \sqrt{3}} \left(\frac{g^2}{4\pi} \right)^3 \frac{\mu^2}{M^2} \varepsilon^4 + \dots,\end{aligned}\tag{4.2}$$

which are both positive. Therefore the asymptotic forms of the distribution functions are both positive. We can also calculate the behaviors near the threshold of the relativistic spectral functions without taking the no-loop approximation. But it turns

out that these are the same as (4.2) except for terms of higher order in the closed-loop mass.

Now, let us compare (4.2) with (4.1). The increases very near the threshold of the relativistic spectral functions are much slower than those of the static ones. But the coefficients of the former are much larger than those of the latter. Hence there may be the possibility that the relativistic spectral functions exceed the corresponding static ones in some region of m^2 (though this of course is not so plausible).

§ 5. Discussions

In the following we summarize the results obtained from our present investigation on the static limits of the 3π -state contribution.

i) The S -wave effect is negligibly small not only for the isovector part but also for the isoscalar part. We may therefore conclude that the relativistic perturbation calculation of nucleon structure, contrary to the scattering problems, does not include the misleading contributions due to the S -wave effect (especially ϕ^2 -term).

ii) The numerical results calculated by the static approximation are extremely sensitive to cutoff methods as was shown in § 3-1 and § 3-2. This is because the static spectral functions increase very rapidly for large m^2 contrary to the relativistic ones. Therefore no quantitative conclusion should be deduced from the static approximation.

iii) The recoil effect will be rather large also for the isoscalar part.

Thus it is desirable to carry out the relativistic calculations of the 3π -state contribution in detail.

Acknowledgment

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Appendix A. Calculation method for the static limit

We first illustrate the method of calculation for the static limits by simple examples of the isovector part. The a. m. m. spectral function of the 2π -state contribution is given by I(3.1), namely

$$\alpha_2^V(m^2) = \frac{1}{\pi} \frac{g^2}{4\pi} M^2 J_1 \quad (\text{A.1})$$

with
$$J_1 \equiv \int x_1^2 \delta(x_1^2 M^2 + V_{st}) dx^{(2)}, \quad (\text{A} \cdot 2)$$

$$V_{st} \equiv (x_2 + x_3) \mu^2 - x_2 x_3 m^2.$$

The integrand of (A·2) is non-vanishing only for $x_1 = O(1/M)$, because otherwise $x_1^2 M^2 + V_{st} > 0$ for $M \rightarrow \infty$. Therefore the static limit is achieved by neglecting higher orders of x_1 , that is to say, we replace

$$dx^{(2)} \equiv \delta(1 - x_1 - x_2 - x_3) dx_1 dx_2 dx_3 \quad (\text{A} \cdot 3)$$

by

$$dx_1 \delta(1 - x_2 - x_3) dx_2 dx_3. \quad (\text{A} \cdot 4)$$

Then, carrying out the integration over x_1 , we get

$$J_1^{(0)} \equiv \lim J_1 = \frac{1}{2M^3} \int \sqrt{-V_{st}} \theta(-V_{st}) \delta(1 - x_2 - x_3) dx_2 dx_3 = \frac{\pi}{16} \frac{m^2 - 4\mu^2}{M^3 m}. \quad (\text{A} \cdot 5)$$

Of course, (A·1) with (A·5) coincides with (3·1), which was calculated directly from the explicit formula I(3·3).

Likewise we calculate the static limit of the charge spectral function of the isovector part:

$$\alpha_1^V(m^2) = \frac{1}{\pi} \frac{g^2}{4\pi} [-M^2 J_1 + \frac{1}{2} J_2] \quad (\text{A} \cdot 6)$$

with
$$J_2 = \int \theta(-x_1^2 M^2 - V_{st}) dx^{(2)}. \quad (\text{A} \cdot 7)$$

Using the above method, we immediately obtain

$$J_2^{(0)} \equiv \lim J_2 = 2M^2 J_1^{(0)}, \quad (\text{A} \cdot 8)$$

which is easily verified also by noticing the identity

$$-\partial J_2 / \partial M = 2M J_1 \quad (\text{A} \cdot 9)$$

and $J_1^{(0)} \propto 1/M^3$ (cf. (A·5)). Therefore the lowest order term of the $1/M$ -expansion of (A·6) vanishes as it should. In order to obtain the static limit of $\alpha_1^V(m^2)$, we must calculate the next orders, $J_1^{(1)}$ and $J_2^{(1)}$.

Using the formal Taylor expansion

$$\delta(1 - x_1 - x_2 - x_3) = \delta(1 - x_2 - x_3) - x_1 \delta'(1 - x_2 - x_3) + \dots, \quad (\text{A} \cdot 10)$$

we have

$$J_2^{(1)} = -\frac{1}{2M^2} \int (-V_{st}) \theta(-V_{st}) \delta'(1 - x_2 - x_3) dx_2 dx_3 = -\frac{m^2 - 2\mu^2}{4M^2 m} \sqrt{m^2 - 4\mu^2}, \quad (\text{A} \cdot 11)$$

and $J_1^{(1)} = J_2^{(1)} / M^2$ because of (A·9) and (A·11). Then we find that

$$\lim a_1^V(m^2) = \frac{1}{\pi} \frac{g^2}{4\pi} [-M^2 J_1^{(1)} + \frac{1}{2} J_2^{(1)}] \quad (\text{A} \cdot 12)$$

coincides with (3.1).

Now, returning to our subject, we calculate the static limits of (2.1). We write

$$I_1 \equiv \int \frac{\xi_0}{U_0^3} \theta(-V_0) dx^{(4)}, \quad (\text{A} \cdot 13)$$

$$I_2 \equiv \int \frac{(-V_0)}{U_0^3} \theta(-V_0) dx^{(4)},$$

$$\text{then} \quad -\partial I_2 / \partial M = 2MI_1 \quad (\text{A} \cdot 14)$$

just as (A.9).

We can easily show, using $\sum_1^5 x_i = 1$, that

$$\xi_0 \geq x_1^2 + x_2^2. \quad (\text{A} \cdot 15)$$

Therefore, the integrands are non-vanishing only for $x_1 = O(1/M)$ and $x_2 = O(1/M)$. Expanding the expressions in (2.2) with respect to x_1 and x_2 , we write

$$\begin{aligned} U_0^{(0)} &= x_3 x_5 + x_3 x_4 + x_4 x_5 = U_{st}, \\ (U_0 \xi_0)^{(0)} &= x_1^2 x_5 + x_2^2 x_3 + (x_1 + x_2)^2 x_4, \\ (U_0 V_0)^{(0)} &= (U_0 \xi_0)^{(0)} M^2 + U_{st} V_{st} \quad (\text{cf. (2.5)}), \end{aligned} \quad (\text{A} \cdot 16)$$

and

$$\begin{aligned} U_0^{(1)} &= x_1 x_5 + x_2 x_3 + x_1 x_4 + x_2 x_4, \\ (U_0 \xi_0)^{(1)} &= x_1^2 x_2 + x_2^2 x_1, \\ (U_0 V_0)^{(1)} &= (U_0 \xi_0)^{(1)} M^2 + U_0^{(1)} (x_3 + x_4 + x_5) \mu^2, \end{aligned} \quad (\text{A} \cdot 17)$$

and

$$\delta(1 - \sum_1^5 x_i) = \delta(1 - \sum_3^5 x_i) - (x_1 + x_2) \cdot \delta'(1 - \sum_3^5 x_i) + \dots \quad (\text{A} \cdot 18)$$

We then have

$$I_2^{(0)} = \int (1/U_{st}^4) \cdot [-(U_0 V_0)^{(0)}] \theta(-(U_0 V_0)^{(0)}) \cdot \prod_1^5 dx_i \delta(1 - \sum_3^5 x_i) \quad (\text{A} \cdot 19)$$

and

$$I_2^{(1)} = I_a + I_b + I_c \quad (\text{A} \cdot 20)$$

with

$$\begin{aligned} I_a &\equiv -4 \int (U_0^{(1)} / U_{st}^5) \cdot [-(U_0 V_0)^{(0)}] \theta(-(U_0 V_0)^{(0)}) \cdot \prod_1^5 dx_i \delta(1 - \sum_3^5 x_i), \\ I_b &\equiv - \int (1/U_{st}^4) \cdot (U_0 V_0)^{(1)} \theta(-(U_0 V_0)^{(0)}) \cdot \prod_1^5 dx_i \delta(1 - \sum_3^5 x_i), \end{aligned} \quad (\text{A} \cdot 21)$$

$$I_c \equiv - \int (1/U_{st}^4) \cdot [-(U_0 V_0)^{(0)}] \theta(-(U_0 V_0)^{(0)}) \cdot (x_1 + x_2) \prod_1^5 dx_i \delta'(1 - \sum_3^5 x_i).$$

Noticing

$$[-(U_0 V_0)^{(0)}] \theta(-(U_0 V_0)^{(0)}) = \int_{-\infty}^{\infty} \theta(-(U_0 V_0)^{(0)}) d(U_{st} V_{st}), \quad (\text{A} \cdot 22)$$

we can carry out the integrations over x_1 and x_2 of (A·19) by elementary methods:

$$I_2^{(0)} = \frac{1}{4M^2} \int \frac{1}{U_{st}^{9/2}} \tan^{-1} \frac{\sqrt{U_{st}}}{x_4} \cdot (-U_{st} V_{st})^2 \theta(-U_{st} V_{st}) dx^{(2)}, \quad (\text{A} \cdot 23)$$

which can be simplified to

$$I_2^{(0)} = \frac{\pi}{12M^2} \int \frac{1}{U_{st}^{5/2}} (-V_{st})^2 \theta(-V_{st}) dx^{(2)} \quad (\text{A} \cdot 24)$$

by means of symmetrical exchanges of x_i and the identity

$$\sum_{i=3}^5 \tan^{-1}(\sqrt{U_{st}}/x_i) = \pi. \quad (\text{A} \cdot 25)$$

The integrals (A·21) are likewise calculated as follows:

$$\begin{aligned} I_a &= -\frac{8}{15M^3} \int \frac{1}{U_{st}^6} \{x_3 \sqrt{x_4 + x_5} + (3 \leftrightarrow 5)\} v^{5/2} \theta(v) dx^{(2)}, \\ I_b &= -\frac{1}{15M^3} \int \left[\frac{1}{U_{st}^6} \left\{ \frac{1}{\sqrt{x_4 + x_5}} (x_3 x_5 + x_3 x_4 - x_4 x_5) + (3 \leftrightarrow 5) \right\} v^{5/2} \right. \\ &\quad \left. + \frac{5}{U_{st}^5} \{x_3 \sqrt{x_4 + x_5} + (3 \leftrightarrow 5)\} (x_3 + x_4 + x_5) \mu^2 v^{3/2} \right] \theta(v) dx^{(2)}, \\ I_c &= -\frac{2}{15M^3} \int \frac{1}{U_{st}^5} \left\{ \frac{x_5}{\sqrt{x_4 + x_5}} + (3 \leftrightarrow 5) \right\} v^{5/2} \theta(v) \\ &\quad \cdot \delta'(1 - x_3 - x_4 - x_5) \prod_3^5 dx_i, \end{aligned} \quad (\text{A} \cdot 26)$$

where

$$v \equiv -U_{st} V_{st}. \quad (\text{A} \cdot 27)$$

It is convenient that the last integral I_c is integrated by parts in the following way:

$$\begin{aligned} I_c &= -\frac{1}{15M^3} \int \left\{ \left(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_3} \right) \left[\frac{\sqrt{x_4 + x_5}}{U_{st}^5} v^{5/2} \right] + (3 \leftrightarrow 5) \right\} \theta(v) dx^{(2)} \\ &= -\frac{1}{15M^3} \int \left[-\frac{10}{U_{st}^6} \{x_3 \sqrt{x_4 + x_5} + (3 \leftrightarrow 5)\} v^{5/2} + \frac{1}{U_{st}^5} \left\{ \frac{1}{\sqrt{x_4 + x_5}} + (3 \leftrightarrow 5) \right\} v^{5/2} \right. \\ &\quad \left. + \frac{5}{2U_{st}^5} \{ \sqrt{x_4 + x_5} \langle (x_3 x_5 + x_3 x_4 - x_4 x_5) m^2 \right. \\ &\quad \left. - (2x_3(x_3 + x_4 + x_5) + U_{st}) \mu^2 \rangle + (3 \leftrightarrow 5) \} v^{3/2} \right] \theta(v) dx^{(2)}. \end{aligned} \quad (\text{A} \cdot 28)$$

Therefore we obtain

$$I_2^{(1)} = -\frac{1}{6M^3} \int \frac{1}{U_{st}^5} \{ \sqrt{x_4 + x_5} [(x_3 x_5 + x_3 x_4 - x_4 x_5) m^2 - U_{st} \mu^2] \\ + (3 \leftrightarrow 5) \} v^{3/2} \theta(v) dx^{(2)}, \quad (\text{A} \cdot 29)$$

where the second term becomes identical to the first by the transformation $x_3 \leftrightarrow x_5$. Furthermore, from (A.14) we get

$$I_1^{(0)} = I_2^{(0)} / M^2, \\ I_1^{(1)} = (3/2) I_2^{(1)} / M^2. \quad (\text{A} \cdot 30)$$

Finally, (2.4) is obtained through

$$\lim \alpha_1^S(m^2) = \frac{3}{32\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{m^2}{M^2} I_1^{(0)}, \\ \lim \alpha_2^S(m^2) = \frac{3}{8\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{1}{M^2} (-M^2 I_1^{(1)} + I_2^{(1)}), \quad (\text{A} \cdot 31) \\ 1/M\text{-correction of } \alpha_1^S(m^2) = \frac{3}{32\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{m^2}{M^2} I_1^{(1)},$$

with (A.24), (A.29) and (A.30).

Appendix B. Calculation of integrals (2.3)

We illustrate the calculation method of integrals (2.3) by an example,

$$I \equiv \int \frac{v^{3/2}}{U_{st}^4} \sqrt{x_4 + x_5} \theta(v) dx^{(2)}, \quad (\text{B} \cdot 1)$$

where v is defined by (A.27), since other integrals appearing in (2.3) can be treated quite analogously.

By a transformation

$$x_3 = 1 - s, \\ x_4 = st, \\ x_5 = s(1 - t), \quad (\text{B} \cdot 2)$$

(B.1) is rewritten as*

$$I = \int_0^1 \int_0^1 \frac{[s(1-s)t(1-t)m^2 - \{1-s+st(1-t)\}\mu^2]^{3/2}}{s[1-s+st(1-t)]^4} \theta(v) ds dt. \quad (\text{B} \cdot 3)$$

Using the symmetry of the integrand, we can replace

* v in the θ -function has the same form as the expression in the square bracket of the numerator in (B.3) as well as in (B.5).

$$\int_0^1 \cdots dt \quad \text{by} \quad 2 \int_0^{1/2} \cdots dt. \quad (\text{B} \cdot 4)$$

We perform the following transformations in turn.

$$\begin{aligned} I &= 16 \int_0^1 ds \int_0^1 dy \frac{[s(1-s)(1-y)m^2 - (4-3s-sy)\mu^2]^{3/2}}{s[4-3s-sy]^4 \sqrt{y}} \theta(v) \\ &\quad \text{for } t(1-t) = (1-y)/4 \\ &= 16 \int_0^1 ds \int_1^\infty dz \frac{z[s(1-s)(z-1)m^2 - \{(4-3s)z-s\}\mu^2]^{3/2}}{s[(4-3s)z-s]^4} \theta(v) \\ &\quad \text{for } y=1/z \\ &= 16 \int_0^1 ds \frac{[s(1-s)m^2 - (4-3s)\mu^2]^{3/2}}{s(4-3s)^4} \theta(s(1-s)m^2 - (4-3s)\mu^2) \cdot J, \end{aligned} \quad (\text{B} \cdot 5)$$

where

$$J \equiv \int_a^\infty \frac{z(z-a)^{3/2} dz}{[z-s/(4-3s)]^4} = \int_0^\infty \frac{(u+a)u^{3/2}}{(u+b)^4} du, \quad \text{for } z=u+a \quad (\text{B} \cdot 6)$$

with

$$\begin{aligned} a &\equiv \frac{s(1-s)m^2 - s\mu^2}{s(1-s)m^2 - (4-3s)\mu^2} \geq 1, \\ b &\equiv a - \frac{s}{4-3s} = \frac{4s(1-s)^2 m^2}{(4-3s)[s(1-s)m^2 - (4-3s)\mu^2]} \geq 0. \end{aligned} \quad (\text{B} \cdot 7)$$

$$J = \frac{1}{b^{1/2}} \int_0^\infty \frac{w^{5/2}}{(1+w)^4} dw + \frac{a}{b^{3/2}} \int_0^\infty \frac{w^{3/2}}{(1+w)^4} dw, \quad \text{for } u=bw. \quad (\text{B} \cdot 8)$$

Using the beta-function formula

$$B(l, m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}} dx, \quad (\text{B} \cdot 9)$$

we get

$$J = \frac{5\pi}{16b^{1/2}} + \frac{\pi a}{16b^{3/2}}. \quad (\text{B} \cdot 10)$$

The substitution of (B.10) in (B.5) leads to

$$\begin{aligned} I &= \pi \int_0^1 ds [s(1-s)m^2 - (4-3s)\mu^2]^{3/2} \left[\frac{5}{2ms^{3/2}(4-3s)^{7/2}(1-s)} \right. \\ &\quad \left. + \frac{s(1-s)m^2 - s\mu^2}{8m^3 s^{5/2}(4-3s)^{5/2}(1-s)^3} \right] \cdot \theta(s(1-s)m^2 - (4-3s)\mu^2). \end{aligned} \quad (\text{B} \cdot 11)$$

We can easily see that

$$0 \leq \alpha \leq \beta \leq 1, \quad (\text{B} \cdot 12)$$

and

$$\begin{aligned} \alpha + \beta &= (m^2 + 3\mu^2)/m^2, \\ \alpha\beta &= 4\mu^2/m^2, \end{aligned} \quad (\text{B} \cdot 13)$$

where α and β are the two roots of the equation

$$s(1-s)m^2 - (4-3s)\mu^2 = 0. \quad (\text{B} \cdot 14)$$

Expanding the numerators of (B·11) in powers of s , we have the following type of integrals:

$$I_{kln} \equiv \int_{\alpha}^{\beta} \frac{ds}{s^{k/2}(4-3s)^{l/2}(1-s)^n}, \quad (\text{B} \cdot 15)$$

where k and l are odd integers and n is an integer. (B·15) is transformed as

$$I_{kln} = \int_{\beta'}^{\alpha'} \frac{(1+x)^{(1/2)(k+l)+n-2}}{x^n(4x+1)^{l/2}} dx \quad (\text{B} \cdot 16)$$

for $s=1/(1+x)$, where

$$\begin{aligned} \alpha' &\equiv (1-\alpha)/\alpha \\ \beta' &\equiv (1-\beta)/\beta. \end{aligned} \quad (\text{B} \cdot 17)$$

(B·16) is reduced to the well-known elementary integrals by expanding its numerator. Calculations should be performed by using (B·17) and (B·13), from which

$$\begin{aligned} (4-3\alpha)(4-3\beta) &= 4, \\ \sqrt{\alpha(4-3\beta)} - \sqrt{\beta(4-3\alpha)} &= -4\mu\kappa/m \quad (\text{cf. (2·7)}), \text{ etc.}, \end{aligned} \quad (\text{B} \cdot 18)$$

follow. After elementary but tedious calculations, we finally obtain

$$I = \frac{\pi\mu}{8m^2} \kappa (m^4 + \mu m^3 - 3\mu^2 m^2 + 5\mu^3 m - 4\mu^4) - \frac{3\pi\mu^2}{4m^3} (m^4 - 4\mu^2 m^2 + \mu^4) \sinh^{-1} \kappa. \quad (\text{B} \cdot 19)$$

Appendix C. Direct proof of $\alpha_{1,st}^S = \lim \alpha_1^S$

Bosco and Alfaro's results²⁾ for the charge distribution function $\rho_{1,st}^S(r)$ (with (1·2)) is

$$\rho_{1,st}^S(r) = -\frac{1}{4\pi^3} \left(\frac{f}{\mu} \right)^6 \iiint \frac{(\mathbf{k}_3 \cdot \mathbf{k}_4 \times \mathbf{k}_5)^2}{\omega_3^2 \omega_4^2 \omega_5^2} e^{i(\mathbf{k}_3 + \mathbf{k}_4 + \mathbf{k}_5) \cdot \mathbf{x}} \prod_3^5 d\mathbf{k}_i \quad (\text{C} \cdot 1)$$

with

$$\omega_i^2 = \mathbf{k}_i^2 + \mu^2, \quad r = \sqrt{\mathbf{x}^2}. \quad (\text{C} \cdot 2)$$

Hence the form factor is

$$F_{1,st}^S(q^2) = \int \rho(r) e^{-iqx} dx$$

$$= -\frac{2}{\pi^5} \left(\frac{f}{\mu} \right)^6 \iiint \frac{(\mathbf{k}_3 \cdot \mathbf{k}_4 \times \mathbf{k}_5)^2}{\omega_3^2 \omega_4^2 \omega_5^2} \delta(\mathbf{k}_3 + \mathbf{k}_4 + \mathbf{k}_5 - \mathbf{q}) d\mathbf{k}_3 d\mathbf{k}_4 d\mathbf{k}_5, \quad (C.3)$$

which has the spectral representation (analytically continued also to $q^2 < 0$)

$$F_{1,st}^S(q^2) = \int \frac{\alpha_{1,st}^S(m^2)}{q^2 + m^2 - i\epsilon} dm^2. \quad (C.4)$$

Consider a function

$$\tilde{\alpha}(q_i, M_i) \equiv \text{Im} \int \frac{1}{(\mathbf{k}_3 + \mathbf{q}_3)^2 + M_i^2 - i\epsilon} \cdot \frac{1}{(\mathbf{k}_4 + \mathbf{q}_4)^2 + M_i^2 - i\epsilon}$$

$$\cdot \frac{1}{(-\mathbf{k}_3 - \mathbf{k}_4 + \mathbf{q}_5)^2 + M_i^2 - i\epsilon} d\mathbf{k}_3 d\mathbf{k}_4. \quad (C.5)$$

Applying the formula presented by one of the authors (N. N.)³⁾ to three-dimensional case, we have

$$\tilde{\alpha}(q_i, M_i) \equiv \pi^3 \int \frac{1}{U_{st}^{3/2}} \theta(-\tilde{V}(q_i, M_i)) dx^{(2)}, \quad (C.6)$$

where

$$\tilde{V}(q_i, M_i) \equiv \sum_{i=3}^5 x_i(M_i^2 + q_i^2) - (1/U_{st}) [x_3(x_4 q_4 - x_5 q_5)^2$$

$$+ x_4(x_3 q_3 - x_5 q_5)^2 + x_5(x_3 q_3 - x_4 q_4)^2]. \quad (C.7)$$

Using the D -operator technique:

$$\mathbf{k}_i + \mathbf{q}_i \rightarrow \mathbf{D}_i \equiv \frac{1}{2} \int_{-\infty}^{M_i} dM_i \frac{\partial}{\partial \mathbf{q}_i}, \quad (C.8)$$

we can write the spectral function $\alpha_{1,st}^S$ as

$$\alpha_{1,st}^S(-q^2) = -\frac{2}{\pi^5} \left(\frac{f}{\mu} \right)^6 [\mathbf{D}_3 \cdot \mathbf{D}_4 \times \mathbf{q}]^2 \tilde{\alpha}(q_i, M_i) \quad (C.9)$$

with $q_3 = q_4 = 0$, $q_5 = q$, $M_i = \mu^2$. After some elementary calculations, we obtain

$$\alpha_{1,st}^S(-q^2) = \frac{2}{\pi^2} \left(\frac{f}{\mu} \right)^6 \int \frac{1}{U_{st}^{3/2}} \cdot \frac{-q^2}{2U_{st}} \cdot \frac{(-V_{st})^2}{2} \theta(-V_{st}) dx^{(2)}, \quad (C.10)$$

which coincides with $\lim \alpha_{1,st}^S(-q^2)$ in (2.3).

Appendix D. Behavior near the threshold

We will present the general method of calculating the behavior near the physical threshold.

The behavior of $V(x)$ near the physical threshold has been fully investigated

by one of the authors (N: N.)⁴⁾* We here quote only the results. In Fig. 7, the lines $\{1, 2, \dots, r\}$ compose an intermediate state, and the "threshold minimum" (i. e. $V = \partial V / \partial x_i = 0$) occurs when

$$-P^2 = \left(\sum_{i=1}^r m_i \right)^2, \quad (\text{D} \cdot 1)$$

and

$$x_i = \frac{1/m_i}{\sum_{k=1}^r 1/m_k} \equiv x_i^0 \quad (i=1, 2, \dots, r),$$

$$\text{other } x_j = 0 \quad (j=r+1, \dots, \alpha),$$

where P stands for the total momentum which flows from A_1 to A_2 , hence $m^2 = -P^2$ in the spectral function. In the neighborhood of the threshold minimum:

$$\begin{aligned} m^2 &= \left(\sum_{i=1}^r m_i \right)^2 \cdot (1 + \delta) \quad (\delta > 0), \\ x_i &= x_i^0 + y_i \quad (i=1, 2, \dots, r), \\ x_j &= y_j \quad (y_j \geq 0), \quad (j=r+1, \dots, \alpha) \\ \sum_{i=1}^{\alpha} y_i &= 0, \end{aligned} \quad (\text{D} \cdot 2)$$

$V(x)$ can be written as

$$V(x) = V_1(y_j) + V_2(y_j) + (1/2) \sum_{i,k=1}^r a_{ik} y_i y_k - \frac{\sum_{k=1}^r m_k}{\sum_{k=1}^r 1/m_k} \cdot \delta. \quad (\text{D} \cdot 3)$$

Here $V_l(y_j)$ ($l=1, 2$) is the V of the subgraph A_l with the external momenta $q_i = (m_i / \sum_{k=1}^r m_k) P$ ($i=1, 2, \dots, r$) (cf. Theorem 9 in N), and is generally of first order with respect to y_j ;

$$\begin{aligned} (1/2) \sum_{i,k=1}^r a_{ik} y_i y_k &\equiv (1/2) \sum_{i,k=1}^r \frac{\partial^2 V}{\partial x_i \partial x_k} \Big|_{x=x^0} \cdot y_i y_k \\ &= \left(\sum_{k=1}^r 1/m_k \right) \left[\left(\sum_{i=1}^r m_i^3 y_i^2 \right) - \left(\sum_{i=1}^r m_i^2 y_i \right)^2 / \sum_{k=1}^r m_k \right] \quad (\text{D} \cdot 4) \\ &\quad (\text{cf. Proof of Theorem 8 in N}); \end{aligned}$$

and the last term of (D·3) is due to N(4·31). Especially, when

$$m_1 = m_2 = \dots = m_r \equiv \mu, \quad (\text{D} \cdot 5)$$

(D·3) is simplified as

$$V(x) = V_1(y_j) + V_2(y_j) + \mu^2 \left[r \sum_{i=1}^r y_i^2 - \left(\sum_{i=1}^r y_i \right)^2 \right] - \mu^2 \delta. \quad (\text{D} \cdot 6)$$

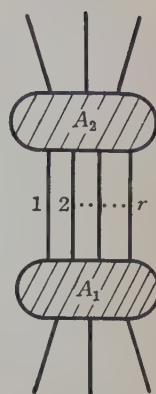


Fig. 7

* Hereafter this paper is referred to as N.

Now, we have

$$\sum_{i,k} a_{ik} y_i y_k \geq 0 \quad (\text{D} \cdot 7)$$

from Theorem 8 in N, and

$$V_1 \geq 0 \quad \text{and} \quad V_2 \geq 0, \quad (\text{D} \cdot 8)$$

if the threshold in question is the lowest threshold. Therefore, since the spectral function is non-vanishing only when

$$-V(x) \geq 0, \quad (\text{D} \cdot 9)$$

we may regard

$$\begin{aligned} y_i &= O(\delta^{1/2}) \quad (i=1, 2, \dots, r), \\ y_i &= O(\delta) \geq 0 \quad (j=r+1, \dots, \alpha). \end{aligned} \quad (\text{D} \cdot 10)$$

In the integral

$$I \equiv \int f(x) \delta(1 - \sum_{i=1}^{\alpha} x_i) \prod_{i=1}^{\alpha} dx_i, \quad (\text{D} \cdot 11)$$

where $f(x)$ is an arbitrary function of x_i involving a factor $\theta(-V)$ or $\delta^{(n)}(-V)$, the integration over x_1 should firstly be carried out, since the restriction for the integration domain then disappears:

$$I = \int f(x) \theta(x_1^0 - \sum_{i=2}^{\alpha} y_i) \prod_{i=2}^{\alpha} dy_i = \int f(x) \prod_{i=2}^{\alpha} dy_i, \quad (\text{D} \cdot 12)$$

because y_i 's are infinitesimal as (D·10). For the special case (D·5), after the integration over x_1 (D·6) becomes (by noticing (D·10))

$$V(x) = V_1(y_j) + V_2(y_j) + r\mu^2 \left[\sum_{i=2}^r y_i^2 + \left(\sum_{i=2}^r y_i \right)^2 \right] - \mu^2 \delta. \quad (\text{D} \cdot 13)$$

If we write $m = r\mu + \varepsilon\mu$, δ is rewritten as

$$\delta = (2/r)\varepsilon + O(\varepsilon^2). \quad (\text{D} \cdot 14)$$

If A_1 or A_2 contains some closed loops, the integrations with respect to these parts can be separately carried out.

Now, we will present some examples which follow from (D·13) immediately. Needless to say, numbering of propagators differs from that in the general theory presented above.

$$\alpha_2^V(m^2) = \frac{1}{\pi} \frac{g^2}{4\pi} \frac{M^2}{\mu^2} \int_0^{\infty} y_1^2 dy_1 \int_{-\infty}^{\infty} dy_2 \delta(-y_1 - 4y_2^2 + \varepsilon) \quad \text{for } m = 2\mu + \varepsilon\mu,$$

where $x_1^0 = 0$, $x_2^0 = x_3^0 = 1/2$ at the threshold of the 2π -state.

$$\lim \alpha_1^s(m^2) = \frac{f^6(3\mu)^2}{2\pi^2\mu^2} 3^{5/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [-3\{y_3^2 + y_4^2 + (y_3 + y_4)^2\} + (2/3)\varepsilon]^2 \cdot \theta(-3\{y_3^2 + y_4^2 + (y_3 + y_4)^2\} + (2/3)\varepsilon) dy_3 dy_4 \quad \text{for } m=3\mu+\varepsilon\mu, \quad (\text{D}\cdot 15)$$

where the coefficient $3^{5/2}$ has come out from $1/U_{st}^{5/2}$ at $x_i=x_i^0=1/3$ ($i=3, 4, 5$). These integrals are calculated very easily, and the results naturally coincide with those calculated from the explicit expressions for $\alpha_2^V(m^2)$ and $\lim \alpha_1^s(m^2)$. Likewise, we can calculate the behaviors of (2.1) near the threshold as follows:

$$\alpha_1^s(m^2) = \frac{3}{32\pi^3} \left(\frac{g^2}{4\pi} \right)^8 \frac{(3\mu)^2}{M^2} 3^3 \int \{y_1^2 + y_2^2 + (y_1 + y_2)^2\} \cdot \theta(-V_0) \prod_{i=1}^4 dy_i, \quad (\text{D}\cdot 16)$$

$$\alpha_2^s(m^2) = \frac{3}{8\pi^3} \left(\frac{g^2}{4\pi} \right)^3 \frac{1}{M^2} \cdot 3^3 \int (-V_0) \theta(-V_0) \prod_{i=1}^4 dy_i$$

with

$$V_0 \equiv [2(y_1 + y_2) + 3\{y_3^2 + y_4^2 + (y_3 + y_4)^2\} - (2/3)\varepsilon]\mu^2$$

and

$$y_1 \geq 0, \quad y_2 \geq 0.$$

The integrals (D.16) are easily calculated, and yield the results (4.2). Finally, consider the spectral functions without taking the no-loop approximation, for which

$$V = V_L + V_0 \quad (\text{D}\cdot 17)$$

and

$$U = (x_6 + x_7 + x_8 + x_9) U_0$$

near the threshold, where V_L denotes the V of Fig. 1. Therefore, using the technique in II, § 6-3, we can reduce those integrals to (D.16), the factor (1.2) being replaced by the matrix element of Fig. 1 at $q_i=q/3$ ($i=3, 4, 5$) and $q^2=-(3\mu)^2$. Thus, near the threshold, the no-loop approximation is uniformly convergent contrary to the static approximation.

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Inconsistency among the Properties of Renormalizability, Analyticity, and Regularity at Zero Charge

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The photon propagator is analyzed by means of general properties of the present theory of electrons and photons. It can be shown, by consideration of the charge renormalization group, that the photon propagator is independent of physical charge in the high energy limit. If, in addition to renormalizability, regularity at vanishing physical charge is assumed, then it follows that the bare charge vanishes. On the other hand, the commonly assumed analyticity properties require that bare charge exceeds physical charge. Thus at least one of the general properties assumed is inadmissible. This relationship of expressions satisfying the charge renormalization group equations to analyticity properties and definite sets of Feynman diagrams (which correspond to an expansion about zero charge) is illustrated by a simple example.

§ 1. Introduction

Gell-Mann and Low¹⁾ were the first to demonstrate clearly the importance of renormalizability as a tool in going beyond the perturbation approach to quantum field theory.²⁾ Bogoliubov and Shirkov³⁾ have applied this tool to a variety of propagators while Blank and Shirkov⁴⁾ have treated vertex functions in this way. The study of analyticity properties of the theory represents another attempt to transcend perturbation solutions. It is natural to try to combine results from these two general lines of investigation:⁵⁾ the charge renormalization group properties and analyticity properties. We shall confine our attention to the photon propagator, which appears to be the simplest object one can study with such methods.

Section 2 contains a derivation of the result we shall need from the charge renormalization group theory of the photon propagator, namely, that the high energy limit is independent of the physical charge when only photons and electrons are considered. The derivation is quite simple because we are here concerned only with this general result, not with the group differential equations or detailed solutions thereof.

In § 3, we consider two common sets of assumptions: (1) The usual analyticity properties are valid. As a consequence, bare charge exceeds physical charge. (2) The theory is renormalizable (so that § 2 is relevant), and regular at vanishing charge in the high energy limit. From these assumptions and the result of § 2, it follows easily that the bare charge vanishes.

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Clearly, (1) and (2) are incompatible. The results thus obtained on the basis of very general properties of the theory is reminiscent of the heuristic argument of Dyson,⁶⁾ who suggested that the quantum electrodynamic vacuum is unstable when $e^2 < 0$,*** which means that the theory is singular at $e^2 = 0$.

Section 4 contains an illustration, drawn from existing explicit calculations, of the relation between expressions satisfying the charge renormalization group equation, analyticity properties, and Feynman diagrams (which correspond to an expansion about zero charge). The illustration behaves as expected from § 3. It should be noted that the incompatibility of assumptions (1) and (2), established generally in § 3 and verified for a specific case in § 4, becomes numerically evident in § 4, only at energies far beyond those for which quantum electrodynamics has received experimental verification. Also, our methods suffice to establish the incompatibility only for the theory of electrons and photons (this point is discussed at the end of § 2).

§ 2. The charge renormalization group

Let us consider quantum electrodynamics. This theory is invariant under the simultaneous transformations^{7),8),9)}

$$e_1^2 \rightarrow e_2^2 = Z_3^{-1} e_1^2, \quad (2.1)$$

$$D_1(e_1^2) \rightarrow D_2(e_2^2) = Z_3 D_1(e_1^2), \quad (2.2)$$

which relate two different renormalizations*** of the electric charge e^2 and photon propagator D . The photon propagator is, in explicitly gauge invariant form,^{†,****}

$$D^{\mu\nu}(k, m^2, e^2) = \frac{1}{ik^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) d(k^2, m^2, e^2). \quad (2.3)$$

We shall concentrate our attention on the unknown d , which contains all the dependence of D on charge and the electron mass m . From (2.1), (2.2), and (2.3) it follows that $e^2 d$ is invariant under renormalization:

$$e_1^2 d_1(k^2, m^2, e_1^2) = e_2^2 d_2(k^2, m^2, e_2^2). \quad (2.4)$$

To each photon 4-momentum $k^2 = \lambda^2$ there corresponds a definite physical value of $e^2 d$. If we call this value e_i^2 , then $d_i(\lambda_i^2, m^2, e_i^2) = 1$. In particular, the physical

* Also W. Thirring, *Helv. Phys. Acta* **26** (1953), 33, C. A. Hurst, *Proc. Cambridge Phil. Soc.* **48** (1952), 625, and M. A. Petermann, *Helv. Phys. Acta* **26** (1953), 291, found that scalar boson theory with $\lambda\phi^3$ coupling diverges for finite momenta when expanded in powers of λ .

** For another case where the solution is not regular at $e^2 = 0$, see the discussion of the Klein paradox on p. 124 of W. Thirring, *Principles of Quantum Electrodynamics*, Academic Press, London and New York (1958).

*** The corresponding subscripts, 1 and 2 in this case, will be explicitly indicated only when necessary: e.g. in (2.1) and (2.2) but not (2.3).

† We use $k^2 = k_0^2 - \vec{k}^2$, $\hbar = c = 1$.

**** We are following here the very clear exposition of Bogoliubov and Shirkov.³⁾

charge e_p^2 corresponds to $\lambda_p^2=0$, and the bare charge e_b^2 corresponds to $\lambda_b^2=\infty$. Since the definition of d depends on λ (z_3 is otherwise undetermined), it is convenient to include the dependence explicitly in (2.4), which allows us to drop the subscript on d :

$$e_1^2 d(k^2/\lambda_1^2, m^2/\lambda_1^2, e_1^2) = e_2^2 d(k^2/\lambda_2^2, m^2/\lambda_2^2, e_2^2) \quad (2.5)$$

with

$$d(1, m^2/\lambda^2, e^2) = 1. \quad (2.6)$$

Equation (2.5) expresses the invariance of the theory under simultaneous changes of charge and momentum scale.

Now let us introduce ultraviolet cutoffs Λ so that $\lim_{k^2 \rightarrow \infty} e^2 d = e_b^2$ is finite*. Then d has a limit $d(k^2/\lambda^2, 0, e^2, \Lambda^2)$, as $m^2/\lambda^2 \rightarrow 0$. Thus, for $k^2, \lambda^2 \gg m^2$ Equation (2.5) becomes asymptotically

$$e_1^2 d(k^2/\lambda_1^2, 0, e_1^2, \Lambda^2) = e_2^2 d(k^2/\lambda_2^2, 0, e_2^2, \Lambda^2). \quad (2.7)$$

The cutoffs Λ have been introduced only to justify setting $m^2/\lambda^2=0$ in the case where $\lim_{k^2 \rightarrow \infty} e^2 d$ diverges. Perhaps this justification is unnecessary because, as remarked by Gell-Mann and Low¹⁾, the smaller of k^2 and λ^2 provides an infrared cutoff on each integral. In any case the use of Λ implies states of negative norm which are inconsistent with the analyticity properties we wish to use in §3, so from now on we consider the limit of (2.7) as $\Lambda \rightarrow \infty$ (and we drop Λ from the explicit notation). Clearly e and λ are not independent; in fact the limit of (2.7) as $\Lambda \rightarrow \infty$ implies that

$$e^2 d(k^2/\lambda^2, 0, e^2) = F\{k^2/\lambda^2 \cdot \phi(e^2)\}. \quad (2.8)$$

Now as k^2 varies ($k^2 \gg \lambda^2 \gg m^2$) we see that the physical charge plays the role of a scale factor (for any finite λ^2 , the e^2 defined by

$$e_p^2 d(\lambda^2, m^2, e_p^2) = e^2$$

is a function of the physical charge e_p^2). The limit $F \rightarrow e_b^2$ as $k^2 \rightarrow \infty$ is independent of this scale factor, so the bare charge e_b^2 (which may be finite or infinite) is independent of physical charge e_p^2 .

Similar arguments could be applied to other quantities X which are renormalization-invariant (i.e., not explicitly dependent on renormalization constants), at least as far as (2.4). If X is dimensionless one can proceed to (2.5). The step which cannot be extended to general X seems to be the dropping of finite momenta (e.g., m^2 in (2.7)) as λ^2 becomes large, which allows us to assert that the charge appears only as a scale factor of the momentum which becomes infinite. In the case of two masses m_1^2 and m_2^2 , for example, X might depend on m_1^2/m_2^2 in a charge-

* Gell-Mann and Low¹⁾, among others, have shown how to do this in a gauge invariant way.

dependent but λ^2 -independent manner. Even in the simple case of the vertex (multiplied by the square of the electron wave function renormalization) connecting two physical electrons (with 4-momenta m^2) and a virtual photon (k^2), an infrared cutoff must be used and cannot be dropped at large k^2 . Likewise the proof of (2.7) for the photon propagator seems to fail when two or more different charged particle masses are introduced. So the conclusions ((2.7), § 3, § 4) reached in this paper for the theory of electrons and photons may break down as soon as more elements are included in the theory.

§ 3. Proof of the contradiction

First, we shall see what happens if $e^2 d$ is assumed to be regular (uniformly convergent) as $e_p^2 \rightarrow 0$. In that case $e^2 d$ and e_b^2 are identically zero. As e_p^2 is increased, e_b^2 remains identically zero because it is independent of e_p^2 (from § 2). So we have

$$e_p^2 > e_b^2 = 0. \quad (3.1)$$

But if the analyticity properties assumed for the spectral representation^{1),10)} of d are correct,

$$e_b^2 \geq e_p^2. \quad (3.2)$$

One or more of the assumptions must therefore be wrong. Although we have not directly proved anything general about the behavior at *finite* momenta, one would certainly expect the discrepancy between (3.1) ($e^2 d = 0$ at $k^2 = \infty$) and (3.2) ($e^2 d \geq e_p^2$ at $k^2 = \infty$) to persist smoothly into a range of finite momenta, which is the behavior exhibited by the examples discussed in § 4.

§ 4. Explicit calculations of the photon propagator

One can be more specific than we have been in § 2, and derive differential or integral equations expressing invariance under the charge renormalization group. These equations can be used to modify any perturbation theory result into an expression invariant under the renormalization group.^{1),3)} The aim of this section is pedagogical: we attempt to provide insight into the general conclusions of the preceding sections by reviewing the relation of some particular expressions for the photon propagator, which satisfy the renormalization group equations, to diagrams and the usual analyticity properties. From § 3 we expect that these two aspects are complementary; a non-trivial expression satisfying the charge renormalization group equations can have the usual analyticity properties,^{5),11),12)} or a power series expansion about zero coupling constant^{3),13)} (diagram description), but not both.

It is well to begin by indicating what sets of diagrams can be treated by the charge renormalization group.* Such sets must conspire to multiply each of the re-

* This subject has already been touched upon by Gell-Mann and Low¹⁾ in their footnote (16).

normalizable quantities (charge, propagator, vertex, wave function) by a consistent amount (renormalization) in each place where that quantity occurs. This condition is satisfied by any set which includes all the possible reducible* and improper* corrections to itself.** Thus to each collection of irreducible, proper parts (employed consistently) there corresponds a renormalizable set of diagrams, obtained by adding all possible reducible and improper parts. In addition, if the set is to exhibit the full symmetry properties of the theory (e.g., gauge invariance), a suitably symmetric choice of irreducible, proper parts should be made.

By way of illustration, consider the quantity d defined by Eq. (2.3). The lowest order correction to the photon propagator is the bubble diagram (Fig. 1). The corresponding value for d at high energies (with correct analyticity properties) is approximately

$$d = 1 + \frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2} \quad (4.1)$$

where $e^2 = 1/137$. The solution of the group equation, regular at $e^2 = 0$, which reduces to (4.1) when expanded in powers of e^2 has been computed by a method which makes no reference to diagrams^{(1),(3)}. The result is

$$d = \frac{1}{1 - \frac{e^2}{3\pi} \ln \left(\frac{4m^2 - k^2}{4m^2} \right)} \quad (4.2)$$

In accordance with (3.1), Eq. (4.2) vanishes as $k^2 \rightarrow \infty$. From the geometrical series expansion of (4.2) in powers of e^2 (valid when $1 > \frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2}$) we



Fig. 1. Photon propagator with lowest order radiative correction.

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \dots$$

Fig. 2. Renormalizable set of diagrams corresponding to Fig. 1.

verify that (4.2) includes all the improper parts (Fig. 2) corresponding to Fig. 1.⁽³⁾ As is well known, (4.2) contains a singularity, at $\frac{e^2}{3\pi} \ln \frac{4m^2 - k^2}{4m^2} = 1$, not in accordance with the general analyticity properties of $d^{(1),(3)}$. In order to remove this

* A skeleton is a diagram with all self-energy and vertex parts omitted⁽⁷⁾. A graph which is its own skeleton is called *irreducible*; all other graphs are called *reducible*. A *proper* part is one which cannot be divided into two parts joined by a single line; all other parts are *improper*.

** Note that the bare photon propagator is itself invariant under the charge renormalization group. But in this trivial case $e^2 d = e_p^2$ is independent of k^2 , so (2.8) is not valid and the conclusions of § 2 and § 3 (which follow from (2.8)) are not fulfilled.

difficulty Redmond^{11)*} has used (4.2) to determine

$$2\pi i I(z) = \lim_{\epsilon \rightarrow 0} [d(z+i\epsilon) - d(z-i\epsilon)] \quad (4.3)$$

in the spectral representation^{1),10)}

$$d = 1 + k^2 \int_0^\infty \frac{I(z)}{k^2 - z + i\epsilon} \frac{dz}{z}. \quad (4.4)$$

The result,

$$e^2 d = \frac{3\pi}{\frac{3\pi}{e^2} - \ln \frac{4m^2 - k^2}{4m^2}} + \frac{3\pi}{1 - \exp \left\{ \frac{3\pi}{e^2} - \ln \frac{4m^2 - k^2}{4m^2} \right\}}, \quad (4.5)$$

has the right analyticity properties as a function of k^2 , exhibits an essential singularity at $e^2=0$, and is finite and independent of e^2 in the high energy limit ($e^2 d(\infty) = 3\pi$). Bogoliubov, Logunov, and Shirkov⁵⁾ have shown that (4.5) can be modified to agree with the charge renormalization group equations, without changing the other properties of (4.5) (in particular $e^2 d(\infty) = 3\pi$ remains independent of e_p^2 in accordance with § 2, and greater than $e_p^2 = 1/137$ in agreement with § 3.). In view of the singularity at $e^2=0$, (4.5) cannot represent an expansion in series of diagrams, although it bears some relationship to Fig. 2. The difference between quantum electrodynamics and meson physics is that (4.5) allows an asymptotic expansion in powers of $e^2 = 1/137$, whereas the corresponding expression in meson physics^{11),12)} probably has no useful expansion in powers of $g^2 \simeq 15$. Equation (4.5) is very well approximated by the lowest order expression, (4.1), for all practical purposes at laboratory energies. The asymptotic expansion of (4.5) in powers of e^2 at moderate energies represents a weak coupling, which builds up to a strong coupling expansion in inverse powers of e^2 at high energies.

To complete the trilogy, consider (4.1), which is *not* a solution of the charge renormalization group equations. It is easily seen that d in (4.1) is regular at $e^2=0$ and satisfies the analyticity condition (4.3), (4.4).

§ 5. Conclusion

In § 3 we found on general grounds that, if only electrons and photons are considered, the photon propagator cannot have all three of the following properties: renormalizability, analyticity, and regularity at $e^2=0$. In § 4, we quoted particular expressions for the propagator, related to the restricted set of diagrams in Fig. 2, which exhibit the behavior predicted in § 3.

The author would like to thank Profs. H. Miyazawa and Y. Munakata for helpful criticism.

* Actually Redmond considered the meson propagator, but his discussion can be adopted to the photon propagator without any difficulty.

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Letters to the Editor

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Note on the Forbidden Processes of the Leptonic Decays

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In this note we assume that the Fermi interactions are invariant when the strong reflection which includes the strong reflection parity is applied to the fermion field, and then we want to study the consequences of this assumption. We assign the strong reflection parities for individual fermions. It is then shown that we have the selection rule for the unwanted decays using the strong reflection parities. This note seems to give a better interpretation on the selection rule of leptonic decay given by Nishijima.¹⁾

The simultaneous reflections of all space-time coordinates,

$$x_\mu \rightarrow x'_\mu = -x_\mu, \quad (1)$$

induce the transformation on the spinor fields:

$$\begin{aligned} \psi_i(x) &\rightarrow \psi'_i(x') = \gamma_5 \gamma_i \psi_i^T(x), \\ \bar{\psi}_i(x) &\rightarrow \bar{\psi}'_i(x') = -\gamma_i^* \bar{\psi}_i^T(x) \gamma_5, \quad (2) \\ \bar{\psi}_i(x) &= \psi_i^+(x) \gamma_4, \end{aligned}$$

where the index i distinguishes the kind of spinor field. The arbitrary phase factor, which we define as "strong reflection parity", should satisfy

$$\eta_i^2 = \pm 1, \quad |\eta_i|^2 = +1. \quad (3)$$

The bilinear covariants transform as

$$\begin{aligned} \frac{1}{2} [\bar{\psi}_i(x), O_I \psi_j(x)] &\rightarrow s^{(I)} \eta_i^* \eta_j \\ &\times \frac{1}{2} [\bar{\psi}_i(x), O_I \psi_j(x)], \quad (4) \end{aligned}$$

where I runs S, T, V, A, P. According to the definition of the ordinary strong reflection,²⁾ the electric current $[\bar{\psi}_i(x), \gamma_\mu \psi_i(x)]$ changes the sign under ordinary R_s^* , and so we expect that $[\bar{\psi}_i(x), O_V \psi_j(x)]$ ($i \neq j$) should change the sign under R_s , too. $s^{(I)}$ takes the value ± 1 according as γ_5 commutes or anticommutes with the operator O_I . We have to take $\eta_i^* \eta_j = +1$ in order that the $[\bar{\psi}_i(x), O_V \psi_j(x)]$ and $[\bar{\psi}_i(x), O_A \psi_j(x)]$ may change the sign under the strong reflection.

We now assume that the Fermi interaction is given by the product of a linear combination of $[\bar{\psi}_i(x), O_V \psi_j(x)]$ and $[\bar{\psi}_i(x), O_A \psi_j(x)]$. For the Universal Fermi interactions, we then take the form of

$$H_{int} = -G j_\mu^F(x)^\dagger \cdot j_\mu^F(x), \quad (5)$$

* We use the abbreviation R_s for the strong reflection. The ordinary R_s means the R_s which does not include parities.

where G is the phenomenological coupling constant, and $j_\mu^F(x) = \frac{1}{2} i \sum_{i,j} [\bar{\psi}_i(x), \gamma_\mu(a + b\gamma_5)\psi_j(x)]$, the constants a and b are arbitrary complex numbers.

We now try to assign the strong reflection parities for individual fermions, by using the processes of β -decay, μ -decay and μ -capture. From our assumptions that the interaction Hamiltonian is invariant under the transformation R_s , and the condition $\eta_i^* \eta_j = +1$ holds, we have the three equations for the above processes:

$$(i') \quad \gamma_p^* \eta_n = +1, \quad \eta_e^* \gamma_\nu = +1;$$

$$\gamma_p^* \eta_n \eta_e^* \gamma_\nu = +1,$$

$$(ii') \quad \eta_e^* \gamma_\nu = +1, \quad \gamma_{\nu'}^* \eta_\mu = +1;$$

$$\eta_e^* \gamma_\nu \gamma_{\nu'}^* \eta_\mu = +1,$$

$$(iii') \quad \gamma_p^* \eta_n = +1, \quad \gamma_{\mu'}^* \eta_{\nu'} = +1;$$

$$\gamma_p^* \eta_n \gamma_{\mu'}^* \eta_{\nu'} = +1.$$

According to the usual scheme, it is, however, difficult to find any reason why there should not be such unwanted processes as $\mu^\pm \rightarrow e^\pm + e^\pm + e^\mp$ and $\mu^- + p \rightarrow p + e^-$. To forbid these processes, Konopinski and others³⁾ assume that e^- and μ^+ are normal particles and distinguish the neutrinos ν accompanied by electrons from the ν' accompanied by μ -mesons. In this note, we also assume that two kinds of neutrinos, ν and ν' , but we treat e^- , μ^- , ν , ν' and nucleons as normal particles, and give the different strong reflection parities to e^- and μ^- , respectively. If we take $+1$ for the strong reflection parity for the nucleon, then the value of η of each particle is given in Table I.

We now investigate the unwanted weak processes using the value of η .

Table I

particle	(anti-particle)	η_i	$(-\eta_i^*)^*$
p	(\bar{p})	$+1$	(-1)
n	(\bar{n})	$+1$	(-1)
e^-	(e^+)	$+1$	(-1)
μ^-	(μ^+)	-1	$(+1)$
ν	$(\bar{\nu})$	$+1$	(-1)
ν'	$(\bar{\nu}')$	-1	$(+1)$

* The transformation of the charge conjugated fields is given by

$$\psi_i^c(x) \rightarrow -\eta_i^* \gamma_5 (\psi_i^c)^T$$

where $\psi_i^c = C \bar{\psi}_i^T$, $C^{-1} \gamma_\mu C = -\gamma_\mu^T$.

From the assignment of the η 's in Table I, the following condition must be satisfied for each of processes, (a) and (b):

$$(a) \quad \mu^\pm \rightarrow e^\pm + e^\pm + e^\mp, \quad \gamma_\mu^* \eta_e = -1,$$

$$\eta_e^* \eta_e = +1; \quad \gamma_\mu^* \eta_e \eta_e^* \eta_e = -1,$$

$$(b) \quad \mu^- + p \rightarrow p + e^-, \quad \eta_\mu^* \eta_p = -1,$$

$$\eta_e^* \gamma_p = +1; \quad \eta_\mu^* \gamma_p \eta_e^* \gamma_p = -1.$$

We, therefore, see that we can forbid these processes if we assume that the Universal Fermi interactions are invariant under the strong reflection.

We wish to thank Prof. M. Sasaki for helpful discussions.

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Lower Levels in Ca^{43}

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In the $f_{7/2}$ shell nuclei, the spin $7/2$ level competes with the spin $5/2$ level. Spin $5/2$ ground state cannot be inter-

preted by Mayer's shell model. Several authors¹⁾ were tempted to interpret this question by using a long-range parameter λ (>1.8) of the nuclear potentials, but it is questionable whether such long-range value has physical meanings.

To interpret this problem, we calculated the lower levels of Ca^{43} by using the method of configuration mixing. The level order of the independent particle model has led us to assume that

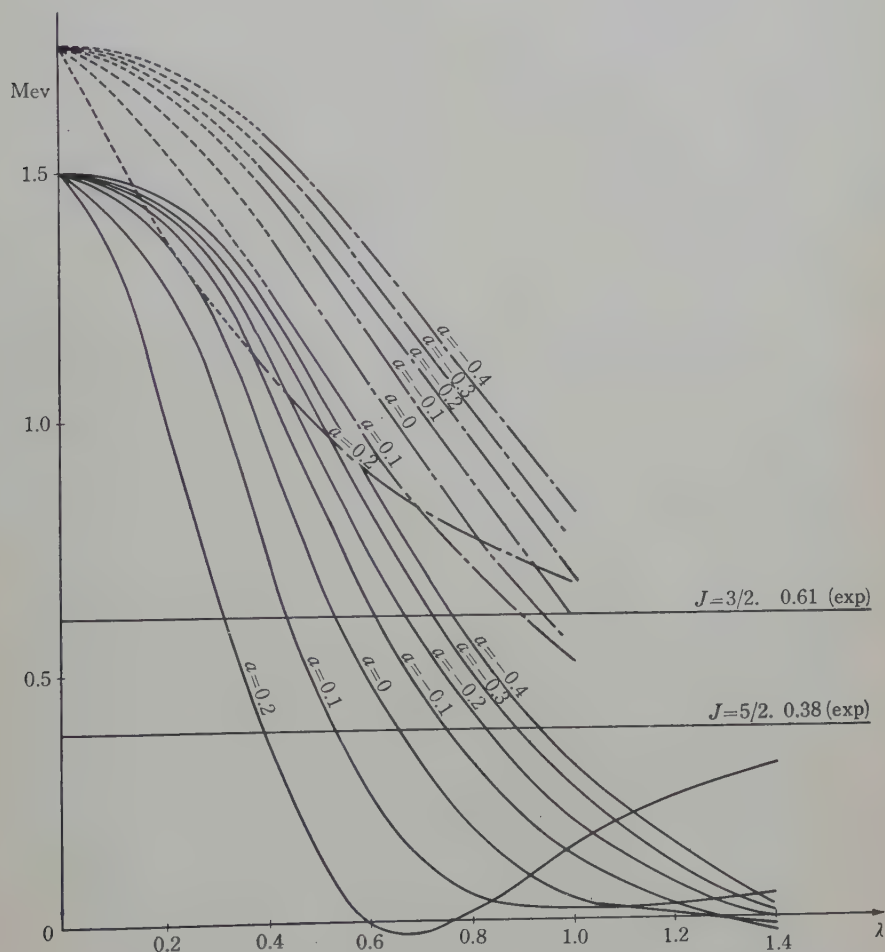


Fig. 1. Calculated values of $J=5/2$ and $3/2$ states in Ca^{43} . The solid curves are $J=5/2$, the broken curves $J=3/2$ states. Energy zero corresponds to the ground state $J=7/2$.

the following 10 configurations are mixed with the main configuration $f_{7/2}^8 J$ in Ca^{43} :

$$J=7/2: f_{7/2}^2(J_1) f_{5/2} 7/2$$

and $f_{7/2}^2(J_2) p_{3/2} 7/2,$

where $J_1=2, 4, 6$ and $J_2=0, 2.$

$$f_{7/2} f_{5/2}^3(J_1) 7/2$$

and $f_{7/2} p_{3/2}^3(J_2) 7/2$

where $J_1=0, 2, 4$ and $J_2=0, 2.$

In the same way, for $J=5/2$ and $3/2$ there are 9 and 8 configurations with one- and two-particle jump, respectively.

The two-body interactions in the unfilled shell were assumed to be

$$G=V(r)(1+a\sigma_1\cdot\sigma_2)$$

where $V(r)=V_0e^{-r^2/r_0^2}.$

As the wave functions, the harmonic oscillator function was used. The

strength V_0 of the interactions was fixed by the requirement that the pairing energy of Ca^{42} is 3 Mev. In Ca^{41} the $f_{5/2}$ level has not yet been found²⁾ and so the single particle energy $\epsilon_{5/2}$ remained free but we noted that $\epsilon_{5/2}$ is 4 Mev or more energetic.

We calculated the lower levels of Ca^{43} as a function of the spin mixture parameter a and the range parameter λ . We considered the mixture parameter to be $a=0.2, 0.1, 0, -0.1, -0.2, -0.3$ and -0.4 . We also calculated the low-lying levels of Ca^{42} and $\text{Ca}^{44\ 3)}$ for such values of the parameter.

In Fig. 1 we show the lower levels of Ca^{43} as a function of the range parameter λ for each value of the spin mixture parameter a . In the region of the range which seems to be suitable from the calculations of the lower levels of Ca^{42} , the $7/2$ and $5/2$ levels are very close and the spin of the

Table I Calculated values $E_J-E_{7/2}=E'_J$ of the first and second excited energies in Ca^{43} (in Mev)

$\lambda \backslash a$	0.2		0		-0.2		-0.4		Experiment	
	$E_{5/2}'$	$E_{3/2}'$	$E_{5/2}'$	$E_{3/2}'$	$E_{5/2}'$	$E_{3/2}'$	$E_{5/2}'$	$E_{3/2}'$	$E_{5/2}'$	$E_{3/2}'$
0.6	0.000	0.890	0.486	1.077	0.755	1.247	0.902	1.351	0.38	0.61
0.8	0.022	0.769	0.194	0.784	0.419	0.961	0.566	1.088		
1.0	0.150	0.678	0.066	0.551	0.204	0.700	0.324	0.834		

Table II. Calculated values $E_2-E_0=E_2'$ of the first excited energy in Ca^{42} assuming the pairing energy to be 3 Mev and the single particle energy $\epsilon_{5/2}$ which we assumed (in Mev).

$\lambda \backslash a$	0.2		0		-0.2		-0.4		Experiment	
	E_2'	$\epsilon_{5/2}$	E_2'	$\epsilon_{5/2}$	E_2'	$\epsilon_{5/2}$	E_2'	$\epsilon_{5/2}$	E_2'	$\epsilon_{5/2}$
0.6	0.628	4.629	1.386	4.474	1.635	4.196	1.743	4.008	1.53	>4
0.8	0.289	4.526	0.978	4.613	1.251	4.351	1.374	4.130		
1.0	0.113	4.459	0.685	4.679	0.934	4.453	1.050	4.227		

second excited state is evidently $3/2$. These are consistent with the experiment.⁴⁾ It was found from the experiment that the spin $7/2$ level is the ground, one but the $5/2$ level also appears to be very close.

In Tables I and II, we listed the calculated values of the lower levels in Ca^{43} together with the levels in Ca^{42} . Furthermore, we listed in Table II the assumed values of the single particle energy $\epsilon_{5/2}$. In the above calculation of Ca^{43} , we used the perturbation method (no perturbation method was used in case of Ca^{42}) but the error included seems to be small, for the off-diagonal elements are very small in case of the spin mixture which we considered.

We have made progress in similar calculations of the Ca isotopes by using the electric computer. Details of the calculations of the low-lying levels in Ca isotopes will be reported in near future.

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Line Shapes of I. M. O.-Absorption in the Semiconductors

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Burstein *et al.*¹⁾ and Zwerdling *et al.*²⁾ showed experimentally that in the presence of a magnetic field there appeared many peaks in the line shape of light absorption which corresponds to the electronic transition from v- to c-band of semiconductors. This is the so-called I. M. O- (interband magneto optical-) absorption. They showed also that the magnetic field caused the absorption edge to shift to higher energies.

We developed the theory of line shape of I. M. O-absorption and got the results which are in good agreement with the observation. Assuming the simple parabolic band in order to treat the electron (hole) near the bottom (top) of c (v)-band, we sought the corresponding line shape to the minimum photon energy. The interaction between carrier and phonon was taken into account after Toyozawa's method³⁾ of generating function. As the momentum component of the carrier along the applied magnetic field $\hbar K_z (\hbar K'_z)$ varies continuously, which is a different situation from Toyozawa's treatment for excitation, many sub-peaks are formed according to each transition from an initial K'_z in a Landau level to a final K_z in the other level corresponding to the photon energy between the both levels, one of which is in v-band

and the other is in c-band.

The energy, the self-energy and the half-width of a sub-peak is denoted respectively by ε_s , Δ_s , $\hbar\Gamma_s$ for a state s of one pair of an electron and a hole, then the coefficient of light absorption is expressed by

$$A(\omega) \approx \frac{R}{\omega} \sum_s (1 + \eta_s) \frac{\Gamma_s/2 + 2\mathcal{Q}_s\Omega_s}{\Omega_s^2 + (\Gamma_s/2)^2} \quad (1)$$

with

$$\mathcal{Q}_s \approx \frac{1}{2} \sum_{s' \neq s} \hbar\Gamma_{ss'}^s / \varepsilon_s - \varepsilon_{s'}, \quad (2)$$

$$\eta_s \approx 2 \sum_{s' \neq s} \Delta_{ss'}^s / \varepsilon_s - \varepsilon_{s'}, \quad (3)$$

$$\Omega_s = \omega - \varepsilon_s / \hbar - \Delta_s / \hbar, \quad (4)$$

where R is a constant, $\Gamma_{ss'}^s$ and $\Delta_{ss'}^s$ are nondiagonal terms of the scattering probability and self-energy respectively. In deriving (1), (2) and (3), we made use of the weak dependency of the oscillator strength of photon perturbation on the wave number of carrier in the case of allowed transitions.

In order to seek Γ_s , Δ_s , \mathcal{Q}_s and η_s , we apply the following two complementary methods of approximation.

(A) *Approximation of strong field.* When the magnetic field is so strong that the energy distance between Landau levels becomes too large to allow the virtual transition of carrier from a Landau level to another one by the phonon perturbation, we can assume that the summation over intermediate states is restricted to the states within one Landau level. However, not only all the intermediate states of which orbital centers of cyclotron motion is different from the initial one but also the phonon energy and the recoil energy of electron (hole) are taken into con-

sideration. In this case we get, near $K_z = K'_z \approx 0$,

$$\hbar\Gamma_s = 4.0 \times 10^{-4} \text{ eV}, \quad (5)$$

$$\Delta_s \approx \Delta^c/20 + \Delta^v/285, \quad (6)$$

$$\mathcal{Q}_s = -3.7 \times 10^{-2}, \quad (7)$$

for germanium at room temperature in the presence of a magnetic field of 5×10^4 Oe.* Δ^c and Δ^v are respectively the energy shift of the bottom of c-band and that of the top of v-band in the absence of the field. Γ_s and Δ_s has complicated dependency on K_z and K'_z but only the value at $K_z = K'_z \approx 0$ is important because of the underlying assumptions.

(B) *Approximation of weak field.* Neglecting the magnetic energy and phonon energy compared to the recoil energy of electron (hole), we can sum up the intermediate states over all Landau levels, then we get

$$\hbar\Gamma_s \approx 1.3 \times 10^{-4} + a(\Delta^c K_z + \Delta^v K'_z) \text{ eV}, \quad (8)$$

$$\Delta_s(K_z, K'_z \approx 0) = \Delta^c(1 + \delta^c) + \Delta^v(1 + \delta^v), \quad (9)$$

$$\mathcal{Q}_s(K_z, K'_z \approx 0) = -8.3 \times 10^{-4}, \quad (10)$$

for the same circumstances as in (A). a in (8) is the lattice constant. The value of Γ_s at the maxima of K_z and K'_z is coincident with the scattering probability in the absence of the field^{5). δ^c and δ^v are complicated functions of effective masses, but these take small positive values at room temperature.}

η_s is negligibly small compared with unity in the both cases.

* We take 10 eV as coupling constant.

Burstein *et al.*¹⁾ experimentally showed

$$\hbar\Gamma_s = 10^{-3} \text{ eV} \quad (11)$$

for the same circumstances as ours. They assumed that the value is independent of K_s and K'_s , but it is really not. When one takes into account the additive contribution from the impurity and imperfection scattering, their value is in good agreement with ours near the absorption edge. \mathcal{O}_s is proved to be so small that $A(\omega)$ is the integration of Lorentzian sub-peaks in the energy range of the first observed peak.

The details will be published shortly with Mr. Nagae's help who studied the matrix element of photon perturbation as well as selection rules.

The authors express their thanks to Prof. Y. Uemura (University of Tokyo) for his helpful discussions.

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On the One-Body Propagator

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The divergences in quantum electrodynamics have been eliminated by the

renormalization method, but there are still difficulties left: the problem of vanishing of the renormalized charge. This is a situation due to the divergence in quantum field theory, which is caused by the singularity of the propagator that does not become less singular than $1/(x-x')^2$ near the light cone.

When the interaction representation exists, the Heisenberg operator $\varphi(x)$ is connected with the free operator $\varphi_{in}(x)$ by a unitary transformation: $\varphi(x) = U^{-1}[\sigma] \varphi_{in}(x) U[\sigma]$. Therefore, when φ_{in} is hermitean, so the Heisenberg operator φ is. Consequently, ρ -function defined by H. Lehmann¹⁾ appears to be positive definite. Then, it is clear that the one-body propagator cannot be less singular than that of the free field. S. Deser²⁾ has pointed out that the introduction of the gravitational field might make the singularity of the one-body propagator weaker. In his case, the quantum fluctuation of the light cone itself plays the essential role to weaken the singularity. It can, however, be shown that his opinion is not always correct. For instance, let us consider the interaction Lagrangean $L' = f \partial_\mu \varphi_1 \partial_\nu \varphi_1 \partial_\mu \varphi_2 \partial_\nu \varphi_2$. For this interaction Hamiltonian is found out,³⁾ and therefore, the singularity of the one-body propagator cannot be weaker than that of the free field, though the light cone fluctuates in Deser's sense. It is aimed here to present an example which illustrates the connection between the canonical formalism and singularity of the propagator. The condition for the existence of the interaction representation has been discussed by Y. Katayama.⁴⁾ We restrict ourselves to the interaction between

spinor and boson fields: $L' = f\bar{O}(\partial)\bar{\psi} \cdot O(\partial)\psi P(\partial)\phi$, $O(\partial)$, $P(\partial)$ being polynomials in ∂ , and the highest order derivatives being $\bar{O}(\partial) \sim \partial^a$, $O(\partial) \sim \partial^a$, $P(\partial) \sim \partial^b$. According to Katayama, the sufficient condition for the existence of Hamiltonian is as follows: $2s_b + 2b \leq 2$, $2s_a + a + a < 2$, where s_a and s_b are spins of spinor and boson fields respectively. When the higher derivatives are operated on field variables, the more independent variables are needed to specify the state at a certain time. This makes it impossible to connect the Heisenberg operator and the free field by a unitary transformation. To simplify the problem, the following Lagrangean is adopted:

$$L' = f \cdot 2 \cdot \{\bar{\psi}\psi \square \phi + \square \phi \cdot \bar{\psi}\psi\}. \quad (1)$$

According to Katayama's condition, there is no Hamiltonian, and this is ascertained by a direct computation. Let us adopt the P^* method⁵⁾ here in order to define the propagator. In our case, it is easily proved that the P^* method and Feynman's quantization by the path integral give a same propagator. S -matrix is given by the following formula:

$$\begin{aligned} S &= P^* \exp \left\{ -i \int L dx \right\} \\ &= \sum \frac{(-i)^n}{n!} \int \dots \int P \{ \bar{\psi}\psi(1) \dots \bar{\psi}\psi(n) \} \\ &\quad \square_1 \dots \square_n P \{ \phi(1) \dots \phi(n) \}. \end{aligned} \quad (2)$$

Then, the boson propagator with radiative correction \mathcal{A}_F is now obtained to be

$$\mathcal{A}_F' = \frac{1}{1 - \Sigma \cdot \mathcal{A}_F} \cdot \mathcal{A}_F \quad (3)$$

where Σ is the boson self energy part.

We calculate Σ by the perturbation method with Feynman cut off Λ . Then,

$$\begin{aligned} \mathcal{A}_F' &= \frac{-i}{(2\pi)^4} \cdot \frac{1}{k^2 + \kappa^2 - i\epsilon} \\ &\quad \cdot \frac{1}{1 - \frac{f^2 k^2}{4\pi} \cdot \left[\Lambda^2 - \frac{k^2}{6} \log \frac{\Lambda^2}{|k^2|} \right]} \\ &\quad \Lambda^2 \gg k^2 \gg \kappa^2. \end{aligned} \quad (4)$$

This leads us to Lehmann's density function as follows:

$$\rho(\lambda^2) = \delta(\lambda^2 - \kappa^2) + \sigma(\lambda^2) - \delta \left(\lambda^2 + \frac{4\pi}{f^2 \Lambda^2} \right).$$

Here,

$$\sigma(\lambda^2) = \begin{cases} 0 & \lambda^2 < (2\kappa)^2 \\ \frac{2\pi\kappa^2}{f^2 \Lambda^4} \cdot \left(\lambda^2 + \frac{4\pi}{f^2 \Lambda^2} \right)^2 & \lambda^2 \gg \kappa^2. \end{cases} \quad (5)$$

We find that ρ is neither positive definite nor zero for $\lambda^2 < 0$. This is a situation quite different from the interaction for which canonical formalism exists: the latter leads us to $\rho = 0$ for $\lambda^2 < 0$, because the Hamiltonian has a lowest eigenvalue. In our example, canonical formalism does not exist, so the above result does not seem so contradictory.

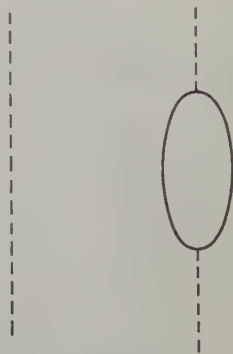


Fig. 1

We shall now construct a simple model in which (4) is the exact solution.⁶⁾ Assuming the interaction Lagrangean to be $L' = \frac{f}{2} \{ \bar{\psi}^+ \phi^+ \cdot \square \phi^- + \square \phi^+ \cdot \bar{\psi}^- \phi^- \}$, we can show that Fig. 1 is the only diagram which we have to take into account. Therefore, (4) is the exact solution.

In conclusion, it seems that we need to have the interaction, for which we cannot find any Hamiltonian, in order to get rid of the difficulty in the renormalization theory⁷⁾. Moreover, the light cone might be deformed in our example like the gravitational field, because the interaction term is higher derivative and it determines the propagation character. But these interactions can appear only at very high energy region, because we have no experiments to confirm them. H. Umezawa has called these interactions the extremety weak interactions.⁷⁾

The author would like to acknowledge his deep gratitude to Prof. Umezawa for his guidance and kind encouragement, and wishes to thank Messrs. S. Sato, H. Ezawa and J. Otokozawa for the most helpful discussions.

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On a Possible Small Pauli Term in Quantum Electrodynamics

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Recently Katayama, Taketani and Ferreira¹⁾ showed that one of the possible ways of explaining the origin of the electron mass is to replace the usual local interaction between electron and photon by a non-local one and to use an appropriate cut-off procedure. They also showed that, as long as one is concerned with the rough calculation of electron mass in the low energy limit of photon, this non-local interaction is equivalent to the usual local interaction plus a small Pauli term

$$\delta(e/4m) \bar{\psi} \sigma_{\lambda\mu} \psi F_{\lambda\mu} \quad (1)$$

where δ is a parameter characterizing the magnitude of the Pauli term. They estimated δ by comparing the additional anomalous magnetic moment of electron due to the Pauli term with the difference between the experimental value of magnetic moment and the theoretical one calculated on the basis of the usual quantum electrodynamics and gave $\delta \sim 10^{-5}$. But if we use more precise experimental value of magnetic moment²⁾ than the one used by them, we get $\delta = (7.4 \pm 5) \times 10^{-6}$.**

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** In ref. 1) it is not explicitly pointed out but it is necessary that δ is negative in order to explain the electron mass. By the way, their Eq. (12) $\delta m^{(m)} \cong \alpha(3/8\pi)(\delta/m_e)\lambda^2$ is in error and should be $-\alpha(3/2\pi)(\delta/m_e)\lambda^2$.

Table I. δ from level shift

Kind of level and element	Exp. value ³⁾ (Mc/sec)	Theor. value ³⁾ (Mc/sec)	Difference (Mc/sec)	Contribution from Pauli term in $m\alpha^4$	δ
$E(2s_{1/2}) - E(2p_{3/2})$ in hydrogen	1057.8 ± 0.1	1057.9 ± 0.2	-0.1 ± 0.2	$\delta/6$	$(-1.7 \pm 3.4) \times 10^{-6}$
$E(3s_{1/2}) - E(3p_{3/2})$ in hydrogen	315 ± 10	314.95 ± 0.05	0.05 ± 10	$4\delta/81$	$(0.029 \pm 5.8) \times 10^{-4}$
$E(2s_{1/2}) - E(2p_{3/2})$ in deuterium	1059.0 ± 0.1	1059.3 ± 0.2	-0.3 ± 0.2	$\delta/6$	$(-5.1 \pm 3.4) \times 10^{-6}$

This Pauli term interaction may have some effects also on other electrodynamic processes. In this note, we shall estimate the limit of δ by making use of the Lamb shift of several atoms and also the fine structure of positronium. We regard the small Pauli term as a phenomenological interaction.

The result of our calculation shows that the most precise limit of δ is given by the Lamb shift in hydrogen atom. The method of our estimation is as follows. We regard the small Pauli term (1) as a small perturbation to the usual interaction and the electromagnetic field $F_{\lambda\mu}$ in (1) is taken to be the Coulomb field of the proton. Its contribution to the level shift $E(2s_{1/2}) - E(2p_{1/2})$ is

$$(1/6)\delta m \alpha^4 = 0.59 \times 10^5 \delta \text{ Mc/sec.} \quad (2)$$

This value must be compared with the difference between the experimental value and the theoretical one based on the usual quantum electrodynamics. The numerical result is shown in Table I, together with the results in cases of other level and element. Thus if we take the level shift in $2s_{1/2}$ and $2p_{1/2}$ states in hydrogen as the case which determine δ most strictly, we get $\delta =$

$(-1.7 \pm 3.4) \times 10^{-6}$ which is to be compared with the value $(7.4 \pm 5) \times 10^{-6}$ obtained from the anomalous magnetic moment of electron. In this connection it must be noted that the effect of other phenomena^{4),5)} to the level shift may exist, but we do not consider these effects in this note.

We also estimate δ from the fine structure of positronium. We first calculate the contribution from the Pauli term to the second order effective Hamiltonian acting between electron and positron and then make non-relativistic approximation. The contribution to the energy splitting $E(^3s) - E(^1s)$ is

$$(7/12)m\alpha^4 \times 2\delta. \quad (3)$$

The difference between the experimental value of this splitting $(2.0338 \pm 0.0004) \times 10^5 \text{ Mc/sec}$ and the theoretical one based on the usual quantum electrodynamics $2.0337 \times 10^5 \text{ Mc/sec}$ is $10 \pm 40 \text{ Mc/sec}^3)$. This gives $\delta = (2.5 \pm 9.8) \times 10^{-5}$, much larger than the value obtained from the Lamb shift.

In conclusion we can say that from the purely experimental point of view there is no definite evidence for or against the existence of a small Pauli term. But it must be noted that its

effect to the various processes is quite similar to that arising from the modification of quantum electrodynamics at short distance discussed by Drell and others⁶⁾. In view of these situation it will be much expected to increase the accuracy of these experimental values by one or two order.

Calculations on scattering phenomena are now in progress.

We wish to thank Prof. S. Goto for his kind discussion. We also would like to express our thanks to Prof. M. Taketani for sending us the preprint of ref. 1) and Profs. T. Toyoda and Y. Katayama for their valuable comments on the manuscript of this note.

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A Note on the Signs of the $K^\pm-p$ Interactions

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Experiments¹⁾ seem to indicate that in the $K^\pm-p$ scatterings at low energies the S -wave scattering is main and the K^+-p (K^-p) interaction is repulsive (attractive). In this note we present some considerations about the signs of the $K^\pm-p$ interactions, assuming the interaction Hamiltonian of the Yukawa type. In the Born approximation the K^+-p interaction is attractive or repulsive according as the (NYK) coupling is of the scalar or pseudo-scalar (ps) type, while the K^-p interaction is repulsive irrespectively of the coupling type. Here Y means A or Σ . The question then arises as to whether or not the sign is changed by higher order effects.

Previously we investigated the case in which both (NAK) and ($N\Sigma K$) couplings are of the scalar type, using static approximation but including higher order effect²⁾. It was shown that the signs of the $K^\pm-p$ interactions are determined by the Born terms and as a result the K^-p elastic cross section is strongly damped by the repulsive rescattering effect. Recently Ferreira³⁾ made the 4-th order perturbation calculation (pion correction) for the K^-p scattering at the threshold and pointed out that, for any combination of the

(*NYK*) couplings, the 4-th order term is opposite in sign with the Born term and large enough to compensate or exceed the Born term. His statement may seem to contradict ours. But the cause of this confusion will be seen as follows.

Let both (*NAK*) and (*NΣK*) be scalar couplings. In ref. 2) it was shown that in the case of scalar couplings relativistic effects are not important in the low energy *K*—*N* scattering. In order to avoid inessential complication we use the static theory and neglect the *Σ*—*A* mass difference. The *T*-matrix elements for the *S*-wave *K*[±]—*p* elastic scatterings are written as $-2\pi(\omega_q\omega_p)^{-1/2} \times h^\pm(\omega_p)$ where *q* and *p* are the initial and final kaon momenta and $\omega_p = (m_K^2 + p^2)^{1/2}$. If the sign of *Re h* is + (−) the interaction is referred to as attractive (repulsive). The Low equation for *h*[−](*ω*) is

$$h^-(\omega) = \frac{-\lambda}{\omega - \Delta} + \frac{1}{\pi} \int_{m_{\pi^+ \Delta}}^{\infty} \frac{\text{Im } h^-(\omega') d\omega'}{\omega' - \omega - i\epsilon} + \frac{1}{\pi} \int_{m_K}^{\infty} \frac{\text{Im } h^+(\omega') d\omega'}{\omega' + \omega}. \quad (1)$$

Cut off factors are dropped. $\lambda = g_A^2 + g_\Sigma^2$; *g_γ* is the renormalized (*NYK*) coupling constant. *Δ* is the *Y*—*N* mass difference. Solving (1) by iteration we have the perturbation expansion with respect to the renormalized coupling constants. For example, the 4-th order term is given by replacing $\text{Im } h^\pm = p(|h^\pm|^2 + \dots)$ in the integrand by their Born terms, and it is easily verified that the 4-th order term is positive and, indeed, considerably large as pointed out by

Ferreira. But such an iteration is quite wrong because the 4-th order term is large. In this case it is preferable to rewrite (1) as⁴⁾

$$h^-(\omega) = \frac{-\lambda}{\omega - \Delta} \cdot [1 + \lambda H^-(\omega)]^{-1},$$

$$H^-(\omega) = \frac{\omega - \Delta}{\pi} \times \left[\int_{m_{\pi^+ \Delta}}^{\infty} d\omega' \frac{\text{Im } h^-(\omega') / |h^-(\omega')|^2}{(\omega' - \Delta)^2 (\omega' - \omega - i\epsilon)} + \int_{m_K}^{\infty} d\omega' \frac{\text{Im } h^+(\omega') / |h^+(\omega')|^2}{(\omega' + \Delta)^2 (\omega' + \omega)} \right]. \quad (2)$$

This is exact provided that *h*[−](*ω*) has no zero on the real axis of the complex *ω*-plane. In the static theory $\text{Im } h^\pm$ are always nonnegative. In the relativistic theory $\text{Im } h^-$ may be negative in the unphysical region $\omega < m_K$, but the contribution from this region will be small. Since *H*[−](*ω*) is positive (for $\omega > \Delta$) *Re h*[−] is negative, i. e. the *K*[−]—*p* interaction is repulsive. It is also shown in the same way that the *K*⁺—*p* interaction is attractive.

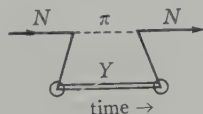
Now it is clear that the model with scalar (*NAK*) and (*NΣK*) couplings gives wrong signs both for the *K*[±]—*p* interactions. Introduction of couplings like *K*⁺*Kπ*² or *K*⁺*Kπ* (relative parity of *K*⁺—*K*⁰ may be odd) can hardly remedy this model because the effects of these couplings are same for *K*[±]—*p* and difference of the sign will not be explained by them.

We now consider the case in which both (*NAK*) and (*NΣK*) couplings are of the *ps* type. In this case we have no simple theory as that of the scalar

coupling. But a similar approach is possible.⁵⁾ Namely we replace the Born term in (1) by an *effective scattering interaction* $V^-(\omega)$. A natural first approximation will be to put

$$V^\pm(\omega) = -\lambda/(m_N + m_Y \mp \omega) \quad (3)$$

where $\lambda = g_A^2 + g_Y^2$; g_Y is the ps coupling constant. In the next step we add the pion correction but do not include the contribution of the (time ordered) diagram of Fig. 1, for it is automatically included in the integral term of (1) or (2). Note that this diagram is the main contributor in the ordinary 4-th order perturbation. By a reasoning analogous to the case of the scalar coupling the signs of the $K^\pm - p$ interactions may be considered as determined by the signs of $V^\pm(\omega)$. We calculated $V^\pm(m_K)$ as Born term plus pion correction (irreducible diagram only) including relativistic effect. The result is shown in Table I. The pion correction



○ indicates an external kaon vertex.

Fig. 1.

Finally we note that the above argument does not depend on the $K^+ - K^0$ relative parity.

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S-Wave Interaction in π -N System and Dispersion Relation

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September 25, 1959

Many remarkable results have been brought by application of dispersion relation to the problems of pion reactions.^{1)~4)} One of them is that the value of $\alpha_1 - \alpha_3$ has been explained, where α_1 and α_3 are the s -wave phase shifts for the states of isotopic spin $I=1/2$ and $3/2$ respectively. But it seems to be difficult to explain the individual values of α_1 and α_3 . As is well known, this is due to the following situation. When the forward scattering amplitudes for $\pi^+ - p$ and $\pi^- - p$ scattering are denoted by $D_+(\omega) + iA_+(\omega)$ and $D_-(\omega) + iA_-(\omega)$ respectively, the dispersion relations are written down as follows,

Table 1

cut-off energy	$10^2 C^+$	$10^2 C^-$
$6 m_\pi$	-0.25	7.87
$8 m_\pi$	-1.03	10.26

$V^\pm(1) = -\lambda B^\pm + (g_A^2 + 5 g_Y^2) C^\pm$, $B^+ = 0.30$, $B^- = 0.19$. The unit is taken as $m_K = 1$. Pion coupling constants (ps) are all assumed as $f^2 = 15$.

is small for $K^+ - p$ but fairly large and attractive for $K^- - p$. Therefore it will be possible that this model gives correct signs both for the $K^\pm - p$ interactions.

$$\begin{aligned} \frac{1}{2}[D_-(\omega) - D_+(\omega)] \\ = -2f^2 \frac{(\mu^2/2M)^2 - \mu^2}{\omega^2 - (\mu^2/2M)^2} \frac{\omega}{\mu^2} + \frac{\omega}{4\pi^2} \\ \times P \int_{\mu}^{\infty} \frac{k'[\sigma_-(\omega') - \sigma_+(\omega')]}{\omega'^2 - \omega^2} d\omega', \quad (1) \end{aligned}$$

$$\begin{aligned} \frac{1}{2}[D_-(\omega) + D_+(\omega)] = F(\omega, \mu, M, f^2) \\ + G(k, \mu, M, f^2), \quad (2) \end{aligned}$$

$$\begin{aligned} F(\omega, \mu, M, f^2) = 2f^2 \frac{(\mu^2/2M)^2 - \mu^2}{\omega^2 - (\mu^2/2M)^2} \\ \times 1/2M, \quad (3) \end{aligned}$$

$$\begin{aligned} G(k, \mu, M, f^2) = \frac{1}{4\pi^2} \\ \times P \int_{\mu}^{\infty} \frac{\omega' k' [\sigma_-(\omega') + \sigma_+(\omega')]}{\omega'^2 - \omega^2} d\omega'. \quad (4) \end{aligned}$$

In these equations, k is the wave number of pion and ω its total energy in the laboratory system, μ and M are the masses of pion and nucleon respectively, and f is equal to $(\mu g/2M)$, where g is the renormalized pseudoscalar coupling constant of the symmetrical pseudoscalar theory. According to the experimental results, in the energy region ≥ 1.9 Bev it seems that $\sigma_-(\omega') \cong \sigma_+(\omega') = \text{const.}$ As the result, the value of $1/2[D_-(\mu) - D_+(\mu)]$ turns out to be finite, while that of $1/2[D_-(\mu) + D_+(\mu)]$ cannot be evaluated because $G(0, \mu, M, f^2)$ in (4) is a divergent quantity. Thus it is impossible to regard Eq. (2) as a correct one. If a significant physical quantity $G_r(k, \mu, M, f^2)$ can be derived from $G(k, \mu, M, f^2)$ by a suitable method, Eq. (2) should be rewritten as follows,

$$\begin{aligned} \frac{1}{2}[D_-(\omega) + D_+(\omega)] = F(\omega, \mu, M, f^2) \\ + G_r(k, \mu, M, f^2). \quad (2') \end{aligned}$$

At present, however, there is no clue to obtain the correct $G_r(k, \mu, M, f^2)$.

In this paper we intend to try to perform the subtraction so that $G_r(k, \mu, M, f^2)$ may satisfy a relation

$$G_r(0, \mu, M, f^2) = G_r(0, 0, M, f^2). \quad (5)$$

Our method can be interpreted in such a meaning that $G_r(0, \mu, M, f^2)$ is renormalized to $G_r(0, 0, M, f^2)$. Then

$$\begin{aligned} \frac{1}{2}[D_-(\omega) + D_+(\omega)] = F(\omega, \mu, M, f^2) \\ + \frac{1}{4\pi^2} P \int_{\mu}^{\infty} \left(\frac{1}{\omega'^2 - \omega^2} - \frac{1}{\omega'^2 - \mu^2} \right) \\ \times \omega' k' [\sigma_-(\omega') + \sigma_+(\omega')] d\omega' \\ + \beta(M, f^2). \quad (6) \end{aligned}$$

It must be noted that, in our case, β in Eq. (6) does not depend on μ because of the relation (5), though β should generally be some function of variables μ, M and f^2 .

An attempt to determine this unknown function $\beta(M, f^2)$ is made on the basis of some consideration about the limiting case of $\mu=0, \omega \rightarrow 0$. Deser, Thirring and Goldberger⁵⁾ have shown the following results at least in the limit of zero pion mass; (i) scattering of pions of zero kinetic energy can be expressed in terms of perturbation theory if only the renormalized coupling constant g_s is adopted, (ii) the forward scattering amplitude can be written down as

$$D_-(0) = D_+(0) = -g_s^2/M, \quad (7)^*$$

where g_s differs from the usual renormalized one. They have also shown

* From the "crossing symmetry" it follows that $D_-(0) = D_+(0)$.

that g_s^2 is connected to the unrenormalized one by the relation

$$g_s^2 \sim g_u^2 [1 + \partial(M^2)/\partial(\alpha'^2)]$$

and that the value of g_s^2 turns out to be very small. (As for the notation, see the reference 5) or 6).) But it seems to be difficult to obtain the correct value of g_s^2 because there is some ambiguity. (It is said that g_s^2 can indeed be negative.⁶⁾) We now introduce the assumption of $g_s^2 = f^2$ without proof.* Then

$$\beta(M, f^2) = G_r(0, 0, M, f^2) = -f^2/M. \quad (8)$$

In the limit as $k \rightarrow 0$,

$$\begin{aligned} D_+(\mu) &= (1/\mu)[1 + (\mu/M)]a_3, \\ D_-(\mu) &= (1/\mu)[1 + (\mu/M)] \\ &\quad \times \{(2/3)a_1 + (1/3)a_3\}, \end{aligned} \quad (9)$$

where $\alpha_1 = a_1\eta$, $\alpha_3 = a_3\eta$. (η is the pion momentum in the center of mass system in units of μ .)

Thus, from (6), (8) and (9), we obtain the following relation,

$$\begin{aligned} [1 + (\mu/M)](a_1 + 2a_3)/3 \\ = -2f^2(\mu/M). \end{aligned} \quad (10)$$

Putting the values of $\mu/M = 0.15$ and $f^2 = 0.08$ into (10),

$$a_1 + 2a_3 \simeq -0.06. \quad (11)$$

By using this together with the result $(a_1 - a_3) = 0.27$ which has been derived from (1)**,

* With regard to this assumption, some discussion will be done in near future.

** So far as the dispersion relation (1) is concerned, it is needless to perform any subtraction procedure by the following reason. Even if a term $H(\mu, M, f^2)$ is added to the right-hand side of (1), this $H(\mu, M, f^2)$ must be equal to zero on account of the relation $D_-(0) = D_+(0)$ based on the crossing symmetry.

$$a_1 = 0.16, \quad a_3 = -0.11. \quad (12)$$

These values agree very well with those which have been obtained from an analysis of low energy pion scattering data such as has been made by Orear.⁷⁾ Although our method may have no theoretical ground enough to be persisted as a unique one, it may be said that our method can be justified to some extent by the important results of (12).

The author should like to express his thanks to Prof. M. L. Goldberger and Prof. D. Ito for their helpful comments.

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Inclusion of Hole Motions in the Brueckner Theory

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September 25, 1959

The purpose of this note is to improve the Brueckner method¹⁾ so as to include the effect of the motion of holes.

In the current Brueckner theory we solve the two-body problem known as the Bethe-Goldstone equation²⁾:

$$(E - T_1 - T_2)\psi(r_1, r_2) = QV(r_1, r_2)\psi(r_1, r_2).$$

In this equation projection operator Q expresses the Pauli principle such that both particles propagate only outside the Fermi sea.

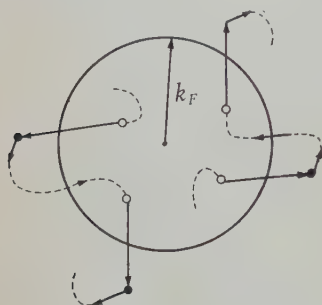


Fig. 1

However, it is possible that the holes happen to move inside the Fermi sea. Also there are such cases that many particles and holes are created, move outside or 'inside' the Fermi sea, and annihilate in a different combination, as is shown in Fig. 1. Fukuda³⁾ and Klein and Prange⁴⁾ once referred to this mechanism. But no one has ever succeeded in treating it in a simple manner. What we want to remark here is that all these processes can be automatically taken into account only if we replace Q by $Q - P$:

$$(E - T_1 - T_2)\psi = (Q - P)V\psi,$$

where the operator P projects inside the Fermi sea. Note $Q = 1 - P$ and $Q - P = 1 - 2P$.

The simplest derivation of this equation would be as follows. Drawing a

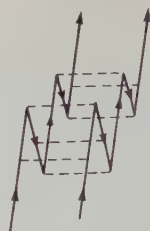


Fig. 2

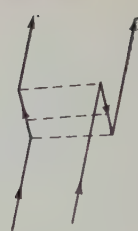


Fig. 3

Feynman diagram we find that our process is represented by two parallel zigzag lines.

In order to get simple formula we have to neglect processes such as represented in Fig. 3. Corresponding to Fig. 2 we can set up an integral equation:

$$\begin{aligned} \psi(r_1 r_2 t) &= \phi(r_1 r_2 t) \\ &- i \int K(r_1 r_2 t; r_1' r_2' t') V(r_1' r_2') \\ &\times \phi(r_1' r_2' t') dr_1' dr_2' dt', \end{aligned}$$

where the two-particle propagator K is simply a product of one-particle propagators to be found in Feynman's theory of positrons⁵⁾:

$$\begin{aligned} K(r_1 r_2 t; r_1' r_2' t') &= \sum_{n, m \gg k_F} \phi_n(r_1) \phi_m(r_2) \phi_n^*(r_1') \phi_m^*(r_2') \\ &\times e^{-i(E_n + E_m)(t - t')} \theta(t - t') \\ &+ \sum_{n, m \ll k_F} \phi_n(r_1) \phi_m(r_2) \phi_n^*(r_1') \phi_m^*(r_2') \\ &\times e^{-i(E_n + E_m)(t - t')} \theta(t' - t). \end{aligned}$$

θ is +1 or 0 according as its argument is positive or negative. Operating $(i\partial/\partial t - T_1 - T_2)$ on the integral equation we obtain immediately the desired equation.

One of the important points would be the interpretation of ψ , for in our case there is no healing distance, whose physical role was especially stres-

sed by Weisskopf and others.⁶⁾ Actually our ψ has somewhat different meaning from that of the B-G equation. In order to see it we must follow a more systematic method similar to Sawada's electron gas theory⁷⁾ or the B. C. S. theory⁸⁾ on superconductivity.

As is well known, fermion pairs

$$c_{K-k} c_{K-k} = \begin{cases} b_K(k), & K \pm k > k_F \\ a_K^*(k), & K \pm k < k_F \end{cases}$$

behave approximately as bosons⁹⁾, if the system is not so different from a degenerate Fermi gas distribution. The Hamiltonian of the system can be expressed as the quadratic form of these boson operators.

In order to transform it into a diagonal form we must solve the eigenvalue equation, which is found to coincide just with our $Q-P$ equation. Using the eigenfunction ψ_λ we can write the normal modes

$$\alpha_\lambda = \sum_K \psi_\lambda(k) (a(k) - b^*(k))$$

$$\beta_\lambda^* = \sum_K \psi_\lambda(k) (-a(k) + b^*(k)),$$

where suffix K is suppressed since the coupling between fermion pairs of different total momentum is neglected in our approximation. From the relation

$$\psi_\lambda^{N-2} = \alpha_\lambda^* \psi_0^N, \quad \psi_\lambda^{N+2} = \beta_\lambda^* \psi_0^N,$$

we can see the meaning of ψ_λ that it is the amplitude of physical hole pair or physical particle pair state compared with physical N particle ground state. Now it is no wonder that ψ has no healing distance, since there are also couplings between pairs on the same energy shell.

It should be noted that fermion pairs

actually satisfy spin commutators¹⁰⁾ rather than boson commutators. The difference would be important in connection with the possible occurrence of energy gap¹¹⁾.

Full accounts of this article will be published shortly.

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Two-Nucleon Problem and Dispersion Relation in Nucleon-Nucleon Scattering

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October 2, 1959

There is no satisfactory method for the two-nucleon problem. The potentials proposed by many authors are phenomenological, and the meson theore-

tical potential still contains energy dependent free parameters. Here we propose a method in which the dispersion relation with unitarity condition is used instead of the ordinary Schrödinger equation with potential. The dispersion relation in nucleon-nucleon scattering has been regarded as a useful tool to determine the two-nucleon scattering. The main obstacle for this effort has been the treatment of the integral over the unphysical region for which no observable quantity corresponds. In principle, however, we can get it by calculating Feynman diagrams. Recently, its important part was calculated.¹⁾ The dispersion relation with this absorptive part in the unphysical region was justified by substituting experimental data into it.¹⁾

Now, we attempt to solve the relation. The relation reads

$$D(\omega) = \frac{P}{\pi} \int_m^\infty \frac{A(\omega')}{\omega' - \omega} d\omega' + U(\omega)$$

(1)

with

$$U(\omega) = \frac{P}{\pi} \int_{-\infty}^m \frac{A(\omega')}{\omega' - \omega} d\omega'$$

which is known theoretically. The relation (1) without any condition has infinitely many solutions. We can suppress this ambiguity by imposing the unitarity condition on $D(\omega)$ and $A(\omega)$ when $\omega > m$. For the sake of simplicity, we do not consider inelastic processes such as pion production. Then, from the unitarity condition, D and A are written with real phase shifts δ_s, \dots . In order to determine δ 's, we must use the dispersion relations for fixed momentum transfer scattering amplitude,

$$D(\omega, \mathcal{A}) = \frac{P}{\pi} \int_{k_\Delta}^\infty \frac{A(\omega', \mathcal{A})}{\omega' - \omega} d\omega' + U(\omega, \mathcal{A}) \quad (2)$$

where

$$U(\omega, \mathcal{A}) = \frac{P}{\pi} \int_{-\infty}^{k_\Delta} \frac{A(\omega', \mathcal{A})}{\omega' - \omega} d\omega',$$

and \mathcal{A} is the transferred momentum between two nucleons,

as well as that for forward scattering amplitude. If we use both (1) and (2), the number of equations is equal to the number of the unknowns. If there are dispersion relations which need subtractions, we must give constants the number of which corresponds to the number of subtractions.

The equations to be solved can be rewritten as follows,

$$\begin{aligned} D_s(\omega, 0) + D_p(\omega, 0) + \dots \\ = \frac{P}{\pi} \int_m^\infty \frac{A_s(\omega', 0) + A_p(\omega', 0) + \dots}{\omega' - \omega} d\omega' \\ + U(\omega, 0), \end{aligned}$$

$$\frac{d}{d\mathcal{A}} (D_s(\omega, \mathcal{A}) + D_p(\omega, \mathcal{A}) + \dots) |_{\mathcal{A}=0} = \dots,$$

$$\frac{d^2}{d\mathcal{A}^2} (D_s(\omega, \mathcal{A}) + D_p(\omega, \mathcal{A}) + \dots) |_{\mathcal{A}=0} = \dots, \quad (3)$$

$$D_s(\omega) = (2\pi/q\omega) \sin \delta_s \cos \delta_s,$$

$$\dots \dots \dots$$

$$A_s(\omega) = (2\pi/q\omega) \sin^2 \delta_s$$

$$\dots \dots \dots$$

It is difficult to solve these coupled equations, and it is not known whether they have a unique solution or not. Here we do not solve (3) completely, but show that we can determine from

(3) the behavior of s -wave in low energies in good approximation when $D(m, 0) = (4\pi/m)a_s$ is given.

We use the dispersion relation for definite spin and isospin scattering amplitude, in which only even waves appear. The relation is

$$\lim_{\omega \rightarrow m} \frac{D(\omega) - D(m)}{\omega - m} = \frac{P}{\pi} \lim_{\omega \rightarrow m} \int_m^{\infty} \frac{A(\omega') d\omega'}{(\omega' - \omega)(\omega' - m)} + \left. \frac{dU(\omega)}{d\omega} \right|_{\omega=m} \quad (4)$$

The left side contains only scattering length a_s and effective range r_s . On the other hand, the first term of the right-hand depends on s -wave and all higher even waves. But we can neglect all contributions except that from low energy s -wave scattering in good approximation. The second term of the right-hand side has been calculated¹⁾ with $f^2/4\pi = 0.08$. Thus, the relation (4) is rewritten in terms of a_s and r_s .

$$(1) \quad S=0, T=1; a_s = -11$$

$$2\pi(a_s^3 - \frac{1}{2}a_s^2 r_s) + 2\pi \frac{a_s^4 - \frac{3}{2}a_s^3 r_s}{\sqrt{a_s^2 - 2a_s r_s}} = -7.1.$$

It has three solutions, but they are nearly the same.

$$r_s = 1.89.$$

$$(2) \quad S=1, T=0; a_s = 3.8$$

$$4\pi(a_s^3 - \frac{1}{2}a_s^2 r_s) = -8.8$$

$$r_s = 7.6.$$

We must not adopt this solution because this set of a_s and r_s gives a pole, so we must add a term corresponding

to a bound state, $4\pi/(1-\gamma r)\gamma^3$ ($\gamma = 1/a_s + (r/2)\gamma^2$) to the relation, then

$$r_s = 1.22.$$

In the same way and in the same approximation we can determine the d -state mixings of α -wave and of deuteron.

Recently, Noyes and Wong²⁾ discussed low energy s -wave scattering using a Chew-Low equation which was in close relation with our Eq. (4). Their equation contains only s -wave, and Eq. (4) becomes an equation for s -wave if higher wave are neglected, but this can be done safely. In this respect their method is superior to ours, but our method is superior to theirs in the following point; that is, our unphysical contribution is calculated exactly, and theirs is an approximate one. The author would like to thank Prof. Miyazawa for discussion.

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An Interpretation of the Peak in the Cross Section of $\pi^- + p \rightarrow K^0 + \Lambda$

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October 5, 1959

According to experiments¹⁾ the cross section of $\pi^- + p \rightarrow K^0 + \Lambda$ seems to show

a peak of about 0.8 mb at a pion bombarding energy slightly below 1 Bev. Previously Adair²⁾ pointed out that if the Σ - A relative parity is even, the S -wave A -production cross section should have an appreciable cusp at the Σ -production threshold ~ 0.9 Bev; hence it has been hoped that the behavior of the A -production cross section near the Σ -threshold offers a clue to the determination of the Σ - A relative parity. But the observed peak seems to take place at an energy somewhat higher than the Σ -threshold. Moreover, the angular distribution is not isotropic but the K^0 is emitted predominantly forward. Therefore the observed peak will not be the cusp shown by Adair.

In this note we present an interpretation of this peak suggested by the fact that the total π - p cross section of $I=1/2$ exhibits a resonance-like peak at about the same energy with that of the A -production. If our interpretation is proved to be correct we think it will give some clues about the mechanism of the relevant resonance of the $I=1/2$ π - p collision.

We assume, on the analogy of the nuclear reaction, that the π - p collision in the resonance region is divided into two steps, the formation of a compound or excited state and its subsequent decay into various channels, and that a cross section of any channel is written as the product of the cross section for the formation of the compound state and the probability that the compound state decays into the channel. Then it follows that the cross sections for all allowed channels exhibit peaks simultaneously at the resonance energy of the compound

state formation. Now, using the one-level formula, we write the cross section of the A -production as

$$\sigma(E) = \sigma(E_0) \cdot \left(\frac{p_0}{p} \right)^2 \frac{\Gamma^2}{4(E-E_0)^2 + \Gamma^2} \quad (1)$$

where E and p are respectively the kinetic energy and pion momentum in the center of mass system. The energy dependence of the total width Γ will be ignored for simplicity. $\sigma(E_0)$, the cross section for the A -production at resonance energy is written as

$$\sigma(E_0) = (2\pi/p_0^2) (2J+1) (\Gamma_A/\Gamma)$$

where J is the angular momentum of the compound state and Γ_A is the partial width. For the present we do not specify J and Γ_A but adjust $\sigma(E_0)$ as a whole to the experimental value.

Experimental analyses^{3),4)} of the $I=1/2$ π - p scattering shows $E_0 \sim 650$ Mev and $\Gamma \sim 100$ Mev. Taking $E_0 = 660$ Mev, $\Gamma = 100$ Mev and $\sigma(E_0) = 0.8$ mb, we plot $\sigma(E)$ in Fig. 1. Experimental details, though yet meager, are not in-

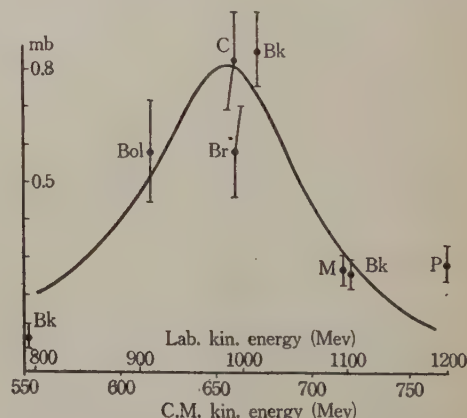


Fig. 1. Bk: Berkeley, Bol: Bologna, Br: Brookhaven, C: Columbia, M: Michigan, P: Pisa.

consistent with our model. Since an appreciable fraction of the cross section may be attributed to a nonresonant collision, a more refined analysis will be necessary in future. Especially the pure one level formula gives symmetric angular distribution about 90° , while the experimental one is not symmetric.

Similar behaviors of cross sections are expected in $\pi^- + p \rightarrow K + \Sigma$ and also in the photoproduction at the equivalent energy, though in the latter such a behavior may be masked by a photoelectric process. As for the $I=3/2$ π - p collision⁴⁾ a resonance-like peak occurs at about 1.3 BeV and it will be interesting to examine the behavior of the cross section of $\pi^+ + p \rightarrow K^+ + \Sigma^+$ near 1.3 BeV from our view-point.

As for the mechanism of the resonance in the $I=1/2$ π - p collision near 1 BeV it has been a important question whether the compound state is formed by the incident pion plus the nucleon core or by the incident pion plus the pion in the nucleon cloud. In our present interpretation the former case has been tacitly assumed. On the other hand a

model with a strong pion-pion interaction proposed by Dyson⁵⁾ and Takeda⁶⁾ corresponds to the latter case. However, if our Eq. (1) fits the experiment, we shall have an evidence against the pion-pion resonance model. The reason is as follows. In the center of mass system of the two pions the kinetic energy is rather small (~ 320 Mev) even at 1 BeV incident energy, hence strange particles, e.g., $K + \bar{K}$, will be scarcely created. Therefore, according to the pion-pion resonance model, a peak of common E_0 and Γ with those of the total π - p cross section is hardly expected in the cross section for the strange particles production.

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Errata

A Lower Limit on the π_0 Lifetime

Steven C. Frautschi

Prog. Theor. Phys. **22** (1959), 544.

- a) p. 546, second paragraph, line 7 should read

"The best choice may be $k \sim 100$ Mev"

instead of 10 Mev.

- b) p. 547, last line before acknowledgements should read

"the residue at $q^2 = m_\pi^2$ "

instead of $q^2 = m^2$.





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